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On Weak Maps of Ternary Matroids

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Let M and N be ternary matroids having the same rank and the same ground set, and assume that every independent set in N is also independent in M. The main result of this paper proves that if M is 3-connected and N is connected and non-binary, then $M = N$. A related result characterizes precisely when a matroid that is obtained by relaxing a circuit-hyperplane of a ternary matroid is also ternary.

1. INTRODUCTION

Let M and N be matroids on a common ground set E. The identity map on E is a *weak map* from M to N if every independent set in N is also independent in M. In this case, N is a *weak-map image* of M. If, moreover, M and N have the same rank, N is a *rank-preserving* weak-map image of M.

Weak maps are very general constructions, and it is not surprising that there are few strong results describing their behavior. A striking exception is Lucas's [4] characterization of weak maps of binary matroids. Amongst other things, he showed that if a connected matroid N is a rank-preserving weak-map image of a binary matroid M, then $M = N$. In this paper, we consider the analogous problem for ternary matroids. In particular, we prove the following:

THEOREM 1.1. *Let M and N be ternary matroids such that N is a rank-preserving weak-map image of M. If M is* 3*-connected, and N is connected and non-binary, then $M = N$.*

In Section 3, we prove Theorem 1.1 in the case when N is 3-connected. The more general case is proved in Section 4. Examples are given at the end of Section 4 to show that Theorem 1.1 is the best possible.

The research which led to Theorem 1.1 arose from our continued attempts to characterize the class of matroids representable over both $GF(3)$ and the rationals. At one stage it appeared that a result like Theorem 1.1 would assist in such a characterization. In fact, weak maps occur frequently in matroid representation problems, albeit often somewhat covertly. A type of weak map which is of particular interest in such problems is matroid relaxation. The matroid $M(E)$ is a *relaxation* of the matroid $N(E)$ if, for some circuit-hyperplane H of N, the set of bases of M is the set of bases of N together with H. In this case, we say that M is obtained by *relaxing* H. Evidently, if M is a relaxation of N, then N is a rank-preserving weak-map image of M. Relaxations abound in matroid representation problems, and in matroid structure theory in general.

In Section 5, we consider the problem of determining when a relaxation of a ternary matroid is ternary. It is an immediate corollary of Theorem 5.3 that if M and N are ternary matroids with N 3-connected, and M is a relaxation of N, then N is the cycle matroid of a wheel and M is the whirl obtained by relaxing the rim of N. A similar result holds when one drops the requirement that N is 3-connected. In that case, N must be a certain type of series-parallel extension of a wheel. We defer the precise statement of this result to Section 5.

2. PRELIMINARIES

For a good survey of the theory of weak maps see Kung and Nguyen [3]. Matroid terminology follows Oxley [5] with the following exceptions. If M is a matroid, we shall write

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$M(E)$ to indicate that E is the ground set of M. We will say that M is *freer* than N if N is a rank-preserving weak-map image of M. If, in addition, $N \neq M$, then M is *strictly* freer than N. Next we note some basic facts about weak maps which we use frequently.

2.1. Let $M(E)$ and $N(E)$ be matroids. The following are equivalent:

(a) N is a weak-map image of M.
(b) Every independent set in N is also independent in M.
(c) Every dependent set in M is also dependent in N.
(d) Every circuit of M contains a circuit of N.
(e) For every subset A of E, $r_M(A) \geq r_N(A)$.

2.2. If M is freer than N, then M^* is freer than N^*.

2.3. If $M(E)$ is freer than $N(E)$, and A is a subset of E for which $r(M|A) = r(N|A)$, then $M|A$ is freer than $N|A$.

We assume the reader is familiar with the theory of connectivity of matroids. For an exposition of this theory, see [5, Chapter 8]. We recall some facts which are of particular importance to this paper. The following result of Seymour [7] is central in the study of 3-connected non-binary matroids.

2.4. If $M(E)$ is a 3-connected non-binary matroid, and a and b are in E, then M has a $U_{2,4}$ minor using both a and b.

A 2-*separation* of a matroid $M(E)$ is a partition $\{X, E-X\}$ of E, where $|X|, |E-X| \geq 2$ and $r(X) + r(E-X) \leq r(M) + 1$. A connected matroid is not 3-connected if and only if it has a 2-separation. The 2-sum of matroids M_1 and M_2 is denoted $M_1 \oplus_2 M_2$ (for a definition of 2-sum, see [5, Section 7.1]). The following connections between 2-sums and 2-separations are fundamental; see [5, Sections 7.1 and 8.3] for proofs and citations. For sets X and Y, the notation $X \sqcup Y$ will refer to the set $X \cup Y$ and will also indicate that X and Y are disjoint.

2.5. If $M = M_1 \oplus_2 M_2$, then M is connected if and only if both M_1 and M_2 are connected.

2.6. A connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

2.7. Let $M(E)$ be a connected matroid and let $\{X, E-X\}$ be a partition of E with $|X|, |E-X| \geq 2$. Then $\{X, E-X\}$ is a 2-separation of M if and only if there are matroids $M_1(X \sqcup p)$ and $M_2((E-X) \sqcup p)$ such that $M = M_1 \oplus_2 M_2$.

Let n be a positive integer. Following Tutte [9, p. 78], we define the *wheel* $\mathcal{W}_n$ to be the graph that is formed from an n-edge cycle C_n by adding a single new vertex and then joining this new vertex to each vertex of C_n by a single new edge. These new edges are the *spokes* of $\mathcal{W}_n$, and the edge set of C_n is the *rim* of $\mathcal{W}_n$. The cycle matroid of the wheel $\mathcal{W}_n$ is also called a *wheel*. The rim C_n is a circuit-hyperplane of $M(\mathcal{W}_n)$, and the *whirl* $\mathcal{W}^n$ is obtained from $M(\mathcal{W}_n)$ by relaxing C_n, that is, by declaring C_n to be a basis and leaving the remaining bases the same. Note that $\mathcal{W}^2$ is the matroid $U_{2,4}$. The terms *rim* and *spoke* will be used in the obvious way in $\mathcal{W}^n$.

The following result is a straightforward consequence of Seymour's Splitter Theorem [6]. For a discussion of this theorem and its consequences see [5, Chapter 11].

2.8. Let $M(E)$ be a non-binary, 3-connected matroid. If M is not a whirl, there exists $x \in E$ such that either $M \setminus x$ or M/x is non-binary and 3-connected.

Finally, we note a link between connectivity and rank-preserving weak maps. The elementary proof of this result is omitted.

2.9. Let n be an integer exceeding one. Suppose that M is freer than N and N is n-connected. Then M is also n-connected.

3. THE 3-CONNECTED CASE OF THEOREM 1.1

In this section we consider the case of Theorem 1.1 which occurs when N is 3-connected. We state this special case as follows.

THEOREM 3.1. *Let M and N be ternary matroids with M freer than N. If N is 3-connected and non-binary, then $M = N$.*

We first establish some lemmas. Often matroids are represented as subsets of points of projective spaces. In this context, cl_P will always denote closure in the ambient projective space. The following lemma is closely related to results of Kahn [2, Section 3].

LEMMA 3.2. *Let M be a 3-connected, non-binary, spanning submatroid of $PG(r-1,3)$. Then, for any pair $\{a,b\}$ of distinct points of $PG(r-1,3)$, there is a hyperplane H of M such that $cl_P(H)$ contains a but not b.*

PROOF. Say M has ground set E, and let $\{a,b,c,d\}$ be the line L of $PG(r-1,3)$ spanned by $\{a,b\}$. Let $M' = PG(r-1,3)|(E \cup \{c,d\})$. Certainly M' is 3-connected, and it follows by (2.4) that M' has a $U_{2,4}$ minor using c and d. A straightforward consequence of this (see [2, Section 3] for identical arguments) is that there is a hyperplane H of M' such that $cl_P(H) \cap L = a$. Now H does not contain either c or d, so H is a hyperplane of M. But H does not contain b and the lemma is proved. □

LEMMA 3.3. *Let M and N be ternary matroids on the ground set $E \sqcup x$. Let N be 3-connected and let M be freer than N. Assume that $M \setminus x = N \setminus x$, and that this matroid is 3-connected and non-binary. Then $M = N$.*

PROOF. Without loss of generality, we may assume that there is a spanning subset E of $PG(r-1,3)$ such that $M \setminus x = PG(r-1,3)|E$. It is well known that, since $M \setminus x$ is connected, there are points x_1 and x_2 of $PG(r-1,3)$ such that the maps f_1 and f_2 that fix all the elements of E and take x to x_1 and x_2, respectively, are isomorphisms between M and $PG(r-1,3)|(E \cup x_1)$ and between N and $PG(r-1,3)|(E \cup x_2)$. (See, for example, [5, Section 10.3].) Assume that x_1 and x_2 are distinct. By Lemma 3.2, there is a hyperplane H of $M \setminus x$ such that $cl_P(H)$ contains x_1 but not x_2. A routine consequence of this is that there is a subset I of H such that $I \cup x_1$ is a circuit of $PG(r-1,3)$. Now $x_2 \notin cl_P(H)$, so $x_2 \notin cl_P(I)$. Hence $I \cup x_2$ is independent in $PG(r-1,3)$. But $E \cup x_1$ represents M and $E \cup x_2$ represents N, so $I \cup x$ is a circuit in M and independent in N. This contradicts the fact that M is freer than N. Therefore $x_1 = x_2$, that is, $M = N$. □

Throughout the rest of this paper we adopt the convention that the elements of $GF(3)$ are written as $\{0, +1, -1\}$. We refer to $+1$ and -1 as the positive and negative elements, respectively, of $GF(3)$.

LEMMA 3.4. *If the matroid M is freer than the rank-r whirl $\mathcal{W}^r$, then $M = \mathcal{W}^r$.*

PROOF. Let B denote the basis of M which forms the rim of $\mathcal{W}^r$. Construct a representation $[I_r|A]$ of $\mathcal{W}^r$ over $GF(3)$, where the columns of I_r correspond to the elements of B, and $A = [a_{ij}]$. It is easily seen that every entry of A is non-zero, and that we may take A to have all the entries on or below the main diagonal being positive, and all the entries above the main diagonal being negative.

Now assume that M is representable over $GF(3)$. Say $[I_r|A']$ represents M over $GF(3)$, where corresponding columns of $[I_r|A]$ and $[I_r|A']$ label the same elements of the common ground set of M and $\mathcal{W}^r$, and $A' = [a'_{ij}]$. It follows from a result of Lucas [4, Proposition 6.7], that if a subdeterminant of A is non-zero, then the corresponding subdeterminant of A' is non-zero. Thus every entry in A' is non-zero, and we may assume that A' agrees with A in row 1 and column 1. Thus the lemma holds if $r = 1$. For $r \geq 2$, consider the subdeterminant of A corresponding to the submatrix

$$\begin{bmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{bmatrix},$$

where $i \geq j$. This subdeterminant is non-zero, so the corresponding subdeterminant of A' is non-zero. Hence $a'_{ij} = 1$. Thus all the entries in A' on or below the main diagonal are positive.

Next consider the determinant of the following submatrix of A:

$$\begin{bmatrix} a_{i1} & a_{ij} \\ a_{r1} & a_{rj} \end{bmatrix},$$

where $j > i$. This subdeterminant is non-zero, so $a'_{ij} = -1$. Thus all the entries of A' above the main diagonal are negative. It follows that $A' = A$, and hence $M = \mathcal{W}^r$. □

PROOF OF THEOREM 3.1. Let $M(E)$ be freer than $N(E)$ where N is 3-connected. The result holds trivially (or by Lemma 3.4) when N is $U_{2,4}$, the unique 3-connected, ternary, non-binary matroid on a ground set of minimum size. Assume that the result holds for any pair of matroids satisfying the conditions of the theorem whose common ground set has cardinality less than $|E|$. If N is a whirl, the result follows from Lemma 3.4. Otherwise, by (2.8), there exists an element x in E such that either $N \setminus x$ or N/x is 3-connected, ternary and non-binary. By (2.2), we may assume the former. Now $r(M \setminus x) = r(N \setminus x)$, so, by (2.3), $M \setminus x$ is freer than $N \setminus x$. Hence, by the induction assumption, $M \setminus x = N \setminus x$. It now follows by Lemma 3.3 that $M = N$. □

4. THE 2-CONNECTED CASE OF THEOREM 1.1

In this section we complete the proof of Theorem 1.1. For convenience, we restate the theorem.

THEOREM 4.1. *Let M and N be ternary matroids with M freer than N. If M is 3-connected, and N is connected and non-binary, then $M = N$.*

This section is structured as follows. Say N is a connected, ternary, non-binary matroid, and M is strictly freer than N. Lemma 4.2 establishes a certain case for which M cannot be ternary. This is used to establish a more general case in Lemma 4.5. The proof of Theorem 4.1 is an application of Lemma 4.5. Much of the argument in this section is devoted to establishing certain properties of 2-sums and 2-separations of matroids. These properties are intuitively not surprising, and we initially felt that they would be well known. However, we could not find them in the literature.

We first develop some terminology associated with 2-separations. Assume that M has a 2-separation $\{S_1, S_2\}$. Then there are matroids $M_1(S_1 \sqcup p)$ and $M_2(S_2 \sqcup p)$ such that $M = M_1 \oplus_2 M_2$. Say $i \in \{1, 2\}$. We call S_i a *binary* or *non-binary part* of M depending on whether M_i is binary or non-binary. Note that $M|S_i$ may be binary even when S_i is a non-binary part of M. A non-binary part S_1 of M is a *minimal* non-binary part of M if every 2-separation $\{T_1, T_2\}$ of M for which T_1 is a proper subset of S_1 has the property that T_1 is a binary part of M.

LEMMA 4.2. *Suppose that M and N are matroids on E such that M is freer than N, and the latter is ternary, non-binary, and connected. Let $\{a, b\}$ be a circuit of N that is independent in M and assume that $E - \{a, b\}$ is a minimal non-binary part of N. Then M is not ternary.*

Before proving Lemma 4.2, we establish some subsidiary lemmas.

LEMMA 4.3. *Let* $\{X, E - X\}$ *and* $\{Y, E - Y\}$ *be* 2*-separations of a connected matroid* M*, and suppose that* $Y \subseteq X$*. Then there are connected matroids* M_1*,* M_2*,* M_3*, and* M_4 *such that* $M = M_1 \oplus_2 M_2 = M_3 \oplus_2 M_4$ *where the basepoints of both 2-sums are labelled by* p*, the ground sets of* M_1 *and* M_3 *are* $X \sqcup p$ *and* $Y \sqcup p$*, and* M_3 *is a minor of* M_1*.*

PROOF. The existence of connected matroids M_1, M_2, M_3, and M_4 so that $E(M_1) = X \sqcup p$, $E(M_3) = Y \sqcup p$, and $M = M_1 \oplus_2 M_2 = M_3 \oplus_2 M_4$ follows from (2.7) and (2.5). To see that M_3 is a minor of M_1, one modifies an argument of Seymour [6, (2.6)], that is reproduced in [5, Proposition 7.1.19]. Let x be an element of $X \cap Y$, and z be an element of $E - X$. Since M is connected, it has a circuit C that contains both x and y. Then, as shown in the two cited sources, M_1 is the matroid $M \setminus (E - X - C)/[(C - X) - z]$ with z renamed as p. Similarly, M_3 is $M \setminus (E - Y - C)/[(C - Y) - z]$ with z renamed as p. Hence M_3 is a minor of M_1. □

LEMMA 4.4. *Let* $\{X_1, Y_1\}$ *and* $\{X_2, Y_2\}$ *be* 2*-separations of a connected matroid* M*. If both* $X_1 \cap X_2$ *and* $Y_1 \cap Y_2$ *are non-empty, then*

$$r(X_1 \cap X_2) + r(Y_1 \cup Y_2) = r(M) + 1$$

and

$$r(Y_1 \cap Y_2) + r(X_1 \cup X_2) = r(M) + 1.$$

Moreover, $\{X_1 \cap X_2, Y_1 \cup Y_2\}$ *is a* 2*-separation of* M *provided that* $|X_1 \cap X_2| \geq 2$*.*

PROOF. We have

$$r(X_1) + r(Y_1) = r(M) + 1$$

and

$$r(X_2) + r(Y_2) = r(M) + 1.$$

Adding these equations and using semimodularity, we obtain

$$r(X_1 \cap X_2) + r(X_1 \cup X_2) + r(Y_1 \cap Y_2) + r(Y_1 \cup Y_2) \leq 2(r(M) + 1).$$

Hence, on regrouping terms, we obtain

$$[r(X_1 \cap X_2) + r(Y_1 \cup Y_2)] + [r(Y_1 \cap Y_2) + r(X_1 \cup X_2)] \leq 2(r(M) + 1). \tag{1}$$

Both $\{X_1 \cap X_2, Y_1 \cup Y_2\}$ and $\{Y_1 \cap Y_2, X_1 \cup X_2\}$ partition $E(M)$ and, since $X_1 \cap X_2$ and $Y_1 \cap Y_2$ are non-empty,

$$r(X_1 \cap X_2) + r(Y_1 \cup Y_2) \geq r(M) + 1 \tag{2}$$

and

$$r(Y_1 \cap Y_2) + r(X_1 \cup X_2) \geq r(M) + 1. \tag{3}$$

On combining (1)–(3), we deduce that equality holds in all three. Moreover, if $|X_1 \cap X_2| \geq 2$, then $\{X_1 \cap X_2, Y_1 \cup Y_2\}$ is a 2-separation of M. □

PROOF OF LEMMA 4.2. Assume that the lemma fails and take a pair of matroids M and N satisfying the hypotheses for which M is ternary and $|E|$ is as small as possible. We show first that

4.2.1. $N \setminus a$ is not 3-connected.

Assume the contrary and suppose also that $N \setminus a, b \neq M \setminus a, b$. Then $M \setminus a$ is strictly freer than $N \setminus a$ and the latter is 3-connected, ternary, and non-binary. Thus, by Theorem 3.1, $M \setminus a$, and hence M, is not ternary; a contradiction. Therefore we may assume that $N \setminus a, b = M \setminus a, b$. Now M is 3-connected since M is freer than N and does not have $\{a, b\}$ as a circuit. Thus M can be represented as a submatroid of $PG(r-1, 3)$. By Lemma 3.2, $M \setminus a$ has a hyperplane whose closure in $PG(r-1, 3)$ contains a but not b. Thus $M \setminus a$ has an independent set I such that $I \cup a$ is a circuit of M and $I \cup b$ is independent in M. Now M is freer than N, so $I \cup a$ is dependent in N. Therefore, since $\{a, b\}$ is a circuit of N, the set $I \cup b$ is dependent in N. Hence $I \cup b$ is dependent in $N \setminus a$ and independent in $M \setminus a$. Thus, as $N \setminus a$ is 3-connected, Theorem 3.1 implies that $M \setminus a$ is not ternary; a contradiction. We conclude that (4.2.1) holds.

We show next that

4.2.2. $N \setminus a$ is simple.

If $N \setminus a$ has a 2-circuit $\{x, y\}$ where $x \neq b$, then, as $N \setminus x$ has fewer elements than N, it follows without difficulty that $M \setminus x$ is non-ternary; a contradiction. Thus (4.2.2) holds.

Next we prove the following:

4.2.3. If $\{T_1, T_2\}$ is a 2-separation of N such that T_2 is a non-binary part and $\{a, b\} \subseteq T_2$, then $a \notin cl_N(T_1)$.

Now N is the 2-sum of two matroids N_1 and N_2 having ground sets $T_1 \sqcup p$ and $T_2 \sqcup p$, respectively. Suppose that $a \in cl_N(T_1)$. Then a and b are parallel to p in N_2, so $\{T_1 \cup \{a, b\}, T_2 - \{a, b\}\}$ is a 2-separation of N having $T_2 - \{a, b\}$ as a non-binary part. This contradicts the choice of $E - \{a, b\}$. Hence (4.2.3) holds.

We know already that N has no 2-circuits other than $\{a, b\}$. We now show that

4.2.4. N has no 2-cocircuits.

Suppose that $\{u, v\}$ is a cocircuit of N. Then $\{\{u, v\}, E - \{u, v\}\}$ is a 2-separation of N, and $\{a, b\} \subseteq E - \{u, v\}$. By (4.2.3), $a \notin cl_N(\{u, v\})$ so $\{u, v, a\}$ is independent in N and hence in M. Thus, as $\{a, b\}$ is independent in M, one of $\{a, b, u\}$ and $\{a, b, v\}$ is independent in M. Without loss of generality, assume the former and let $N' = N/u$ and $M' = M/u$. Evidently $\{a, b\}$ is a circuit of N' and an independent set of M'. Moreover, if $\{T_1, T_2\}$ is a 2-separation of N' such that T_1 is a non-binary part and T_2 properly contains $\{a, b\}$, then one easily checks that $\{T_i \cup u, T_j\}$ is a 2-separation of N where $\{i, j\} = \{1, 2\}$ and $v \in T_i$. This contradicts the choice of $E - \{a, b\}$. Hence $E(N') - \{a, b\}$ is a minimal non-binary part of N' and it follows by the choice of the pair (M, N) that M' is non-ternary. This contradiction to the fact that M is ternary implies that (4.2.4) holds.

By (4.2.1), $N \setminus a$ is not 3-connected. Hence $N \setminus a$ has a 2-separation $\{S_1, S_2\}$ where $b \in S_2$. Thus $\{S_1, S_2 \cup a\}$ is a 2-separation of N which, by the choice of $E - \{a, b\}$, has S_1 as a binary part and $S_2 \cup a$ as a non-binary part. Therefore N is the 2-sum of two matroids N_1 and N_2 having ground sets $S_1 \sqcup p$ and $S_2 \sqcup a \sqcup p$, respectively. Since N has no 2-cocircuits and no 2-circuits other than $\{a, b\}$, it follows that

4.2.5. $|S_1| \geq 3$.

We show next that

4.2.6. S_1 contains an element x for which $N \setminus x$ is connected.

Suppose that S_1 does not contain such an element. If S_1 contains a circuit of N, then, by [5, Lemma 10.2.1], S_1 contains a 2-cocircuit of N. But N has no 2-cocircuits. Thus S_1 contains no circuits of N. Therefore N_1 is a circuit, so S_1 is contained in a series class of N, and, again, S_1 contains a 2-cocircuit of N. We conclude that (4.2.6) holds.

Now let $N' = N \backslash x$ and $M' = M \backslash x$. Then N' is ternary, non-binary, and connected, and M' is freer than N'. Moreover, $\{a, b\}$ is a circuit of N' and an independent set of M'. Indeed, $E - \{a, b, x\}$ is a non-binary part of N'. If $E - \{a, b, x\}$ is a minimal non-binary part of N', then, by the choice of (M, N), it follows that M' is non-ternary. This implies the contradiction that M is non-ternary. Thus N' has a 2-separation $\{T_1, T_2\}$ where T_1 is a non-binary part of N', and T_2 properly contains $\{a, b\}$. Thus

$$r(T_1) + r(T_2) = r(N') + 1. \tag{4}$$

Moreover, as neither $T_1 \cup x$ nor T_1 is a non-binary part of N,

$$r(T_1 \cup x) = r(T_1) + 1 \tag{5}$$

and

$$r(T_2 \cup x) = r(T_2) + 1. \tag{6}$$

Since $r(S_1) + r(S_2 \cup a) = r(N) + 1$ and N' is connected,

$$r(S_1 - x) + r(S_2 \cup a) = r(N') + 1 \tag{7}$$

and

$$r(S_1 - x) = r(S_1). \tag{8}$$

Thus, by (4) and (4.2.5), $\{S_1 - x, S_2 \cup a\}$ is a 2-separation of N' having $S_1 - x$ as a binary part and $S_2 \cup a$ as a non-binary part. By (2), (3) and (5), neither T_1 nor T_2 contains $S_1 - x$. Hence

$$T_2 \cap (S_1 - x) \neq \emptyset \tag{9}$$

and

$$T_1 \cap (S_1 - x) \neq \emptyset. \tag{10}$$

Moreover, T_2 contains $\{a, b\}$ so

$$T_2 \cap (S_2 \cup a) \neq \emptyset. \tag{11}$$

Finally,

$$T_1 \cap (S_2 \cup a) \neq \emptyset \tag{12}$$

otherwise $T_1 \subseteq S_1 - x$ which contradicts Lemma 4.3 since T_1 is a non-binary part of N', and $S_1 - x$ is a binary part of N'.

Statements (6)–(9) enable us to apply Lemma 4.4 twice to the 2-separations $\{T_1, T_2\}$ and $\{S_1 - x, S_2 \cup a\}$ of N'. This yields four equations, two of which are

$$r(T_2 \cap (S_2 \cup a)) + r(T_1 \cup (S_1 - x)) = r(N') + 1 \tag{13}$$

and

$$r(T_1 \cap (S_2 \cup a)) + r(T_2 \cup (S_1 - x)) = r(N') + 1. \tag{14}$$

As $\{T_2 \cap (S_2 \cup a), T_1 \cup (S_1 - x)\}$ is a partition of $E(N')$ and $|T_2 \cap (S_2 \cup a)| \geq 2$, this partition is a 2-separation of N'. By (5), it follows that $\{T_2 \cap (S_2 \cup a), T_1 \cup S_1\}$ is a 2-separation of N. Since T_1 is a non-binary part of N', Lemma 4.3 implies that $T_1 \cup S_1$ is a non-binary part of N. Since $T_2 \cap (S_2 \cup a) \supseteq \{a, b\}$, the choice of $E - \{a, b\}$ means that equality must hold here. Thus $T_1 \cap (S_2 \cup a) = S_2 - b$ and $T_2 \cup (S_1 - x) = (S_1 - x) \cup \{a, b\}$. As $S_2 \cup a$ is a non-binary part of N, $|S_2 - b| \geq 2$. Hence, by (5) and (11), $\{S_2 - b, S_1 \cup \{a, b\}\}$ is a partition of $E(N)$ that is a 2-separation of N.

We conclude that both $\{S_2 - b, S_1 \cup b\}$ and $\{S_2, S_1\}$ are 2-separations of $N \backslash a$. Moreover, S_2 is a non-binary part of $N \backslash a$ and, by the choice of $E - \{a, b\}$, the set $S_2 - b$ is a binary part of N and hence of $N \backslash a$.

Now, by (4.2.3), $a \notin cl_N(S_1)$, so $b \notin cl_N(S_1)$. Hence $r(S_1 \cup b) = r(S_1) + 1$, so

$$r(S_2 - b) = r(S_2) - 1. \tag{15}$$

By Lemma 4.3, there are connected matroids M_1, M_2, M_3, and M_4 such that $N \setminus a = M_1 \oplus_2 M_2 = M_3 \oplus_2 M_4$ where M_1 and M_3 have ground sets $S_2 \sqcup p$ and $(S_2 - b) \sqcup p$, and M_3 is a minor of M_1. From above, M_1 is non-binary and M_3 is binary. Since $M_1 \setminus p = (N \setminus a)|S_2$ and $M_3 \setminus p = (N \setminus a)|(S_2 - b)$, and both M_1 and M_3 are connected, $r(M_1) = r(M_1 \setminus p) = r(S_2)$ and $r(M_3) = r(M_3 \setminus p) = r(S_2 - b)$. Thus, by (12), $r(M_1) = r(M_3) + 1$. But

$$M_1 \setminus p, b = [(N \setminus a)|S_2] \setminus b = (N \setminus a)|(S_2 - b) = M_3 \setminus p.$$

Hence $\{p, b\}$ is a cocircuit of M_1. Since M_3 is a minor of M_1 and $E(M_1) - E(M_3) = \{b\}$, it follows that $M_3 = M_1/b$. As M_1 is non-binary, M_3 is too. This contradiction completes the proof of Lemma 4.2. □

Lemma 4.2 establishes the base case for the inductive argument which proves the following lemma.

LEMMA 4.5. *Let N be a ternary, non-binary, connected matroid with a 2-separation $\{S_1, S_2\}$ for which S_1 is a minimal non-binary part. Let M be freer than N, and assume that $r_N(S_2) < r_M(S_2)$. Then M is not ternary.*

Again we first prove some subsidiary lemmas.

LEMMA 4.6. *Let S_1 be a minimal non-binary part of the 2-separation $\{S_1, S_2\}$ of the connected matroid N. Let N' be a connected minor of the form $N \setminus x$ or N/x for some x in S_2. Then $\{S_1, S_2 - x\}$ is a 2-separation of N', and S_1 is a minimal non-binary part of this 2-separation.*

PROOF. Assume that $N' = N \setminus x$. Since N is connected, $r(N') = r(N)$, and, since N' is connected, $r_{N'}(S_1) + r_{N'}(S_2 - x) > r(N')$. It follows that $\{S_1, S_2 - x\}$ is a 2-separation of N'. Evidently, S_1 is a non-binary part of this 2-separation. It also follows that $r_N(S_2) = r_{N'}(S_2 - x)$. Now assume that T_1 is a non-binary part of the 2-separation $\{T_1, T_2\}$ of N', where $T_1 \subseteq S_1$. Then $T_2 \supseteq S_2 - x$, so $r_{N'}(T_2) = r_N(T_2 \cup x)$. Hence $\{T_1, T_2 \cup x\}$ is a 2-separation of N having T_1 as a non-binary part. Thus $T_1 = S_1$ and so S_1 is a minimal non-binary part of N'.

If $N' = N/x$, then a straightforward dualization of the above argument establishes the lemma. □

LEMMA 4.7. *Let $N'(E \sqcup p)$ be a connected matroid and $M(E)$ be a matroid. Let $N = N' \setminus p$. Assume that $|E| \geq 3$, that $r(N) < r(M)$, and that N is a weak-map image of M. Then there exists x in E such that either $N' \setminus x$ is connected and $r(N \setminus x) < r(M \setminus x)$, or N'/x is connected and $r(N/x) < r(M/x)$.*

PROOF. If $r(M) > r(N) + 1$, the result is clear. So assume that $r(M) = r(N) + 1$. Assume that M is free. Then N' is a connected matroid whose rank is 2 less than the cardinality of its ground set. Hence $(N')^*$ has rank 2. A routine argument shows that, since $|E \cup p| \geq 4$, there exists $x \in E$ such that $(N')^* \setminus x$ is connected, that is, N'/x is connected. Clearly $r(N/x) < r(M/x)$.

Assume then that M is not free. Let x belong to a circuit of M. Then $r(M \setminus x) = r(M)$, and $r(N \setminus x) \leq r(N)$. Hence $r(N \setminus x) < r(M \setminus x)$. Since x is not a loop of N, we must have that $r(N/x) < r(M/x)$. The result follows when it is observed that either $N' \setminus x$ or N'/x is connected. □

PROOF OF LEMMA 4.5. The proof is by induction on the cardinality of S_2. Assume that $|S_2| = 2$, say $S_2 = \{a, b\}$. Then $\{a, b\}$ is a circuit in N and an independent set in M, and it follows from Lemma 4.2 that M is not ternary.

Now suppose that $|S_2| > 2$. Assume that the lemma holds for all pairs of matroids N, M satisfying the conditions of the lemma and having $|S_2| < n$, and let N, M be such a pair with $|S_2| = n$. Since $\{S_1, S_2\}$ is a 2-separation of N, this matroid is the 2-sum of matroids $N_1(S_1 \sqcup p)$ and $N_2(S_2 \sqcup p)$. Now $N_2 \setminus p = N|S_2$, and $N|S_2$ is a weak-map image of $M|S_2$ with $r(N|S_2) < r(M|S_2)$. By Lemma 4.7, there exists $x \in S_2$ such that either (i) $N_2 \setminus x$ is connected and $r((N|S_2) \setminus x) < r((M|S_2) \setminus x)$, or (ii) N_2/x is connected and $r((N|S_2)/x) < r((M|S_2)/x)$. Choose such an x. In case (i), we let $M' = M \setminus x$ and $N' = N \setminus x$, so $N' = N_1 \oplus_2 (N_2 \setminus x)$. In case (ii), we let $M' = M/x$ and $N' = N/x$, so $N' = N_1 \oplus_2 (N_2/x)$. In both cases, N' is the 2-sum of connected matroids, so N' is connected. Moreover, by Lemma 4.6, S_1 is a minimal non-binary part of the 2-separation $\{S_1, S_2 - x\}$ of N'. Finally, $r_{N'}(S_2 - x) < r_{M'}(S_2 - x)$. It now follows by the induction assumption that M', and hence M, is not ternary. □

LEMMA 4.8. *Let $M(E)$ be freer than $N(E)$, and let A be a subset of E. Then $r_M(A) = r_N(A)$ if and only if $r_{M^*}(E - A) = r_{N^*}(E - A)$.*

PROOF. Recall that, for any subset X of E,

$$r_{M^*}(X) = |X| + r_M(E - X) - r_M(E).$$

It follows that $r_{M^*}(E - A) = r_{N^*}(E - A)$ if and only if

$$|E - A| + r_M(A) - r_M(E) = |E - A| + r_N(A) - r_N(E).$$

Clearly the last equation holds if and only if $r_M(A) = r_N(A)$. □

PROOF OF THEOREM 4.1. Assume that the ternary matroid M is freer than N where N is ternary, non-binary, and connected. Assume that N is not 3-connected. Then there is a 2-separation $\{S_1, S_2\}$ of N for which S_1 is a minimal non-binary part. Now M is 3-connected, so $\{S_1, S_2\}$ is not a 2-separation of M. Hence either $r_N(S_1) < r_M(S_1)$ or $r_N(S_2) < r_M(S_2)$. It then follows from Lemma 4.8 that either $r_N(S_2) < r_M(S_2)$ or $r_{N^*}(S_2) < r_{M^*}(S_2)$. But it is evident that $\{S_1, S_2\}$ is a 2-separation of N^* for which S_1 is a minimal non-binary part. All other properties of M and N relevant to the conditions of the theorem are also preserved under duality. It follows that we may assume, without loss of generality, that $r_N(S_2) < r_M(S_2)$. But then, by Lemma 4.5, M is not ternary; a contradiction. Hence N is 3-connected. It then follows by Theorem 3.1 that $M = N$. □

The following examples show that Theorem 4.1 is best possible. For full definitions of the matroids referred to see [5, Appendix].

A rank-preserving weak-map image of a 3-connected ternary matroid need not be ternary. For example, the Fano matroid F_7 is non-ternary, and is a rank-preserving weak-map image of the non-Fano matroid F_7^-, a ternary matroid. Indeed, a rank-preserving weak-map image of a 3-connected ternary matroid need not be representable over any field. The matroid $AG(3, 2)'$, which is obtained from the binary affine cube $AG(3, 2)$ by relaxing a circuit-hyperplane, is non-representable. But $AG(3, 2)'$ is a rank-preserving weak-map image of the real affine cube R_8, a ternary matroid. These examples show that the condition that N be ternary cannot be dropped from Theorem 4.1.

We now consider weak-map images of ternary matroids that are connected but not 3-connected. The matroid R_6 is the rank-3 matroid consisting of two disjoint 3-point lines, that is, $R_6 = U_{2,4} \oplus_2 U_{2,4}$. Let N denote the matroid that is obtained from R_6 by declaring two points on one of the 3-point lines to be parallel. Then N is a non-trivial rank-preserving weak-map image of R_6, and it is evident that both these matroids are connected and non-binary. This shows that the condition that M be 3-connected cannot be dropped from Theorem 4.1.

Also, a rank-preserving weak-map image of a 3-connected ternary matroid may be both ternary and binary, that is, it may be regular, so that the condition that N be non-binary cannot be dropped from Theorem 4.1. Examples of this abound. The following section is devoted to showing that, for the special case of relaxations, such examples can occur in only limited ways.

5. RELAXATIONS OF TERNARY MATROIDS

In this section, we characterize precisely when a matroid that is obtained by relaxing a circuit-hyperplane of a ternary matroid is also ternary.

A $\{0, 1\}$-matrix $[a_{ij}]$ is a *solid staircase matrix* if it has the property that, whenever $a_{ij} = 1$, all entries $a_{i'j'}$ with $i' \geq i$ and $j' \leq j$ are also 1. The *number of stairs* in such a matrix A is zero if A is empty or zero, and otherwise equals the number of non-zero rows of A that differ from their immediate successors, where we always view the last row as differing from its successor.

The following lemma has a straightforward inductive proof (see, for example, Ding [1] or Truemper [8, p. 304]).

LEMMA 5.1. *Let A be a $\{0, 1\}$-matrix. Then A has neither $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ nor $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ as a submatrix if and only if a solid staircase matrix can be obtained from A by permuting rows and permuting columns.*

A graph G is an *enlarged k-wheel* if G can be obtained from a wheel with k spokes by the following operations:

(i) subdividing some set of rim edges, thereby forming the *rim* of the enlarged k-wheel; and
(ii) adding edges in parallel with some set of spokes.

It is straightforward to show that a relaxation of a binary matroid M is binary if and only if $M \cong U_{k-1,k} \oplus U_{1,m}$ for some positive integers k and m. For comparison with the corresponding result for ternary matroids, we restate this as follows.

PROPOSITION 5.2. *Let M be a binary matroid, H be a circuit-hyperplane of M, and M' be obtained from M by relaxing H. Then M' is binary if and only if there is an enlarged 1-wheel G having rim H such that $M = M(G)$.*

The next theorem is the main result of this section. Its proof, which is much more difficult than that of Proposition 5.2, occupies the rest of the section.

THEOREM 5.3. *Let M be a ternary matroid, H be a circuit-hyperplane of M, and M' be obtained from M by relaxing H. Then M' is ternary if and only if, for some $k \geq 1$, there is an enlarged k-wheel G having rim H such that $M = M(G)$.*

The matrix $A(\alpha, D)$, which appears throughout the proof of the theorem, is given by

$$A(\alpha, D) = \begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & \cdots & x_r & y_1 & y_2 & y_3 & \cdots & y_{r^*} \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & \cdots & 0 & \alpha & 1 & 1 & \cdots & 1 \\ 0 & & & & & 1 & & & & \\ 0 & & & & & 1 & & & & \\ \vdots & & & I_{r-1} & & \vdots & & & D & \\ 0 & & & & & 1 & & & & \end{array}\right] \end{array}.$$

LEMMA 5.4. *A matroid N is the cycle matroid of an enlarged k-wheel G with rim H if and only if there is a solid staircase matrix D with $k-1$ stairs such that $A(0, D)$ is a $GF(3)$-representation for N with $\{x_2, x_3, \ldots, x_r, y_1\} = H$. Moreover, if D is a solid staircase matrix with $k-1$ stairs and $H = \{x_2, x_3, \ldots, x_r, y_1\}$, then $A(-1, D)$ is a $GF(3)$-representation for the matroid that is obtained from $M[A(0, D)]$ by relaxing the circuit-hyperplane H.*

PROOF. Truemper [8, p. 304] noted that $M(\mathcal{W}_k)$ is represented over $GF(2)$ by $A(0, D_k)$ where $r = r^* = k$ and D_k is the $(k-1) \times (k-1)$ matrix having ones on or below the main diagonal and zeros elsewhere. Moreover, $\{x_2, x_3, \dots, x_k, y_1\}$ corresponds to the rim R of $\mathcal{W}_k$. It is straightforward to show that $A(0, D_k)$ also represents $M(\mathcal{W}_k)$ over $GF(3)$ and that $A(-1, D_k)$ represents $\mathcal{W}^k$ over $GF(3)$.

To complete the proof of the lemma, one needs only to combine these facts with the following three elementary observations. First, a matroid is the cycle matroid of an enlarged k-wheel if and only if it can be obtained from $M(\mathcal{W}_k)$ by adding elements in parallel to some of the spokes of $\mathcal{W}_k$ and adding elements in series with some members of the rim. Second, a matroid is represented over $GF(3)$ by $A(0, D)$ where D is a solid staircase matrix with $k-1$ stairs if and only if this matroid can be obtained from $M[A(0, D_k)]$ by adding elements in series with some members of $\{x_2, x_3, \dots, x_k, y_1\}$ and adding elements in parallel with some of the other elements. The third observation will be stated just for series extensions but we shall also need the dual statement. If we add an element e in series with an element f of a circuit-hyperplane X of a matroid M_1 and then relax the circuit-hyperplane $X \cup e$ of the resulting matroid, we obtain the same matroid as if we had relaxed X in M_1 and then added e in series with f in the relaxed matroid. □

PROOF OF THEOREM 5.3. By the last lemma, if, for some $k \geq 1$, there is an enlarged k-wheel G with rim H such that $M = M(G)$, then M' is ternary. To prove the converse, we argue by induction on $|E(M)|$. Assume that M' is ternary. Since M has a circuit-hyperplane, $|E(M)| \geq 2$. Moreover, if equality holds here, then $M \cong M(\mathcal{W}_1)$ and the theorem holds. Assume the theorem holds for $|E(M)| < n$ and let $|E(M)| = n$.

Suppose first that M has a unique element e that is not in H. Then e is a coloop of M and M can be obtained from $M(\mathcal{W}_1)$ by performing a sequence of series extensions on the rim. Again the required result holds. We may now assume that M has more than one element that is not in H. Let e be such an element. Then $M \setminus e$ has H as a circuit-hyperplane, and $M' \setminus e$ is ternary. Thus, by the induction assumption, $M \setminus e = M(G)$ where G is an enlarged k-wheel with rim H. By Lemma 5.4, we can label the edges of G so that there is a solid staircase matrix D such that $A(0, D)$ and $A(-1, D)$ represent $M \setminus e$ and $M' \setminus e$, respectively, over $GF(3)$, and $\{x_2, x_3, \dots, x_r, y_1\} = H$.

Now, by the unique representability of ternary matroids, we know that we can adjoin columns $\boldsymbol{v}$ and $\boldsymbol{v}'$ corresponding to e to each of $A(0, D)$ and $A(-1, D)$ to obtain ternary representations $A(0, D)+\boldsymbol{v}$ and $A(-1, D)+\boldsymbol{v}'$ for M and M', respectively. Let $\overline{A}(0, D)+\boldsymbol{v}$ and $\overline{A}(-1, D)+\boldsymbol{v}'$ be obtained from $A(0, D)+\boldsymbol{v}$ and $A(-1, D)+\boldsymbol{v}'$, respectively, by deleting the first r columns. Since the fundamental circuit of e with respect to $\{x_1, x_2, \dots, x_r\}$ is the same in M as it is in M', the columns $\boldsymbol{v}$ and $\boldsymbol{v}'$ must have precisely the same set of zero entries. Moreover, since H is a circuit-hyperplane of M, the first entry in each of $\boldsymbol{v}$ and $\boldsymbol{v}'$ is non-zero, so by scaling, we may assume it to be one. If $\boldsymbol{v}$ has no other non-zero entries, then e is added in parallel to a spoke of $\mathcal{W}_k$ and the result follows easily. Thus we may assume that $\boldsymbol{v}$ has some other non-zero entry.

Since M' is obtained from M by relaxing H, a set B is a basis of M if and only if B is a basis of M' and $B \neq H$. Thus a square submatrix of $\overline{A}(0, D)+\boldsymbol{v}$ has zero determinant if and only if the corresponding submatrix U of $\overline{A}(-1, D)+\boldsymbol{v}'$ has zero determinant and U is not the 1×1 matrix with row labelled by x_1 and column labelled by y_1.

Suppose next that $\boldsymbol{v}'$ has the entry in its ith row equal to -1. Then $\overline{A}(-1, D)+\boldsymbol{v}'$ has $\left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ as the submatrix corresponding to rows 1 and i and columns y_1 and e. This submatrix has zero determinant, whereas the corresponding submatrix of $\overline{A}(0, D)+\boldsymbol{v}$, which is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right]$ for some a in $\{1, -1\}$, has non-zero determinant. We conclude that every non-zero entry in $\boldsymbol{v}'$ is positive.

Next consider $\boldsymbol{v}$. If its entries in rows s and t are of opposite sign for some s and t exceeding one, then the submatrix of $\overline{A}(0, D)+\boldsymbol{v}$ corresponding to rows s and t and columns y_1 and e has non-zero determinant. But the corresponding submatrix of $\overline{A}(-1, D)+\boldsymbol{v}'$ has zero determinant. Hence all the non-zero entries in rows $2, 3, \ldots, r$ of $\boldsymbol{v}$ have the same sign.

We show next that all the entries of $\boldsymbol{v}$ are non-negative. Assume, to the contrary, that $\boldsymbol{v}$ has a negative entry in row s, say. Then all the entries in $\boldsymbol{v}$, other than the first, are non-positive. Suppose that, for some $j \geq 2$, column y_j has a non-zero entry in row s. Then the submatrix of $\overline{A}(0, D)+\boldsymbol{v}$ corresponding to rows 1 and s and columns y_j and e has non-zero determinant. The corresponding submatrix of $\overline{A}(-1, D) + \boldsymbol{v}'$ has zero determinant. This contradiction implies that, for each i in $\{2, 3, \ldots, r\}$, if the entry in row i of $\boldsymbol{v}$ is non-zero, then all of the columns $y_2, y_3, \ldots, y_{r^*}$ have zero entries in row i. Thus we may reorder rows $2, 3, \ldots, r$ of $A(0, D)+\boldsymbol{v}$ so that the rows in which $\boldsymbol{v}$ is non-zero occur together at the top but the positions of the ones in the solid staircase matrix D remain unchanged. On reordering the columns in the first part of the matrix to restore an identity, we obtain the matrix

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & & & & & 1 & & & & & -1 \\
\vdots & & & & & \vdots & & & & & \vdots \\
0 & & & & & 1 & & & & & -1 \\
0 & & I_{r-1} & & & 1 & & & D & & 0 \\
\vdots & & & & & \vdots & & & & & \vdots \\
0 & & & & & 1 & & & & & 0
\end{bmatrix}.$$

In this matrix, columns $2, 3, \ldots, r+1$ correspond to the set H, and, if an entry in the last column is -1, all the entries in the same row that are in D are 0. Let the first zero entry in $\boldsymbol{v}$ be in row t. Now, pivoting on the second entry of column $r+1$ and interchanging columns 2 and $r+1$, we obtain the following matrix where the second 1 in the last column occurs in row t:

$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & & & & & 1 & & & & & -1 \\
0 & & & & & -1 & & & & & 0 \\
\vdots & & & & & \vdots & & & & & \vdots \\
0 & & I_{r-1} & & & -1 & & & D & & 0 \\
0 & & & & & -1 & & & & & 1 \\
\vdots & & & & & \vdots & & & & & \vdots \\
0 & & & & & -1 & & & & & 1
\end{bmatrix}.$$

On multiplying row 2 and columns 2 and $r+1$ by -1, we obtain a matrix with all non-negative entries. Swapping column $r+2$ and the last column, and then rows 2 and $t-1$, and finally columns 2 and $t-1$, we obtain the matrix

$$\begin{bmatrix}
 & 0 & 1 & 1 & \cdots & 1 \\
 & 1 & & & & \\
I_r & 1 & & & D^+ & \\
 & \vdots & & & & \\
 & 1 & & & &
\end{bmatrix}$$

as a $GF(3)$ representation for M where D^+ is a solid staircase matrix and H corresponds to columns $2, 3, \ldots, r+1$. Thus the theorem follows in this case by Lemma 5.4.

We may now assume that all the entries in $\boldsymbol{v}$ are non-negative. Hence $\boldsymbol{v} = v'$. Let D'' be the matrix that is obtained from D by adjoining the column that is equal to $\boldsymbol{v}$ with its first entry deleted. Assume that $\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ is a submatrix of D''. Then, by permuting rows if necessary, we obtain that $\overline{A}(\alpha, D) + \boldsymbol{v}$ has

$$\begin{bmatrix} \alpha & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

as a submatrix with the first row and first column corresponding to the elements x_1 and y_1, respectively. This submatrix has determinant $\alpha - 2$. Thus, in $\overline{A}(-1, D) + \boldsymbol{v}$, this determinant is zero, while in $\overline{A}(0, D) + \boldsymbol{v}$, it is non-zero. This contradiction implies that neither $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ nor $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ occurs as a submatrix of D''. Hence, by Lemma 5.1, one can permute rows and permute columns in D'' to obtain a solid staircase matrix. But, in that case, the required result follows immediately by Lemma 5.4. □

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