Combinatorics, Probability and Computing (1993) 2, 519–530

Recognizing Polymatroids Associated with Hypergraphs

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Received 20 December 1992; revised 22 June 1993

For Paul Erdős on his 80th birthday

Two natural classes of polymatroids can be associated with hypergraphs: the so-called Boolean and hypergraphic polymatroids. Boolean polymatroids carry virtually all the structure of hypergraphs; hypergraphic polymatroids generalize graphic matroids. This paper considers algorithmic problems associated with recognizing members of these classes. Let k be a fixed positive integer and assume that the k-polymatroid ρ is presented via a rank oracle. We present an algorithm that determines in polynomial time whether ρ is Boolean, and if it is, finds the hypergraph. We also give an algorithm that decides in polynomial time whether ρ is the hypergraphic polymatroid associated with a given hypergraph. Other structure-theoretic results are also given.

1. Introduction

Consider the problem of deciding whether a matroid belongs to a given class. Using an independence or a rank oracle this is, in general, a hard problem. However, Seymour [6] has shown that graphic matroids can be recognized in polynomial time. This paper considers analogous problems for polymatroids.

Associated with a hypergraph are two natural polymatroids: its *Boolean* polymatroid, and its *hypergraphic* polymatroid (for definitions see Section 2). Boolean polymatroids carry virtually all the structure of hypergraphs – this class becomes trivial in the matroid case. Hypergraphic polymatroids are natural generalizations of graphic matroids. Both Boolean and hypergraphic polymatroids are important classes of polymatroids. The research that led to this paper was motivated by the question of whether membership of

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[*] This author's research was supported by a grant from the Commonwealth of Australia through the Australian Research Council.

these classes could be recognized, and members of these classes realized as hypergraphs, in polynomial time.

In Section 3 we present some algorithmic results for Boolean polymatroids. It is shown that given a k-polymatroid it can be decided if it is Boolean (and if it is, the corresponding hypergraph given) using at most $1+nk^2+n2^k$ calls to a rank oracle. If k is fixed, this is a polynomial time algorithm. If k is allowed to vary, it is not. This shows that the problem of recognizing a Boolean polymatroid is fixed parameter tractable in the sense of [1]. Example 4.2 shows that if k is not fixed, no polynomial time algorithm for recognizing a Boolean polymatroid exists. Section 4 also includes a number of other results on Boolean polymatroids.

Section 5 considers hypergraphic polymatroids. Theorem 5.2 is, essentially, a generalization of the main result of Seymour [6]. Unfortunately, this does not lead to a polynomial time algorithm for recognizing when a k-polymatroid is hypergraphic, because we lack the necessary supporting results. We regard it as an interesting open problem to establish these results if possible.

2. Preliminaries

Let E be a finite set and let ρ be a function from the power set of E into the integers. Then ρ is *normalised* if $\rho(\emptyset)=0$; ρ is *increasing* if $\rho(A)\leqslant\rho(B)$ whenever $A\subseteq B\subseteq E$; and ρ is *submodular* if $\rho(A)+\rho(B)\geqslant\rho(A\cup B)+\rho(A\cap B)$ for all subsets A and B of E. If ρ is normalised, increasing and submodular, then ρ is a *polymatroid on E*. We say that E is the *ground set* of ρ and $\rho(E)$ is the *rank* of ρ. Let k be a positive integer. Then the polymatroid ρ is a *k-polymatroid* if $\rho(e)\leqslant k$ for all $e\in E$. A 1-polymatroid is a matroid.

A *hypergraph* is a triple $H=(V,E,I)$, where V and E are finite sets whose members are called *vertices* and *edges* respectively, and $I\subseteq V\times E$ is its incidence relation. Two hypergraphs with the same edge sets are *equal* if there exists an isomorphism between them that is the identity on the edge set. In other words, by equality we mean equality up to vertex labelling. For a subset F of E, define $\bar{F}$ to be the set of vertices incident with at least one member of F, and for a subset W of V, define $\bar{W}$ to be the set of edges incident with at least one member of W. If, for all $e\in E, |\bar{e}|\leqslant k$, then H is a *k-hypergraph*. It is assumed that hypergraphs have no isolated vertices: that is, it is assumed that if $v\in V$, then $|\bar{v}|\geqslant 1$. This condition has the effect of simplifying a number of statements in this paper.

Let A be a subset of E. Then the *restriction of H to A*, denoted $H|A$, is defined by $H|A=(\bar{A},A,I')$, where $I'=I\cap(\bar{A}\times A)$. A *component* of H is a minimal non-empty subset V' of V with the property that if e is an edge of H, then either $\bar{e}\cap V'=\emptyset$ or $\bar{e}\subseteq V'$. Evidently the components of H partition V. The number of components of H is denoted by $\kappa(H)$. If $\kappa(H)=1$, then H is *connected.*

We now consider polymatroids defined on the edge sets of hypergraphs. Define the set function $\beta_H:2^E\to Z$ by $\beta_H(A)=|\bar{A}|$ for all $A\subseteq E$. It is well known, and easily seen, that β_H is a polymatroid. A polymatroid ρ is *Boolean* if $\rho=\beta_H$ for some hypergraph H. It is shown in Lemma 3.4(ii) that Boolean polymatroids carry all the structure of hypergraphs. In other words, if ρ is Boolean, there is a unique hypergraph H such that $\rho=\beta_H$. The correspondence with hypergraphs means that Boolean polymatroids form a significant class.

Now define the set function χ_H on the edges of the hypergraph H by

$$\chi_H(A) = |\bar{A}| - \kappa(H|A).$$

It is well known that χ_H is a polymatroid (see, for example, [8]). A polymatroid ρ is *hypergraphic* if $\rho = \chi_H$ for some hypergraph H. Evidently, if H is a graph, then χ_H is the cycle matroid of the graph. Much of the information carried by graphic matroids is also carried in more generality by hypergraphic polymatroids. This applies, in particular, to vertex colouring [2, 9].

We assume throughout that polymatroids are given by rank oracles. For good discussions on oracles see [4, Box 11A], [1, Section 1.2] or [5]. Note that in the literature, matroids are often given by independence oracles. It is straightforward to show that rank oracles and independence oracles for matroids are polynomially equivalent [5].

3. Recognizing and realising Boolean polymatroids

In this section we give polynomial time (for fixed k) algorithms for the following problems:

Problem 3.1. Given a Boolean k-polymatroid ρ, find a k-hypergraph H such that $\rho = \beta_H$.

Problem 3.2. Given a k-hypergraph H and a k-polymatroid ρ defined on the edge set of H, determine whether $\rho = \beta_H$.

Problem 3.3. Given a k-polymatroid ρ, decide whether or not ρ is Boolean. In the case that ρ is Boolean, find a hypergraph H such that $\rho = \beta_H$.

We first consider Problem 3.1. For a hypergraph $H = (V, E, I)$, and $A \subseteq E$, let $m_H(A) = |\{v \in V : \bar{v} = A\}|$: that is, $m_H(A)$ is the number of vertices v with $\bar{v} = A$. (Note that $m_H(\emptyset) = 0$, as there are no isolated vertices.) Clearly $m_H : 2^E \to \mathbb{Z}$ uniquely determines H, since by the convention set in Section 2, vertices of a hypergraph can be labelled arbitrarily. For $V_1 \subseteq V$, let $H \| V_1 = (E, V_1, I')$, where $I' = I \cap (V_1 \times E)$, that is, $H \| V_1$ is obtained from H by removing the vertices in $V - V_1$. Recall that β_H denotes the Boolean polymatroid associated with H.

Lemma 3.4.

(*i*) *Let A be a subset of E. Then $E-A$ is a maximal non-spanning set of β_H if and only if A is a minimal set for which $m(A) \neq 0$. If these hold, $m(A) = \beta_H(E) - \beta_H(E-A)$.*
(*ii*) *The hypergraph $H = (V, E, I)$ is uniquely determined by $\beta_H : 2^E \to \mathbb{Z}$.*
(*iii*) *For $A, B \subseteq E$ define*

$$\beta_A(B) = \begin{cases} 0 \text{ if } B \subseteq E - A \\ 1 \text{ otherwise.} \end{cases}$$

Then

$$\beta_H(B) = \sum_{v \in V} \beta_{\bar{v}}(B).$$

(iv) If V_1 and V_2 partition V, then for all $A \subseteq E$,

$$\beta_H(A) = \beta_{H \| V_1}(A) + \beta_{H \| V_2}(A).$$

Proof. Let $l_H(A) = \{v \in V : \bar{v} \subseteq A\}$. Then

$$l_H(A) = \sum_{B \subseteq A} m_H(B).$$

Now a vertex v is incident with no edge in $E - A$ if and only if $\bar{v} \subseteq A$, so it follows from the definition of β_H that

$$l_H(A) = \beta_H(E) - \beta_H(E - A).$$

Therefore $E - A$ is a maximal non-spanning set of β_H if and only if A is a minimal set for which $l_H(A) \neq 0$, if and only if A is a minimal set for which $m_H(A) \neq 0$. If these hold, then $m_H(A) = l_H(A)$ and (i) follows.

By Möbius inversion

$$m_H(A) = \sum_{B \subseteq A} (-1)^{|A-B|} l_H(B)$$

$$= \sum_{B \subseteq A} (-1)^{|A-B|} (\beta_H(E) - \beta_H(E - B)),$$

and (ii) follows.

For $A \subseteq E, \beta_A$ is simply the Boolean polymatroid of the hypergraph $(\{v\}, E, \{v\} \times A)$ – a hypergraph with edge set E and single vertex v such that $\bar{v} = A$. The expression in (iii) for $\beta_H(B)$ then easily follows from the definition of β_H. Finally (iv) follows immediately from (iii). □

Let ρ be a Boolean k-polymatroid on E, with $|E| = n$ and $\rho(E) = m$. Note that $m \leqslant kn$, so $1 + mn \leqslant 1 + kn^2$.

Theorem 3.5. *The following algorithm solves Problem* 3.1. *It runs in time polynomial in n and k and uses at most* $1 + mn$ *calls to the oracle.*

Algorithm 3.6. First find $m = \rho(E)$ and assign an arbitrary order to the members of E, say $E = \{e_1, \ldots, e_n\}$. With respect to this order one can find the lexicographically first maximal non-spanning set of any increasing function σ on E as follows. Set $A_0 = \varnothing$. For $1 \leqslant i \leqslant n$, set $A_i = A_{i-1} \cup \{e_i\}$ if $\sigma(A_{i-1} \cup \{e_i\}) < \sigma(E)$, and $A_i = A_{i-1}$ otherwise. The algorithm proceeds as follows:

Variables: V_1 a set, $I_1 \subseteq V_1 \times E$, $H_1 = (V_1, E, I_1)$.
Set $V_1 := \varnothing, I_1 := \varnothing$. (Thus initially $H_1 = (\varnothing, E, \varnothing)$ and $\beta_{H_1} \equiv 0$.)
While $\rho(E) > \beta_{H_1}(E)$ do the following loop:
  begin of loop
  Let $\sigma = \rho - \beta_{H_1}$.

Find the lexicographically first maximal non-spanning set of σ, call it A.
Add a set V' of $\sigma(E)-\sigma(A)$ *new* vertices to H_1 where each vertex $v \in V'$ satisfies $\bar{v} = E-A$; that is, $V_1 := V_1 \cup V'$ and $I_1 := I_1 \cup (V' \times (E-A))$.
end of loop.

Output H_1.

Proof. It is easily seen that the procedure for finding a maximal non-spanning set for an increasing function works. Since ρ is Boolean, there exists a hypergraph $H = (V,E,I)$ such that $\rho = \beta_H$. By Lemma 3.4(ii), H is unique (up to vertex labelling). Assume for induction that at the beginning of some iteration of the loop, $H_1 = H \| V_1$ for some $V_1 \subseteq V$. This is certainly true initially, when $V_1 = \varnothing$. By Lemma 3.4(iv), $\sigma = \rho - \beta_{H_1} = \beta_H - \beta_{H\|V_1} = \beta_{H\|V_2}$, where $V_2 = V - V_1$. Since σ is an increasing function, the algorithm finds a maximal non-spanning set of σ, and hence by Lemma 3.4(i) it finds vertices of $H \| V_2$; and these are vertices of H not already found in $H \| V_1$. We may assume that the labels chosen by the algorithm for these new vertices (the elements of V') coincide with their labels in H. These new vertices are added to V_1 and H_1, and it remains true for the new H_1 (at the end of the loop) that $H_1 = H \| V_1$ (at the end of the loop). By induction, $H_1 = H \| V_1$ for some V_1 after the last iteration of the loop. But at this time $|V| = \rho(E) \leqslant \beta_H(E) = |V_1|$, so $V_1 = V$, and hence $H_1 = H$ as required.

To compute $\sigma(A) = \rho(A) - \beta_{H_1}(A)$ find $\rho(A)$ from one call to the oracle for ρ, and compute $\beta_{H_1}(A)$ in time polynomial in n and k. One iteration of the loop requires n calls to the oracle for ρ, and there are at most m iterations of the loop. One call to the oracle is required to find $m = \rho(E)$, so altogether at most $1+mn$ calls to the oracle are required. It is clear that the running time is polynomial in n and k. □

We now consider Problem 3.2. While we state the following lemma in terms of polymatroids, it clearly holds for all set functions.

Lemma 3.7. *Let ρ_1 and ρ_2 be polymatroids on E such that $\rho_1(E) = \rho_2(E)$. If $\rho_1 \neq \rho_2$, there exists $B \subseteq E$ and $e \in E-B$ such that*

$$\rho_1(B \cup e) - \rho_1(B) < \rho_2(B \cup e) - \rho_2(B).$$

Proof. Assume that $\rho_1 \neq \rho_2$. Order the elements of E, say $E = \{e_1, \ldots, e_n\}$, letting $E_i = \{e_1, \ldots, e_i\}$, so that for some $i, \rho_1(E_i) \neq \rho_2(E_i)$. For $1 < i \leqslant n$, let $s_i = \rho_1(E_i) - \rho_1(E_{i-1})$, and let $t_i = \rho_2(E_i) - \rho_2(E_{i-1})$. Let $s_1 = \rho_1(\{e_1\})$ and $t_1 = \rho_2(\{e_1\})$. Then

$$s_1 + \cdots + s_n = \rho_1(E) = \rho_2(E) = t_1 + \cdots + t_n.$$

But the two sequences differ, since $s_1 + \cdots + s_i = \rho_1(E_i)$ and $t_1 + \cdots + t_i = \rho_2(E_i)$. By elementary arithmetic there exists a j such that $s_j < t_j$ (and, of course, a k such that $s_k > t_k$). The result then follows by setting $B = E_{j-1}$ and $e = e_j$. □

Let ρ be a k-polymatroid on E with $|E| = n$, and let $H = (V,E,I)$ be a k-hypergraph.

Theorem 3.8. *The following algorithm solves Problem 3.2. It runs in time polynomial in n and 2^k. The number of oracle calls is at most $n2^{k+1}$.*

Algorithm 3.9. The input is a polymatroid ρ on E and a k-hypergraph $H = (V,E,I)$. For an edge e of E and subset of vertices $W \subseteq \bar{e}$, set $A(e,W) = \{f \in E - e \mid \bar{f} \cap W = \varnothing\}$. The algorithm proceeds by checking for each edge e of E and each $W \subseteq \bar{e}$, that $\rho(A(e,W)) = \beta_H(A(e,W))$ and $\rho(A(e,W) \cup e) = \beta_H(A(e,W) \cup e)$. If equality holds in all cases, the output is YES; otherwise the output is NO.

Proof. The bounds on running time and oracle calls are clear. Suppose the algorithm is incorrect. Then there exists a polymatroid ρ on E and a k-hypergraph $H = (V,E,I)$ such that $\rho \neq \beta_H$, but $\rho(X) = \beta_H(X)$ whenever X is a set examined by the algorithm. Now, for any edge e we have $E = A(e,\phi) \cup e$, so $\rho(E) = \beta_H(E)$. Hence, by Lemma 3.7, there exists a subset B of E and an edge $e \in E - B$ such that

$$\rho(B \cup e) - \rho(B) < \beta_H(B \cup e) - \beta_H(B).$$

Let $W = \bar{e} - \bar{B}$ and $A = A(e,W)$, so that $B \subseteq A$. Then

$$\beta_H(B \cup e) - \beta_H(B) = |W| = \beta_H(A \cup e) - \beta_H(A),$$

and by submodularity,

$$\rho(A \cup e) - \rho(A) \leqslant \rho(B \cup e) - \rho(B).$$

This contradicts the fact that

$$\rho(A \cup e) - \rho(A) = \beta_H(A \cup e) - \beta_H(A). \qquad \square$$

It follows from the above proof that it suffices to check ranks in cases where $\bar{e} - \overline{A(e, W)} = W$, and small improvements to the $n2^{k+1}$ bound could be made.

Now consider Problem 3.3. For non-Boolean input, Algorithm 3.6 always terminates within the bounds on time and oracle calls stated in Theorem 3.5. (We do not prove this fact, since it is not necessary that the algorithm terminates. Even if it did not, one could make it stop when the bounds on running time are reached.) With an input of a polymatroid ρ, this algorithm will construct a hypergraph H. If ρ is Boolean, then $\rho = \beta_H$. If ρ is not Boolean, then certainly $\rho \neq \beta_H$, and we can test for this using Algorithm 3.9.

The proof of Theorem 3.10 below is now immediate. Let ρ be a k-polymatroid on E with $|E| = n$ and $\rho(E) = m$.

Theorem 3.10. *The following algorithm solves Problem 3.3. It runs in time polynomial in n and 2^k. There are at most $1 + mm + n2^{k+1}$ calls to the oracle.*

Algorithm 3.11. First apply Algorithm 3.6 to ρ. The algorithm constructs a k-hypergraph H. Apply Algorithm 3.9 with input ρ and H. If the output is YES, then output YES, H. Otherwise output NO.

Associated with a polymatroid ρ on E is a matroid r on E, the so-called *induced matroid*. The independent sets of r are the subsets I with the property that $\rho(I') \geqslant |I'|$ for all subsets I' of I. If ρ is Boolean, this is the standard way of obtaining transversal matroids. (If ρ is Boolean, then $\rho = \beta_H$ for some hypergraph H. Interpret H as a bipartite graph with vertices

$V \cup E$. The transversal matroid on E obtained in the usual way is r.) Given that Boolean polymatroids are easily recognized, one might conjecture that transversal matroids are easily recognized within the class of matroids. However, this is not the case as the following example shows.

Example 3.12. Let $n \geqslant 3$ be a positive integer, and let E be a set with $2n$ elements. Let M_1 be the rank n uniform matroid on E. Now let $\mathscr{A}$ be a collection of subsets of E with the following properties: $|\mathscr{A}\| = n+1$; if $A \in \mathscr{A}$, then $|A| = n$; and if A and B are distinct members of $\mathscr{A}$, then $|A \triangle B| \geqslant 4$ (where $A \triangle B$ denotes the symmetric difference of A and B). It is easily seen that the collection of n-element subsets of E with $\mathscr{A}$ deleted is the collection of bases of a matroid M_2 on E. Moreover M_2 is not transversal, since it has $n+1$ connected hyperplanes (the members of $\mathscr{A}$), and a transversal matroid has at most n connected hyperplanes (see for example Ingleton [3]).

Now assume that we are trying to distinguish between M_1 and M_2 via a rank oracle. How many calls to the oracle will be needed before we can guarantee that we have M_1 rather than M_2? Clearly we need only check the rank of the n-element subsets. For each such subset X there are n^2+1 subsets Y with $|X \triangle Y| < 4$. Also, it is easily seen that so long as at least $(n+1)(n^2+1)$ subsets remain unchecked, it is possible that the oracle has M_2 rather than M_1 in mind. It follows that at least

$$\binom{2n}{n} - (n+1)(n^2+1)$$

calls to the oracle are needed to be sure that we have M_1 rather than M_2. This is clearly not bounded by any polynomial in $|E| = 2n$.

Note also that $U_{n,2n}$ and $U_{n,2n}$ minus a base are two transversal matroids needing $\binom{2n}{n}$ calls to the oracle to distinguish, so transversal matroids can be neither recognized nor distinguished in time polynomial in the size of the ground set.

4. Some structure theorems for Boolean polymatroids

The theorems in this section were found while researching the problem of Section 3. We believe them to be of interest in their own right.

Algorithm 3.9 operates on the principle that once the ranks of certain subsets in a polymatroid ρ are known to agree with their ranks in a given Boolean polymatroid, ρ must be that Boolean polymatroid. In terms of the subsets examined, Algorithm 3.9 looks from the 'top down'. The following theorem shows that one can also look from the 'bottom up'.

Theorem 4.1. *Let ρ be a k-polymatroid on E and $H = (V,E,I)$ be a hypergraph. If $\rho(A) = \beta_H(A)$ for all $A \subseteq E$ with $|A| \leqslant k+1$, and $\rho(E) = \beta_H(E)$, then $\rho = \beta_H$.*

Proof. (This proof essentially dualises the argument of the proof of Theorem 3.8.) Let ρ be a polymatroid on E, and $H = (V,E,I)$ be a hypergraph such that $\rho(E) = \rho_H(E)$, and $\rho(A) = \beta_H(A)$ for all $A \subseteq E$ with $|A| \leqslant k+1$. Assume that $\rho \neq \beta_H$. By Lemma 3.7, there exists $B \subseteq E$ and $e \in E - B$, such that

$$\rho(B \cup e) - \rho(B) > \beta_H(B \cup e) - \beta_H(B). \tag{1}$$

Let $W = \bar{e} - \bar{B}$ and choose a subset A of B that is minimal, with the property that $\bar{e} - \bar{A} = W$. Clearly $|A| \leqslant k$, so

$$\rho(A \cup E) - \rho(A) = \beta_H(A \cup e) - \beta_H(A).$$

But

$$\beta_H(A \cup e) - \beta_H(A) = |W| = \beta_H(B \cup e) - \beta_H(B),$$

and by submodularity,

$$\rho(A \cup e) - \rho(A) \geqslant \rho(B \cup e) - \rho(B).$$

Hence,

$$\rho(B \cup e) - \rho(B) \leqslant \beta_H(B \cup e) - \beta_H(B);$$

contradicting (1). □

Theorem 4.1 gives an alternative algorithm for Problem 3.2. For fixed k it requires $1 + \sum_{i=1}^{k+1} \binom{n}{i}$ calls to the oracle: that is, it requires $0(n^{k+1})$ calls. This algorithm is not as efficient as Algorithm 3.9.

Also note that examination of the proof of Theorem 4.1 shows that not all subsets of size less than or equal to $k+1$ need to be checked. A closer examination (the details of which are omitted here) shows that it is sufficient to check a certain $0(n^k)$ of them. However, the following example shows that no further improvement is possible. It also shows that if k is not fixed, no polynomial time algorithm exists for solving Problems 3.2 or 3.3.

Example 4.2. Let $k \geqslant 1$ and $n \geqslant k+1$ be integers, and let $E = \{e_1, \ldots, e_n\}$. We now define a hypergraph $H = (V, E, I)$ with $|V| = \binom{k+1}{2} + k$. For each distinct pair $\{i, j\}$ with $i, j \leqslant k+1$, there is one vertex incident with exactly e_i and e_j. (This part of H is the complete graph on $k+1$ vertices with the roles of edge and vertex interchanged.) Each of the other k vertices is incident with exactly the edges $e_{k+1}, \ldots, e_n$.

Now let ρ be defined by letting $\rho(A) = \beta_H(A)$ for all $A \subseteq E$ except for the set $X = \{e_1, \ldots, e_{k+1}\}$, where we let $\rho(X) = \binom{k+1}{2} + 1$. It is routine to check that ρ is a polymatroid. But $\beta_H(X) = \binom{k+1}{2} = \rho(X) - 1$, so $\rho \neq \beta_H$. We have constructed a k-polymatroid ρ on E and a hypergraph $H = (V, E, I)$ such that $\rho(A) = \beta_H(A)$ whenever $|A| \leqslant k$ and when $A = E$ but $\rho \neq \beta_H$.

Now let $n = k+1$. As before we define a polymatroid agreeing with β_H on all but a single subset. Let X be any non-empty subset of E. It is routine to check that if $1 \leqslant |X| \leqslant k$, the rank of X can be increased by one, the resulting set function being a polymatroid, while if $2 \leqslant |X| \leqslant k+1$, the rank of X can be decreased by one. That is, for any subset X of E, there exists a polymatroid on E agreeing with β_H on all subsets of E except X. Assuming that one has a reasonable definition of "size" for a hypergraph, this example shows that if k is not bounded, there exists no polynomial time algorithm for solving Problem 3.2 or 3.3.

Apart from β_H, the polymatroids constructed in the above example are not Boolean. Theorem 4.3 below shows that the bound of Theorem 4.1 can be sharpened if one is testing for equality of Boolean polymatroids.

Theorem 4.3. *Let ρ_1 and ρ_2 be Boolean k-polymatroids on E. If $\rho_1(A) = \rho_2(A)$ for all $A \subseteq E$ with $|A| \leqslant 2 + \log_2 k$, then $\rho_1 = \rho_2$.*

Proof. For $i \in \{1, 2\}$, let the hypergraph $H_i = (V_i, E, I_i)$ satisfy $\beta_{H_i} = \rho_i$, and for $A \subseteq E$, define $m_i(A) = |\{v \in V_i : \bar{v} = A\}|$ and $l_i(A) = |\{v \in V_i : \bar{v} \subseteq A|$. Suppose ρ_1 and ρ_2 provide a counterexample to the theorem on a minimum sized ground set. Then, without loss of generality, we can assume that $\rho_1(A) = \rho_2(A)$ for all proper subsets A of E, and that $\rho_1(E) - \rho_2(E) = w$, for some $w \geqslant 1$. Now $l_i(A) = \rho_i(E) - \rho_i(E - A)$. Therefore,

$$l_1(A) - l_2(A) = \begin{cases} w & \text{if } A \neq \varnothing \\ 0 & \text{if } A = \varnothing. \end{cases}$$

Now

$$l_1(A) - l_2(A) = \sum_{B \subseteq A} (m_1(B) - m_2(B)).$$

Hence, by Möbius inversion,

$$\begin{aligned} m_1(A) - m_2(A) &= \sum_{B \subseteq A} (-1)^{|A-B|}(l_1(B) - l_2(B)) \\ &= \begin{cases} -1(-1)^{|A|}w & \text{if } A \neq \varnothing \\ 0 & \text{if } A = \varnothing. \end{cases} \end{aligned}$$

Thus, $m_1(A) - m_2(A) = w$ for all subsets of E with odd cardinality (and $-w$ for all nonempty subsets of E with even cardinality). Therefore H_1 has (at least) w vertices v with $\bar{v} = A$ for each subset A of E with odd cardinality. Let $|E| = n$. It now follows that the number of vertices incident with each edge of H_1 is at least

$$w \sum_{\substack{i \leq n-1 \\ i \text{ odd}}} \binom{n-1}{i-1} = w2^{n-2}.$$

Therefore, $k \geqslant w2^{n-2}$; that is, $k \geqslant 2^{n-2}$. Hence, $n \leqslant 2 + \log_2 k$; that is $|E| \leqslant 2 + \log_2 k$. This means $\rho_1(E) = \rho_2(E)$, contradicting the assumption that $\rho_1(E) \neq \rho_2(E)$. □

The following example shows that Theorem 4.3 is, in a sense, best possible. It shows that for each pair of integers n and k, with $n \geqslant 2$ and $k \geqslant 2^{n-2}$, there exist k-hypergraphs H_1 and H_2 with edge set E, where $|E| = n$ with the property that $\beta_{H_1}(A) = \beta_{H_2}(A)$ for all proper subsets A of E, but $\beta_{H_1}(E) \neq \beta_{H_2}(E)$.

Example 4.4. Let $|E| = n \geqslant 2$ and $k = 2^{n-2}$. Define hypergraphs H_1 and H_2 with edge sets E as follows. In H_1, for each subset A of E with odd cardinality, there is one vertex v with $\bar{v} = A$ and there are no other vertices. In H_2, for each nonempty subset A of E with even cardinality, there is one vertex v with $\bar{v} = A$ and there are no other vertices. Now H_1 and H_2 are k-hypergraphs, and if A is a proper subset of E, then

$$\beta_{H_1}(A) = \beta_{H_2}(A) = 2^{n-1} - 2^{n-|A|-1}.$$

But $\beta_{H_1}(E) = 2^{n-1}$, whereas $\beta_{H_2}(E) = 2^{n-1} - 1$.

In the above example, when $n = 3$, $k = 2$, so H_1 and H_2 are graphs. Here H_1 is the 3-edge star and H_2 is the triangle. Note that Theorem 4.3 does not help in the graph reconstruction problem (see for example [8, Chapter 5]), since edges here are labelled.

5. Hypergraphic polymatroids

In this section we give a polynomial time algorithm for the following problem.

Problem 5.1. Given a $(k-1)$-polymatroid ρ on E and a k-hypergraph $H = (V,E,I)$, determine whether $\rho = \chi_H$.

Theorem 5.2. *Algorithm 5.3 below solves Problem 5.1 in time polynomial in 2^k and n. The number of oracle calls is at most $n2^{k+1}$.*

Algorithm 5.3. As in Algorithm 3.9, for an edge e of E and subset of vertices $W \subseteq \bar{e}$, we set $A(e, W) = \{f \in E - e \mid \bar{f} \cap W = \varnothing\}$. The algorithm then proceeds by checking for each edge e of E, and each $W \subseteq \bar{e}$, that $\rho(A(e, W)) = \chi_H(A(e, W))$ and $\rho(A(e, W) \cup e) = \chi_H(A(e, W) \cup e)$. If equality holds in all cases, the output is YES, otherwise NO.

Proof. The bounds on running time and oracle calls are clear. Now assume that the algorithm outputs YES when the connected hypergraph $H = (V,E,I)$ and the k-polymatroid ρ on E are given as input. We first prove three lemmas.

Lemma 5.4. *If C is a subset of E, then $\rho(C) \geqslant \chi_H(C)$.*

Proof. Order the elements $e_1, \ldots, e_i$ of C, letting $C_j = \{e_1, \ldots, e_j\}$ such that $\kappa(H \mid C_j)$ is non-decreasing for increasing j. (It is routinely seen that such an order exists.) Let $W_j = \bar{e}_{j+1} - \bar{C}_j$, and $A = A(e_{j+1}, W_j)$. By the assumption about $\kappa(H \mid C_j)$, it follows that e_{j+1} meets the same number c of components, where $c = 0$ or $c = 1$, of both $H \mid C_j$ and $H \mid A$. Then by the definition of χ_H, it follows that

$$\begin{aligned} \chi_H(C_j \cup e_{j+1}) - \chi_H(C_j) &= |W_j| - 1 + c \\ &= \chi_H(A \cup e_{j+1}) - \chi_H(A). \end{aligned}$$

Now, by submodularity,

$$\rho(C_j \cup e_{j+1}) - \rho(C_j) \geqslant \rho(A \cup e_{j+1}) - \rho(A),$$

and, since $A \cup e_{j+1}$ and A are checked by the algorithm,

$$\rho(A \cup e_{j+1})-\rho(A)=\chi_H(A \cup e_{j+1})-\chi_H(A).$$

Therefore,

$$\rho(C_j \cup e_{j+1})-\rho(C_j) \geqslant \chi_H(C_j \cup e_{j+1})-\chi_H(C_j).$$

A straightforward induction using the above fact then establishes the lemma. □

Lemma 5.5. *If C is a subset of E such that $H|C$ is connected, then $\rho(C)=\chi_H(C)$.*

Proof. Say $|C|=i$. Order the elements $e_1,\dots,e_n$ of E, letting $E_j=\{e_1,\dots,e_j\}$, such that $C=E_i$, and $H|E_j$ is connected for all $j \geqslant 1$. Since H is connected, it is easily seen that such an order exists. By Lemma 5.4, $\rho(C) \geqslant \chi_H(C)$, and the argument of Lemma 5.4 shows that, for $1 \leqslant j < n$,

$$\rho(E_j \cup e_{j+1})-\rho(E_j) \geqslant \chi_H(E_j \cup e_{j+1})-\chi_H(E_j).$$

Suppose $\rho(C) > \chi_H(C)$. Then, by induction using the above inequality, it follows that $\rho(E) > \chi_H(E)$. But E is checked by the algorithm, so $\rho(E)=\chi_H(E)$. Hence $\rho(C)=\chi_H(C)$. □

Lemma 5.6. $\rho(C)=\chi_H(C)$ *for all* $C \subseteq E$.

Proof. Let $C_1,\dots,C_l$ be the connected components of $H|C$. Now by Lemma 5.4, $\rho(C) \geqslant \chi_H(C)$; by the definition of χ_H, $\chi_H(C)=\sum_{i=1}^{l}\chi_H(C_i)$; by Lemma 5.5, $\sum_{i=1}^{l}\chi_H(C_i)=\sum_{i=1}^{l}\rho(C_i)$; and by the submodularity of ρ, $\sum_{i=1}^{l}\rho(C_i) \geqslant \rho(C)$. Hence $\rho(C) \geqslant \chi_H(C) \geqslant \rho(C)$, so $\rho(C)=\chi_H(C)$, and the lemma follows. □

It follows from Lemma 5.6 that the theorem holds when H is connected. The extension to the case when H is not connected is evident. □

It follows from the proof of Theorem 5.2 that not all the subsets checked by Algorithm 5.3. need to be. At the expense of a somewhat more complicated statement, this algorithm could be refined somewhat, but the bounds on running time and oracle calls would not be changed to an interesting extent.

The key result of Seymour [6] is a corollary that gives sufficient conditions for a matroid to be equal to the cycle matroid of a graph. This result is similar to the case $k=2$ of Theorem 5.2, except that, for an edge e and subset W of $\bar{e}$, Seymour does not compare ranks if $W=\varnothing$ or $W=\bar{e}$. It is not hard to see that, in general, these cases can be omitted, provided that $\rho(E)=\chi_H(E)$ is checked; which Seymour does.

Unfortunately, we cannot give a polynomial time algorithm for determining whether a k-polymatroid is hypergraphic. Such an algorithm would required polymatroid-theoretic generalizations of results in, for example, [7]. Alternatively, a characterization of when two hypergraphs have the same hypergraphic polymatroid would be useful. This would be an extension of Whitney's 2-isomorphism theorem [10].

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