

q -LIFTS OF TANGENTIAL k -BLOCKS

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ABSTRACT

A tangential k -block over $\text{GF}(q)$ is a geometry representable over $\text{GF}(q)$ with critical exponent $k+1$ for which every proper loopless minor has critical exponent at most k . We define what is meant by a q -lift of a matroid representable over $\text{GF}(q)$ and show that, if M is a tangential k -block over $\text{GF}(q)$ and M' is a q -lift of M , then M' is a tangential $k+1$ -block over $\text{GF}(q)$. This enables us to extend the class of known tangential k -blocks and to answer some natural questions concerning the existence of modular hyperplanes in tangential k -blocks.

1. Introduction

This paper continues the study of tangential k -blocks over $\text{GF}(q)$ begun in [9, 10, 11]. To some extent it completes a picture. We define what is meant by a q -lift of a matroid representable over $\text{GF}(q)$, and show that, if M is a tangential k -block over $\text{GF}(q)$ and M' is a q -lift of M , then M' is also a tangential k -block over $\text{GF}(q)$. While the construction is general, interest is focused on tangential k -blocks over $\text{GF}(2)$; in particular those obtained by successively taking q -lifts of $M^*(P_{10})$ (the cocycle matroid of the Petersen graph).

We assume that the reader is familiar with the basic concepts of matroid theory, particularly those of the characteristic polynomial $P(M; \lambda)$ of a matroid and the critical exponent $c(M; q)$ of a matroid representable over $\text{GF}(q)$. Welsh [7, Chapter 16] provides a good introduction to these topics; free use will be made of results from this chapter. The terminology used here for matroids will in general follow Welsh [7]. Some differences follow. A geometry is a matroid without loops or parallel elements. If M is a matroid with ground set E and $S \subseteq E$, the restriction of M to $E \setminus S$ will be denoted by $M|(E \setminus S)$ or by $M \setminus S$ according to convenience and the contraction of M to $E \setminus S$ will be denoted by M/S . The closure and rank of S in M will be denoted by $\text{cl}_M(S)$ and $r_M(S)$, respectively or if no danger of ambiguity exists by $\text{cl}(S)$ and $r(S)$, respectively. The geometry whose lattice of flats is isomorphic to that of M will be denoted by \bar{M} .

Some familiarity with properties of modular flats of matroids is assumed. Modular flats are studied extensively in [1].

A tangential k -block over $\text{GF}(q)$ is a geometry representable over $\text{GF}(q)$ with critical exponent $k+1$ for which every proper loopless minor has critical exponent less than $k+1$. If M is a geometry representable over $\text{GF}(q)$ with critical exponent $k+1$, then it is straightforward to show that M is a tangential k -block over $\text{GF}(q)$ if $c(M/F; q) \leq k$ whenever F is a proper non-empty flat of M .

Let S be a set of points of $\text{PG}(r-1, q)$. To avoid unwieldy notation we denote $\text{cl}_{\text{PG}(r-1, q)}(S)$ by $\text{cl}_P(S)$.

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2. q -lifts of tangential k -blocks

Let M be a rank r matroid representable over $\text{GF}(q)$; then the geometry M' is a q -lift of M with base E and apex p if the following conditions hold:

- (i) E is a set of points of $\text{PG}(r, q)$ with $\text{PG}(r, q) \setminus E \cong \bar{M}$;
- (ii) p is a point of $\text{PG}(r, q)$ not contained in $\text{cl}_p(E)$;
- (iii) $M' = \text{PG}(r, q) \setminus \{\bigcup_{x \in E} \text{cl}_p(\{p, x\})\}$.

In other words, one embeds \bar{M} in a hyperplane H of $\text{PG}(r, q)$ and then obtains M' by restricting $\text{PG}(r, q)$ to the set of points on lines joining points of the embedding of \bar{M} to a fixed point of $\text{PG}(r, q)$ not on H .

The object of this section is to prove the following result.

THEOREM 2.1. *If M is a tangential k -block over $\text{GF}(q)$ and M' is a q -lift of M then M' is a tangential $k+1$ -block over $\text{GF}(q)$.*

Before proving Theorem 2.1 we need some lemmas on the structure of q -lifts. Throughout this section it is assumed that M is representable over $\text{GF}(q)$ and that M' is a q -lift of M with base E and apex p . The ground set of M' is denoted by S .

The following lemma is straightforward.

- LEMMA 2.2.** (i) *Any line of M' containing p is isomorphic to $\text{PG}(1, q)$.*
(ii) $\overline{M'/p} \cong \bar{M}$.

- LEMMA 2.3.** (i) *If H is a hyperplane of $\text{PG}(r, q)$ disjoint from p , then $H \cap S$ is a hyperplane of M' .*
(ii) *If H' is a hyperplane of M' disjoint from p , then $M' \setminus H' \cong \bar{M}$.*
(iii) *If H' is a hyperplane of M' disjoint from p , then M' is also a q -lift of M with base H' and apex p .*

Proof. We first show that, if H is a hyperplane of $\text{PG}(r, q)$ not containing p , then $M' \setminus (H \cap S) \cong \bar{M}$. Let H be such a hyperplane. Since $p \notin H$, no line of M' containing p is contained in H . By Lemma 2.2, each of these lines is isomorphic to $\text{PG}(1, q)$, so none of these lines is disjoint from H . Therefore H meets each line of M' containing p in a single point, so $(M'/p) \setminus (H \cap S) \cong \overline{M'/p}$. But p is a coloop of $M' \setminus ((H \cap S) \cup p)$, so $(M'/p) \setminus (H \cap S) \cong M' \setminus (H \cap S)$ and, by Lemma 2.2, $\overline{M'/p} \cong \bar{M}$ so $M' \setminus (H \cap S) \cong \bar{M}$.

Part (i) of the lemma follows since $r(M') = r(M) + 1$. Part (ii) follows from the observation that every hyperplane H' of M' disjoint from p is of the form $H \cap S$ for some hyperplane H of $\text{PG}(r, q)$ disjoint from p . Part (iii) follows since $S = \{\bigcup_{x \in H \cap S} \text{cl}(\{p, x\})\}$.

LEMMA 2.4. *If x is an element of S distinct from p , then $\overline{M'/x}$ is isomorphic to a q -lift of a proper minor of M .*

Proof. Let H' be a hyperplane of M' containing x but disjoint from p ; then, by Lemma 2.3(iii), M' is a q -lift of M with base H' and apex p . Therefore we may assume

without loss of generality that $x \in E$. Let H be a hyperplane of $\text{PG}(r, q)$ containing p but missing x . Let

$$S_1 = \text{PG}(r, q) \mid \left\{ \bigcup_{y \in S} \{\text{cl}_p(\{x, y\}) \cap H\} \right\}$$

and

$$S_2 = \text{PG}(r, q) \mid \left\{ \bigcup_{y \in E} \{\text{cl}_p(\{x, y\}) \cap H\} \right\}.$$

Clearly $\text{PG}(r, q) \mid S_1 \cong \overline{M'/x}$ and $\text{PG}(r, q) \mid S_2 \cong \overline{(M' \mid E)/x}$. Therefore $\text{PG}(r, q) \mid S_2$ is isomorphic to a proper minor of M . Since $p \in H$ the theorem is proved if we can show that $S_1 = \{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\}$. Assume that $a \in S_1$; then there exists $b \in S \setminus x$ with the property that $\text{cl}_p(\{x, b\}) \cap H = a$. If $b = p$ or $b \in E$, then $a \in \{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\}$, so we assume that $b \in S \setminus \{p \cup E\}$. In this case, $b \in \text{cl}_p(\{p, e\})$ for some point $e \in E$. But then $a \in \text{cl}_p(\{p, \text{cl}_p(\{x, e\}) \cap H\})$ and therefore $a \in \{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\}$. Hence $S_1 \subseteq \{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\}$. Assume that $a \in \{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\}$. If $a \in S_2$ or $a = p$, then $a \in S_1$; so we assume that $a \notin \{S_2 \cup p\}$. In this case there exists $e' \in S_2$ with the property that $a \in \text{cl}_p(\{p, e'\})$. But $e' = \text{cl}_p(\{x, e\}) \cap H$ for some point $e \in E$ and there exists a point $b \in \text{cl}_p(\{p, e\})$ with the property that $\text{cl}_p(\{x, b\}) \cap H = a$. But $b \in S$ so $a \in S_1$. Hence $\{\bigcup_{y \in S_2} \text{cl}_p(\{p, y\})\} \subseteq S_1$. The two sets are therefore equal, and the lemma is proved.

LEMMA 2.5. *Let F be a non-empty flat of M' .*

- (i) *If $p \in F$, then $\overline{M'/F}$ is isomorphic to a minor of M .*
- (ii) *If $p \notin F$, then $\overline{M'/F}$ is isomorphic to a q -lift of a proper minor of M .*

Proof. Parts (i) and (ii) are routine consequences of Lemmas 2.2 and 2.4 respectively.

LEMMA 2.6. *If M is loopless and $c(M; q) = k$, then $c(M'; q) = k + 1$.*

Proof. Say $c(M'; q) = j$ and let (H_1, \dots, H_j) be a j -tuple of hyperplanes of $\text{PG}(r, q)$ which distinguishes S . At least one of these hyperplanes is disjoint from p , so assume without loss of generality that H_j misses p . Then

$$j - 1 = c(\text{PG}(r, q) \mid (H_j \cap S); q).$$

But, by Lemma 2.3, $\text{PG}(r, q) \mid (H_j \cap S) \cong \overline{M}$, and since M is loopless, it follows that $c(\overline{M}; q) = c(M; q)$; hence $j - 1 = c(M; q) = k$. Therefore $j = k + 1$ and $c(M'; q) = k + 1$.

We are now in a position to prove our main theorem.

Proof of Theorem 2.1. Certainly M' is a geometry and is representable over $\text{GF}(q)$. Since M is a tangential k -block over $\text{GF}(q)$ it follows that $c(M; q) = k + 1$. Therefore, by Lemma 2.6, $c(M'; q) = k + 2$. Let F be a non-empty flat of M' . If $p \in F$, then, by Lemma 2.5(i), $\overline{M'/F}$ is isomorphic to a minor M'' of M . Therefore, since M is a tangential k -block, $c(\overline{M'/F}; q) < k + 2$ and also $c(M'/F; q) < k + 2$. If $p \notin F$, then, by Lemma 2.5(ii), $\overline{M'/F}$ is isomorphic to a q -lift of a proper minor M'' of M . We may assume that M'' is loopless; then, since M'' is a proper loopless minor of a tangential k -block over $\text{GF}(q)$ it follows that $c(M''; q) < k + 1$. Therefore, by Lemma 2.6, $c(M'/F; q) = c(M''; q) + 1 < k + 2$.

We conclude that M' is a geometry representable over $\text{GF}(q)$ with critical exponent $k+2$ and, whenever F is a non-empty flat of M' , $c(M'/F) < k+2$; hence M' is a tangential $k+1$ -block over $\text{GF}(q)$.

By repeatedly taking q -lifts, it is possible, given a tangential k -block over $\text{GF}(q)$, to construct tangential k' -blocks over $\text{GF}(q)$ for all $k' > k$. The extent to which this construction is interesting depends, in part, on the properties inherited by q -lifts. It is to this question that we now turn our attention.

3. Modularity in q -lifts

In [9] it is shown that tangential k -blocks with modular hyperplanes form a well-behaved class. It is therefore of interest to know to what extent modularity is preserved in q -lifts. The following lemma is a routine consequence of results in [1]. It is proved in [8].

LEMMA 3.1. *Let M be a loopless rank r matroid and let F be a rank k flat of M ; then F is modular in M if and only if $F \cap F' \neq \emptyset$ for every flat F' of M with $r(F') = r - k + 1$.*

As before, we assume that M' is a q -lift of M with base E and apex p , and that the ground set of M' is S . Since our interest is in tangential k -blocks, we lose no generality in assuming that M is the geometry $\text{PG}(r-1, q) | E$.

LEMMA 3.2. *Let F be a flat of M ; then F is modular in M if and only if $\text{cl}_M(F \cup p)$ is modular in M' .*

Proof. Let F be a rank k flat. Assume that $\text{cl}_M(F \cup p)$ is modular in M' ; then if F' is a flat of M with $r(F') = r - k + 1$, we have that $F' \cap \text{cl}_M(F \cup p) \neq \emptyset$. But $F' \subseteq E$ and $\text{cl}_M(F \cup p) \cap E = F$; so $F' \cap F \neq \emptyset$. Therefore F is modular in M .

Assume that F is modular in M . Now $r(\text{cl}_M(F \cup p)) = k+1$, so let F' be a flat of M' with $r(F'_1) = r+1 - (k+1) + 1 = r - k + 1$. If $p \in F'$, then $F' \cap \text{cl}_M(F \cup p) \neq \emptyset$; so we assume that $p \notin F'$. Let H be a hyperplane of M' containing F' but missing p . By Lemma 2.3(ii), $\text{PG}(r, q) | H \cong M$. Furthermore, it is easily seen that the function $\psi(x) = \text{cl}_p(\{p, x\}) \cap E$ is an isomorphism from $\text{PG}(r, q) | H$ to M . Now F is modular in M and $r(F') = r - k + 1$ so $\psi(F') \cap F \neq \emptyset$, and therefore $F' \cap \psi^{-1}(F) \neq \emptyset$. But $\psi^{-1}(F) \subseteq \text{cl}_M(F \cup p)$, so $\text{cl}_M(F \cup p)$ is modular in M' , and the result follows.

A matroid M is supersolvable if it contains a saturated chain of modular flats; that is, a chain of flats $F_0 \subset F_1 \subset \dots \subset F_{r-1} \subset F_r$ with the property that for $0 \leq i \leq r$, F_i is a rank i modular flat of M . Stanley [3, 4] shows that if M is a supersolvable geometry then

$$P(M; \lambda) = \prod_{i=1}^r (\lambda - |F_i \setminus F_{i-1}|).$$

THEOREM 3.3. *If M is supersolvable with saturated chain of modular flats*

$$F_0 \subset F_1 \subset \dots \subset F_{r-1} \subset F_r,$$

then M' is supersolvable and

$$P(M'; \lambda) = (\lambda - 1) \prod_{i=2}^{r+1} (\lambda - q | F_{i-1} \setminus F_{i-2}).$$

Proof. Let $F'_0 = \emptyset$ and for $1 \leq i \leq r+1$ let $F'_i = \text{cl}_{M'}(\{F_{i-1} \cup p\})$. Then, using Lemma 3.2, we obtain a saturated chain $F'_0 \subset F'_1 \subset \dots \subset F'_r \subset F'_{r+1}$ of modular flats of M' ; hence M' is supersolvable. A routine argument then shows that for $i > 1$, $|F'_i \setminus F'_{i-1}| = q |F_{i-1} \setminus F_{i-2}|$. The result follows from this observation.

'Nice' tangential k -blocks are ones which are supersolvable or at least contain modular hyperplanes. Lemma 3.2 and Theorem 3.3 show that these properties are inherited by q -lifts. 'Nasty' tangential k -blocks are ones which contain no modular hyperplanes. I do not hide the fact that my initial interest in q -lifts was due to the hope that q -lifts of tangential k -blocks would always contain modular hyperplanes. For fixed k and q this would enable bounds to be placed on the rank of tangential k -blocks over $\text{GF}(q)$. In dealing with the critical problem naive hopes such as these are invariably futile.

THEOREM 3.4. *M contains no modular hyperplanes if and only if M' contains no modular hyperplanes.*

Proof. If M contains a modular hyperplane then, by Lemma 3.2, M' contains a modular hyperplane so assume that M contains no modular hyperplanes. If H is a hyperplane of M' containing p , then $H = \text{cl}_{M'}(F \cup p)$ for some flat F of M . But since $r(\text{cl}_M(F \cup p)) = r(F) + 1$, F is a hyperplane of M . But F is not modular in M so, by Lemma 3.2, H is not modular in M' . If H is disjoint from p then, by Lemma 2.3, $M' | H \cong M$ and M' is also a q -lift of M with base H and apex p . Therefore we may assume without loss of generality that $H = E$. Let $C \subseteq E$ be a coline of M' . Then C is a hyperplane of M and since C is not modular in M there exists $\{x, y\} \subseteq E$ with $\text{cl}_M(\{x, y\}) \cap C = \emptyset$. One routinely shows that

$$r(\text{cl}_p(\{x, y\}) \cap \text{cl}_p(C)) = 1,$$

so let $a = \text{cl}_p(\{x, y\}) \cap \text{cl}_p(C)$. Let b be a point of $\text{cl}_p(\{p, x\}) \setminus \{p, x\}$ and let $c = \text{cl}_p(\{a, b\}) \cap \text{cl}_p(\{p, y\})$. Since $\{x, y\} \subseteq E, \{b, c\} \subseteq S$. Now $\text{cl}_p(\{b, c\}) \cap \text{cl}_p(E) = a$ and $a \notin E$ so $\text{cl}_M(\{b, c\}) \cap E = \emptyset$. A modular hyperplane of a geometry must meet every line of that geometry so E is not modular in M' and the theorem is proved.

By the preceding theorem, q -lifts of nasty tangential k -blocks are also nasty. Now $M^*(P_{10})$, is a tangential 2-block over $\text{GF}(2)$ containing no modular hyperplanes. By taking successive q -lifts of $M^*(P_{10})$ one obtains (in the light of Theorems 2.1 and 3.4) tangential k -blocks over $\text{GF}(2)$ containing no modular hyperplanes for all $k \geq 2$. In [11] it is shown that whenever q is a prime power exceeding two, then there exist tangential k -blocks over $\text{GF}(q)$ containing no modular hyperplanes for all positive integers k except in the special case when both $q = 3$ and $k = 1$. In [5] it is shown that the only tangential 1-block over $\text{GF}(2)$ is the three point line $U_{2,3}$, and in [6] it is

shown that the only tangential 1-blocks over GF(3) are the four point line $U_{2,4}$ and the cycle matroid of K_4 , $M(K_4)$. Therefore there exist tangential k -blocks over GF(q) containing no modular hyperplanes for all prime powers q and positive integers k except in the case when $k = 1$ and $q = 2$ or 3. That is, nasty tangential k -blocks abound.

It is of interest to note that $k = 1$ and $q = 2$ or $q = 3$ are the only values of k and q for which all tangential k -blocks over GF(q) are known (or at least known to be known). The existence of tangential k -blocks containing no modular hyperplanes would seem to lie at the heart of the difficulty of the critical problem.

We conclude with an example.

4. Tangential 3-blocks over GF(2)

Tutte's tangential 2-block conjecture states that the only tangential 2-blocks over GF(2) are PG(2, 2), $M(K_5)$ and $M^*(P_{10})$. As yet this conjecture is unresolved (but see [2]). A similar conjecture for tangential 3-blocks over GF(2) requires a purported complete list of such blocks.

In [10] a number of tangential 3-blocks over GF(2) is given with their matrix representations over GF(2). These blocks are all supersolvable and have the characteristic polynomials shown in Table 1.

TABLE 1

Rank	Characteristic polynomial	
8	$(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)$	$M(K_9)$
7	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)$	
6	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-7)(\lambda-8)$	
5	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-7)(\lambda-8)$	
4	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-8)$	PG(3, 2)

Any 2-lift of a tangential 2-block over GF(2) is a tangential 3-block over GF(2). A 2-lift of PG(2, 2) is just PG(3, 2) which is already in the above list. By Theorem 3.3, a 2-lift of $M(K_5)$ is supersolvable and has characteristic polynomial $(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-8)$, so there exist at least two non-isomorphic supersolvable rank 5 tangential 3-blocks over GF(2). Since $M^*(P_{10})$ has rank 6 a 2-lift of $M^*(P_{10})$ has rank 7. This geometry is not supersolvable, so there exist two non-isomorphic rank 7 tangential 3-blocks over GF(2). This gives a list of seven tangential 3-blocks over GF(2). I have no idea whether or not this list is likely to be complete.

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