TANGLES, TREE-DECOMPOSITIONS AND GRIDS IN MATROIDS

JIM GEELEN, BERT GERARDS, AND GEOFF WHITTLE

Abstract. A tangle in a matroid is an obstruction to small branch-width. In particular, the maximum order of a tangle is equal to the branch-width. We prove that: (i) there is a tree-decomposition of a matroid that "displays" all of the maximal tangles, and (ii) when M is representable over a finite field, each tangle of sufficiently large order "dominates" a large grid-minor. This extends results of Robertson and Seymour concerning Graph Minors.

1. Introduction

Robertson and Seymour [7] introduced branch-width for graphs and showed that this parameter is characterized by "tangles". Robertson and Seymour also stated that their results extend to matroids [7, p. 190]; the details were later given by Dharmatilake [1] (see, also, [3]). Here we use the definitions given in [3]; we defer these definitions until Section 3. For the purpose of this introduction, a tangle of order θ in M can be thought of as a "θ-connected component" of M. We prove the following two results.

1.1. *Each matroid has a tree-decomposition that "displays" all its maximal tangles.*

This will be made precise in Theorem 9.1, which extends a result in Graph Minors X [7, (10.3)].

Theorem 1.2. *For each finite field $\mathbb{F}$ and positive integer k there exists an integer θ such that, if M is an $\mathbb{F}$-representable matroid and $\mathcal{T}$ is a tangle in M of order θ, then $\mathcal{T}$ dominates a minor N that is isomorphic to the cycle matroid of a k by k grid.*

Date: October 2, 2007.

1991 *Mathematics Subject Classification.* 05B35.

Key words and phrases. branch-width, tangles, tree-decomposition, matroids, graph minors.

This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada and the Marsden Fund of New Zealand.

The proof is given in Section 7. Theorem 1.2 extends a result of Robertson, Seymour, and Thomas [6, (2.3)]. The term "dominates" is used specifically with respect to grid-minors and is defined in Section 7. To prove Theorem 1.2 we will use the main result of [4] which says that an $\mathbb{F}$-representable matroid with huge branch-width contains a large grid-minor.

These results are technical, but the motivation is to, hopefully, use them in extending the Graph Minors Structure Theorem [8]. For example, for certain fixed binary matroids N, we are interested in the class of binary matroids that do not contain an N-minor. Typically we choose N to be a highly structured matroid, such as: the cycle matroid of a grid, the cycle matroid of a complete graph, or a projective geometry. In such cases N has a unique maximal tangle $\mathcal{T}_N$. Now, if N is a minor of some binary matroid M, then the tangle $\mathcal{T}_N$ "induces" a tangle $\mathcal{T}_M$ in M. Any tangle in M that contains $\mathcal{T}_M$ is said to "dominate" N. Now 1.1 shows that the maximal tangles in M are composed in a tree-like way. This tree structure essentially localizes each maximal tangle in M and shows how M is composed from these local parts. So, to determine the structure of binary matroids with no N-minor, it suffices to determine the local structure of each maximal tangle in M that does not dominate an N-minor. Unfortunately the local structure of tangles that do not dominate N is complicated. This is partly overcome by considering only tangles whose order is much larger than the order of $\mathcal{T}_N$. By Theorem 1.2, each such tangle dominates a huge grid. Supposing that our tangle does not dominate an N-minor, the hope then is that this huge grid-minor will impose local structure on M.

2. Connectivity and branch-width

We assume that the reader is familiar with matroid theory; we use the notation of Oxley [5].

Let λ be a function that assigns an integer value to each subset of a finite set E. We call λ *symmetric* if $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$. We call λ *submodular* if $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$ for all $X, Y \subseteq E$. If λ is integer-valued, symmetric, and submodular, then we call λ a *connectivity function on E*. A *connectivity system* is a pair $K = (E, \lambda)$ where λ is a connectivity function on E. A partition (A, B) of $E(K)$ is called a *separation of order* $\lambda_K(A)$.

For a matroid M and $X \subseteq E(M)$, we let $\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) + 1$. It is straightforward to prove that $K_M = (E(M), \lambda_M)$ is a connectivity system. For a graph G and $X \subseteq E(G)$, we let $\lambda_G(X)$ denote the number of vertices of G that are incident with both an edge of X and an edge of $E(G) - X$. It is also straightforward to prove that $K_G = (E(G), \lambda_G)$ is a connectivity system. Moreover, if G is connected we have for each $X \subseteq E(G)$ that $\lambda_{M(G)}(X) \leq \lambda_G(X)$.

Branch-width plays only a minor role in this paper, but we include a definition for completeness. Let K be a connectivity system. A tree is *cubic* if its internal vertices all have degree 3. A *branch-decomposition* of K is a cubic tree T whose leaves are labeled by elements of $E(K)$ such that each element in $E(K)$ labels exactly one leaf of T and each leaf of T receives at most one label from $E(K)$. If T' is a subgraph of T and $X \subseteq E(K)$ is the set of labels of T', then we say that T' *displays* X. The *width* of an edge e of T is defined to be $\lambda_K(X)$ where X is the set displayed by one of the components of $T - \{e\}$. The *width* of T is the maximum among the widths of its edges. The *branch-width* of K is the minimum among the widths of all branch-decompositions of K.

The *branch-width* of a matroid M is the branch-width of its connectivity system $K_M = (E(M), \lambda_M)$.

We remark that there are some trivial graphs G, such as trees, for which K_G and $K_{M(G)}$ have different branch-width. It is, however, conjectured that, if G has a circuit of length at least 2, then K_G and $K_{M(G)}$ have the same branch-width. In Section 6 we prove that this is at least true for n by n grids.

3. Tangles

In this section we review results and definitions from [3].

Let K be a connectivity system. A *tangle* in K of *order* θ is a collection $\mathcal{T}$ of subsets of $E(K)$ such that:

- $(T1)$ For each $B \in \mathcal{T}$, $\lambda_K(B) < \theta$.
- $(T2)$ For each separation (A, B) of order less than θ, $\mathcal{T}$ contains either A or B.
- $(T3)$ If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E(K)$.
- $(T4)$ For each $e \in E(K)$, $E(K) - \{e\} \notin \mathcal{T}$.

It is proved in [3, Lemma 3.1] that, to verify that $\mathcal{T}$ is a tangle, we may replace $(T3)$ by the following weaker conditions:

- $(T3a)$ If $B \in \mathcal{T}$, $A \subseteq B$, and $\lambda_K(A) < \theta$, then $A \in \mathcal{T}$.
- $(T3b)$ If (A_1, A_2, A_3) is a partition of $E(K)$, then $\mathcal{T}$ does not contain all three of A_1, A_2, and A_3.

Note that throughout this text partitions may have empty members; in particular, $(T3b)$ also says that no two members of $\mathcal{T}$ partition $E(K)$.

The following slight variation of [7, (3.5)] was proved in [3, Theorem 3.2].

Theorem 3.1. *Let K be a connectivity system. Then, the maximum order of a tangle in K is equal to the branch-width of K.*

A *tangle* in a matroid M is a tangle in its connectivity system K_M. The following fact is used in the proof of (7.3.1).

Lemma 3.2. *Let $\mathcal{T}$ be a tangle of order θ at least 3 in a matroid M. Then each subset of $E(M)$ with rank less than $\theta - 1$ is in $\mathcal{T}$.*

Proof. Let X be a smallest possible subset in $E(M)$ that is not in $\mathcal{T}$. As $\theta \geq 3$ it follows from $(T2)$ and $(T4)$ that singletons are in $\mathcal{T}$. So X can be partitioned into two smaller sets. By the choice of X these two sets are in $\mathcal{T}$. Hence by $(T3)$, $E(M) - X$ is not in $\mathcal{T}$. Thus by $(T2)$, $\lambda_M(X) \geq \theta$. Note that, for any $Y \subseteq E(M)$, the rank of Y is at least $\lambda_M(Y) - 1$. So X has rank at least $\theta - 1$; as required. □

Let $\mathcal{T}$ be a tangle of order θ in matroid M. For $X \subseteq E(M)$, if X is a subset of a set in $\mathcal{T}$, then we let

$$\phi_{\mathcal{T}}(X) = \min(\lambda_M(A) - 1 \,:\, X \subseteq A \in \mathcal{T}),$$

otherwise we let $\phi_{\mathcal{T}}(X) = \theta - 1$. The following result was proved in [3, Lemma 4.3].

Lemma 3.3. *Let M be a matroid and let $\mathcal{T}$ be a tangle in M of order θ. Then $\phi_{\mathcal{T}}$ is the rank function of a matroid of rank $\theta - 1$.*

This matroid is referred to as the *tangle matroid* of $\mathcal{T}$.

4. New tangles from old

In this section we look at different constructions for tangles. Let $\mathcal{T}$ be a tangle of order θ in a connectivity system K and let $\theta' \leq \theta$. Now let $\mathcal{T}'$ be the collection of all sets $A \in \mathcal{T}$ with $\lambda_K(A) < \theta'$. It is straightforward to verify that:

Lemma 4.1. *$\mathcal{T}'$ is a tangle in K of order θ'.*

We say that $\mathcal{T}'$ is the *truncation* of $\mathcal{T}$ to order θ'. Note that if $\mathcal{T}'$ and $\mathcal{T}$ are tangles in K, then $\mathcal{T}'$ is a truncation of $\mathcal{T}$ if and only if $\mathcal{T}' \subseteq \mathcal{T}$.

Let $K = (E, \lambda)$ be a connectivity system and let $X \subseteq E$. We let $K \circ X = ((E - X) \cup \{e_X\}, \lambda')$ where, for each $A \subseteq E - X$, $\lambda'(A) = \lambda(A)$ and $\lambda'(A \cup \{e_X\}) = \lambda(A \cup X)$. It is straightforward to verify that:

Lemma 4.2. *If K is a connectivity system and $X \subseteq E(K)$, then $K \circ X$ is a connectivity system.*

We can also obtain a tangle in $K \circ X$ from a tangle in K.

Lemma 4.3. *Let $\mathcal{T}$ be a tangle of order θ in the connectivity system K and let $X \in \mathcal{T}$. Now let $\mathcal{T}'$ be the collection of subsets of $E(K \circ X)$ such that, for $A \subseteq E(K) - X$, $A \in \mathcal{T}'$ if and only if $A \in \mathcal{T}$; and $A \cup \{e_X\} \in \mathcal{T}'$ if and only if $A \cup X \in \mathcal{T}$. Then $\mathcal{T}'$ is a tangle of order θ in $K \circ X$.*

Proof. Each of the conditions $(T1)-(T4)$ for $\mathcal{T}'$ to be a tangle follows directly from the corresponding condition for $\mathcal{T}$. □

A set X of elements in a connectivity system K is called *titanic* if each partition (A_1, A_2, A_3) of X satisfies $\lambda_K(A_i) \geq \lambda_K(X)$ for at least one $i = 1, 2, 3$.

The following result is a partial converse of Lemma 4.3; it generalizes a result in Graph Minors X [7, (8.3)].

Lemma 4.4. *Let K be a connectivity system, let $X \subseteq E(K)$ be titanic with $\lambda_K(X) < \theta$, and let $\mathcal{T}'$ be a tangle of order θ in $K \circ X$. Now let $\mathcal{T}$ be the collection of all $A \subseteq E(K)$ such that $\lambda_K(A) < \theta$ and either $A - X \in \mathcal{T}'$ or $(A - X) \cup \{e_X\} \in \mathcal{T}'$. Then $\mathcal{T}$ is a tangle of order θ in K.*

Proof. Let $Y = E(K) - X$ and $L = K \circ X$. Note that $\lambda_L(\{e_X\}) = \lambda_L(Y) = \lambda_K(Y) = \lambda_K(X) < \theta$, so $\{e_X\} \in \mathcal{T}'$. By definition, $\mathcal{T}$ satisfies $(T1)$.

We next prove that $\mathcal{T}$ satisfies $(T2)$. Consider a separation (A, B) of order less than θ in K. Since X is titanic in K, either $\lambda_K(X \cap A) \geq \lambda_K(X)$ or $\lambda_K(X \cap B) \geq \lambda_K(X)$. By symmetry between A and B, we may assume that $\lambda_K(X \cap A) \geq \lambda_K(X)$. Then, by submodularity and symmetry of λ_K, we see that $\lambda_L(Y \cap B) = \lambda_K(Y \cap B) = \lambda_K(A \cup X) \leq \lambda_K(A) + \lambda_K(X) - \lambda_K(A \cap X) \leq \lambda_K(A) < \theta$. Therefore, as $\mathcal{T}'$ satisfies $(T2)$, one of $Y \cap B = B - X$ or $(Y \cap A) \cup \{e_X\} = (A - X) \cup \{e_X\}$ is in $\mathcal{T}'$. Thus, $\mathcal{T}$ contains B or A, as required. So $\mathcal{T}$ satisfies $(T2)$.

Next consider $(T3a)$. Let $B \in \mathcal{T}$ and $A \subseteq B$ with $\lambda_K(A) < \theta$. Then, by definition, $B - X$ is contained in a set in $\mathcal{T}'$. Since $A \subseteq B$, the union of $(E(K) - A) - X$, $B - X$ and $\{e_X\}$ is $E(L)$. As $\{e_X\}$ in $\mathcal{T}'$ and as $\mathcal{T}'$ satisfies $(T3)$, this implies that $(E(K) - A) - X$ is not contained in a set of $\mathcal{T}'$. So, $E(K) - A \notin \mathcal{T}$. As $\lambda_K(A) < \theta$ and as $\mathcal{T}$ does satisfy $(T2)$ this implies that $A \in \mathcal{T}$, as required. So $\mathcal{T}$ satisfies $(T3a)$.

We next prove by contradiction that $\mathcal{T}$ satisfies $(T3b)$. Let A_1, A_2, and A_3 be members of $\mathcal{T}$ that partition $E(K)$. Then each of $A_1 - X$, $A_2 - X$ and $A_3 - X$ is contained in a set in $\mathcal{T}'$. So, since $E(L)$ cannot be covered by three sets in $\mathcal{T}'$, none of the sets $(A_1 \cap Y) \cup \{e_X\}$, $(A_2 \cap Y) \cup \{e_X\}$, or $(A_3 \cap Y) \cup \{e_X\}$ is in $\mathcal{T}'$. Thus $\mathcal{T}'$ contains each of $A_1 \cap Y$, $A_2 \cap Y$, and $A_3 \cap Y$. Since $A_1 \cap Y$ and $\{e_X\}$ lie in $\mathcal{T}'$, $\mathcal{T}'$

does not contain $Y - A_1$. Now since $\mathcal{T}'$ contains neither $Y - A_1$ nor $(A_1 \cap Y) \cup \{e_X\}$, we have $\lambda_K(Y - A_1) = \lambda_L(Y - A_1) \geq \theta > \lambda_K(A_1)$. So, by submodularity and symmetry of λ_K, we get that $\lambda_K(X \cap A_1) \leq \lambda_K(X) + \lambda_K(A_1) - \lambda_K(X \cup A_1) = \lambda_K(X) + \lambda_K(A_1) - \lambda_K(Y - A_1) < \lambda_K(X)$. Similarly $\lambda_K(X \cap A_2) < \lambda_K(X)$ and $\lambda_K(X \cap A_2) < \lambda_K(X)$. However this contradicts the fact that X is titanic. Thus $\mathcal{T}$ satisfies $(T3b)$ and, hence, $\mathcal{T}$ is a tangle of order θ in K.

Finally we prove by contradiction that $\mathcal{T}$ satisfies $(T4)$. Suppose $e \in E(K)$ with $E(K) - \{e\} \in \mathcal{T}$. Then at least one of $E(L) - \{e, e_X\} = E(K) - \{e\} - X$ or $E(L) - \{e\} = (E(K) - \{e\} - X) \cup \{e_X\}$ is in $\mathcal{T}'$. As $\mathcal{T}'$ satisfies $(T4)$, this means $E(L) - \{e, e_X\} \in \mathcal{T}'$ and $e \in E(L) - \{e_X\}$. Now we have, as $E(K) - \{e\} \in \mathcal{T}$, that $\lambda_L(\{e\}) = \lambda_K(\{e\}) = \lambda_K(E(K) - \{e\}) < \theta$. So, as $\mathcal{T}'$ satisfies $(T4)$, the singleton $\{e\}$ is in $\mathcal{T}'$. But since also $\{e_X\}$ and $E(L) - \{e, e_X\}$ are in $\mathcal{T}'$, this contradicts that $\mathcal{T}'$ satisfies $(T3)$. So $\mathcal{T}$ does indeed satisfy $(T4)$. □

5. Minors and tangles

Let N be a minor of M and let $\mathcal{T}_N$ be a tangle in N of order θ. Now let $\mathcal{T}_M$ be the collection of all sets $A \subseteq E(M)$ where $\lambda_M(A) < \theta$ and $A \cap E(N) \in \mathcal{T}_N$. The following result is an immediate consequence of definitions.

Lemma 5.1. *$\mathcal{T}_M$ is a tangle in M of order θ.*

We say that $\mathcal{T}_M$ is the *tangle in M induced by $\mathcal{T}_N$*.

Let $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a function and $m \in \mathbb{Z}_+$. A matroid M is called *(m, f)-connected* if whenever (A, B) is a separation of order ℓ where $\ell < m$ we have either $|A| \leq f(\ell)$ or $|B| \leq f(\ell)$.

Let $g(n) = (6^{n-1} - 1)/5$. Note that $g(1) = 0$ and $g(n) = 6g(n-1) + 1$ for all $n > 1$. The main result in this section is the following.

Theorem 5.2. *Let $\mathcal{T}$ be a tangle of order θ in a matroid M. Then there exists a (θ, g)-connected minor N of M and a tangle $\mathcal{T}'$ of order θ in N such that $\mathcal{T}$ is the tangle in M induced by $\mathcal{T}'$.*

We will use the following result from [2, Lemma 3.1].

Lemma 5.3. *Let $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a nondecreasing function. If e is an element of an (m, f)-connected matroid M, then $M \setminus e$ or M/e is $(m, 2f)$-connected.*

5.4. *Proof of Theorem 5.2.* The proof is by induction on $|E(M)|$ with θ fixed; the root of this induction lies in the (θ, g)-connected matroids. Let $\mathcal{T}$ be a tangle of order θ in a matroid M and assume M is not (θ, g)-connected. Choose $m \in \{1, \ldots, \theta - 1\}$ as small as possible such that M is not $(m+1,g)$-connected. Then there exists a separation (A,B) of order m with $|A|,|B| > g(m)$. By symmetry we may assume that $A \in \mathcal{T}$. Now let $e \in A$. By Lemma 5.3 and duality, we may assume that M/e is $(m,2g)$-connected.

5.4.1. *$A - \{e\}$ is titanic in M/e.*

Subproof. When $m = 1$ this is vacuously true. Suppose that $m > 1$ and consider any partition (A_1, A_2, A_3) of $A - \{e\}$. Since $|A| > g(m) = 6g(m-1)+1$, we have $|A_i| > 2g(m-1)$ for some $i \in \{1,2,3\}$. Then, since M/e is $(m,2g)$-connected, $\lambda_{M/e}(A_i) \geq m \geq \lambda_{M/e}(A - \{e\})$. Hence $A - \{e\}$ is indeed titanic in M/e. □

5.4.2. *For each $X \subseteq B$, $\lambda_M(X) = \lambda_{M/e}(X)$.*

Subproof. Since M/e is $(m,2g)$-connected, $\lambda_M(B) = \lambda_{M/e}(B)$. Hence $e \notin \mathrm{cl}_M(B)$. Therefore, for each $X \subseteq B$, $e \notin \mathrm{cl}_M(X)$. So $\lambda_M(X) = \lambda_{M/e}(X)$; as required. □

5.4.3. *For each $X \subseteq E(M)$ with $\lambda_M(X) < \theta$ we have that $X \in \mathcal{T}$ if and only if $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$.*

Subproof. Let $X \subseteq E(M)$ with $\lambda_M(X) < \theta$. First assume that $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$. Then, as $A \in \mathcal{T}$, it follows from $(T3)$ that $E(M) - X \notin \mathcal{T}$. Hence $X \in \mathcal{T}$.

For the reverse implication assume now that $X \in \mathcal{T}$. By 5.4.2, $\lambda_M(A) = \lambda_M(B) = \lambda_{M/e}(B - \{e\}) = \lambda_{M/e}(A - \{e\})$. So as A is titanic in M/e either $\lambda_M(A - X) \geq \lambda_{M/e}(A - X) \geq \lambda_M(A)$ or $\lambda_M(A \cup X) \geq \lambda_{M/e}(A \cup X) \geq \lambda_M(A)$. If $\lambda_M(A - X) \geq \lambda_M(A)$, then by symmetry and submodularity of λ_M we have that $\lambda_M(X - A) = \lambda_M(X \cap B) \leq \lambda_M(X) + \lambda_M(B) - \lambda_M(X \cup B) = \lambda_M(X) + \lambda_M(A) - \lambda_M(A - X) \leq \lambda_M(X) < \theta$. Hence, if $\lambda_M(A - X) \geq \lambda_M(A)$ then it follows from $(T3a)$ that $X - A \in \mathcal{T}$. If $\lambda_M(A \cap X) \geq \lambda_M(A)$, then, again by submodularity, $\lambda_M(A \cup X) \leq \lambda_M(X) + \lambda_M(A) - \lambda_M(A \cap X) \leq \lambda_M(X) < \theta$. So by $(T2)$ either $A \cup X \in \mathcal{T}$ or $B - X \in \mathcal{T}$. However, as $A \in \mathcal{T}$ and $X \in \mathcal{T}$ it follows from $(T3)$ that $B - X \notin \mathcal{T}$. So $A \cup X \in \mathcal{T}$. We conclude that if $X \in \mathcal{T}$ then $X - A \in \mathcal{T}$ or $X \cup A \in \mathcal{T}$. □

Let $\mathcal{T}_1$ be the tangle in $K_M \circ A$ of order θ obtained from $\mathcal{T}$ via Lemma 4.3. By 5.4.2, there is a natural isomorphism between $K_M \circ A$ and $K_{M/e} \circ (A - \{e\})$; let $\mathcal{T}_2$ be the tangle in $K_{M/e} \circ (A - \{e\})$ of order θ that is obtained from $\mathcal{T}_1$ via this isomorphism. In both $K_M \circ A$ and $K_{M/e} \circ (A - \{e\})$ denote the element that is not in B by e'.

Let $\mathcal{T}_3$ be the tangle in M/e of order θ that is obtained from $\mathcal{T}_2$ via Lemma 4.4. Finally let $\mathcal{T}_4$ be the tangle in M that is induced by $\mathcal{T}_3$.

5.4.4. $\mathcal{T} = \mathcal{T}_4$.

Subproof. Let (X, Y) be a separation of M of order less than θ with $e \in Y$. Then each of the following sequence of equivalences follows directly from definitions.

$$\begin{aligned} X \in \mathcal{T}_4 &\iff X \in \mathcal{T}_3 \\ &\iff X - (A - \{e\}) \in \mathcal{T}_2 \text{ or } (X - (A - \{e\})) \cup \{e'\} \in \mathcal{T}_2 \\ &\iff X - A \in \mathcal{T}_1 \text{ or } (X - A) \cup \{e'\} \in \mathcal{T}_1 \\ &\iff X - A \in \mathcal{T} \text{ or } X \cup A \in \mathcal{T}. \end{aligned}$$

So by 5.4.3, $X \in \mathcal{T}_4$ if and only if $X \in \mathcal{T}$; as required. □

The result now follows easily by applying induction to the tangle $\mathcal{T}_3$ in M/e. □

6. A tangle in a grid

An *n by n grid* is a graph G_n with vertex set $V = \{(i, j) : i, j \in \{1, \ldots, n\}\}$ where vertices (i, j) and (i', j') are adjacent if and only if either $i = i'$ and $|j - j'| = 1$, or $j = j'$ and $|i - i'| = 1$.

The goal of this section is to prove the existence of a natural tangle of order n in $M(G_n)$. For $i \in \{1, \ldots, n\}$ let P_i denote the path in G_n on vertices $(i, 1), \ldots, (i, n)$ and let Q_i denote the path in G_n on vertices $(1, i), \ldots, (n, i)$. Now we let $\mathcal{T}_n$ denote the collection of all subsets $A \subseteq E(G_n)$ such that $\lambda_{M(G_n)}(A) < n$ and A does not contain any $E(P_i)$ for $i \in \{1, \ldots, n\}$. We will prove, for $n \geq 3$:

Lemma 6.1. *$\mathcal{T}_n$ is a tangle in $M(G_n)$ of order n.*

A similar result was proved by Kleitman and Saks; see [7, (7.3)]. They considered tangles in K_{G_n}, whereas we consider tangles in $K_{M(G_n)}$. Our proof follows that of Kleitman and Saks; we need some preliminary results on connectivity.

Let X and Y be disjoint subsets of $E(M)$, we define $\kappa_M(X, Y) = \min(\lambda_M(A) : X \subseteq A \subseteq E(M) - Y)$. The following result, due to Tutte [9], is an extension of Menger's Theorem.

Theorem 6.2 (Tutte's Linking Theorem)**.** *If S and T are disjoint sets of elements in a matroid M, then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = \kappa_M(S, T)$.*

The following result was proved in [4].

Lemma 6.3. *Let S and T be disjoint sets of elements of a matroid M. Then there exist sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| + 1 = |T_1| + 1 = \kappa_M(S_1, T_1) = \kappa_M(S, T)$.*

In order to prove Lemma 6.1, we first need to establish that certain sets of edges in a grid are "highly connected".

Lemma 6.4. *Let* $i \in \{1, \dots, n\}$ *and, for each* $j \in \{1, \dots, n\} - \{i\}$*, let* e_j *and* f_j *be disjoint edges of* P_j*. Now let* $X = \{e_j : j \in \{1, \dots, n\} - \{i\}\}$ *and let* $Y = \{f_j : j \in \{1, \dots, n\} - \{i\}\}$*. Then* $\kappa_{M(G_n)}(X, Y) = n$*.*

Proof. Let $D = E(Q_2) \cup \cdots \cup E(Q_{n-1})$ and let $C = E(Q_1) \cup E(Q_n) \cup ((E(P_1) \cup \cdots \cup E(P_n)) - (X \cup Y))$. Now let $H = G_n \setminus D/C$. Note that $H[X]$ and $H[Y]$ are disjoint spanning trees of H. Therefore $n = \lambda_{M(H)}(X) = \kappa_{M(H)}(X, Y) \leq \kappa_{M(G_n)}(X, Y) \leq |X| + 1 = n$. Thus $\kappa_{M(G_n)}(X, Y) = n$, as required. □

The proofs of the following two results are similar to that of Lemma 6.4; we leave these to the reader.

Lemma 6.5. *Let* $i, j \in \{1, \dots, n\}$*. Then* $\kappa_{M(G_n)}(P_i, Q_j) = n$*. Also, if* $i \neq j$*, then* $\kappa_{M(G_n)}(P_i, P_j) = n$ *and* $\kappa_{M(G_n)}(Q_i, Q_j) = n$*.*

Lemma 6.6. *Let* $X \subseteq E(P_1) \cup E(P_n)$ *with* $|X| \geq n - 1$ *and let* $j \in \{1, \dots, n\}$*. Then* $\kappa_{M(G_n)}(X, Q_j) = n$*.*

We call a set $A \subseteq E(G_n)$ *small* if $\lambda_{M(G_n)}(A) < n$ and A does not contain any of $E(P_1), \dots, E(P_n)$ or $E(Q_1), \dots, E(Q_n)$.

Lemma 6.7. *Let* (A, B) *be a separation of* $M(G_n)$ *of order less than* n*. Then one of* A *and* B *is small. Moreover, if* B *is small, then* A *contains one of* $E(P_1), \dots, E(P_n)$ *and one of* $E(Q_1), \dots, E(Q_n)$*.*

Proof. By Lemma 6.4, either A or B must contain one of $E(P_1), \dots, E(P_n)$. Then, by symmetry, either A or B must contain one of $E(Q_1), \dots, E(Q_n)$. However, by Lemma 6.5, A and B cannot both contain one of $E(P_1), \dots, E(P_n), E(Q_1), \dots, E(Q_n)$. □

Note that $\mathcal{T}_n$ trivially satisfies conditions $(T1)$, $(T3a)$, and $(T4)$. By Lemma 6.7, $\mathcal{T}_n$ also satisfies $(T2)$. Thus in order to complete the proof of Lemma 6.1, we need only verify $(T3b)$; this is achieved by the following result.

Lemma 6.8. *For* $n \geq 3$*,* $E(G_n)$ *cannot be partitioned into three small sets.*

Proof. The proof is by induction on n. The case $n = 3$ is trivial; suppose then that $n \geq 4$ and that the result holds for G_{n-1}. Now assume (A_1, A_2, A_3) is a partition of $E(G_n)$ into three small sets.

By symmetry we may assume that Q_n meets A_1 and A_2. (That is, $A_1 \cap E(Q_n)$ and $A_2 \cap E(Q_n)$ are nonempty.) By Lemma 6.7, there is a path Q_j disjoint from A_1. Note that $\kappa_{M(G_n)}(A_1 \cap (E(P_1) \cup E(P_n)), Q_j) \leq$

$\lambda_{M(G_n)}(A_1) < n$. Then, by Lemma 6.6, $|A_1 \cap (E(P_1) \cup E(P_n))| < n-1$. Similarly $|A_2 \cap (E(P_1) \cup E(P_n))| < n-1$. Therefore either P_1 or P_n meets A_3; by symmetry, we may assume that P_n meets A_3. Therefore $E(P_n) \cup E(Q_n)$ meets each of A_1, A_2, and A_3.

Note that $G_{n-1} = G_n - (V(P_n) \cup V(Q_n))$. For each $i \in \{1, 2, 3\}$, let $A'_i = E(G_{n-1}) \cap A_i$.

6.8.1. *There exists* $k \in \{1, 2, 3\}$ *such that* $\lambda_{M(G_{n-1})}(A'_k) \geq n - 1$.

Subproof. By the induction hypothesis, there exists $k \in \{1, 2, 3\}$ such that A'_k is not small in G_{n-1}. Suppose that $\lambda_{M(G_{n-1})}(A'_k) < n - 1$. Then A'_k contains one of $E(P_1) \cap E(G_{n-1}), \ldots, E(P_{n-1}) \cap E(G_{n-1})$ or one of $E(Q_1) \cap E(G_{n-1}), \ldots, E(Q_{n-1}) \cap E(G_{n-1})$. By Lemma 6.7, A_k avoids some path P_i and some path Q_j. Since $E(P_n) \cup E(Q_n)$ meets each of A_1, A_2, and A_3, either $i \neq n$ or $j \neq n$. Thus A'_k avoids one of $E(P_1) \cap E(G_{n-1}), \ldots, E(P_{n-1}) \cap E(G_{n-1})$ or one of $E(Q_1) \cap E(G_{n-1}), \ldots, E(Q_{n-1}) \cap E(G_{n-1})$. So, applying Lemma 6.7 to G_{n-1}, we contradict the assumption that $\lambda_{M(G_{n-1})}(A'_k) < n - 1$. □

By Lemma 6.3, there exists $S \subseteq A'_k$ and $T \subseteq E(G_{n-1}) - A'_k$ such that $|S| + 1 = |T| + 1 = \kappa_{M(G_{n-1})}(S, T) \geq n - 1$. Now, by Tutte's Linking Theorem, there exists a minor H of G_{n-1} such that $E(H) = S \cup T$ and $\lambda_{M(H)}(S) \geq n$. Suppose that $H = G_{n-1} \setminus D/C$; we may choose D and C such that D does not contain a cut of G_n. Thus H is connected and S and T are disjoint spanning trees of H; thus $|V(H)| \geq n - 1$. Now let $H' = G_n \setminus D/H$. Vertices $(1, n)$ and $(n, 1)$ both have a neighbour in $V(H)$ in H'. Note that there exist $e \in (E(P_n) \cup E(Q_n)) \cap A_k$ and $f \in (E(P_n) \cup E(Q_n)) - A_k$. Now there exists a minor H'' of H' such that $S \cup \{e\}$ and $T \cup \{f\}$ are disjoint spanning trees of H''. Thus $\lambda_{M(H'')}(S \cup \{f\}) \geq n$. However, this contradicts the fact that $\lambda_M(A_k) < n$. □

7. A grid in a tangle

Let M be a matroid and let N be a minor of M that is isomorphic to the cycle matroid of the n by n grid. Now let $\mathcal{T}_N$ be the tangle in N of order n given by Lemma 6.1 and let $\mathcal{T}_M$ be the tangle in M of order n that is induced by $\mathcal{T}_N$. (We recall that the term "induced" was defined at the start of Section 5 and the term "truncation" was defined at the start of Section 4.) A tangle $\mathcal{T}$ in M is said to *dominate* N if $\mathcal{T}_M$ is a truncation of $\mathcal{T}$. In this section we prove Theorem 1.2. We need the following lemma. (We use the "tangle matroid" which is defined at the end of Section 3.)

Lemma 7.1. *Let $\mathcal{T}$ be a tangle in a matroid M and let $M_{\mathcal{T}}$ be the tangle matroid of $\mathcal{T}$. Now let G_n be the n by n grid and suppose that $N = M(G_n)$ is a minor of M. Then $\mathcal{T}$ dominates N if and only if each of the sets $E(P_1), \ldots, E(P_n)$ is independent in $M_{\mathcal{T}}$.*

Proof. Note that, if $\mathcal{T}'$ is the truncation of $\mathcal{T}$ to order n, then $M_{\mathcal{T}'}$ is the truncation of $M_{\mathcal{T}}$ to rank $n-1$. Thus, by possibly truncating, we may assume that $\mathcal{T}$ has order n. Now let $\mathcal{T}_n$ be the tangle in N of order n given by Lemma 6.1 and let $\mathcal{T}_M$ be the tangle in M of order n that is induced by $\mathcal{T}_N$. Thus $\mathcal{T}$ dominates N if and only if $\mathcal{T} = \mathcal{T}_M$. Now $\mathcal{T} \neq \mathcal{T}_M$ if and only if there exists a set $A \in \mathcal{T}$ that contains one of $E(P_1), \ldots, E(P_n)$. On the other hand, $E(P_i)$ is independent in $M_{\mathcal{T}}$ if and only if there does not exist $A \in \mathcal{T}$ such that $E(P_i) \subseteq A$. □

We also need the following result from [4].

Theorem 7.2. *There exists an integer-valued function $f(k, q)$ such that for any positive integer k and prime-power q, if M is a $GF(q)$-representable matroid with branch-width at least $f(k, q)$, then M contains a minor isomorphic to $M(G_k)$.*

Note that, if M has a tangle of high order, then M has large branch-width and, hence by Theorem 7.2, M has a big grid as a minor. Unfortunately, this grid-minor need not be dominated by the tangle.

7.3. *Proof of Theorem 1.2.* Let $g(t) = (6^t - 1)/5$ for any integer $t \geq 0$. Let $n = g(k-1) + 2$, let q be the order of $\mathbb{F}$, and let $\theta = f(n, q)$. Now let M be an $\mathbb{F}$-representable matroid and let $\mathcal{T}$ be a tangle in M of order θ. By Theorem 5.2, there exists a (θ, g)-connected minor M_1 of M and a tangle $\mathcal{T}_1$ in M_1 of order θ such that $\mathcal{T}$ is the tangle in M that is induced by $\mathcal{T}_1$. By Theorem 3.1 and Theorem 7.2, there exists a minor N of M_1 that is isomorphic to $M(G_n)$. By possibly relabeling, we may assume that $N = M(G_n)$. Now let $P_1, \ldots, P_n$ be the vertical paths in G_n, let $M_{\mathcal{T}_1}$ be the tangle matroid of $\mathcal{T}_1$, and let ϕ_1 be the rank-function of $M_{\mathcal{T}_1}$.

7.3.1. $\phi_1(E(P_i)) \geq k - 1$ *for each* $i \in \{1, \ldots, n\}$.

Subproof. Suppose to the contrary that $\phi_1(E(P_i)) < k - 1$ for some i. Thus there exists $A \in \mathcal{T}_1$ such that $E(P_i) \subseteq A$ and $\lambda_{M_1}(A) \leq k - 1$. By definition $|A| \geq |E(P_i)| = n - 1 > g(k-1)$. Therefore, since M_1 is (θ, g)-connected, $|E(M_1) - A| \leq g(k-1) = n - 2 \leq f(n, q) - 2 < \theta - 1$. Moreover, as $k \geq 1$, we have that $\theta \geq 3$. Hence by Lemma 3.2, $E(M_1) - A \in \mathcal{T}_1$; contradicting $(T3)$. □

For each $i \in \{1, \ldots, k\}$, let A_i be an $M_{\mathcal{T}_1}$-independent subset of $E(P_{1+(i-1)k})$ with $|A_i| = k - 1$; as $k^2 - k + 1 \leq n$ these sets A_i exist.

Now there exists a minor H of G_n such that H is isomorphic to G_k and such that $A_1, \dots, A_k$ are the edge-sets of the vertical paths in H. By Lemma 7.1, $\mathcal{T}_1$ dominates H. Then, since $\mathcal{T}$ is induced by $\mathcal{T}_1$, $\mathcal{T}$ also dominates H. □

8. Tree-decompositions and laminar families

We begin by reviewing some elementary results on laminar families and tree-decompositions. Let E be a set. A partition of E into two sets is called a *separation* of E. Two separations (A_1, A_2) and (B_1, B_2) of a set E are said to *cross* if $A_i \cap B_j \neq \emptyset$ for each i and j in $\{1, 2\}$. A collection $\mathcal{S}$ of separations of E is *laminar* if no two separations in $\mathcal{S}$ cross.

A *tree-decomposition* of E consists of a pair $(T, \mathcal{P})$ where T is a tree and $\mathcal{P} = (P_v : v \in V(T))$ is a partition of E (where one or more of the P_v may be empty). For any $X \subseteq V(T)$, we let $\mathcal{P}[X]$ denote the set $\cup_{v \in X} P_v$. Now, for any $e \in E(T)$, the *separation of E displayed by e* is $(\mathcal{P}[V(T_1)], \mathcal{P}[V(T_2)])$ where T_1 and T_2 are the two components of $T - e$. The following result is both easy and well-known.

Lemma 8.1. *If $(T, \mathcal{P})$ is a tree-decomposition of E, then the set of all separations displayed by $(T, \mathcal{P})$ is laminar.*

Let $(T, \mathcal{P})$ be a tree-decomposition of E and let $\mathcal{S}$ be a set of separations of E. We say that $(T, \mathcal{P})$ *represents* $\mathcal{S}$ if $\mathcal{S}$ is the set of separations displayed by $(T, \mathcal{P})$. The following converse to Lemma 8.1 is also well-known.

Lemma 8.2. *If $\mathcal{S}$ is a laminar set of separations of E, then there is a tree-decomposition of E that represents $\mathcal{S}$.*

Let K be a connectivity system. A set $X \subseteq E(K)$ is *robust* if for each proper partition (X_1, X_2) of X either $\lambda_K(X_1) > \lambda_K(X)$ or $\lambda_K(X_2) > \lambda_K(X)$. (A partition is *proper* if all its members are nonempty.) A separation (X, Y) of K is *robust* if X and Y are both robust.

Lemma 8.3. *Let K be a connectivity system and let $\mathcal{S}$ be the set of all robust separations of K. Then $\mathcal{S}$ is laminar.*

Proof. Suppose that $(A_1, A_2), (B_1, B_2) \in \mathcal{S}$ cross. By symmetry, we may assume that $\lambda_K(A_1) \leq \lambda_K(B_1)$. As λ_K is symmetric, we may assume that $\lambda_K(A_2 \cap B_2) \geq \lambda_K(A_1 \cap B_2)$; otherwise swap A_1 and A_2. Then, since B_2 is robust, $\lambda_K(A_2 \cap B_2) > \lambda_K(B_2)$. So symmetry and submodularity of λ_K yield $\lambda_K(A_1 \cap B_1) \leq \lambda_K(A_1) + \lambda_K(B_1) - \lambda_K(A_1 \cup B_1) = \lambda_K(A_1) + \lambda_K(B_2) - \lambda_K(A_2 \cap B_2) < \lambda_K(A_1)$. So, since A_1 is robust, $\lambda_K(A_1 \cap B_2) > \lambda_K(A_1)$. Also, as $\lambda_K(B_1) \geq \lambda_K(A_1) \geq \lambda_K(A_1 \cap B_1)$ and as B_1 are robust, $\lambda_K(A_2 \cap B_1) > \lambda_K(B_1)$. Combining the last two strict inequalities we get $\lambda_K(A_1 \cap B_2) + \lambda_K(A_2 \cap B_1) > \lambda_K(A_1) + \lambda_K(B_1) = \lambda_K(A_1) + \lambda_K(B_2)$. As $\lambda_K(A_2 \cap B_1) = \lambda_K(A_1 \cup B_2)$, this contradicts submodularity. □

9. Tree-representations of maximal tangles

The main result of this section is Theorem 9.1; when applied to the maximal tangles $\mathcal{T}_1, \dots, \mathcal{T}_n$ of the matroid, those that are not truncations of others, it is the result alluded to in the introduction by 1.1.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two tangles in a connectivity system K, neither of which is a truncation of the other, then there exists a *distinguishing separation* (X_1, X_2) with $X_1 \in \mathcal{T}_1$ and $X_2 \in \mathcal{T}_2$.

Theorem 9.1. *Let K be a connectivity system and let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be tangles in K, none of which is a truncation of another. Then there exists a tree-decomposition $(T, \mathcal{P})$ of $E(K)$ such that $V(T) = \{1, \dots, n\}$ and such that the following hold:*
(i) For each $i \in V(T)$ and $e \in E(T)$ if T' is the component of $T - e$ containing i then $\mathcal{P}[V(T')]$ is not in $\mathcal{T}_i$.
(ii) For each pair of distinct vertices i and j of T, there exists a minimum-order distinguishing separation for $\mathcal{T}_i$ and $\mathcal{T}_j$ that is displayed by T.

Let K and K' be connectivity systems with $E(K) = E(K')$. We call K' a *tie-breaker* for K if for each $X, Y \subseteq E(K)$:

(i) $\lambda_{K'}(X) \neq \lambda_{K'}(Y)$ unless $X = Y$ or $X = E(K) - Y$,
(ii) $\lambda_{K'}(X) < \lambda_{K'}(Y)$ if $\lambda_K(X) < \lambda_K(Y)$.

Lemma 9.2. *Each connectivity system has a tie-breaker.*

Proof. Let K be a connectivity system. We may assume that $E(K) = \{1, \dots, n\}$. Now, for $X \subseteq \{1, \dots, n-1\}$, let $\lambda_L(X) = \sum_{i \in X} 2^i$ and let $\lambda_L(E(K) - X) = \lambda_L(X)$. We leave it to the reader to verify that $L = (E(K), \lambda_L)$ is indeed a connectivity system. Now, for each $X \subseteq E(K)$, we let $\lambda_{K'}(X) = 2^n \lambda_K(X) + \lambda_L(X)$. It is easy to check that $K' = (E(K), \lambda_{K'})$ has the desired properties. □

It is evident that a tangle in a connectivity system K is a tangle in any tie-breaker for K.

Lemma 9.3. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be tangles in a connectivity system K that are incomparable by truncation, let K' be a tie-breaker for K, and let (X_1, X_2) be a distinguishing separation for $\mathcal{T}_1$ and $\mathcal{T}_2$ with minimum order in K'. Then (X_1, X_2) is a robust separation of K'.*

Proof. Suppose otherwise. Then, by symmetry, we may assume that there exists a proper partition (A, B) of X_1 such that $\lambda_{K'}(A) \le \lambda_{K'}(X_1)$ and $\lambda_{K'}(B) \le \lambda_{K'}(X_1)$. Since K' is a tie-breaker, $\lambda_{K'}(A) < \lambda_{K'}(X_1)$ and $\lambda_{K'}(B) < \lambda_{K'}(X_1)$. Condition $(T3a)$ for $\mathcal{T}_1$ implies that $A, B \in \mathcal{T}_1$. Then, by our choice of the distinguishing separation (X_1, X_2), $\mathcal{T}_2$ contains neither $E(K) - A$ nor $E(K) - B$. Then, by $(T2)$, $A, B \in \mathcal{T}_2$. But then $\mathcal{T}_2$ contains each of A, B, and X_2; contrary to $(T3)$. □

Proof of Theorem 9.1. Let K' be a tie-breaker for K. As $\mathcal{T}_1, \dots, \mathcal{T}_n$ are tangles in K', we may assume that $K = K'$. For each $i, j \in \{1, \dots, n\}$ with $i \ne j$ let (X_{ij}, Y_{ij}) be the minimum-order separation of K distinguishing $\mathcal{T}_i$ and $\mathcal{T}_j$ (where we assume that $X_{ij} \in \mathcal{T}_i$). By Lemma 9.3, (X_{ij}, Y_{ij}) is a robust separation of K. Now let $\mathcal{S}$ be the collection of all of these distinguishing separations. By Lemma 8.3, $\mathcal{S}$ is laminar. Then, by Lemma 8.2, there is a tree-decomposition $(T, \mathcal{P})$ of $E(K)$ that represents $\mathcal{S}$. We may assume that if v is a vertex of T with degree 1 or 2, then $P_v \ne \emptyset$ (since, otherwise, we could find a smaller tree-decomposition representing $\mathcal{S}$). This means that the edges of T display proper and distinct separations. It remains to show that there is a bijection between $\mathcal{T}_1, \dots, \mathcal{T}_n$ and $V(T)$ satisfying the conclusion of Theorem 9.1.

For $i = \{1, \dots, n\}$, consider the collection $\mathcal{X}_i$ of nonempty subsets X of $V(T)$ such that $E(K) - \mathcal{P}[X] \in \mathcal{T}_i$ and such that $(\mathcal{P}[X], E(K) - \mathcal{P}[X])$ is displayed by T. Each member of $\mathcal{X}_i$ induces a subtree of T and by $(T3)$ each two members of $\mathcal{X}_i$ intersect. As any collection of pairwise intersecting subtrees of a tree has a common vertex, the members of $\mathcal{X}_i$ have a nonempty intersection. Call that intersection V_i.

Note that by construction of V_i each edge of T that leaves V_i displays a separation (A, B) with $\mathcal{P}[V_i] \subseteq A$ and $B \in \mathcal{T}_i$. From this, $(T2)$, $(T3)$ and the fact that each separation in $\mathcal{S}$ is displayed by T it is straightforward to see that to prove Theorem 9.1 it suffices to show that $(V_1, \dots, V_n)$ is a partition of $V(T)$ into singletons.

The sets $V_1, \dots, V_n$ are pairwise disjoint as for each $i \ne j$ the set $\mathcal{P}[V_i]$ lies in Y_{ij} and the set $\mathcal{P}[V_j]$ lies in $Y_{ji} = X_{ij}$.

It remains to prove that if w in $V(T)$ then $\{w\} = V_i$ for some i. Among the edges incident with w take the one that displays the separation, (X_{ij}, Y_{ij}) say, of largest order. So that order is at most the order of $\mathcal{T}_i$ and of $\mathcal{T}_j$. We may assume that $\mathcal{P}_w \subseteq Y_{ij}$. As no two edges of T display the same separation, all other edges incident with w display a separation of order less than those of $\mathcal{T}_i$ and $\mathcal{T}_j$. By the definition of (X_{ij}, Y_{ij}) these separations do not distinguish $\mathcal{T}_i$ from $\mathcal{T}_j$. Combining that with $(T3)$ for $\mathcal{T}_j$, we see that for each of these separations $\mathcal{P}_w$ is not part of the side that is in $\mathcal{T}_i$. Hence $V_i \subseteq \{w\}$. As V_i is not empty, $\{w\} = V_i$ as claimed. □

We conclude with a simple corollary to Theorem 9.1.

Corollary 9.4. *An m-element connectivity system has at most $\frac{m-2}{2}$ maximal tangles.*

Proof. Let K be an m-element connectivity system and let $\mathcal{T}_1, \dots, \mathcal{T}_n$ be the maximal tangles in K. Now let $(T, \mathcal{P})$ be the tree-decomposition of $E(M)$ given by Theorem 9.1. Let v be a vertex of T of degree d_v. By $(T3)$ and $(T4)$, $d_v + |P_v| \geq 4$. Now $4n \leq \sum_{i=1}^{n}(d_i + |P_i|) = 2|E(T)| + |E(M)| = 2(n-1) + m$. So $n \leq \frac{m-2}{2}$ as claimed. □

Acknowledgement

We thank the referees for carefully reading this paper.

References

[1] J.S. Dharmatilake, *A min-max theorem using matroid separations*, Matroid Theory (Seattle, WA, 1995), 333-342, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.

[2] J.F. Geelen, A.M.H. Gerards, N. Robertson, and G.P. Whittle, *On the excluded-minors for the matroids of branch-width k*, J. Combin. Theory, Ser. B **88** (2003), 261-265.

[3] J. Geelen, B. Gerards, N. Robertson, and G. Whittle, *Obstructions to branch-decomposition of matroids*, J. Combin. Theory, Ser. B **96** (2006), 560-570.

[4] J. Geelen, B. Gerards, and G. Whittle, *Excluding a planar graph from GF(q)-representable matroids*, J. Combin. Theory, Ser. B **97** (2007), 971-998.

[5] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[6] N. Robertson, P.D. Seymour, and R. Thomas, *Quickly excluding a planar graph*, J. Combin. Theory, Ser. B **62** (1994), 323-348.

[7] N. Robertson, and P.D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory, Ser. B **52** (1991), 153-190.

[8] N. Robertson, and P.D. Seymour, *Graph minors. XVI. Excluding a non-planar graph*, J. Combin. Theory, Ser. B **89** (2003), 43-76.

[9] W.T. Tutte, *Menger's theorem for matroids*, Journal of Research of the National Bureau of Standards–B. Mathematics and Mathematical Physics, **69B** (1965), 49-53.

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada

Centrum voor Wiskunde en Informatica, Amsterdam, The Netherlands and Technische Universiteit Eindhoven, Eindhoven, The Netherlands

School of Mathematical and Computing Sciences, Victoria University, Wellington, New Zealand