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Partial Fields and Matroid Representation

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A partial field $\mathbf{P}$ is an algebraic structure that behaves very much like a field except that addition is a partial binary operation, that is, for some $a, b \in \mathbf{P}$, $a + b$ may not be defined. We develop a theory of matroid representation over partial fields. It is shown that many important classes of matroids arise as the class of matroids representable over a partial field. The matroids representable over a partial field are closed under standard matroid operations such as the taking of minors, duals, direct sums, and 2-sums. Homomorphisms of partial fields are defined. It is shown that if $\varphi\colon \mathbf{P_1} \to \mathbf{P_2}$ is a non-trivial partial-field homomorphism, then every matroid representable over $\mathbf{P_1}$ is representable over $\mathbf{P_2}$. The connection with Dowling group geometries is examined. It is shown that if G is a finite abelian group, and $r > 2$, then there exists a partial field over which the rank-r Dowling group geometry is representable if and only if G has at most one element of order 2, that is, if G is a group in which the identity has at most two square roots.

1. INTRODUCTION

It follows from a classical result of Tutte [19] that a matroid is representable over $GF(2)$ and some field of characteristic other than 2 if and only if it can be represented over the rationals by the columns of a totally unimodular matrix, that is, by a matrix over the rationals all of whose non-zero subdeterminants are in $\{1, -1\}$. Consider the analogous problem for matroids representable over $GF(3)$ and other fields. It is shown in [22, 23] that essentially three new classes arise. Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if all non-zero subdeterminants are in $\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}$. A *near-regular* matroid is one that can be represented over the rationals by a near-unimodular matrix. A matrix over the rationals is *dyadic* if all non-zero subdeterminants are in $\{\pm 2^i : i \in Z\}$. A *dyadic matroid* is one that can be represented over the rationals by a dyadic

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matrix. A matrix over the complex numbers is a $\sqrt[6]{1}$*-matrix* if all non-zero subdeterminants are complex sixth roots of unity. A *matroid* is one that can be represented over the complex numbers by a $\sqrt[6]{1}$-matrix. It is shown in [22, 23] that if $\mathbf{F}$ is a field other than $GF(2)$ whose characteristic is not 3, then the class of matroids representable over $GF(3)$ and $\mathbf{F}$ is the class of near-regular matroids, the class of dyadic matroids, the class of $\sqrt[6]{1}$-matroids, or the class of matroids obtained by taking direct sums and 2-sums of dyadic matroids and $\sqrt[6]{1}$-matroids.

The striking thing about the above classes is that they are all obtained by restricting the values of non-zero subdeterminants in a particular way. Let G be a subgroup of the multiplicative group of a field $\mathbf{F}$ with the property that for all $g \in G$, $-g \in G$. A $(G, \mathbf{F})$*-matroid* is one that can be represented over $\mathbf{F}$ by a matrix, all of whose non-zero subdeterminants are in G. For appropriate choices of field and subgroup, the classes of regular, near-regular, dyadic, and $\sqrt[6]{1}$-matroids are all $(G, \mathbf{F})$-matroids. Given the significance of these classes it is clear that a general study of $(G, \mathbf{F})$-matroids is justified, particularly when one considers the natural conjecture that the matroids representable over all members of any given set of fields can be obtained by taking direct sums and 2-sums of members of appropriate classes of $(G, \mathbf{F})$-matroids. In fact the research that led to this paper began as a study of $(G, \mathbf{F})$-matroids, but it soon became apparent that a further level of generality was appropriate.

Consider a field $\mathbf{F}$ and a subgroup G of $\mathbf{F}^*$ such that $-g \in G$ for all $g \in G$. Then $G \cup \{0\}$ with the induced operations from $\mathbf{F}$ behaves very much like a field except for the fact that, for some $a, b \in G$, $a + b$ may not be defined. We axiomatise such structures via the notion of "partial field" in Section 2. Subgroups of fields give rise to partial fields in the way described above, but many partial fields cannot be embedded in a field. In Section 3 we consider determinants of matrices over partial fields. In general the determinant of a square matrix need not be defined. It is shown that if A is a matrix over a partial field that has the property that all of its square submatrices have defined determinants, then a well-defined matroid can be associated with A. A matroid is representable over the partial field if it can be obtained in such a way. The classes of regular, near-regular, dyadic, and $\sqrt[6]{1}$-matroids can all be interpreted as classes of matroids representable over a partial field. It is also shown that the class of matroids representable over a given partial field is minor-closed and is closed under the taking of duals, direct sums, and 2-sums.

Section 5 considers homomorphisms. There are several ways to define a homomorphism of a partial algebra. It turns out that the weakest is strong enough to give significant information about the matroids representable over partial fields. It is shown that if $\varphi: \mathbf{P_1} \rightarrow \mathbf{P_2}$ is a non-trivial partial-field homomorphism, then the class of matroids representable over $\mathbf{P}_1$ is contained in the class of matroids representable over $\mathbf{P}_2$. In Section 6 it is shown that a theory for equivalence of representations over partial fields can be developed that is similar to that for fields.

In Section 7 the connection with Dowling group geometries is considered. Among other things it is shown that if G is a finite abelian group, and $r > 2$, then there exists a partial field over which the rank-r Dowling group geometry is representable if and only if G has at most one element of order 2, that is, if G is a group in which the identity has at most two square roots.

It is almost certainly possible that this theory could be generalised to non-commutative structures, that is, to partial division rings. The theory of determinants of these structures could be based on the theory of determinants of division rings, see for example [1, 5]. However, such a theory would involve several additional technicalities. Since we do not know of any major combinatorial motivation to extend the theory to non-commutative structures it was felt that the generalisation was not justified.

The real motivation for developing the theory of this paper is the desire to solve some of the many outstanding problems in matroid representation theory. With current techniques it seems we can, at best, chip away at the fringes of these problems: new techniques are desperately needed. We hope that the theory of partial fields will assist in the development of such techniques.

2. PARTIAL FIELDS

Recall that a *partial function* on a set S is a function whose domain is a subset of S. It follows that a *partial binary operation* on S is a function $+: A \to S$ whose domain is a subset A of $S \times S$. If $(a, b) \in A$ then $a + b$ is defined, otherwise $a + b$ is not defined.

Let G be a subgroup of the multiplicative group of a field $\mathbf{F}$ with the property that for all $g \in G$, $-g \in G$, and consider $G \cup \{0\}$ together with the induced operations from $\mathbf{F}$. It was noted in the Introduction that, except for the fact that $+$ is a partial operation, $G \cup \{0\}$ behaves very much like a field. We have an additive identity, additive inverses, and versions of the distributive and associative laws. In seeking to axiomatise such structures independently of the embedding field only the version of the associative law causes difficulty. A natural way to attempt such a law is as follows. If $a + (b + c)$ is defined and $a + b$ is defined, then $(a + b) + c$ is defined and $(a + b) + c = a + (b + c)$. This is all very well but one does not just want three term sums to associate. Consider the expressions $(a + b) + (c + d)$ and $(a + c) + (b + d)$. Assume that $(a + b) + (c + d)$

is defined (this means that all sums in the expression are defined) and assume that $a + c$ and $b + d$ are defined. One would certainly want this to imply that $(a + c) + (b + d)$ is defined and to have $(a + b) + (c + d) = (a + c) + (b + d)$. For fields this is an immediate consequence of the associative law for sums with three terms, but if $+$ is a partial operation this is not the case. For this reason we need a more complicated associative law.

Let S be a set with a commutative partial binary operation $+$. Say S' is a finite multiset of elements of S. An *association* of the multiset S' is a way of unambiguously defining sums to obtain an expression that is a version of the sum of the elements of S'. (This definition does not purport to be even vaguely precise. The reader can easily see how a precise definition could be given in terms of rooted binary trees with their terminal vertices labelled by the members of S'.) An association of S' is *defined* if all of the sums in the expression are defined. We now define what it means for the associative law to hold. Let S' be any finite multiset of elements of S and consider any defined association of S', the result of performing the sums being s. Consider any other association of S' that has the property that all sums apart possibly from the final sum are defined. The *associative law* holds if in all such cases the final sum is indeed defined, the result being equal to s. Assume that the associative law holds for $+$. To say that $a_1 + a_2 + \cdots + a_n$ is defined means that some association of $\{a_1, a_2, \ldots, a_n\}$ has all sums defined.

Let $\mathbf{P}$ be a set with a distinguished element called 0, and set $\mathbf{P}^* = \mathbf{P} - \{0\}$. Let $\circ$ be a binary operation on $\mathbf{P}$, and $+$ be a partial binary operation on $\mathbf{P}$. Then $\mathbf{P}$ is a partial field if the following properties are satisfied:

(P1) $\mathbf{P}^*$ is an abelian group under $\circ$.

(P2) For all $a \in \mathbf{P}$, $a + 0 = a$.

(P3) For all $a \in \mathbf{P}$, there exists an element $-a \in \mathbf{P}$ with the property that $a + (-a) = 0$.

(P4) For all $a, b \in \mathbf{P}$, if $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(P5) For all $a, b, c \in \mathbf{P}$, $a(b + c)$ is defined if and only if $ab + ac$ is defined; in this case $a(b + c) = ab + ac$.

(P6) The associative law holds for $+$.

The above definition already uses some standard ring-theoretic notational conventions; for example, ab denotes $a \circ b$. We will continue to use such conventions without comment. The terminology adopted here more or less agrees with that of Grätzer [8, Chapter 2]. In particular our notion of partial binary operation agrees with [8]. Grätzer defines the notion of "partial algebra." Our partial fields are special cases of partial algebras.

Certain elementary properties of rings hold for partial fields. The proofs are essentially the same as for rings. The only difficulty is that one has to ensure at each stage that sums are defined. In particular we have:

PROPOSITION 2.1. *If a and b are elements of a partial field, then* $a0 = 0 = 0a$, $(-a)b = a(-b) = -(ab)$, *and* $(-a)(-b) = ab$.

The motivation for studying partial fields arose from examples obtained from subsets of fields. We certainly need to show that if G is a subgroup of the multiplicative group of a field such that $-g \in G$ for all $g \in G$, then $G \cup \{0\}$ with the induced operations is a partial field. More generally we have

PROPOSITION 2.2. *Let* $\mathbf{P}$ *be a partial field, and let G be a subgroup of* $\mathbf{P}^*$ *with the property that* $-a \in G$ *for all* $a \in G$. *Then* $G \cup \{0\}$ *with the induced operations from* $\mathbf{P}$ *is a partial field.*

Proof. The only property that is not immediate is **P5**. Say $a, b, c \in (G \cup \{0\})$. Then $ab + ac \in (G \cup \{0\})$ if and only if $a(b + c)$ is, and $a(b + c) \in G \cup \{0\}$ if and only if $a^{-1}a(b + c) = b + c$ is. It follows routinely from this observation that **P5** holds. ∎

Of course Proposition 2.2 holds when $\mathbf{P}$ is a field. The partial field obtained in Proposition 2.2 is denoted $(G, \mathbf{P})$.

With Proposition 2.2 in hand we simplify terminology for some of our most fundamental classes. The partial field $(\{1, -1\}, \mathbf{Q})$ leads to the class of regular matroids. Set $\mathbf{Reg} = (\{1, -1\}, \mathbf{Q})$. The partial field $(\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$ leads to the class of near-regular matroids. Set $\mathbf{NR} = (\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$. The partial field $(\{\pm 2^i : i \in Z\}, \mathbf{Q})$ leads to the class of dyadic matroids. Set $\mathbf{D} = (\{\pm 2^i : i \in Z\}, \mathbf{Q})$. Let G_6 denote the group of complex sixth roots of unity. The partial field $(G_6, \mathbf{C})$ leads to the class of $\sqrt[6]{1}$-matroids. Since the multiplicative group of $(G_6, \mathbf{C})$ has order 6, set $\mathbf{P}_6 = (G_6, \mathbf{C})$. Anticipating the definition of "isomorphism" given in Section 5 we will use **Reg**, **NR**, **D**, and $\mathbf{P_6}$ to denote any member of their respective isomorphism classes.

It is time for some examples to illustrate some elementary, but important, facts. Note that the partial fields obtained from fields via Proposition 2.2 depend on both the group and the field. For example $(\{-1, 1\}, GF(3)) = GF(3)$, while $\mathbf{Reg} = (\{-1, 1\}, \mathbf{Q})$ is quite a different structure. The point is that $1 + 1$ and $-1 - 1$ are not defined in **Reg**.

For a less trivial example consider possible partial fields having $G_6 = \{a : a^6 = 1\}$ as their multiplicative group. Define $\mathbf{P_T}$ as follows. Let $-1 = a^3$, $-a = a^4$, and $-a^2 = a^5$. Then, in $\mathbf{P_T}$, $x + y$ is defined if and only if $x = -y$, in which case, $x + y = 0$. Of course we define $x + 0 = 0 + x = x$. Routine checking shows that $\mathbf{P_T}$ is a partial field. Note that the operation of addition in $\mathbf{P_T}$ is as trivial as it could be. Another partial field is $\mathbf{P_6}$, which is obtained by embedding G_6 as a subgroup of the complex numbers. Clearly, $\mathbf{P_6}$ has a partial addition that is an augmented version of the partial addition of $\mathbf{P_T}$. We now also have $a^2 + 1 = a$, and $a^4 + 1 = a^5$. Yet another partial field is obtained by embedding G_6 as a subgroup of the multiplicative group of $GF(7)$. Of course, this partial field is just $GF(7)$ itself. Note that whenever G_6 is embedded as a subgroup of the multiplicative group of a field the relations $a^2 + 1 = a$ and $a^4 + 1 = a^5$ hold so that $\mathbf{P_T}$ cannot be embedded in any field.

3. PARTIAL FIELDS AND MATROIDS

Our interest in partial fields is essentially due to the fact that classes of matroids can be associated with them. Let $\mathbf{P}$ be a partial field. Consider column vectors with entries in $\mathbf{P}$. If $\mathbf{x} = (x_1, x_2, \dots, x_m)^t$ is such a vector and $a \in \mathbf{P}$, then, obviously, we define the *scalar multiple* $a\mathbf{x}$ of $\mathbf{x}$ by $a\mathbf{x} = (ax_1, ax_2, \dots, ax_m)^t$. One can also define the sum of two vectors in the obvious way; such a sum will be defined only if the sum is defined for each coordinate. To associate matroids with partial fields, one needs to have a criterion to decide whether a set of vectors is independent. The familiar way from vector spaces is to use linear combinations. Such a notion for partial fields will be a partial operation, and it is unclear how one deals with the existence of undefined linear combinations in attempting to decide whether a set of vectors is independent. An alternative approach is to use determinants. A set of vectors $\{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}\}$ in a vector space over a field is independent if and only if at least one of the $n \times n$ submatrices of the matrix $[\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}]$ has a non-zero determinant. This is the approach we generalise to partial fields.

Let A be an $n \times n$ square matrix with entries in a partial field $\mathbf{P}$. Just as with fields we define the determinant to be a signed sum of products determined by permutations. Let p be an element of S_n, the group of permutations of $\{1, 2, \dots, n\}$. Then $\varepsilon(p)$ denotes the sign of p. Formally, the *determinant* of A is defined by

$$\det(A) = \sum_{p \in S_n} \varepsilon(p) a_{1p(1)} a_{2p(2)} \cdots a_{np(n)},$$

if this sum is defined. The arguments that prove the following proposition are essentially the same as those for fields.

PROPOSITION 3.1. *Let A be a square matrix with entries in a partial field $\mathbf{P}$.*

(i) *If B is obtained from A by interchanging a pair of rows or columns, then $\det(B)$ is defined if and only if $\det(A)$ is defined, in which case $\det(B) = -\det(A)$.*

(ii) *If B is obtained from A by multiplying each entry of a row or a column by a non-zero element k of* $\mathbf{P}$, *then* $\det(B)$ *is defined if and only if* $\det(A)$ *is defined, in which case* $\det(B) = k\det(A)$.

(iii) *If* $\det(A)$ *is defined and B is obtained from A by adding two rows or two columns whose sum is defined, then* $\det(B)$ *is defined and* $\det(B) = \det(A)$.

Other elementary properties of determinants generalise straightforwardly. For example we have

PROPOSITION 3.2. *Let A be a square a matrix with entries in a partial field* $\mathbf{P}$. *Let A_{ij} denote the submatrix obtained by deleting row i and column j from A.*

(i) *If A has a row or a column of zeros, then* $\det(A) = 0$.

(ii) *If a_{ij} is the only non-zero entry in its row or column, then* $\det(A)$ *is defined if and only if* $\det(A_{ij})$ *is defined, in which case* $\det(A) = (-1)^{i+j}a_{ij}\det(A_{ij})a_{ij}\det(A_{ij})$.

With familiar classes such as totally unimodular, near-unimodular, or dyadic matrices a condition is placed on *all* subdeterminants of a matrix. Generalising to partial fields, we require that all subdeterminants be *defined*. An $m \times n$ matrix A over a partial field $\mathbf{P}$ is a $\mathbf{P}$-*matrix* if $\det(A')$ is defined for every square submatrix A' of A. Say A is a $\mathbf{P}$-*matrix*; then a non-empty set of columns $\{\mathbf{c}_{\mathbf{i}_1}, \mathbf{c}_{\mathbf{i}_2}, \ldots, \mathbf{c}_{\mathbf{i}_k}\}$ of A is *independent* if $k \le m$, and at least one of the $k \times k$ submatrices of A with columns indexed by $\{i_1, i_2, \ldots, i_k\}$ has a non-zero determinant. Also an empty set of columns is independent. We aim to show that the independent sets of vectors of a $\mathbf{P}$-matrix are the independent sets of a matroid. In what follows we consider matrices whose columns are labelled by the elements of a set E. A subset of E is independent if the set of columns it labels is independent. We first show that the property of being a $\mathbf{P}$-matrix is preserved under some standard operations. It is assumed that labels are fixed under these operations apart from the operation of interchanging columns where labels are interchanged with the columns. Let x_{st} be a non-zero entry of a matrix A. Recall that a *pivot* on x_{st} is obtained by multiplying row s by $1/x_{st}$ and, for i in $\{1, 2, \ldots, s-1, s+1, \ldots, m\}$, replacing x_{ij} by $x_{st}^{-1}\begin{vmatrix} x_{st} & x_{sj} \\ x_{it} & x_{ij} \end{vmatrix}$.

PROPOSITION 3.3. *Let A be a* $\mathbf{P}$-*matrix. If the matrix B is obtained from A by one of the following operations, then B is a* $\mathbf{P}$-*matrix*:

(i) *interchanging a pair of rows or columns*;

(ii) *replacing a row or column by a non-zero scalar multiple of that row or column*;

(iii) *performing a pivot on a non-zero entry of A.*

Proof. If B is obtained from A by interchanging rows or columns, or by multiplying rows or columns by a non-zero scalar, it follows immediately from Proposition 3.1 that the proposition holds. Assume that B is obtained from A by pivoting on a non-zero entry of A. By (i), we may assume without loss of generality that the entry is a_{11}. Since A is a $\mathbf{P}$-matrix, and since all entries of B are, up to a scalar multiple, equal to subdeterminants of A, it follows that B is defined. We now show that B is a $\mathbf{P}$-matrix.

Let A' and B' be corresponding square submatrices of A and B, respectively, each having their rows and columns indexed by the sets J_{R} and J_{C}, respectively. We want to show that $\det(B')$ is defined. If $1 \in J_{\mathrm{R}}$, then it follows from Proposition 3.1 that $\det(B')$ is defined. Hence we may assume that $1 \notin J_{\mathrm{R}}$. In this case, if $1 \in J_{\mathrm{C}}$, then B' has a zero column, and by Proposition 3.2, $\det(B')$ is defined with $\det(B') = 0$. Thus we may also assume that $1 \notin J_{\mathrm{C}}$. Now let A'' and B'' be the submatrices of A and B whose rows and columns are indexed by $J_{\mathrm{R}} \cup \{1\}$ and $J_{\mathrm{C}} \cup \{1\}$. By the above, $\det(B'')$ is defined. The only non-zero entry in column 1 of B'' is b''_{11}. Hence, by Proposition 3.2, $\det(B')$ is defined. In all cases $\det(B')$ is defined and we conclude that B is a $\mathbf{P}$-matrix. ∎

The following lemma has an obvious geometric interpretation for matrices over fields and a straightforward inductive proof. This proof generalises immediately to partial fields.

LEMMA 3.4. *Let A be an $(n+1) \times n$ $\mathbf{P}$-matrix, where $n \geq 2$, and assume that each row of A has a non-zero entry. Let B be an $n \times n$ submatrix of A. If all other $n \times n$ submatrices of A have zero determinant, then* $\det(B) = 0$.

PROPOSITION 3.5. *The independent sets of a $\mathbf{P}$-matrix are preserved under the operations of interchanging a pair of rows or columns, multiplying a column or a row by a non-zero scalar, and performing a pivot on a non-zero entry of the matrix.*

Proof. Say B is obtained from the $\mathbf{P}$-matrix A by one of the above operations. By Proposition 3.3, B is a $\mathbf{P}$-matrix so the independent sets of B are defined. If B is obtained by interchanging rows or columns, or by multiplying a row or a column by a non-zero scalar, the result is clear. Assume that B is obtained by performing a pivot on a non-zero entry of A. Without loss of generality assume that this entry is a_{11}. Let A' and B' be corresponding submatrices of A and B with the assumption that A' meets all rows of A. In other words, A' and B' consist of columns of A and B, respectively. Say $|J_{\mathrm{C}}| = k$. Assume that the columns of A' are independent. Then some $k \times k$ submatrix A'' of A' has a non-zero determinant. If this submatrix contains the first row of A'', then by Proposition 3.1 the corresponding submatrix B'' of B' also has a non-zero determinant and the columns of B' are independent. Assume that A'' does not contain the first row of A'. If the first row of A' consists of zeros, then the pivot has no effect on A', and again the columns of B' are independent. Assume that there is a non-zero entry in the first row of A'. Let A^+ denote the matrix obtained by adjoining the first row of A' to A''. By Lemma 3.4, A'' is not the only $k \times k$ submatrix of A^+ with a non-zero determinant. Hence A' has a $k \times k$ submatrix with a non-zero determinant that contains the first row of A' and we are in a case that has been covered. It follows that if the columns of A' are independent, then the columns of B' are independent. The argument in the case that the columns of A' are dependent is similar and is omitted. ∎

THEOREM 3.6. *Let A be a* **P**-*matrix whose columns are labelled by a set S. Then the independent subsets of S are the independent sets of a matroid on S.*

Proof. Evidently the empty set is independent. Say I is a nonempty independent subset of S with $|I| = k$. By pivoting, taking scalar multiples, interchanging rows and columns, and applying Proposition 3.5, we may assume without loss of generality that the first k rows of the submatrix of columns labelled by I form an identity matrix. All other rows of this submatrix consist of zeros. It follows immediately that all subsets of I are independent. Now say J is an independent subset of S with $|J| > |I|$. It is easily seen that at least one of the columns labelled by $x \in J$ has a non-zero entry in a row other than the first k rows. Certainly $x \notin I$. It now follows readily that $I \cup \{x\}$ is independent and the theorem is proved. ∎

If A is a **P**-matrix for some partial field **P**, then the matroid obtained via Theorem 3.6 is denoted by $M[A]$. A matroid M is *representable* over **P** or is **P**-*representable* if it is equal to $M[A]$ for some **P**-matrix A; in this case A is said to be a *representation* of M.

4. BASIC PROPERTIES

In this section we show that the class of matroids representable over a fixed partial field shares some of the properties enjoyed by the matroids representable over a field. Let **P** be a partial field. A routine application of Proposition 3.5 proves

PROPOSITION 4.1. *If the matroid M is representable over* **P** *then M can be represented by a* **P**-*matrix of the form* $[I|A]$, *where I is an identity matrix.*

A representation of the form $[I|A]$ is said to be in *standard form*.

PROPOSITION 4.2. *Let M and N be matroids representable over* **P**.

(i) *M^* is representable over* **P**.

(ii) *All minors of M are representable over* **P**.

(iii) *The direct sum of M and N is representable over* **P**.

(iv) *The parallel connection $P(M,N)$ and series connection $S(M,N)$ of M and N relative to any chosen basepoint are* **P**-*representable*.

(v) *The* 2-*sum of M and N is representable over* **P**.

Proof. The proof of (i) is a matter of showing that the standard proof for matroids representable over a field (see [14, Theorem 2.2.8]) works in the more general setting of partial fields. The proofs of (ii) and (iii) are straightforward. Essentially, the proof of (iv) is a matter of showing that a matrix A of the form

$$\left[\begin{array}{c|c|c} A_1 & \begin{matrix}0\\0\\\vdots\\0\end{matrix} & 0 \\ \cline{2-3} & 1 & \\ \cline{1-2} 0 & \begin{matrix}0\\\vdots\\0\\0\end{matrix} & A_2 \end{array}\right],$$

is a **P**-matrix if and only if the matrices

$$B_1 = \left[\begin{array}{c|c} A_1 & \begin{matrix}0\\0\\\vdots\\0\\1\end{matrix}\end{array}\right] \text{ and } B_2 = \left[\begin{array}{c|c} \begin{matrix}1\\0\\\vdots\\0\\0\end{matrix} & A_2\end{array}\right]$$

are both **P**-matrices. Moreover, $M[A]$ represents $P(M[B_1], M[B_2])$. For full details see [16]. It follows immediately from (iv) that (v) holds. ∎

Let G be a subgroup of the multiplicative group of a field **F** with the property that, for all g in G, $-g$ is in G. Recall that a $(G, \mathbf{F})$-matroid is a matroid that can be represented over **F** by a matrix A over **F** with the property that all non-zero subdeterminants of A are in G. Since these classes of matroids form the motivation for the development of partial fields and their associated matroids we certainly need to show that being representable over the partial field $(G, \mathbf{F})$ and being a $(G, \mathbf{F})$-matroid coincide.

PROPOSITION 4.3. *The matroid M is a $(G, \mathbf{F})$-matroid if and only if it is representable over $(G, \mathbf{F})$.*

Proof. It only needs to be shown that a matrix A with entries in $G \cup \{0\}$ is a $(G, \mathbf{F})$-matrix if and only if, regarded as a matrix over $\mathbf{F}$, all non-zero subdeterminants of A are in G. Certainly, if A is a $(G, \mathbf{F})$-matrix, then all non-zero subdeterminants of A are in G. Consider the converse. Assume that A is a matrix over $\mathbf{F}$ such that all non-zero subdeterminants are in G. If A is 1×1 or 2×2, then it is clear that A is a $(G, \mathbf{F})$-matrix. Say $n > 2$ and make the obvious induction assumption. If A is the zero matrix then it is clear that A is a $(G, \mathbf{F})$-matrix. Otherwise perform a pivot on a non-zero element a_{ij} of A. One routinely checks that the resulting matrix A' also has the property that all non-zero subdeterminants are in G. Moreover, by induction, the matrix obtained by deleting row i and column j from A is a $(G, \mathbf{F})$-matrix. It is now easily seen that A' is a $(G, \mathbf{F})$-matrix. But A' is obtained from A by a pivot, so A is also a $(G, \mathbf{F})$-matrix. ∎

It follows from Proposition 4.3 that, as expected, the classes of matroids representable over the partial fields **Reg**, **NR**, **D**, and $\mathbf{P_6}$ are the classes of regular, near-regular, dyadic, and $\sqrt[6]{1}$-matroids, respectively.

5. HOMOMORPHISMS

The study of homomorphisms of partial fields is motivated by the desire to understand the relationships between the classes of matroids representable over them. Our terminology follows that of Grätzer [8, Chapter 2].

Let $\mathbf{P_1}$ and $\mathbf{P_2}$ be partial fields. A function $\varphi\colon \mathbf{P_1} \to \mathbf{P_2}$ is a *homomorphism* if, for all $a, b \in \mathbf{P_1}$, $\varphi(ab) = \varphi(a)\varphi(b)$, and whenever $a + b$ is defined, then $\varphi(a) + \varphi(b)$ is defined, and $\varphi(a + b) = \varphi(a) + \varphi(b)$. Of course, it may be the case that $\varphi(a) + \varphi(b)$ is defined when $a + b$ is not (insisting that $\varphi(a) + \varphi(b)$ is defined if and only if $a + b$ is defined leads to a strictly stronger notion of "homomorphism"). We are interested in the effect that homomorphisms have on represented matroids. For a matrix A over $\mathbf{P_1}$, $\varphi(A)$ denotes the matrix over $\mathbf{P_2}$ whose (i, j)-th entry is $\varphi(a_{ij})$. Of course, the *kernel* of a homomorphism φ is the set $\{a \in \mathbf{P_1} : \varphi(a) = 0\}$. The homomorphism φ is *trivial* if its kernel is equal to $\mathbf{P_1}$.

Grätzer defines three distinct types of homomorphism for partial algebras. The one we have used is the weakest of these. It turns out that this is sufficient for our purposes. Evidently the kernel of a homomorphism φ

contains 0. Say that φ is non-trivial. Then $\varphi(1) = 1$. Moreover, if $a \neq 0$, then $\varphi(a)\varphi(a^{-1}) = \varphi(1) = 1$, so $\varphi(a) \neq 0$. Therefore the kernel of a non-trivial homomorphism contains only 0. The proof of Proposition 5.1 below follows from this observation and from the definitions of "determinant" and "homomorphism."

PROPOSITION 5.1. *Let* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *be partial fields and let* $\varphi{:}\mathbf{P_1} \to \mathbf{P_2}$ *be a homomorphism. Let* A *be a* $\mathbf{P_1}$*-matrix.* (*Recall that this means that determinants are defined for all square submatrices of* A.)

(i) $\varphi(A)$ *is a* $\mathbf{P_2}$*-matrix.*

(ii) *If* A *is square and* $\det(A) = 0$, *then* $\det(\varphi(A)) = 0$.

(iii) *If* A *is square and* φ *is non-trivial*, *then* $\det(A) = 0$ *if and only if* $\det(\varphi(A)) = 0$.

As an immediate consequence of Proposition 5.1 we have

COROLLARY 5.2. *Let* $\mathbf{P_1}$ *and* $\mathbf{P_2}$ *be partial fields and let* $\varphi\colon \mathbf{P_1} \to \mathbf{P_2}$ *be a non-trivial homomorphism. If* A *is a* $\mathbf{P_1}$*-matrix, then* $M[\varphi(A)] = M[A]$.

This in turn immediately implies

COROLLARY 5.3. *If there exists a non-trivial homomorphism* $\varphi\colon \mathbf{P_1} \to \mathbf{P_2}$, *then every matroid representable over* $\mathbf{P_1}$ *is also representable over* $\mathbf{P_2}$.

We now give some examples to illustrate these ideas. The function $\varphi\colon \mathbf{Reg} \to GF(2)$ defined by $\varphi(-1) = \varphi(1) = 1$ and $\varphi(0) = 0$ is easily seen to be a non-trivial homomorphism. The well-known fact that regular matroids are binary follows from the existence of this homomorphism. Consider also the function $\varphi\colon \mathbf{D} \to GF(3)$ defined by $\varphi(\pm 2^i) = \pm(-1)^i$ and $\varphi(0) = 0$. Again it is easily checked that φ is a homomorphism. One can conclude from the existence of this homomorphism that dyadic matroids are ternary. Recall the partial field $\mathbf{P_T}$ defined in Section 2. One readily checks that the identity maps $\iota\colon \mathbf{P_T} \to \mathbf{P_6}$ and $\iota\colon \mathbf{P_6} \to GF(7)$ are homomorphisms. Hence the matroids representable over $\mathbf{P_T}$ are contained in the matroids representable over $\mathbf{P_6}$ and these in turn are contained in the matroids representable over $GF(7)$. It is also easily checked that these containments are proper.

The homomorphism $\varphi\colon \mathbf{P_1} \to \mathbf{P_2}$ is an *isomorphism* if it is a bijection and has the property that $a + b$ is defined if and only if $\varphi(a) + \varphi(b)$ is defined. Note that being an isomorphism is stronger than being a bijective homomorphism. Of course, if $\mathbf{P_1}$ and $\mathbf{P_2}$ are isomorphic, then the class of $\mathbf{P_1}$-matroids is equal to the class of $\mathbf{P_2}$-matroids, but the converse does not hold. A strictly weaker condition that guarantees that two partial fields $\mathbf{P_1}$ and $\mathbf{P_2}$ carry the same class of matroids is that there exists a non-trivial homomorphism $\varphi_1\colon \mathbf{P_1} \to \mathbf{P_2}$ and a non-trivial homomorphism $\varphi_2\colon \mathbf{P_2} \to \mathbf{P_1}$.

It is easily checked that such homomorphisms exist for the partial fields **Reg** and $(\{\pm 3^i : i \in Z\}, \mathbf{Q})$, and these partial fields are certainly not isomorphic.

It follows that a matroid is representable over $(\{\pm 3^i : i \in Z\}, \mathbf{Q})$ if and only if it is regular. (This fact has also been noted in [13].) It is clear that very little is going on in the partial addition of $(\{\pm 3^i : i \in Z\}, \mathbf{Q})$. We make this notion precise. A partial field $\mathbf{P}$ has *trivial* addition if $1 \neq -1$ and, for all $x \in \mathbf{P}$, $x - 1$ is defined if and only if $x \in \{0, 1\}$. The definition of trivial is justified by

PROPOSITION 5.4. *If* $\mathbf{P}$ *has trivial addition, then, for all* $x, y \in \mathbf{P}^*$, $x + y$ *is defined if and only if* $y = -x$.

Proof. Say that $\mathbf{P}$ has trivial addition. Then $x + y$ is defined if and only if $-y(-xy^{-1} - 1)$ is defined, and the latter expression is defined if and only if $-xy^{-1} - 1$ is defined that is, $y = -x$. ∎

The following theorem shows that both regular and near-regular matroids are particularly significant in the study of matroids representable over partial fields.

THEOREM 5.5. *Let* $\mathbf{P}$ *be a partial field.*

(i) *If* $\mathbf{P}$ *has trivial addition, then the class of* $\mathbf{P}$*-representable matroids is the class of regular matroids.*

(ii) *If* $-1 = 1$ *in* $\mathbf{P}$, *then the class of* $\mathbf{P}$*-representable matroids contains the class of binary matroids.*

(iii) *If there exists an element* $a \in (\mathbf{P} - \{0, 1\})$ *such that* $(a - 1) \in \mathbf{P}$, *then the class of* $\mathbf{P}$*-representable matroids contains the class of near-regular matroids.*

Proof. Assume that $\mathbf{P}$ has trivial addition. It is straightforward to check that neither $U_{2,4}$ nor the Fano-matroid F_7 is representable over $\mathbf{P}$. Since the class of $\mathbf{P}$-matroids is minor-closed and closed under duality we deduce that the class of $\mathbf{P}$-representable matroids is contained in the class of regular matroids. Now define φ: $(\{1, -1\}, \mathbf{Q}) \to \mathbf{P}$ by $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi(-1) = -1$. One readily checks that φ is a non-trivial homomorphism and it follows that a matroid is representable over $\mathbf{P}$ if and only if it is regular.

Assume that $\mathbf{P}$ satisfies (ii). Define φ: $GF(2) \to \mathbf{P}$ by $\varphi(0) = 0$ and $\varphi(1) = 1$. Evidently φ is a non-trivial homomorphism and it follows by Corollary 5.3 that all binary matroids are $\mathbf{P}$-representable.

Assume that $\mathbf{P}$ satisfies (iii). A matroid is near-regular if it is representable over the partial field $\mathbf{NR} = (\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}, \mathbf{Q}(\alpha))$. Define φ: $\mathbf{NR} \to \mathbf{P}$ by $\varphi(0) = 0$, and, for $i, j \in Z$, $\varphi(\pm\alpha^i(\alpha - 1)^j) =$

$\pm a^i(a-1)^j$. Again, it is straightforward to check that φ is a non-trivial homomorphism. It follows that near-regular matroids are **P**-representable. ∎

Of course a partial field can simultaneously satisfy conditions (ii) and (iii) of Theorem 5.5. It is shown in [23] that near-regular matroids are the matroids representable over all fields except perhaps $GF(2)$. Also, regular matroids are the matroids representable over all fields [19, 20]. From these facts and Theorem 5.5 we have

COROLLARY 5.6. *A matroid is representable over all partial fields if and only if it is regular. A matroid is representable over all non-trivial partial fields except possibly* $GF(2)$ *if and only if it is near-regular.*

6. EQUIVALENT REPRESENTATIONS

An *automorphism* of a partial field **P** is an isomorphism $\varphi: \mathbf{P} \to \mathbf{P}$. From a matroid-theoretic point of view the main interest in automorphisms is the role that they play in determining whether representations of a matroid are equivalent. For partial fields we define equivalence of representations just as for fields (see [14, Chapter 6.3]). Two matrix representations of a matroid M over a partial field **P** are *equivalent* if one can be obtained from the other by a sequence of the following operations: interchanging two rows, interchanging two columns (together with their labels), pivoting on a non-zero element, multiplying a row or a column by a non-zero member of **P**, and replacing each entry of the matrix by its image under some automorphism of **P**. A matroid is *uniquely representable* over **P** if all representations of M over **P** are equivalent.

Equivalent representations of matroids over fields have been quite well studied. It is easily seen that matroids are uniquely representable over $GF(2)$. In fact Brylawski and Lucas [4] show that representations of binary matroids are unique over any field. They also show that representations of matroids over $GF(3)$ are unique, although note that ternary matroids may have inequivalent representations over other fields. Kahn [10] has shown that representations of 3-connected matroids over $GF(4)$ are unique. Unfortunately, if M is the 2-sum of non-binary matroids then one can apply the non-trivial automorphism of $GF(4)$ to one part of the sum to obtain a strictly inequivalent representation of the same matroid. This situation occurs for any field (or partial field) that has a non-trivial automorphism. The main reason that equivalence of representations has been studied is that strong results in matroid representation theory are generally obtainable only when matroids are uniquely representable: for example all known proofs of the excluded-minor characterisations of binary, ternary, and regular matroids use unique representability in an essential way [3, 9, 12, 17, 19, 7]. Note also that while ternary matroids generally have inequivalent representations over fields other than $GF(3)$, the precise way such representations occur is understood. This understanding is essential to the results of [22, 23].

Given the above, it is certainly of interest to understand the behaviour of representations over partial fields. We initiate such a study by looking at some fundamental classes. We first note that the techniques of [4] can be extended to prove

PROPOSITION 6.1. *If M is a binary matroid representable over the partial field* $\mathbf{P}$, *then M is uniquely representable over* $\mathbf{P}$.

Now consider representations over the partial fields $\mathbf{NR}$, $\mathbf{D}$, and $\mathbf{P_6}$. The following lemma is a straightforward generalisation of results in [21] to partial fields.

LEMMA 6.2. *If the rank-3 whirl $\mathscr{W}^3$ has a finite number k of inequivalent representations over the partial field* $\mathbf{P}$, *then any 3-connected, ternary matroid that is representable over* $\mathbf{P}$ *has at most k inequivalent representations over* $\mathbf{P}$.

In Lemma 6.2 the rank-3 whirl could have been replaced by the rank-2 whirl $U_{2,4}$, but there is some ambiguity in the literature regarding the criteria for equivalence of rank-2 matroids. Use of $\mathscr{W}^3$ avoids this problem. The following lemma is clear.

LEMMA 6.3. *Let* $\mathbf{P}$ *be a partial field and let* $\mathbf{P}'$ *be a subset of* $\mathbf{P}$ *that is a partial field with the induced operations. If φ is an automorphism of* $\mathbf{P}$ *with the property that the restriction of φ to* $\mathbf{P}'$ *is a bijection of* $\mathbf{P}'$, *then* $\varphi|_{\mathbf{P}'}$ *is an automorphism of* $\mathbf{P}'$.

THEOREM 6.4. *Let M be a 3-connected matroid.*

(i) *If M is representable over* $\mathbf{NR}$, *then M is uniquely representable over* $\mathbf{NR}$.

(ii) *If M is representable over* $\mathbf{D}$, *then M either has three inequivalent representations over* $\mathbf{D}$ *or is uniquely representable over* $\mathbf{D}$. *The former case occurs if M is non-binary and near-regular.*

(iii) *If M is representable over* $\mathbf{P_6}$, *then M is uniquely representable over* $\mathbf{P_6}$.

Proof. Say that $\mathscr{W}^3$ is representable over the partial field **P**. Then it is straightforward to show that any representation of $\mathscr{W}^3$ is equivalent to one of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -x \end{bmatrix},$$

where $x \in (\mathbf{P} - \{0,1\})$, and $x - 1$ is defined. Note that matroids representable over **NR**, **D**, or $\mathbf{P_6}$ are ternary so we may apply Lemma 6.2.

Now say that $\mathbf{P} = \mathbf{NR}$. It is routine to check that if x is chosen so that the above matrix represents $\mathscr{W}^3$ over **NR**, then

$$x \in \{\alpha, -(\alpha - 1), \alpha/(\alpha - 1), -1/(\alpha - 1), 1/\alpha, (\alpha - 1)/\alpha\}.$$

This gives six representations of $\mathscr{W}^3$. We now show that these representations are equivalent. Consider automorphisms of $\mathbf{Q}(\alpha)$. Such automorphisms are determined by their action on α, since the image of the rational function $r(\alpha)$ under an automorphism φ is just $r(\varphi(\alpha))$. It is well known (see for example [5, Proposition 5.2.3]) that all automorphisms of $\mathbf{Q}(\alpha)$ have the following action on α:

$$\alpha \to \frac{a\alpha + b}{c\alpha + d};$$

where $a, b, c, d \in \mathbf{Q}$, and $ad - bc \neq 0$. It follows that $\alpha \to -(\alpha - 1)$, $\alpha \to \alpha/(\alpha - 1)$, $\alpha \to -1/(\alpha - 1)$, $\alpha \to 1/\alpha$, and $\alpha \to (\alpha - 1)/\alpha$ each generate automorphisms of $\mathbf{Q}(\alpha)$. It is straightforward to check that the restriction of each of these automorphisms to $\{\pm\alpha^i(\alpha - 1)^j : i, j \in Z\}$ is a bijection. It now follows from Lemma 6.3 that these restrictions are all automorphisms of **NR**. We deduce that the above representations of $\mathscr{W}^3$ over **NR** are indeed equivalent. We conclude by Lemma 6.2 that (i) holds.

Say that $\mathbf{P} = \mathbf{D}$. We first show that **D** has no non-trivial automorphisms. Assume that φ is an automorphism of **D**. Then

$$1 = \varphi(1) = \varphi(2 - 1) = \varphi(2) - \varphi(1) = \varphi(2) - 1.$$

Hence $\varphi(2) = 2$, and it follows that φ is the identity map. It is easily checked that the possible choices for x in the above matrix to obtain a representation of $\mathscr{W}^3$ are 2, $\frac{1}{2}$, and -1. We conclude that $\mathscr{W}^3$ has three inequivalent representations over **D**. This establishes part of (ii). The remaining claims in (ii) follow from an application of [22, Theorems 5.11 and 7.2].

Now say that $\mathbf{P} = \mathbf{P_6}$. In this case the choices for x are $(1 \pm \sqrt{3}\,i)/2$. But conjugation in the complex numbers clearly induces an automorphism of $\mathbf{P_6}$. Hence $\mathscr{W}^3$ is uniquely representable over $\mathbf{P_6}$, and it now follows from Lemma 6.2 that (iii) holds. ∎

7. DOWLING GROUP GEOMETRIES

For this section it is assumed that the reader has some familiarity with Dowling group geometries. These are introduced in [6]. Other useful references are [2, 11]. See also [24, 25] for a graph-theoretic perspective.

Consider a finite group G. We denote the rank-r Dowling group geometry associated with G by $Q_r(G)$. A matroid is a *G-matroid* if it is isomorphic to a minor of $Q_r(G)$ for some positive integer r. It is natural to ask whether there exists a partial field $\mathbf{P}$ such that the class of $\mathbf{P}$-matroids contains the class of G-matroids. Of course there may be many such partial fields. For example, if G is the trivial group, then the class of G-matroids is just the class of graphic matroids, and, since graphic matroids are regular, they are representable over every partial field. Of these, the minimal partial field (in a natural sense) is **Reg**. A similar situation holds if G is the 2-element group. Here the natural minimal partial field $\mathbf{P}$ with the property that the $\mathbf{P}$-matroids contain the G-matroids is $\mathbf{D}$. In what follows we generalise these ideas in a way that we make precise.

A partial field $\mathbf{P}$ *supports* a group G if $\mathbf{P}^*$ has a subgroup G' isomorphic to G with the property that $g - 1$ is defined for all $g \in G'$.

THEOREM 7.1. *Let G be a group and $\mathbf{P}$ be a partial field. Then the class of G-matroids is contained in the class of $\mathbf{P}$-matroids if and only if $\mathbf{P}^*$ supports a subgroup isomorphic to G.*

Proof. Assume that $\mathbf{P}$ supports G. We show that for each rank r, $Q_r(G)$ is representable over $\mathbf{P}$. By relabelling if necessary, we may assume that G is a subgroup of $\mathbf{P}^*$. Say $r \geq 2$. A column vector $\mathbf{x} = (x_1, x_2, \ldots, x_r)^t$ is a *G-vector* if it has the following properties. There are exactly two non-zero entries, x_i and x_j, where $i < j$. Moreover, $x_i = 1$, and $x_j = (-1)^{j-i-1}g$ for some $g \in G$. Let A be the $r \times \binom{r}{2}|G|$ matrix consisting of all possible G-vectors. Consider the matrix $[I|A]$.

7.1.1. $[I|A]$ is a $\mathbf{P}$-matrix.

Proof. It is straightforward to check that all square submatrices of $[I|A]$ will have a defined determinant so long as ones that—up to a permutation of rows and columns—are of the form

$$D = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ h_1 & 1 & \cdots & 0 & 0 \\ 0 & h_2 & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & h_{n-1} & h_n \end{bmatrix}$$

have a defined determinant. Evidently this matrix will have a defined determinant if and only if

$$h_n + (-1)^{n-1} h_1 h_2 \cdots h_{n-1} = h_n\big(1 + (-1)^{n-1} h_n^{-1} h_1 h_2 \cdots h_{n-1}\big)$$

is defined. For $1 \le k \le n$, consider the column of A corresponding to the kth column of D. This column has non-zero entries in rows k_i and k_j. Let $d_k = k_j - k_i - 1$. Then $h_k = (-1)^{d_k} g_k$ for some $g_k \in G$. Thus

$$\begin{aligned} &1 + (-1)^{n-1} h_n^{-1} h_1 h_2 \cdots h_{n-1} \\ &\qquad = 1 + (-1)^{n-1-d_n+d_1+d_2+\cdots+d_{n-1}} g_n^{-1} g_1 g_2 \cdots g_{n-1}. \end{aligned}$$

The above sum—and hence $\det(D)$—will certainly be defined if $n - 1 - d_n + d_1 + d_2 + \cdots + d_{n-1}$ is odd. We now show that this is the case. The sum $d_1 + d_2 + \cdots + d_n$ is $(-1)n$ minus twice the index in A of row 1 of D plus twice the index of row n (all indices except the first and last appear once as a k_i and once as a k_j and so cancel). Therefore $d_1 + d_2 + \cdots + d_n$ has the same parity as n so that $-d_n + d_1 + d_2 + \cdots + d_{n-1}$ has the same parity as n. It follows that $n - 1 - d_n + d_1 + d_2 + \cdots + d_{n-1}$ is odd as claimed. ∎

Since $[I|A]$ is a **P**-matrix, the matroid $M[I|A]$ is well-defined.

7.1.2. $M[I|A] = Q_r(G)$.

Proof. Consider a G-vector with non-zero entries in the ith and jth coordinates, the jth coordinate being h. Then $h = (-1)^{j-i-1} g$ for some $g \in G$. Call g the *underlying* group element of the G-vector. Return attention to the matrix D examined in 7.1.1. It follows from the analysis in the proof of 7.1.1 that $\det(D) = 0$ if and only if the underlying group elements of the columns of D satisfy $g_n^{-1} g_1 g_2 \cdots g_{n-1} = 1$. Consider the columns of A that meet D. Evidently, these columns are independent if and only if $\det(D) \neq 0$. (Note that dependency of the columns is determined entirely by group multiplication in G.) The proof of the fact that

$M[I|A] = Q_r(G)$ now follows without much difficulty. We omit details and merely outline one way (of several) that such a proof could be completed. Associated with $[I|A]$ is a *biased graph* in the sense of [24]. Edges of this graph are labelled by elements of G. Here $[I|A]$ can be regarded as being —up to the signs of entries—a weighted incidence matrix of this biased graph. It is straightforward to show that $M[I|A]$ is equal to the *bias matroid* of this biased graph. But it is known [25] that the bias matroid of the graph we have constructed is just $Q_r(G)$. ▮

We conclude that if **P** supports G, then $Q_r(G)$ is representable over **P**. Consider the converse. Assume that, for $r \geq 2$, $Q_r(G)$ is representable over **P**. Then, in particular $Q_3(G)$ is representable over **P**. Let $[I_3|A]$ be a representation of $Q_3(G)$ in standard form where the columns of I_3 represent the *joints* of $Q_3(G)$. In such a representation, if $\mathbf{c}$ is a column of A, then for some $x \in \mathbf{P}^*$, $\mathbf{c}$ is equal to $(1, x, 0)^t$, $(1, 0, x)^t$, or $(0, 1, x)^t$. An argument, essentially identical to that of [6, Theorem 9], shows that the set $S = \{x : (1, x, 0)^t$ is a column of $A\}$ is a coset of a subgroup G' of $\mathbf{P}^*$ that is isomorphic to G. To show that **P** supports G, all that remains is to show that $g - 1$ is defined for all $g \in G'$. Since S is a coset of G, there exists $k \in \mathbf{P}^*$ such that $S = \{kg : g \in G'\}$. For $g \in G'$,

$$D = \begin{bmatrix} 1 & 1 \\ k & kg \end{bmatrix}$$

is a submatrix of the **P**-matrix A, so D has a defined determinant. But $\det(D) = k(g - 1)$ is defined if and only if $g - 1$ is defined. We deduce that **P** indeed supports G. ▮

By Theorem 7.1, for a given group G, deciding whether there exists a partial field **P** such that the **P**-matroids contain the G-matroids reduces to deciding whether there exists a partial field that supports G. Of course, if G is not abelian, such a partial field does not exist. But even being abelian is not enough, a fact that initially surprised us.

THEOREM 7.2. *Let G be an abelian group. Then there exists a partial field that supports G if and only if G has at most one element of order* 2 (*that is, there exists at most one element* $g \neq 1$ *such that* $g^2 = 1$).

Proof. Let **P** be a partial field supporting the group G. Evidently we may regard G as a subgroup of **P**. Say $g \in G$, $g \neq 1$, and $g^2 = 1$. Since **P** supports G, $(g - 1) \in \mathbf{P}$. Now

$$g(g-1) = g^2 - g = 1 - g = -(g-1).$$

Since $g \neq 1$, $(g-1)^{-1}$ is defined, and it follows that $g = -1$. We conclude that there is at most one element of G of order 2.

Consider the converse. In what follows we use multiplicative notation for all groups—even free abelian groups. Let G be a group with at most one element of order 2. We proceed by constructing a canonical partial field associated with G that supports G. We need to be able to subtract 1 from elements of G. With this in mind we define the set S by $S = \{g - 1 : g \in G,\ g \neq 1\}$. Note that $g - 1$ is just a name for an element of S; we do not yet have a notion of subtraction. Let G_S denote the free abelian group generated by S. We also need to be able to negate elements. With this in mind we let G_2 be the 2-element group defined by $G_2 = \{1, -1\}$. Now put the groups together and let G' be the direct product of G, G_S, and G_2. Evidently, the elements of G' all have the form

$$\pm g(g_1 - 1)^{i_1}(g_2 - 1)^{i_2} \cdots (g_n - 1)^{i_n} : g \in G, g_1, g_2, \ldots, g_n \in (G - \{1\}).$$

As yet G' is not appropriate as the multiplicative group of a partial field **P**. In such a group we would want to interpret $g - 1$ as $g + (-1)$. Also, in **P** we require the distributive law to hold. This means that we need to have $g - 1 = g(1 - g^{-1})$, that is, $g - 1 = -g(g^{-1} - 1)$. But this does not hold in G'. We solve this problem by imposing this relation on G'. Let $G'' = \langle -g(g^{-1} - 1)(g-1)^{-1} : g \in (G - \{1\}) \rangle$, and set $\mathbf{P}^* = G'/G''$. Obviously we intend to make $\mathbf{P}^*$ the multiplicative group of our partial field, but there is little point in doing this if, in the process of factoring out G'', we have lost information about G. We now show that this has not happened.

Evidently, the elements of G correspond to distinct elements of $\mathbf{P}^*$ if and only if 1 is the only member of G in G''. First note that

$$\begin{aligned}\left(-g(g^{-1}-1)(g-1)^{-1}\right)^{-1} &= -g^{-1}(g^{-1}-1)^{-1}(g-1) \\ &= -h(h^{-1}-1)(h-1)^{-1},\end{aligned}$$

where $h = g^{-1}$. It follows that if $g'' \neq 1$ is a member of G'', then g'' has the form

$$(-1)^n g_1(g_1^{-1} - 1)(g_1 - 1)^{-1} g_2(g_2^{-1} - 1) \cdot (g_2 - 1)^{-1} \cdots g_n(g_n^{-1} - 1)(g_n - 1)^{-1},$$

for some $g_1, g_2 \cdots g_n \in (G - \{1\})$. In what circumstances can g'' be in G? Certainly, if $n = 1$, $g'' \notin G$. Say $n = 2$; that is, say

$$g'' = g_1\big(g_1^{-1} - 1\big)(g_1 - 1)^{-1} g_2\big(g_2^{-1} - 1\big)(g_2 - 1)^{-1}.$$

If $g'' \in G$, then either $g_1 = g_2^{-1}$ or $g_1^2 = g_2^2 = 1$. In the former case $g'' = 1$. Consider the latter. Since G does not have two distinct elements of order 2, we also have $g_1 = g_2$, and again $g'' = 1$. A straightforward inductive argument based on these two cases shows that for all n, $g'' \in G$ if and only if $g'' = 1$. It follows that if we can represent $\mathbf{P}^*$ as the multiplicative group of a partial field $\mathbf{P}$ in such a way that $g - 1 = g + (-1)$, then $\mathbf{P}$ will support G. We turn to this question now.

Let $\mathbf{P} = \mathbf{P}^* \cup \{0\}$. Define the multiplication of $\mathbf{P}$ to be that of $\mathbf{P}^*$ with, of course, $a0 = 0a = 0$ for all $a \in \mathbf{P}$. Now define $+$ as follows:

(i) For all $a \in \mathbf{P}^*$, $a + (-a) = 0$ and $a + 0 = 0 + a = 0$.

(ii) For all $g, h \in G$, $g + (-h) = h(gh^{-1} - 1)$, in particular, $g - 1 = g + (-1)$.

(iii) For all $g, h \in G$, $(g - 1) + (-(h - 1)) = g - h$.

(iv) For all $a, b \in \mathbf{P}^*$, $a + b$ is defined if there exists $x \in \mathbf{P}$ such that for some $h, g \in G$, either $a = xg$ and $b = -xh$ or $a = x(g - 1)$ and $b = -x(h - 1)$. In either case, $a + b = x(g - h)$.

In any case not covered by (i)–(iv), $+$ is not defined. It remains to show that with this definition $\mathbf{P}$ is indeed a partial field. Evidently, 0 is the additive identity. Consider the commutative law. For $g \in G$, $1 - g = g(g^{-1} - 1)$. But we know that $-g(g^{-1} - 1) = g - 1$. Therefore $1 - g = -(g - 1)$. The fact that $a + b$ is defined if and only if $b + a$ is defined, in which case $a + b = b + a$ now follows routinely.

Consider the distributive law. Say that $a, b, c \in \mathbf{P}$, and that $a(b + c)$ is defined. Then for some $g, h \in G$, either $b = xg$ and $c = -xh$ or $b = x(g - 1)$ and $c = x(h - 1)$. In the former case $ab = axg$ and $ac = -axh$, and in the latter case $ab = ax(g - 1)$ and $ac = -ax(h - 1)$. In either case we have $a(b + c) = ax(g - h) = ab + ac$. The converse follows from an obvious reversal of this argument.

Finally consider the associative law. We use the theory of group rings. Let $\mathbf{F}$ be a field and $H = \{h_i : i \in I\}$ be a multiplicative group, the *group ring* $\mathbf{F}[H]$ consists of all formal sums $\sum_{i \in I} a_i g_i$ for $a_i \in \mathbf{F}$, and $h_i \in H$, where all but a finite number of the a_i are 0. The sum of two elements of $\mathbf{F}[H]$ is defined by

$$\Big(\sum_{i \in I} a_i h_i\Big) + \Big(\sum_{i \in I} b_i h_i\Big) = \sum_{i \in I} (a_i + b_i) h_i,$$

and the product is defined by

$$\left(\sum_{i\in I} a_i h_i\right)\left(\sum_{i\in I} b_i h_i\right) = \sum_{i\in I}\left(\sum_{h_j h_k = h_i} a_j b_k\right) h_i .$$

A comprehensive treatment of group rings is given in [15]. All that we need to know here is that group rings are indeed rings. Now consider the group ring $\mathbf{Q}[G]$. Let $\mathbf{P}^+$ denote the elements of P that can be written in the form

$$\pm g(g_1 - 1)^{i_1}(g_2 - 1)^{i_2} \cdots (g_n - 1)^{i_n},$$

where, for $1 \le j \le n$, $i_j \ge 0$. There is a natural embedding of $\mathbf{P}^+$ into $\mathbf{Q}[G]$. Moreover it is easily checked that this embedding preserves sums. We now show that the associative law holds for elements of $\mathbf{P}^+$. Say $a_1, a_2, \dots, a_n$ are elements of $\mathbf{P}^+$. Assume that some association of $a_1, a_2, \dots, a_n$ is defined, the result of this sum being a. Consider some other association in which all sums are defined except possibly the final sum. Denote this sum by $b + c$. It follows from the embedding of $\mathbf{P}^+$ into $\mathbf{Q}[G]$ that if $b + c$ is defined, then $a = b + c$. We show that $b + c$ is indeed defined. Regarding a, b, and c as elements of $\mathbf{Q}[G]$ we have $a = b + c$, so that $c = a - b$. But these are all well-defined as elements of $\mathbf{P}$. Hence in $\mathbf{P}$ we also have $c = a - b$. But

$$(a - b) + b = b\big((ab^{-1} - 1) - (0 - 1)\big) = b(ab^{-1}) = a.$$

It follows that $b + c$ is indeed defined in $\mathbf{P}$. Now say that $a_1, a_2, \dots, a_n$ are in $\mathbf{P}$. Then for some non-zero element $x, a_1 x, a_2 x, \dots, a_n x$ are all in $\mathbf{P}^*$. We know that the associative law holds for these elements. It then follows from the distributive law that the associative law holds for $a_1, a_2, \dots, a_n$. We conclude that the associative law holds in general and that $\mathbf{P}$ is a partial field. ∎

Let G be an abelian group with at most one element of order 2. Denote the partial field supporting G constructed via the technique of Theorem 7.2 by $\mathbf{P_G}$. We now show that $\mathbf{P_G}$ is in some sense a minimum partial field supporting G.

THEOREM 7.3. *Let* $\mathbf{P}$ *be a partial field supporting the group* G. *Then there exists a non-trivial homomorphism* $\varphi\colon \mathbf{P_G} \to \mathbf{P}$.

Proof. Regard G as a subgroup of $\mathbf{P_G^*}$. Say that $\mathbf{P}$ supports G', where $G' \cong G$. Let $\phi\colon G \to G'$ be an isomorphism. If $x \in \mathbf{P_G}$, then for some $g, g_1, g_2, \dots, g_n \in G$, and $i_1, i_2, \dots, i_n \in Z$,

$$x = \pm g(g_1 - 1)^{i_1}(g_2 - 1)^{i_2} \cdots (g_n - 1)^{i_n}.$$

Define $\varphi: \mathbf{P_G} \to \mathbf{P}$ by

$$\varphi\left(\pm g(g_1 - 1)^{i_1}(g_2 - 1)^{i_2} \cdots (g_n - 1)^{i_n}\right)$$
$$= \pm\phi(g)(\phi(g_1) - 1)^{i_1}(\phi(g_2) - 1)^{i_2} \cdots (\phi(g_n) - 1)^{i_n}.$$

The details of the straightforward argument that shows φ is a homomorphism are omitted. ▮

We immediately obtain

COROLLARY 7.4. *If, for $r \geq 3$, the partial field* $\mathbf{P}$ *supports the group G, then the class of matroids representable over* $\mathbf{P}$ *contains the class of matroids representable over* $\mathbf{P_G}$.

If G is the trivial group, then $\mathbf{P_G} = \mathbf{Reg}$. It is also readily checked that if G is the 2-element group, then $\mathbf{P_G} = \mathbf{D}$, so that the dyadic matroids form the smallest class of matroids representable over a partial field that contain the matroids representable over the 2-element group. For both these groups, $\mathbf{P_G}$ can be embedded in a field. Say $G_3 = \{a : a^3 = 1\}$. Here, in $\mathbf{P_{G_3}}$, $a + 1$ is not defined. But whenever G_3 is embedded as a subgroup of the multiplicative group of a field, $a + 1 = -a^2$, so $\mathbf{P_{G_3}}$ cannot be embedded in any field.

It is well known (and easily seen) that a finite subgroup of the multiplicative group of a field is cyclic. It follows from this fact that no partial field that supports a non-cyclic group can be embedded in a field.

While knowledge of the groups supported by a partial field give insight into its structure, this is by no means the whole story. The only group supported by the partial fields **Reg**, **NR**, $\mathbf{P_6}$, and $GF(2)$ is the trivial group, yet the classes of matroids representable over these partial fields are very different.

We now test the reader's patience with a final general comment. Let G be an abelian group with at most one element of order 2. It is of interest to compare the class of G-matroids with the class of $\mathbf{P_G}$-representable matroids. Which class has the most satisfying structure theory? In their interesting paper [11], Kahn and Kung show that the class of G-matroids forms a variety. In particular this means that the class is minor-closed and is closed under the taking of direct sums. Moreover, for each rank r, there is a universal model, namely $Q_r(G)$. This means that every simple rank-r G-matroid is a restriction of $Q_r(G)$. What about the class of $\mathbf{P_G}$-representable matroids? This class is certainly closed under direct sums and the taking of minors. However, there does not exist a universal model, so that the class is not a variety. On the other hand, the class of $\mathbf{P_G}$-representable matroids is closed under duality and 2-sums, and neither of these properties is enjoyed by the class of G-matroids. In balance we believe that the loss of the universal model is adequately compensated by the gain of 2-sums and duality.

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