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On excluded minors for real-representability ☆

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ABSTRACT

We show that for any infinite field $\mathbb{K}$ and any $\mathbb{K}$-representable matroid N there is an excluded minor for $\mathbb{K}$-representability that has N as a minor.

1. Introduction

In [2] it is proved that an excluded minor for the class of $GF(q)$-representable matroids cannot contain a large projective geometry over $GF(q)$ as a minor. But what if the field is infinite? In contrast to the behaviour for finite fields Geelen [1] made the striking conjecture that if N is any matroid representable over $\mathbb{R}$, then there is an excluded minor for $\mathbb{R}$-representability that contains N as a minor. In this paper we resolve Geelen's conjecture in the affirmative by proving the following theorem.

Theorem 1.1. *Let $\mathbb{K}$ be an infinite field, and N be a matroid representable over $\mathbb{K}$. Then there exists an excluded minor for the class of $\mathbb{K}$-representable matroids that is not representable over any field and has N as a minor.*

Perhaps the most famous open problem in matroid theory is Rota's conjecture, which states that if $\mathbb{F}$ is a finite field, then there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids. If true, this would imply that, up to isomorphism, only a finite number of $\mathbb{F}$-representable matroids are minors of an excluded minor for $\mathbb{F}$-representability, making the contrast between the behaviour for finite and infinite fields even sharper.

Geelen raised a number of other interesting questions in [1]. Here is one. An example given by Seymour [6] shows that, for a matroid given by a rank oracle, it requires exponentially many rank evaluations to decide if a matroid is binary. It is straightforward to give similar examples for all other fields. On the other hand, for a prime field $GF(p)$, certifying non-$GF(p)$-representability requires only

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$O(n^2)$ rank evaluations [3]. Indeed, if Rota's conjecture were true, certifying non-$\mathbb{K}$-representability would require only $O(1)$ rank evaluations for any finite field $\mathbb{K}$. Geelen asked the following question: "Can non-$\mathbb{R}$-representability be certified using a polynomial number of rank evaluations?" We suspect that the answer to Geelen's question is "no." It may be tempting to think that Theorem 1.1 sheds some light on this question, but this is not the case. Each of the excluded minors we construct in Theorem 1.1 violates the Ingleton condition—discussed below—so each can be proved to be non-representable with only 10 rank evaluations.

2. The Proof

We first deal with some preliminaries. Let $\mathbb{K}$ be a field. We denote the rank-r projective space over $\mathbb{K}$ by $PG(r-1,\mathbb{K})$. Recall that a rank-r matroid M is *representable* over $\mathbb{K}$ if its associated simple matroid is isomorphic to $PG(r-1,\mathbb{K}) \mid E$ for some subset E of $PG(r-1,\mathbb{K})$. For a set of points A in a projective space, define $\langle A\rangle$ to be the subspace spanned by A.

Let E be a set of points of $PG(r-1,\mathbb{K})$ and let U be a subspace of $PG(r-1,\mathbb{K})$. A set $X \subseteq U$ is *freely placed in* U *relative to* E if, for all $x \in X$, and all $Z \subseteq E \cup X - \{x\}$, we have $x \in \langle Z\rangle$ if and only if $U \subseteq \langle Z\rangle$. We now consider the situation where we wish to add more than one set of elements freely. Let $(U_1, U_2, \dots, U_n)$ be subspaces of $PG(r-1,q)$, and let $(X_1,\dots,X_n)$ be sets such that $X_i \subseteq U_i$ for all $i \in \{1,\dots,n\}$. Then $(X_1,\dots,X_n)$ is *independently freely placed in* $(U_1,\dots,U_n)$ *relative to* E if X_i is freely placed in U_i relative to $E \cup X_1 \cup \dots \cup X_{i-1} \cup X_{i+1} \cup \dots \cup X_n$, for all $i \in \{1,\dots,n\}$. The next lemma seems to be well-known, but hard to pin down in the literature so we outline a proof. The case of the lemma when $n=1$ simply says that it is possible to add an arbitrary number of elements freely to a given subspace relative to any given finite set of points, and is certainly well-known.

Lemma 2.1. *Let $\mathbb{K}$ be an infinite field, let E be a finite set of points of $PG(r-1,\mathbb{K})$, let $(U_1,\dots,U_n)$ be a collection of subspaces of $PG(r-1,\mathbb{K})$ each having rank at least 2, and let $s_1,\dots,s_n$ be non-negative integers. Then there exist sets $(X_1,\dots,X_n)$ such that, for all $i \in \{1,\dots,n\}$, $|X_i| = s_i$, and $(X_1,\dots,X_n)$ is independently freely placed in $(U_1,\dots,U_n)$ relative to E.*

Proof. Note that placing $X = \{x_1,\dots,x_t\}$ freely on U relative to E is the same as placing $(\{x_1\},\dots,\{x_t\})$ independently freely on $(U,U,\dots,U)$, so it suffices to prove the lemma in the case that each $s_i = 1$ for all i. We prove the lemma for the case $n=2$. The general case follows from a routine induction. Let B_1 and B_2 be bases for U_1 and U_2. It is easily seen that U_1 is not a union of a finite number of its proper subspaces and it follows from this that there is an element $x_1 \in U_1$ that is freely placed in U_1 relative to $E \cup B_2$. Now let x_2 be freely placed in U_2 relative to $E \cup B \cup \{x_1\}$. It is easily checked that $(\{x_1\},\{x_2\})$ is independently freely placed in (U_1,U_2) relative to E. □

If N is a matroid represented over $\mathbb{K}$ by a set E, then a special case of the above operation occurs when X is added freely in $\langle E\rangle$ relative to E. It is well-known that the resulting matroid N' on $E \cup X$ is independent of the choice of representation or infinite field and we say that N' has been obtained by extending N *freely* by the set X.

The next lemma shows that to prove Theorem 1.1 we may restrict attention to a specific subclass of matroids.

Lemma 2.2. *Let N be a matroid representable over an infinite field $\mathbb{K}$. Then N is a minor of a $\mathbb{K}$-representable matroid whose ground set can be partitioned into two independent hyperplanes.*

Proof. Let B be a basis for N, $F = E(N) - B$, let A be a maximum-sized independent set in F and let $m = |F - A|$. We construct a matroid N' from N as follows. First extend N by adding a set C of $r - |A|$ points freely to N. Now replace each point x_i in $F - A$ with a series pair x'_i, x''_i. The sets $B_1 = B \cup \{x'_1,\dots,x'_m\}$ and $B_2 = A \cup C \cup \{x''_1,\dots,x''_m\}$ are bases which partition the ground set of N' and N' is $\mathbb{K}$-representable. Moreover N' certainly has an N-minor.

Say $r(N') = n$. We may assume that $E(N') = E$ is a representation of N' in $PG(n+1,\mathbb{K})$. Let $\{y_0, z_0\}$ be freely placed in $PG(n+1,\mathbb{K})$ relative to E. Note that y_0 and z_0 are coloops of $PG(n+1,\mathbb{K}) \mid (E \cup \{y_0, z_0\})$. Say $B_1 = \{y_1, \dots, y_n\}$ and $B_2 = \{z_1, \dots, z_n\}$. By Lemma 2.1, we may let $(\{y'_1\}, \dots, \{y'_n\}, \{z'_1\}, \dots, \{z'_n\})$ be independently freely placed in $(\langle\{y_0, y_1\}\rangle, \dots, \langle\{y_0, y_n\}\rangle, \langle\{z_0, z_1\}\rangle, \dots, \langle\{z_0, z_n\}\rangle)$ relative to E. Let $B'_1 = \{y_0, y'_1, \dots, y'_n\}$ and $B'_2 = \{z_0, z'_1, \dots, z'_n\}$. Let $N'' = PG(n+1,\mathbb{K}) \mid (B'_1 \cup B'_2)$.

It is easily seen that $N''/y_0, z_0 \cong N'$ so that N'' has an N-minor. Moreover B'_1 and B'_2 are independent hyperplanes of N''. □

A *circuit-hyperplane* of a matroid M is a subset of $E(M)$ that is both a circuit and a hyperplane. It is well-known and easily seen that, if Z is a circuit-hyperplane of M and $\mathcal{B}$ is the collection of bases of M, then $\mathcal{B} \cup \{Z\}$ is also the collection of bases of a matroid M'. We say that M' is obtained by *relaxing* the circuit-hyperplane Z. The next lemma is elementary.

Lemma 2.3. *Let Z be a circuit-hyperplane of the matroid M and M' be the matroid obtained by relaxing Z.*

(i) *If $x \in Z$, then $M \setminus x = M' \setminus x$.*
(ii) *If $x \notin Z$, then $M/x = M'/x$.*

What follows is not necessary for the proof, but may aid intuition. Let A, B, C and D be disjoint 2-element sets. Then there is a unique simple, rank-4 matroid M on $A \cup B \cup C \cup D$ whose non-spanning circuits are precisely the sets $X \cup Y$, where X and Y are distinct elements of $\{A, B, C, D\}$. Geometrically M is obtained by taking a set of four copunctual lines in rank 4 and placing a pair of points freely on each line. Let V_8 be the matroid obtained by relaxing the circuit-hyperplane $C \cup D$. Then V_8 is the *Vámos matroid* and it is known that V_8 is not representable over any field [7]. This is the simplest example of the construction that we present in the proof of Theorem 1.1.

As a final preliminary we recall a necessary condition for representability over any field, established by Ingleton [4].

Theorem 2.4 *(Ingleton's condition). For any subsets X_1, X_2, X_3, X_4 of a representable matroid,*

$$\begin{aligned} &r(X_1) + r(X_2) + r(X_1 \cup X_2 \cup X_3) + r(X_1 \cup X_2 \cup X_4) + r(X_3 \cup X_4) \\ &\quad \leqslant r(X_1 \cup X_2) + r(X_1 \cup X_3) + r(X_1 \cup X_4) + r(X_2 \cup X_3) + r(X_2 \cup X_4). \end{aligned}$$

Proof of Theorem 1.1. By Lemma 2.2 we lose no generality in assuming that $E(N)$ has a partition into disjoint independent hyperplanes. Say N has rank r. If $r \leqslant 2$, then $N \cong U_{2,2}$ and every excluded minor for $\mathbb{K}$ representability has N as a minor. Thus we may assume that $r \geqslant 3$. We may also assume that $N = PG(r,\mathbb{K}) \mid E$ for some subset E of $PG(r,\mathbb{K})$. Let $P = PG(r,\mathbb{K})$. Observe that E spans a hyperplane of P. Let (A, B) be a partition of E into two independent hyperplanes of N.

We proceed by extending N to obtain a representable matroid M_0 that contains N as a restriction. We will then relax a circuit-hyperplane of M_0 to obtain an excluded minor containing N as a restriction. We use Lemma 2.1 freely.

Let $\{p, q\}$ be a pair of points that is freely placed in P relative to E, and let $V = \langle A\rangle \cap \langle B\rangle$. Choose c with $2 \leqslant c \leqslant r-1$ (such a choice is possible for c because $r \geqslant 3$). By Lemma 2.1 we may let C be a set of c points and D be a set of $r+1-c$ points such that (C, D) is independently freely placed in $(\langle V \cup \{p\}\rangle, \langle V \cup \{q\}\rangle)$ relative to E.

Let $M_0 = P \mid (A \cup B \cup C \cup D)$. The following facts about M_0 are elementary consequences of the above constructions of C and D.

2.4.1.

(i) If X and Y are distinct members of $\{A, B, C, D\}$, then $r(X \cup Y) = r$.
(ii) If X, Y and Z are distinct members of $\{A, B, C, D\}$, then $r(X \cup Y \cup Z) = r+1 = r(M_0)$.
(iii) $C \cup D$ is a circuit-hyperplane of M_0.

Let M be the matroid obtained from M_0 by relaxing the circuit-hyperplane $C \cup D$. Note that M is not representable over any field as it follows from 2.4.1 that the partition (A, B, C, D) of $E(M)$ violates the Ingleton condition. As M contains an N-minor, to complete the proof it suffices to show that every proper minor of M is $\mathbb{K}$-representable. We have symmetry between A and B and symmetry between C and D. Thus it suffices to show that any matroid obtained by deleting or contracting an element $x \in A$ or $y \in C$ is $\mathbb{K}$-representable.

Recall that the set of non-spanning circuits of a matroid together with its rank determine the matroid uniquely [5].

2.4.2. A set Z is a non-spanning circuit of M if and only if either Z is a circuit of N, or $|Z| = r + 1$ and, for some $R \in \{A, B\}$ and $S \in \{C, D\}$, we have $Z \subseteq R \cup S$.

Proof of Claim. We find the non-spanning circuits of M_0. Assume that $Z \subseteq A \cup C$. As A is independent, there is an element $c \in C \cap Z$. As $c \in \mathrm{cl}(Z - \{c\})$, it follows from the fact that the elements of C are freely placed that $C \subseteq \mathrm{cl}(Z)$. Thus $\langle V \cup \{p\}\rangle \subseteq \mathrm{cl}(Z)$. However, as Z contains at least one element of A and $V \cap A = \emptyset$, we see that $\mathrm{cl}(Z)$ properly contains $\langle V \cup \{p\}\rangle$. But $r(A \cup C) = r\langle V \cup \{p\}\rangle + 1$. Hence Z spans $A \cup C$, so that $|Z| = r + 1$.

From the above argument and symmetry, we deduce that all sets of the form described in the claim are circuits of M_0. Assume that Z is a circuit of M_0 that meets C, D, and A. Then $\langle V \cup \{p\}\rangle \subseteq \mathrm{cl}(Z)$ and $\langle V \cup \{q\}\rangle \subseteq \mathrm{cl}(Z)$. But $\langle V \cup \{p, q\}\rangle \cap A = \emptyset$, and we deduce that Z is spanning. A similar argument shows that Z is spanning if Z meets A, B and C. It follows that the only other non-spanning circuit of M_0 is $C \cup D$. The claim now follows from the definition of relaxation. $\square$

If $x \in A$ then $M/x = M_0/x$, and is $\mathbb{K}$-representable by Lemma 2.3. Likewise, if $y \in C$ then $M\backslash y = M_0\backslash y$, and so is $\mathbb{K}$-representable.

2.4.3. If $x \in A$, then $M \backslash x$ is $\mathbb{K}$-representable.

Proof of Claim. Consider the subset $(E - \{x\}) \cup \{p, q\}$ of P. Let $V' = \langle A - \{x\}\rangle \cap \langle B\rangle$. Note that $r(V') = r(V) - 1 = r - 3$. Let V_C and V_D be distinct rank-$(r-2)$ subspaces of P such that $V' \subseteq V_C \subseteq \langle B\rangle$ and $V' \subseteq V_D \subseteq \langle B\rangle$. Such subspaces clearly exist. Let C' be a set of c points and D' be a set of $r + 1 - c$ points such that (C', D') is independently freely placed in $(\langle V_C \cup \{p\}\rangle, \langle V_D \cup \{q\}\rangle)$ relative to $E - \{x\}$. Let $M' = P \mid (A \cup B \cup C' \cup D')$. We prove that $M' = M \backslash x$.

Observe that $r(\langle V_C \cup \{p\}\rangle \cap \langle V_D \cup \{q\}\rangle) = r - 3$; so that $r(\langle V_C \cup \{p\}\rangle \cup \langle V_D \cup \{q\}\rangle) = r + 1 = |C' \cup D'|$. As C' and D' are freely placed in these subspaces, we see that $C' \cup D'$ is independent. We may now argue, just as in 2.4.2, that a set Z is a non-spanning circuit of M' if and only if Z is an $r+1$-element subset of $(A - \{x\}) \cup C'$, $(A - \{x\}) \cup D'$, $B \cup C'$ or $B \cup D'$. The claim follows from these observations. $\square$

2.4.4. If $y \in C$, then M/y is $\mathbb{K}$-representable.

Proof of Claim. Start with the representation E of N over $\mathbb{K}$, but regard it as a representation in the rank-r projective space $PG(r-1, \mathbb{K})$. As before, let $V = \langle A\rangle \cap \langle B\rangle$. Let C' be a set of $c - 1$ points and D' be a set of $r - c + 1$ points such that (C', D') is independently freely placed in $(V, PG(r-1, \mathbb{K}))$ relative to E. In other words the elements of C' are freely placed in $\langle A\rangle \cap \langle B\rangle$ and the elements of D' are freely placed in the projective space. Let $M' = PG(r-1, q) \mid (A \cup B \cup C' \cup D')$. Evidently $C' \cup D'$ is independent in M'. Moreover, a set Z is a non-spanning circuit of M' if and only if Z is an r-element subset of either $A \cup C'$ or $B \cup C'$. Hence $M' \cong M/y$. $\square$

The theorem follows from 2.4.3 and 2.4.4. $\square$

Finally we observe that it is routine to adapt the techniques of this paper to prove that if M is a matroid representable over a finite field $\mathbb{F}$, then there is an excluded minor for a finite extension field of $\mathbb{F}$ that has M as a minor.

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