

Modularity in Tangential k -Blocks

GEOFF WHITTLE

*Department of Mathematics, University of Tasmania,
GPO Box 252C, Hobart, Tasmania 7001*

Communicated by the Editors

Received July 23, 1984

This paper examines the role of modularity in tangential k -blocks over $\text{GF}(q)$. It is shown that if M is a tangential k -block over $\text{GF}(q)$ and F is a modular flat of M which is affine over $\text{GF}(q)$ then the simple matroid associated with the complete Brown truncation of M by F is also a tangential k -block over $\text{GF}(q)$. This enables us to construct tangential k -blocks over $\text{GF}(q)$ of all ranks r where $q^k - q + 2 \leq r \leq q^k$. We also consider tangential k -blocks which have modular hyperplanes; bounds are placed on the rank of members of this class and some of their minors are exhibited. © 1987 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to study the role of modularity in tangential k -blocks over $\text{GF}(q)$. In so doing we extend the class of known tangential k -blocks and show a conjecture of Welsh [11] to be false. For good discussions of the problem see [1, 11, 12].

In Section 3 we show that if F is an affine modular flat of a matroid M representable over $\text{GF}(q)$, the complete Brown truncation of M by F , denoted $T_F(M)$, is also representable over $\text{GF}(q)$. In the case that F is not affine or not modular we obtain extension fields of $\text{GF}(q)$ over which $T_F(M)$ is representable. We then show that if M is a tangential k -block over $\text{GF}(q)$ and F an affine modular flat of M , the simple matroid associated with $T_F(M)$ is also a tangential k -block over $\text{GF}(q)$. This enables us to find tangential k -blocks over $\text{GF}(q)$ of all ranks r where $q^k - q + 2 \leq r \leq q^k$. As a consequence a conjecture of Welsh [11], namely that the only tangential 1-blocks over the prime field $\text{GF}(p)$ are the cycle matroid of the graph K_{p+1} and the uniform geometry $U_{2,p+1}$, is shown to be false for all p not equal to 2 or 3. Tutte [9] and Walton and Welsh [10] show the conjecture to be true for $p=2$ and $p=3$, respectively.

Most known tangential k -blocks are supersolvable in the sense of Stanley [7, 8] and therefore have modular hyperplanes. In Section 4 we study the

class of tangential k -blocks over $\text{GF}(q)$ with modular hyperplanes. We show that this class coincides with the class of tangential k -blocks over $\text{GF}(q)$ which have a cocircuit C^* with $|C^*| = q^k$, that the rank of any member of the class is less than or equal to q^k and that if the rank r matroid M belongs to the class, M contains the cycle matroid of the graph K_{r+1} as a restriction minor.

In Section 5 we give examples of tangential 1-blocks over $\text{GF}(4)$ and $\text{GF}(5)$.

2. DEFINITIONS AND PRELIMINARY RESULTS

We assume that the reader is familiar with the basic concepts of matroid theory, particularly that of the characteristic polynomial $P(M; \lambda)$ of a matroid and the critical exponent $c(M; q)$ of a matroid representable over $\text{GF}(q)$. Welsh [13, Chap. 16] provides a good introduction to these topics; free use will be made of results in this chapter. The terminology used here for matroids will in general follow Welsh [13]. If M is a matroid with ground set E and $S \subseteq E$, the restriction of M to $E \setminus S$ will be denoted by $M|(E \setminus S)$ or by $M \setminus S$ and the contraction of M to $E \setminus S$ will be denoted by $M \cdot (E \setminus S)$ or by M/S in either case according to convenience. The closure and the rank of S in M will be denoted by $\text{cl}_M(S)$ and $r_M(S)$, respectively, or if no danger of ambiguity exists by $\text{cl}(S)$ and $r(S)$, respectively. The simple matroid associated with M will be denoted by \bar{M} . If H is a hyperplane of M then the cocircuit $E \setminus H$ will be called the cocircuit corresponding to H .

Tangential k -Blocks over $\text{GF}(q)$

As defined in [10], for $1 \leq k \leq r-1$, a set X of points of the projective geometry $PG(r-1, q)$ is a *tangential k -block over $\text{GF}(q)$* if the following conditions hold:

- (i) $X \cap W \neq \emptyset$ for every subspace W of $PG(r-1, q)$ which has rank $r(W) = r - k$.
- (ii) For every proper nonempty flat F of $PG(r-1, q) \setminus X$ with $r(F) \leq r - k$ there exists a subspace W of $PG(r-1, q)$ with $r(W) = r - k$ and $W \cap X = F$.

Alternatively we have, in the language of characteristic polynomials; a matroid M is a *tangential k -block over $\text{GF}(q)$* if the following conditions hold:

- (a) M is simple and representable over $\text{GF}(q)$,
- (b) $P(M; q^k) = 0$,

(c) $P(M/F; q^k) > 0$ whenever F is a proper nonempty flat of M .

Note that condition (c) above is equivalent to the apparently stronger,

(c') $P(M'; q^k) > 0$ whenever M' is a proper loopless minor of M .

We say that M is a *tangential k -block* if there exists a prime power q such that M is a tangential k -block over $\text{GF}(q)$.

Modular Flats

A flat F of a matroid M is *modular* if $r(F) + r(F') = r(F \cup F') + r(F \cap F')$ for every flat F' of M . Modular flats are studied extensively in [2] where Propositions 2.1, 2.2, and 2.3 are proved.

PROPOSITION 2.1. *If F is a modular flat of the matroid M with ground set E then for any $A \subseteq E \setminus F$, F is a modular flat of $M \setminus A$.*

PROPOSITION 2.2. *If F is a modular flat of the matroid M and F' is a flat of M disjoint from F then $\text{cl}_{M/F}(F)$ is modular in M/F' and $M|F = (M/F')|F$.*

PROPOSITION 2.3. *H is a modular hyperplane of the loopless matroid M if and only if H meets every rank 2 flat of M .*

PROPOSITION 2.4. *If F is a rank k flat of the loopless matroid M and there exists $X \subset F$ such that $F \setminus X$ is a rank k modular flat of $M \setminus X$ then F is a modular flat of M .*

Proof. In [2] Brylawski shows that a flat F is modular if and only if for all flats F' disjoint from F , $r(F) + r(F') = r(F \cup F')$. The result follows from this observation.

The Complete Brown Truncation

Throughout this section M will denote a rank r simple matroid with ground set E and F will denote a rank k flat of M . The *complete Brown truncation of M by F* , denoted $T_r(M)$ is the matroid whose bases are subsets of E of the form B or $B' \cup \{x\}$, $x \in F$ where B and B' are independent subsets (in M) of $E \setminus F$ with $|B| = r - k + 1$, $|B'| = r - k$ and $r(B \cup F) = r(B' \cup F) = r$.

This definition is slightly idiosyncratic in that the complete Brown truncation as defined in [3] is equal to $\overline{T_r(M)}$; that is, the simple matroid associated with $T_r(M)$.

Let M' be a matroid and $M = M' \setminus P$. If F is a flat of M , then the set P of points is *freely placed on F* if $P \subseteq \text{cl}_{M'}(F)$ and whenever C is a circuit of M' meeting P , $F \subseteq \text{cl}_{M'}(C)$. As pointed out in [3] we have the following equivalent definition of the complete Brown truncation. $T_r(M) = M'/P$

where M' is the matroid obtained from M by putting a set P of $k-1$ independent points freely on the flat F .

If S is any subset of E then the complete Brown truncation of M by S is equal to $T_{\text{cl}_M(S)}(M)$.

Most applications of $T_F(M)$ occur when F is modular. In this case the bases of $T_F(M)$ can be characterised more simply.

PROPOSITION 2.5. *If F is modular, the bases of $T_F(M)$ are the subsets B , independent in M with $|B| = r - k + 1$ and $r(B \cup F) = r$.*

Proof. Say B is a set of $r - k + 1$ independent points with $r(B \cup F) = r$. If B contains no point of F then B is a basis of $T_F(M)$ so assume $B \cap F \neq \emptyset$. Since F is modular in M , $r(\text{cl}(B) \cap F) = 1$ so B contains exactly one point of F , say x . Since $x \notin \text{cl}(B \setminus \{x\})$, $r(\text{cl}(B \setminus \{x\}) \cap F) = 0$ and therefore $r((B \setminus \{x\}) \cup F) = r(F) + r(B \setminus \{x\}) = r$ which implies that $(B \setminus \{x\}) \cup \{y\}$ is a basis of $T_F(M)$ for any $y \in F$; in particular B is a basis of $T_F(M)$.

On the other hand it is routine to show that all bases of $T_F(M)$ satisfy the conditions of the proposition.

The following proposition is proved in [2].

PROPOSITION 2.6. *If F is modular then A is a flat of $T_F(M)$ if and only if A is a flat of M which either contains F or is disjoint from F . If A is disjoint from F , $T_F(M)|_A = M|_A$.*

LEMMA 2.7. *If F is any rank k flat of M , $T_F(M)/F = M/F$.*

Proof. If B is a basis of $T_F(M)/F$ then since F is a rank 1 flat of $T_F(M)$, $B \cup \{x\}$ is a basis of $T_F(M)$ for some $x \in F$. But this implies from the definition of $T_F(M)$ that B is a set of $r - k$ independent points (in M) of $E \setminus F$ with $r(B \cup F) = r$ and this is the requirement for B to be a basis of M/F .

On the other hand if B is a basis of M/F , B is a set of $r - k$ independent points of $E \setminus F$ with $r(B \cup F) = r$, hence $B \cup \{x\}$ is a basis of $T_F(M)$ for any $x \in F$ and B is therefore a basis of $T_F(M)/F$.

PROPOSITION 2.8. *Let F be modular in M and A be a flat of $T_F(M)$. If F is contained in A , $T_F(M)/A = M/A$ and if F is disjoint from A , $T_F(M)/A = T_F(M/A)$.*

Proof. If F is contained in A , the result follows from Lemma 2.7 so assume that F is disjoint from A . Let M' be the matroid obtained from M by placing a set P of $k-1$ independent points freely on the flat F . By Proposition 2.4, $F \cup P$ is a modular flat of M' . Also A is a flat of M' with

$A \cap (F \cup P) = \emptyset$, hence by Proposition 2.2, $\text{cl}_{M'/A}(F \cup P)$ is modular in M'/A and $(M'/A)|(F \cup P) = M'| (F \cup P)$. This shows that P is a set of $k-1$ independent points of $\text{cl}_{M'/A}(F \cup P)$ in M'/A . Say C is a circuit of M'/A with $C \cap P \neq \emptyset$ then there exists $A' \subseteq A$ such that $C \cup A'$ is a circuit of M' and therefore $F \subseteq \text{cl}_{M'}(C \cup A') \subseteq \text{cl}_{M'}(C \cup A)$. Now if $x \in E \setminus A$, $x \in \text{cl}_{M'/A}(C)$ if and only if $x \in \text{cl}_{M'}(C \cup A)$ so $F \subseteq \text{cl}_{M'/A}(C)$ and therefore $\text{cl}_{M'/A}(F) = \text{cl}_{M'/A}(F \cup P) \subseteq \text{cl}_{M'/A}(C)$. That is, P is freely placed on $\text{cl}_{M'/A}(F \cup P)$. Hence $T_F(M/A) = (M'/A)/P = (M'/P)/A = T_F(M)/A$.

Proposition 2.9 is a fundamental result connecting modular flats, the complete Brown truncation and characteristic polynomials; it first appeared in [2].

PROPOSITION 2.9. *If F is modular in M , $P(M; \lambda) = P(T_F(M); \lambda) P(M|F; \lambda)/(\lambda - 1)$.*

3. REPRESENTABILITY OF $T_F(M)$ AND CONSTRUCTION OF TANGENTIAL k -BLOCKS OVER $\text{GF}(q)$

THEOREM 3.1. *If M is a matroid representable over $\text{GF}(q)$ and F is a modular flat of M which is affine over $\text{GF}(q)$ then $T_F(M)$ is representable over $\text{GF}(q)$.*

Proof. We lose no generality in assuming that M is simple and that for some set of points E of $PG(r-1, q)$, $M = PG(r-1, q)|E$. Let r be the rank function of $PG(r-1, q)$. Say F has rank k , then $\text{cl}_{PG(r-1, q)}(F)$ is a rank k subspace of $PG(r-1, q)$ and since F is affine this subspace contains a rank $k-1$ subspace, with set of points say F' , which is disjoint from F . Let $M' = PG(r-1, q)|(E \cup F')$. We show that $T_F(M) = M'/F'$.

Say B is a basis of $T_F(M)$ then since F is modular in M we know by Proposition 2.5 that B is a set of $r-k+1$ independent points of E with $r(B \cup F) = r$. $F \cup F'$ is a rank k flat of M' and by Proposition 2.4 $F \cup F'$ is modular in M' and hence $r((F \cup F') \cap \text{cl}_{M'}(B)) = r(F \cup F') + r(B) - r((F \cup F') \cup B) = 1$ and since F is modular in M , $r(F \cap \text{cl}_M(B)) = r(F) + r(B) - r(F \cup B) = 1$. But $\text{cl}_{M'}(B) \supseteq \text{cl}_M(B)$ so $(F \cup F') \cap \text{cl}_{M'}(B) \supseteq F \cap \text{cl}_M(B)$ and since both these sets are single points we have $(F \cup F') \cap \text{cl}_{M'}(B) = F \cap \text{cl}_M(B)$ and therefore $\text{cl}_{M'}(B) \cap F' = \emptyset$. F' is a modular flat of $PG(r-1, q)$ and is therefore by Proposition 2.1 a modular flat of M' so $r(B \cup F') = r(B) + r(F') - r(\text{cl}_{M'}(B) \cap F') = r$ and therefore B is a basis of M'/F' .

On the other hand if B is a basis of M'/F' , B is a set of $r-k+1$ independent points of E with $r(B \cup F') = r$ and therefore $r(B \cup (F \cup F')) = r$. Since $F \cup F'$ is modular, $r(\text{cl}_{M'}(B) \cap (F \cup F')) = 1$ and therefore $r(\text{cl}_M(B) \cap F) \leq 1$.

But $r(B \cup F) = r(B) + r(F) - r(\text{cl}_M(B) \cap F)$ so $r(B \cup F) \geq r$; that is, $r(B \cup F) = r$ and therefore B is a basis of $T_F(M)$. So $T_F(M) = M'/F'$ and is representable over $\text{GF}(q)$.

COROLLARY 3.2. *If M is representable over $\text{GF}(q)$ and F is a modular flat of M with $c(M|F; q) = l$ then $T_F(M)$ is representable over $\text{GF}(q^l)$.*

Proof. M is representable over $\text{GF}(q^l)$ and $M|F$ is affine over $\text{GF}(q^l)$.

COROLLARY 3.3. *If M is representable over $\text{GF}(q)$ and F is a rank k flat of M then $T_F(M)$ is representable over $\text{GF}(q^k)$.*

Proof. Say $M = PG(r-1, q)|E$ and K is the set of points of the rank k subspace of $PG(r-1, q)$ spanned by F . Let $M' = PG(r-1, q)|(E \cup K)$; K is a rank k modular flat of M' and $c(M'|K; q) = k$ so $T_K(M')$ is representable over $\text{GF}(q^k)$. It is readily seen that $T_F(M) = T_K(M')|E$ and therefore $T_F(M)$ is representable over $\text{GF}(q^k)$.

Although we do not use Corollaries 3.2 and 3.3 they are included because they follow nicely from Theorem 3.1. The following theorem is our main result.

THEOREM 3.4. *If M is a tangential k -block over $\text{GF}(q)$ and F is a proper nonempty modular flat of M which is affine over $\text{GF}(q)$ then $\overline{T_F(M)}$ is a tangential k -block over $\text{GF}(q)$.*

Proof. Certainly $\overline{T_F(M)}$ is simple and since F is modular in M and $M|F$ is affine over $\text{GF}(q)$, $\overline{T_F(M)}$ is representable over $\text{GF}(q)$ by Theorem 3.1.

$$P(M; q^k) = 0 \quad \text{and} \quad P(M|F; q^k) > 0$$

so by Proposition 2.9, $P(\overline{T_F(M)}; q^k) = P(T_F(M); q^k) = (q^k - 1) P(M; q^k) / P(M|F; q^k) = 0$.

Let A be a proper nonempty flat of $T_F(M)$ then by Propositions 2.6 and 2.8 either A is a flat of M containing F , in which case $T_F(M)/A = M/A$ or A is a flat of M disjoint from F in which case $T_F(M)/A = T_F(M/A)$. In the case $T_F(M)/A = M/A$, $P(T_F(M)/A; q^k)$ is certainly positive so assume that F is disjoint from A and hence $T_F(M)/A = T_F(M/A)$. By Proposition 2.2, $\text{cl}_{M/A}(F)$ is modular in M/A and therefore $P(T_F(M)/A; q^k) = P(T_F(M/A); q^k) = (q^k - 1) P(M/A; q^k) / P((M/A)|\text{cl}_{M/A}(F); q^k) > 0$. But minors obtained by contracting out flats in $\overline{T_F(M)}$ differ from minors obtained by contracting out corresponding flats in $T_F(M)$ only in respect of parallel elements and therefore have identical characteristic polynomials, so if M' is a minor obtained by contracting out a proper nonempty flat of $\overline{T_F(M)}$, $P(M'; q^k) > 0$ and the theorem is proved.

COROLLARY 3.5. *If M is a tangential 1-block over $\text{GF}(q)$ and F is a proper nonempty modular flat of M then $\overline{T_F(M)}$ is a tangential 1-block over $\text{GF}(q)$.*

Proof. All proper nonempty flats of a tangential 1-block over $\text{GF}(q)$ are affine over $\text{GF}(q)$.

The cycle matroid of the complete graph on $q^k + 1$ vertices, denoted $M(K_{q^k+1})$ is a tangential k -block over $\text{GF}(q)$ for any prime power q (see, e.g., Welsh [11]) and for $1 \leq l \leq q^k + 1$, $M(K_{q^k+1})$ contains modular flats isomorphic to $M(K_l)$ (see, e.g., Brylawski [2]) and for $2 \leq n \leq q$, $M(K_n)$ is affine over $\text{GF}(q)$. Hence for $2 \leq n \leq q$, $\overline{T_{M(K_n)}(M(K_{q^k+1}))}$ is a tangential k -block over $\text{GF}(q)$ but $r(\overline{T_{M(K_n)}(M(K_{q^k+1}))}) = r(M(K_{q^k+1})) - r(M(K_n)) + 1 = q^k - (n-1) + 1 = q^k - n + 2$ and therefore we have:

COROLLARY 3.6. *If q is a prime power and $q^k - q + 2 \leq r \leq q^k$ then there exists a tangential k -block over $\text{GF}(q)$ of rank r ;*

and in particular;

COROLLARY 3.7. *There exist tangential 1-blocks over $\text{GF}(q)$ of all ranks r where $2 \leq r \leq q$.*

In [11] Welsh conjectures that the only tangential 1-blocks over the prime field $\text{GF}(p)$ are $M(K_{p+1})$ and $U_{2,p+1}$. It is an immediate consequence of Corollary 3.7 that this conjecture is false for all primes other than 2 and 3.

4. TANGENTIAL k -BLOCKS WITH MODULAR HYPERPLANES

In this section we study the structure of tangential k -blocks with modular hyperplanes; we first show that this class coincides with the class of tangential k -blocks which have cocircuits of cardinality q^k . The following result appears in [4, 5, 6].

LEMMA 4.1 (Oxley). *Let M be a matroid with hyperplane H and corresponding cocircuit $\{e_1, e_2, \dots, e_m\}$ then the characteristic polynomial of M satisfies the identity*

$$P(M; \lambda) = (\lambda - m) P(M \setminus \{e_1, e_2, \dots, e_m\}; \lambda) \\ + \sum_{i=2}^m \sum_{j=1}^{i-1} P(M \setminus \{e_i, \dots, e_{j-1}, e_{j+1}, \dots, e_{i-1}\} / \{e_j, e_i\}, \lambda).$$

Theorem 4.2 appears without proof, credited to Oxley, in [3, Exercise 15(b), p. 141].

THEOREM 4.2 (Oxley). *A hyperplane of a tangential k -block over $\text{GF}(q)$ is modular if and only if its corresponding cocircuit has cardinality q^k .*

Proof. Say H is a modular hyperplane of the tangential k -block M over $\text{GF}(q)$. Let m be the cardinality of the cocircuit corresponding to H . We have, since H is modular, $P(M; q^k) = (q^k - m) P(M|H; q^k)$, but $P(M; q^k) = 0$ and $P(M|H; q^k) > 0$ so $m = q^k$.

Say M has a cocircuit C^* with $|C^*| = q^k$. Assume that the corresponding hyperplane H is not modular; then by Proposition 2.3 there exist points x and y in C^* with $\text{cl}(\{x, y\}) \cap H = \emptyset$. Let $C^* = \{e_1, e_2, \dots, e_{q^k}\}$ where $e_{q^k-1} = x$ and $e_{q^k} = y$; $\{e_{q^k-1}, e_{q^k}\}$ is a flat of $M \setminus \{e_1, e_2, \dots, e_{q^k-2}\}$ and therefore $M' = M \setminus \{e_1, e_2, \dots, e_{q^k-2}\} / \{e_{q^k-1}, e_{q^k}\}$ is a loopless proper minor of M so $P(M'; q) > 0$. But by Lemma 4.1, $P(M; q^k) \geq P(M'; q^k) > 0$ which is a contradiction and H is therefore modular.

As a straightforward consequence of Theorem 4.2 we observe that if C^* is a cocircuit of a tangential k -block over $\text{GF}(q)$ with no modular hyperplanes then $|C^*| > q^k$.

In general it is very difficult to place a bound on the number of tangential k -blocks over a given field. We now show that in the restricted class of tangential k -blocks with modular hyperplanes it is always possible to do this.

LEMMA 4.3. *Let M be a tangential k -block over $\text{GF}(q)$ with modular hyperplane H and corresponding cocircuit C^* . If $x \in H$ then x belongs to a line determined by points of C^* .*

Proof. If $r(M) = 2$ the results is certainly true so assume that $r(M) > 2$. In [2] Brylawski shows that a modular flat of a connected matroid is connected and it is routine to show that any tangential k -block over $\text{GF}(q)$ is connected, hence H is connected. Assume $x \in H$ and x belongs to no line determined by points of C^* . Since H is connected and $r(H) \geq 2$, x is not a coloop of $M|H$ and therefore $H \setminus \{x\}$ is a hyperplane of $M \setminus \{x\}$. But $H \setminus \{x\}$ meets every line determined by points of the cocircuit C^* of $M \setminus \{x\}$ so $H \setminus \{x\}$ is modular in $M \setminus \{x\}$. Hence $P(M \setminus \{x\}; \lambda) = P(H \setminus \{x\}; \lambda)(\lambda - q^k)$ and therefore $P(M \setminus \{x\}; q^k) = 0$ which contradicts the assumption that M is a tangential k -block over $\text{GF}(q)$. So x belongs to a line determined by points of C^* and the Lemma is proved.

COROLLARY 4.3. *If M is a tangential k -block over $\text{GF}(q)$ with cocircuit C^* corresponding to the modular hyperplane H then $r(C^*) = r(M)$.*

Proof. Since every point of H belongs to a line determined by points in C^* , $H \subseteq \text{cl}(C^*)$ and therefore $r(C^*) = r(M)$.

As an immediate corollary we have

COROLLARY 4.4. *If M is a tangential k -block $\text{GF}(q)$ with a modular hyperplane (equivalently a cocircuit of cardinality q^k) then $r(M) \leq q^k$.*

Thus there are only a finite number of tangential k -blocks with modular hyperplanes over a given finite field. The persistence of Tutte's tangential 2-block conjecture indicates the difficulty of finding a similar result for arbitrary tangential k -blocks.

Although, in general, it is difficult to find all tangential k -blocks over a given field many critical problems could be solved if it could be shown that all tangential k -blocks possess certain "interesting" minors. In particular I know of no counterexample to the conjecture that any tangential k -block has a minor isomorphic to $M(K_{k+2})$; the conjecture is certainly true for all tangential 1-blocks and for tangential 2-blocks over $\text{GF}(2)$. While this conjecture seems difficult to resolve in general we can show that for the restricted class of tangential k -blocks with modular hyperplanes a stronger form is true; that is, any tangential k -block over $\text{GF}(q)$ with a modular hyperplane contains $M(K_{k+2})$ as a restriction minor.

LEMMA 4.5. *Let M be a matroid representable over $\text{GF}(q)$, H be a modular hyperplane of M and C^* the cocircuit corresponding to H . If $r(C^*) = m$ then M contains a restriction minor isomorphic to $M(K_{m+1})$.*

Proof. If M is not simple consider \bar{M} ; that is, we may assume without loss of generality that M is simple. Since $r(C^*) = m$, C^* contains an independent set B' with $|B'| = m$. Since H is modular, each line determined by a pair of points of B' meets H . Let P be the set of points of intersection of these lines with H and let $M' = M|(B' \cup P)$. We show that M' is isomorphic to $M(K_{m+1})$.

Let $B' = \{b_1, \dots, b_m\}$. Since M is representable over $\text{GF}(q)$, M' is and since B' is a basis of M' there exists a matrix representation of M' over $\text{GF}(q)$ where (for $1 \leq i \leq m$) the i th column of I_m (the $m \times m$ identity matrix over $\text{GF}(q)$) corresponds to b_i . In this representation, P is represented by a subset of the set of vectors of a hyperplane of $V(r, q)$ which contains none of the column vectors of I_m . A hyperplane, with equation $a_1 x_1 + \dots + a_m x_m = 0$ misses the column vectors of I_m if and only if for $1 \leq i \leq m$, $a_i \neq 0$. A vector common to such a hyperplane and the flat of $V(r, q)$ spanned by the i th and j th columns of I_m is the vector $(x_1, \dots, x_m)^T$ where $x_i = a_j$, $x_j = -a_i$ and otherwise $x_k = 0$. That is, there exists an m -tuple (a_1, \dots, a_m) of nonzero elements of $\text{GF}(q)$ such that the $m \times \binom{m+1}{2}$

matrix S described below is a representation of M' over $\text{GF}(q)$. S is the matrix whose first m columns form I_m while the remaining $\binom{m}{2}$ columns are all possible distinct vectors of the form $(x_1, \dots, x_m)^T$ where for $i < j$, $x_i = a_j$, $x_j = -a_i$ and otherwise $x_k = 0$. Let $R = (r_{ij})$ be the $m \times m$ matrix over $\text{GF}(q)$ defined by

$$r_{ij} = \begin{cases} a_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

One routinely sees that by multiplication of the columns of RS by appropriate nonzero scalars one obtains a matrix T of the following form. The first m columns of T form I_m while the remaining columns of T consist of the $\binom{m}{2}$ distinct vectors of the form $(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)^T$. R is nonsingular so T is a representation of M' over $\text{GF}(q)$. But T is well known to be a unimodular representation of $M(K_{m+1})$; that is a representation of $M(K_{m+1})$ over any field and in particular a representation of $M(K_{m+1})$ over $\text{GF}(q)$. Hence $M' \cong M(K_{m+1})$.

COROLLARY 4.6. *If M is a rank r tangential k -block over $\text{GF}(q)$ with a modular hyperplane then M contains a restriction minor isomorphic to $M(K_{r+1})$.*

Proof. By Corollary 4.3, the cocircuit corresponding to any modular hyperplane in M is a spanning and hence has rank r .

COROLLARY 4.7. *If M is a tangential k -block over $\text{GF}(q)$ with a modular hyperplane then M contains a restriction minor isomorphic to $M(K_{k+2})$.*

Proof. Say M is simple and representable over $\text{GF}(q)$ with $r(M) < k + 1$. Then M is isomorphic to a restriction of $PG(k - 1, q)$ and therefore $P(M; q^k) \geq P(PG(k - 1, q); q^k) > 0$. So no tangential k -block has rank less than $k + 1$ and the result follows from Corollary 4.6.

The following corollary is an immediate consequence of Corollary 4.6.

COROLLARY 4.8. *$M(K_{q^k+1})$ is the unique tangential k -block over $\text{GF}(q)$ with a modular hyperplane and rank q^k .*

5. TANGENTIAL 1-BLOCKS OVER $\text{GF}(4)$ AND $\text{GF}(5)$

5.1. Tangential 1-Blocks over $\text{GF}(4)$

As pointed out in [10], $M(K_5)$, $U_{2,5}$, $PG(2, 2)$, $AG(2, 3)$ and the cocycle matroid of the Petersen graph, $M(P_{10}^*)$ are all tangential 1-blocks over $\text{GF}(4)$. In addition to this, $T_{M(K_3)}(M(K_5))$ is a tangential 1-blocks over

$\text{GF}(4)$ and it is readily verified that $\overline{T_{M(K_3)}(M(P_{10}^*))}$ is well defined and is a tangential 1-block over $\text{GF}(4)$.

5.2. Tangential 1-Blocks over $\text{GF}(5)$

Over $\text{GF}(5)$ we have as tangential 1-blocks $M(K_6)$, $\overline{T_{M(K_3)}(M(K_6))}$, $\overline{T_{M(K_4)}(M(K_6))}$, and $\overline{T_{M(K_5)}(M(K_6))}$ which is isomorphic to $U_{2,6}$. But these are by no means the only ones as the following example shows.

Consider the matroids M_1 , M_2 , and M_3 whose Euclidean representations are shown in Figure 1. M_1 is isomorphic to $\overline{T_{M(K_4)}(M(K_6))}$ and is a tangential 1-block over $\text{GF}(5)$. It is routine to show that M_2 and M_3 are representable over $\text{GF}(5)$ (either by arguing geometrically or by finding a direct representation), and that neither has a minor isomorphic to $U_{2,6}$. Since H_2 and H_3 are modular hyperplanes of M_2 and M_3 respectively we have, $P(M_2; \lambda) = (\lambda - 1)(\lambda - 4)(\lambda - 5)$ and $P(M_3; \lambda) = (\lambda - 1)(\lambda - 3)(\lambda - 5)$ and therefore both M_2 and M_3 are tangential 1-blocks over $\text{GF}(5)$. Note that $P(M_1; \lambda) = (\lambda - 1)(\lambda - 4)(\lambda - 5)$ and M_1 and M_2 are therefore nonisomorphic tangential 1-blocks over $\text{GF}(5)$ with identical characteristic polynomials.

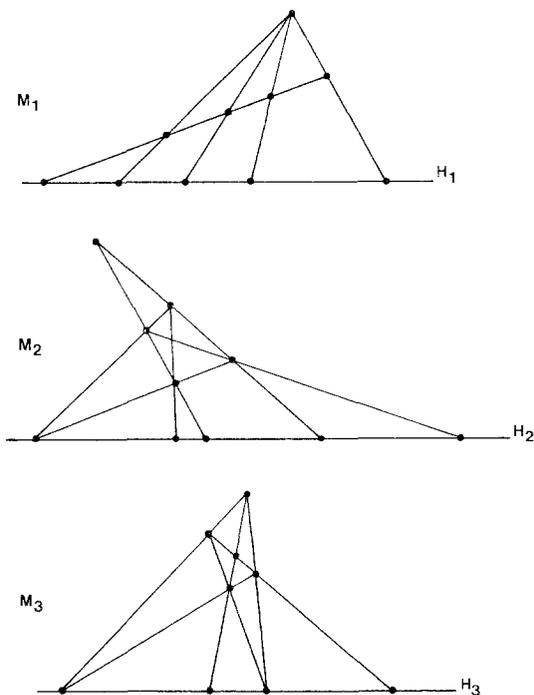


FIGURE 1

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