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Modularity in Tangential k-Blocks

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This paper examines the role of modularity in tangential k-blocks over GF(q). It is shown that if M is a tangential k-block over GF(q) and F is a modular flat of M which is affine over GF(q) then the simple matroid associated with the complete Brown truncation of M by F is also a tangential k-block over GF(q). This enables us to construct tangential k-blocks over GF(q) of all ranks r where $q^k - q + 2 \leqslant r \leqslant q^k$. We also consider tangential k-blocks which have modular hyperplanes; bounds are placed on the rank of members of this class and some of their minors are exhibited. © 1987 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to study the role of modularity in tangential k-blocks over GF(q). In so doing we extend the class of known tangential k-blocks and show a conjecture of Welsh [11] to be false. For good discussions of the problem see [1, 11, 12].

In Section 3 we show that if F is an affine modular flat of a matroid M representable over GF(q), the complete Brown truncation of M by F, denoted $T_F(M)$, is also representable over GF(q). In the case that F is not affine or not modular we obtain extension fields of GF(q) over which $T_F(M)$ is representable. We then show that if M is a tangential k-block over GF(q) and F an affine modular flat of M, the simple matroid associated with $T_F(M)$ is also a tangential k-block over $GF(q)$. This enables us to find tangential k-blocks over GF(q) of all ranks r where $q^k - q + 2 \leqslant r \leqslant q^k$. As a consequence a conjecture of Welsh [11], namely that the only tangential 1-blocks over the prime field GF(p) are the cycle matroid of the graph K_{p+1} and the uniform geometry $U_{2,p+1}$, is shown to be false for all p not equal to 2 or 3. Tutte [9] and Walton and Welsh [10] show the conjecture to be true for $p = 2$ and $p = 3$, respectively.

Most known tangential k-blocks are supersolvable in the sense of Stanley [7, 8] and therefore have modular hyperplanes. In Section 4 we study the

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class of tangential k-blocks over GF(q) with modular hyperplanes. We show that this class coincides with the class of tangential k-blocks over GF(q) which have a cocircuit C^* with $|C^*| = q^k$, that the rank of any member of the class is less than or equal to q^k and that if the rank r matroid M belongs to the class, M contains the cycle matroid of the graph K_{r+1} as a restriction minor.

In Section 5 we give examples of tangential 1-blocks over GF(4) and GF(5).

2. Definitions and Preliminary Results

We assume that the reader is familiar with the basic concepts of matroid theory, particularly that of the characteristic polynomial $P(M; \lambda)$ of a matroid and the critical exponent $c(M; q)$ of a matroid representable over GF(q). Welsh [13, Chap. 16] provides a good introduction to these topics; free use will be made of results in this chapter. The terminology used here for matroids will in general follow Welsh [13]. If M is a matroid with ground set E and $S \subseteq E$, the restriction of M to $E \backslash S$ will be denoted by $M | (E \backslash S)$ or by $M \backslash S$ and the contraction of M to $E \backslash S$ will be denoted by $M \cdot (E \backslash S)$ or by M/S in either case according to convenience. The closure and the rank of S in M will be denoted by $\mathrm{cl}_M(S)$ and $r_M(S)$, respectively, or if no danger of ambiguity exists by $\mathrm{cl}(S)$ and $r(S)$, respectively. The simple matroid associated with M will be denoted by $\bar{M}$. If H is a hyperplane of M then the cocircuit $E \backslash H$ will be called the cocircuit corresponding to H.

Tangential k-Blocks over GF(q)

As defined in [10], for $1 \leqslant k \leqslant r-1$, a set X of points of the projective geometry $PG(r-1, q)$ is a *tangential k-block over* GF(q) if the following conditions hold:

(i) $X \cap W \neq \varnothing$ for every subspace W of $PG(r-1, q)$ which has rank $r(W) = r-k$.

(ii) For every proper nonempty flat F of $PG(r-1, q) | X$ with $r(F) \leqslant r-k$ there exists a subspace W of $PG(r-1, q)$ with $r(W) = r-k$ and $W \cap X = F$.

Alernatively we have, in the language of characteristic polynomials; a matroid M is a *tangential k-block over* GF(q) if the following conditions hold:

(a) M is simple and representable over GF(q),

(b) $P(M; q^k) = 0$,

(c) $P(M/F; q^k) > 0$ whenever F is a proper nonempty flat of M.

Note that condition (c) above is equivalent to the apparently stronger,

(c)′ $P(M'; q^k) > 0$ whenever M' is a proper loopless minor of M.

We say that M is a *tangential k-block* if there exists a prime power q such that M is a tangential k-block over GF(q).

Modular Flats

A flat F of a matroid M is *modular* if $r(F) + r(F') = r(F \cup F') + r(F \cap F')$ for every flat F' of M. Modular flats are studied extensively in [2] where Propositions 2.1, 2.2, and 2.3 are proved.

PROPOSITION 2.1. *If F is a modular flat of the matroid M with ground set E then for any $A \subseteq E\backslash F$, F is a modular flat of $M \backslash A$.*

PROPOSITION 2.2. *If F is a modular flat of the matroid M and F' is a flat of M disjoint from F then $\mathrm{cl}_{M/F'}(F)$ is modular in M/F' and $M|F = (M/F')|F$.*

PROPOSITION 2.3. *H is a modular hyperplane of the loopless matroid M if and only if H meets every rank 2 flat of M.*

PROPOSITION 2.4. *If F is a rank k flat of the loopless matroid M and there exists $X \subset F$ such that $F\backslash X$ is a rank k modular flat of $M\backslash X$ then F is a modular flat of M.*

Proof. In [2] Brylawski shows that a flat F is modular if and only if for all flats F' disjoint from F, $r(F) + r(F') = r(F \cup F')$. The result follows from this observation.

The Complete Brown Truncation

Throughout this section M will denote a rank r simple matroid with ground set E and F will denote a rank k flat of M. The *complete Brown truncation of M by F*, denoted $T_F(M)$ is the matroid whose bases are subsets of E of the form B or $B' \cup \{x\}$, $x \in F$ where B and B' are independent subsets (in M) of $E\backslash F$ with $|B| = r - k + 1$, $|B'| = r - k$ and $r(B \cup F) = r(B' \cup F) = r$.

This definition is slightly idiosyncratic in that the complete Brown truncation as defined in [3] is equal to $\overline{T_F(M)}$; that is, the simple matroid associated with $T_F(M)$.

Let M' be a matroid and $M = M' \backslash P$. If F is a flat of M, then the set P of points is *freely placed on* F if $P \subseteq \mathrm{cl}_{M'}(F)$ and whenever C is a circuit of M' meeting P, $F \subseteq \mathrm{cl}_{M'}(C)$. As pointed out in [3] we have the following equivalent definition of the complete Brown truncation. $T_F(M) = M'/P$

where M' is the matroid obtained from M by putting a set P of $k-1$ independent points freely on the flat F.

If S is any subset of E then the complete Brown truncation of M by S is equal to $T_{\mathrm{cl}_M(S)}(M)$.

Most applications of $T_F(M)$ occur when F is modular. In this case the bases of $T_F(M)$ can be characterised more simply.

PROPOSITION 2.5. *If F is modular, the bases of $T_F(M)$ are the subsets B, independent in M with $|B| = r-k+1$ and $r(B \cup F) = r$.*

Proof. Say B is a set of $r-k+1$ independent points with $r(B \cup F) = r$. If B contains no point of F then B is a basis of $T_F(M)$ so assume $B \cap F \neq \emptyset$. Since F is modular in M, $r(\mathrm{cl}(B) \cap F) = 1$ so B contains exactly one point of F, say x. Since $x \notin \mathrm{cl}(B \backslash \{x\})$, $r(\mathrm{cl}(B \backslash \{x\}) \cap F) = 0$ and therefore $r((B \backslash \{x\}) \cup F) = r(F) + r(B \backslash \{x\}) = r$ which implies that $(B \backslash \{x\}) \cup \{y\}$ is a basis of $T_F(M)$ for any $y \in F$; in particular B is a basis of $T_F(M)$.

On the other hand it is routine to show that all bases of $T_F(M)$ satisfy the conditions of the proposition.

The following proposition is proved in [2].

PROPOSITION 2.6. *If F is modular then A is a flat of $T_F(M)$ if and only if A is a flat of M which either contains F or is disjoint from F. If A is disjoint from F, $T_F(M) | A = M | A$.*

LEMMA 27. *If F is any rank k flat of M, $T_F(M)/F = M/F$.*

Proof. If B is a basis of $T_F(M)/F$ then since F is a rank 1 flat of $T_F(M)$, $B \cup \{x\}$ is a basis of $T_F(M)$ for some $x \in F$. But this implies from the definition of $T_F(M)$ that B is a set of $r-k$ independent points (in M) of $E \backslash F$ with $r(B \cup F) = r$ and this is the requirement for B to be a basis of M/F.

On the other hand if B is a basis of M/F, B is a set of $r-k$ independent points of $E \backslash F$ with $r(B \cup F) = r$, hence $B \cup \{x\}$ is a basis of $T_F(M)$ for any $x \in F$ and B is therefore a basis of $T_F(M)/F$.

PROPOSITION 2.8. *Let F be modular in M and A be a flat of $T_F(M)$. If F is contained in A, $T_F(M)/A = M/A$ and if F is disjoint from A, $T_F(M)/A = T_F(M/A)$.*

Proof. If F is contained in A, the result follows from Lemma 2.7 so assume that F is disjoint from A. Let M' be the matroid obtained from M by placing a set P of $k-1$ independent points freely on the flat F. By Proposition 2.4, $F \cup P$ is a modular flat of M'. Also A is a flat of M' with

$A \cap (F \cup P) = \phi$, hence by Proposition 2.2, $\mathrm{cl}_{M'/A}(F \cup P)$ is modular in M'/A and $(M'/A)|(F \cup P) = M'|(F \cup P)$. This shows that P is a set of $k-1$ independent points of $\mathrm{cl}_{M'/A}(F \cup P)$ in M'/A. Say C is a circuit of M'/A with $C \cap P \neq \emptyset$ then there exists $A' \subseteq A$ such that $C \cup A'$ is a circuit of M' and therefore $F \subseteq \mathrm{cl}_{M'}(C \cup A') \subseteq \mathrm{cl}_{M'}(C \cup A)$. Now if $x \in E \backslash A$, $x \in \mathrm{cl}_{M'/A}(C)$ if and only if $x \in \mathrm{cl}_{M'}(C \cup A)$ so $F \subseteq \mathrm{cl}_{M'/A}(C)$ and therefore $\mathrm{cl}_{M'/A}(F) = \mathrm{cl}_{M'/A}(F \cup P) \subseteq \mathrm{cl}_{M'/A}(C)$. That is, P is freely placed on $\mathrm{cl}_{M'/A}(F \cup P)$. Hence $T_F(M/A) = (M'/A)/P = (M'/P)/A = T_F(M)/A$.

Proposition 2.9 is a fundamental result connecting modular flats, the complete Brown truncation and characteristic polynomials; it first appeared in [2].

PROPOSITION 2.9. *If F is modular in M, $P(M; \lambda) = P(T_F(M); \lambda) P(M|F; \lambda)/(\lambda - 1)$.*

3. REPRESENTABILITY OF $T_F(M)$ AND CONSTRUCTION OF TANGENTIAL k-BLOCKS OVER GF(q)

THEOREM 3.1. *If M is a matroid representable over* GF(q) *and F is a modular flat of M which is affine over* GF(q) *then $T_F(M)$ is representable over* GF(q).

Proof. We lose no generality in assuming that M is simple and that for some set of points E of $PG(r-1, q)$, $M = PG(r-1, q)|E$. Let r be the rank function of $PG(r-1, q)$. Say F has rank k, then $\mathrm{cl}_{PG(r-1,q)}(F)$ is a rank k subspace of $PG(r-1, q)$ and since F is affine this subspace contains a rank $k-1$ subspace, with set of points say F', which is disjoint from F. Let $M' = PG(r-1, q)|(E \cup F')$. We show that $T_F(M) = M'/F'$.

Say B is a basis of $T_F(M)$ then since F is modular in M we know by Proposition 2.5 that B is a set of $r-k+1$ independent points of E with $r(B \cup F) = r$. $F \cup F'$ is a rank k flat of M' and by Proposition 2.4 $F \cup F'$ is modular in M' and hence $r((F \cup F') \cap \mathrm{cl}_{M'}(B)) = r(F \cup F') + r(B) - r((F \cup F') \cup B) = 1$ and since F is modular in M, $r(F \cap \mathrm{cl}_M(B)) = r(F) + r(B) - r(F \cup B) = 1$. But $\mathrm{cl}_{M'}(B) \supseteq \mathrm{cl}_M(B)$ so $(F \cup F') \cap \mathrm{cl}_{M'}(B) \supseteq F \cap \mathrm{cl}_M(B)$ and since both these sets are single points we have $(F \cup F') \cap \mathrm{cl}_{M'}(B) = F \cap \mathrm{cl}_M(B)$ and therefore $\mathrm{cl}_{M'}(B) \cap F' = \emptyset$. F' is a modular flat of $PG(r-1, q)$ and is therefore by Proposition 2.1 a modular flat of M' so $r(B \cup F') = r(B) + r(F') - r(\mathrm{cl}_{M'}(B) \cap F') = r$ and therefore B is a basis of M'/F'.

On the other hand if B is a basis of M'/F', B is a set of $r-k+1$ independent points of E with $r(B \cup F') = r$ and therefore $r(B \cup (F \cup F')) = r$. Since $F \cup F'$ is modular, $r(\mathrm{cl}_{M'}(B) \cap (F \cup F')) = 1$ and therefore $r(\mathrm{cl}_M(B) \cap F) \leqslant 1$.

But $r(B \cup F) = r(B) + r(F) - r(\mathrm{cl}_M(B) \cap F)$ so $r(B \cup F) \geqslant r$; that is, $r(B \cup F) = r$ and therefore B is a basis of $T_F(M)$. So $T_F(M) = M'/F'$ and is representable over $\mathrm{GF}(q)$.

COROLLARY 3.2. *If M is representable over* $\mathrm{GF}(q)$ *and F is a modular flat of M with $c(M \mid F; q) = l$ then $T_F(M)$ is representable over* $\mathrm{GF}(q^l)$.

Proof. M is representable over $\mathrm{GF}(q^l)$ and $M \mid F$ is affine over $\mathrm{GF}(q^l)$.

COROLLARY 3.3. *If M is representable over* $\mathrm{GF}(q)$ *and F is a rank k flat of M then $T_F(M)$ is representable over* $\mathrm{GF}(q^k)$.

Proof. Say $M = PG(r-1, q) \mid E$ and K is the set of points of the rank k subspace of $PG(r-1, q)$ spanned by F. Let $M' = PG(r-1, q) \mid (E \cup K)$; K is a rank k modular flat of M' and $c(M' \mid K; q) = k$ so $T_K(M')$ is representable over $\mathrm{GF}(q^k)$. It is readily seen that $T_F(M) = T_K(M') \mid E$ and therefore $T_F(M)$ is representable over $\mathrm{GF}(q^k)$.

Although we do not use Corollaries 3.2 and 3.3 they are included because they follow nicely from Theorem 3.1. The following theorem is our main result.

THEOREM 3.4. *If M is a tangential k-block over* $\mathrm{GF}(q)$ *and F is a proper nonempty modular flat of M which is affine over* $\mathrm{GF}(q)$ *then $\overline{T_F(M)}$ is a tangential k-block over* $\mathrm{GF}(q)$.

Proof. Certainly $\overline{T_F(M)}$ is simple and since F is modular in M and $M \mid F$ is affine over $GF(q)$, $\overline{T_F(M)}$ is representable over $\mathrm{GF}(q)$ by Theorem 3.1.

$$P(M; q^k) = 0 \qquad \text{and} \qquad P(M \mid F; q^k) > 0$$

so by Proposition 2.9, $P(\overline{T_F(M)}; q^k) = P(T_F(M); q^k) = (q^k - 1)\, P(M; q^k)/P(M \mid F; q^k) = 0$.

Let A be a proper nonempty flat of $T_F(M)$ then by Propositions 2.6 and 2.8 either A is a flat of M containing F, in which case $T_F(M)/A = M/A$ or A is a flat of M disjoint from F in which case $T_F(M)/A = T_F(M/A)$. In the case $T_F(M)/A = M/A$, $P(T_F(M)/A; q^k)$ is certainly positive so assume that F is disjoint from A and hence $T_F(M)/A = T_F(M/A)$. By Proposition 2.2, $\mathrm{cl}_{M/A}(F)$ is modular in M/A and therefore $P(T_F(M)/A; q^k) = P(T_F(M/A); q^k) = (q^k - 1)\ P(M/A; q^k)/P((M/A) \mid \mathrm{cl}_{M/A}(F); q^k) > 0$. But minors obtained by contracting out flats in $\overline{T_F(M)}$ differ from minors obtained by contracting out corresponding flats in $T_F(M)$ only in respect of parallel elements and therefore have identical characteristic polynomials, so if M' is a minor obtained by contracting out a proper nonempty flat of $\overline{T_F(M)}$, $P(M'; q^k) > 0$ and the theorem is proved.

COROLLARY 3.5. *If M is a tangential* 1*-block over* GF(q) *and F is a proper nonempty modular flat of M then $\overline{T_F(M)}$ is a tangential* 1*-block over* GF(q).

Proof. All proper nonempty flats of a tangential 1-block over GF(q) are affine over GF(q).

The cycle matroid of the complete graph on q^k+1 vertices, denoted $M(K_{q^k+1})$ is a tangential k-block over GF(q) for any prime power q (see, e.g., Welsh [11]) and for $1 \leqslant l \leqslant q^k+1$, $M(K_{q^k+1})$ contains modular flats isomorphic to $M(K_l)$ (see, e.g., Brylawski [2]) and for $2 \leqslant n \leqslant q$, $M(K_n)$ is affine over GF(q). Hence for $2 \leqslant n \leqslant q$, $\overline{T_{M(K_n)}(M(K_{q^k+1}))}$ is a tangential k-block over GF(q) but $r(\overline{T_{M(K_n)}(M(K_{q^k+1}))}) = r(M(K_{q^k+1})) - r(M(K_n)) + 1 = q^k - (n-1) + 1 = q^k - n + 2$ and therefore we have:

COROLLARY 3.6. *If q is a prime power and $q^k - q + 2 \leqslant r \leqslant q^k$ then there exists a tangential k-block over* GF(q) *of rank r*;

and in particular;

COROLLARY 3.7. *There exist tangential* 1*-blocks over* GF(q) *of all ranks r where $2 \leqslant r \leqslant q$.*

In [11] Welsh conjectures that the only tangential 1-blocks over the prime field GF(p) are $M(K_{p+1})$ and $U_{2,p+1}$. It is an immediate consequence of Corollary 3.7 that this conjecture is false for all primes other than 2 and 3.

4. TANGENTIAL k-BLOCKS WITH MODULAR HYPERPLANES

In this section we study the structure of tangential k-blocks with modular hyperplanes; we first show that this class coincides with the class of tangential k-blocks which have cocircuits of cardinality q^k. The following result appears in [4, 5, 6].

LEMMA 4.1 (Oxley). *Let M be a matroid with hyperplane H and corresponding cocircuit $\{e_1, e_2, \ldots, e_m\}$ then the characteristic polynomial of M satisfies the identity*

$$P(M; \lambda) = (\lambda - m)\, P(M \backslash \{e_1, e_2, \ldots, e_m\}; \lambda) + \sum_{i=2}^{m} \sum_{j=1}^{i-1} P(M \backslash \{e_i, \ldots, e_{j-1}, e_{j+1}, \ldots, e_{i-1}\}/\{e_j, e_i\}, \lambda).$$

Theorem 4.2 appears without proof, credited to Oxley, in [3, Exercise 15(b), p. 141].

THEOREM 4.2 (Oxley). *A hyperplane of a tangential k-block over* GF(q) *is modular if and only if its corresponding cocircuit has cardinality* q^k.

Proof. Say H is a modular hyperplane of the tangential k-block M over GF(q). Let m be the cardinality of the cocircuit corresponding to H. We have, since H is modular, $P(M; q^k) = (q^k - m)\ P(M \mid H; q^k)$, but $P(M; q^k) = 0$ and $P(M \mid H; q^k) > 0$ so $m = q^k$.

Say M has a cocircuit C^* with $|C^*| = q^k$. Assume that the corresponding hyperplane H is not modular; then by Proposition 2.3 there exist points x and y in C^* with $\text{cl}(\{x, y\}) \cap H = \emptyset$. Let $C^* = \{e_1, e_2, ..., e_{q^k}\}$ where $e_{q^k - 1} = x$ and $e_{q^k} = y$; $\{e_{q^k-1}, e_{q^k}\}$ is a flat of $M \backslash \{e_1, e_2, ..., e_{q^k-2}\}$ and therefore $M' = M \backslash \{e_1, e_2, ..., e_{q^k-2}\} / \{e_{q^k-1}, e_{q^k}\}$ is a loopless proper minor of M so $P(M'; q) > 0$. But by Lemma 4.1, $P(M; q^k) \geqslant P(M'; q^k) > 0$ which is a contradiction and H is therefore modular.

As a straightforward consequence of Theorem 4.2 we observe that if C^* is a cocircuit of a tangential k-block over GF(q) with no modular hyperplanes then $|C^*| > q^k$.

In general it is very difficult to place a bound on the number of tangential k-blocks over a given field. We now show that in the restricted class of tangential k-blocks with modular hyperplanes it is always possible to do this.

LEMMA 4.3. *Let M be a tangential k-block over* GF(q) *with modular hyperplane H and corresponding cocircuit C^*. If $x \in H$ then x belongs to a line determined by points of C^*.*

Proof. If $r(M) = 2$ the results is certainly true so assume that $r(M) > 2$. In [2] Brylawski shows that a modular flat of a connected matroid is connected and it is routine to show that any tangential k-block over GF(q) is connected, hence H is connected. Assume $x \in H$ and x belongs to no line determined by points of C^*. Since H is connected and $r(H) \geqslant 2$, x is not a coloop of $M \mid H$ and therefore $H \backslash \{x\}$ is a hyperplane of $M \backslash \{x\}$. But $H \backslash \{x\}$ meets every line determined by points of the cocircuit C^* of $M \backslash \{x\}$ so $H \backslash \{x\}$ is modular in $M \backslash \{x\}$. Hence $P(M \backslash \{x\}; \lambda) = P(H \backslash \{x\}; \lambda)(\lambda - q^k)$ and therefore $P(M \backslash \{x\}; q^k) = 0$ which contradicts the assumption that M is a tangential k-block over GF(q). So x belongs to a line determined by points of C^* and the Lemma is proved.

COROLLARY 4.3. *If M is a tangential k-block over* GF(q) *with cocircuit C^* corresponding to the modular hyperplane H then $r(C^*) = r(M)$.*

Proof. Since every point of H belongs to a line determined by points in C^*, $H \subseteq \text{cl}(C^*)$ and therefore $r(C^*) = r(M)$.

As an immediate corollary we have

COROLLARY 4.4. *If M is a tangential k-block* GF(q) *with a modular hyperplane (equivalently a cocircuit of cardinality q^k) then $r(M) \leqslant q^k$.*

Thus there are only a finite number of tangential k-blocks with modular hyperplanes over a given finite field. The persistence of Tutte's tangential 2-block conjecture indicates the difficulty of finding a similar result for arbitrary tangential k-blocks.

Although, in general, it is difficult to find all tangential k-blocks over a given field many critical problems could be solved if it could be shown that all tangential k-blocks possess certain "interesting" minors. In particular I know of no counterexample to the conjecture that any tangential k-block has a minor isomorphic to $M(K_{k+2})$; the conjecture is certainly true for all tangential 1-blocks and for tangential 2-blocks over GF(2). While this conjecture seems difficult to resolve in general we can show that for the restricted class of tangential k-blocks with modular hyperplanes a stronger form is true; that is, any tangential k-block over GF(q) with a modular hyperplane contains $M(K_{k+2})$ as a restriction minor.

LEMMA 4.5. *Let M be a matroid representable over* GF(q), *H be a modular hyperplane of M and C^* the cocircuit corresponding to H. If $r(C^*) = m$ then M contains a restriction minor isomorphic to $M(K_{m+1})$.*

Proof. If M is not simple consider $\bar{M}$; that is, we may assume without loss of generality that M is simple. Since $r(C^*) = m$, C^* contains an independent set B' with $|B'| = m$. Since H is modular, each line determined by a pair of points of B' meets H. Let P be the set of points of intersection of these lines with H and let $M' = M|(B' \cup P)$. We show that M' is isomorphic to $M(K_{m+1})$.

Let $B' = \{b_1, ..., b_m\}$. Since M is representable over GF(q), M' is and since B' is a basis of M' there exists a matrix representation of M' over GF(q) where (for $1 \leqslant i \leqslant m$) the ith column of I_m (the $m \times m$ identity matrix over GF(q)) corresponds to b_i. In this representation, P is represented by a subset of the set of vectors of a hyperplane of $V(r, q)$ which contains none of the column vectors of I_m. A hyperplane, with equation $a_1 x_1 + \cdots + a_m x_m = 0$ misses the column vectors of I_m if and only if for $1 \leqslant i \leqslant m$, $a_i \neq 0$. A vector common to such a hyperplane and the flat of $V(r, q)$ spanned by the ith and jth columns of I_m is the vector $(x_1, ..., x_m)^{\mathrm{T}}$ where $x_i = a_j$, $x_j = -a_i$ and otherwise $x_k = 0$. That is, there exists an m-tuple $(a_1, ..., a_m)$ of nonzero elements of GF(q) such that the $m \times \binom{m+1}{2}$

matrix S described below is a representation of M' over GF(q). S is the matrix whose first m columns form I_m while the remaining $\binom{m}{2}$ columns are all possible distinct vectors of the form $(x_1,..., x_m)^T$ where for $i<j$, $x_i = a_j$, $x_j = -a_i$ and otherwise $x_k = 0$. Let $R = (r_{ij})$ be the $m \times m$ matrix over GF(q) defined by

$$r_{ij} = \begin{cases} a_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

One routinely sees that by multiplication of the columns of RS by appropriate nonzero scalars one obtains a matrix T of the following form. The first m columns of T form I_m while the remaining columns of T consist of the $\binom{m}{2}$ distinct vectors of the form $(0,..., 0, 1, 0,..., 0, -1, 0,..., 0)^T$. R is nonsingular so T is a representation of M' over GF(q). But T is well known to be a unimodular representation of $M(K_{m+1})$; that is a representation of $M(K_{m+1})$ over any field and in particular a representation of $M(K_{m+1})$ over GF(q). Hence $M' \cong M(K_{m+1})$.

COROLLARY 4.6. *If M is a rank r tangential k-block over* GF(q) *with a modular hyperplane then M contains a restriction minor isomorphic to $M(K_{r+1})$.*

Proof. By Corollary 4.3, the cocircuit corresponding to any modular hyperplane in M is a spanning and hence has rank r.

COROLLARY 4.7. *If M is a tangential k-block over* GF(q) *with a modular hyperplane then M contains a restriction minor isomorphic to $M(K_{k+2})$.*

Proof. Say M is simple and representable over GF(q) with $r(M) < k+1$. Then M is isomorphic to a restriction of $PG(k-1, q)$ and therefore $P(M; q^k) \geq P(PG(k-1, q); q^k) > 0$. So no tangential k-block has rank less than $k+1$ and the result follows from Corollary 4.6.

The following corollary is an immediate consequence of Corollary 4.6.

COROLLARY 4.8. *$M(K_{q^k+1})$ is the unique tangential k-block over* GF(q) *with a modular hyperplane and rank q^k.*

5. TANGENTIAL 1-BLOCKS OVER GF(4) AND GF(5)

5.1. *Tangential 1-Blocks over* GF(4)

As pointed out in [10], $M(K_5)$, $U_{2,5}$, $PG(2, 2)$, $AG(2, 3)$ and the cocycle matroid of the Petersen graph, $M(P_{10}^*)$ are all tangential 1-blocks over GF(4). In addition to this, $\overline{T_{M(K_3)}(M(K_5))}$ is a tangential 1-blocks over

GF(4) and it is readily verified that $\overline{T_{M(K_3)}(M(P_{10}^*))}$ is well defined and is a tangential 1-block over GF(4).

5.2. *Tangential* 1-*Blocks over* GF(5)

Over GF(5) we have as tangential 1-blocks $M(K_6)$, $\overline{T_{M(K_3)}(M(K_6))}$,, $\overline{T_{M(K_4)}(M(K_6))}$, and $\overline{T_{M(K_5)}(M(K_6))}$ which is isomorphic to $U_{2,6}$. But these are by no means the only ones as the following example shows.

Consider the matroids M_1, M_2, and M_3 whose Euclidean representations are shown in Figure 1. M_1 is isomorphic to $\overline{T_{M(K_4)}(M(K_6))}$ and is a tangential 1-block over GF(5). It is routine to show that M_2 and M_3 are representable over GF(5) (either by arguing geometrically or by finding a direct representation), and that neither has a minor isomorphic to $U_{2,6}$. Since H_2 and H_3 are modular hyperplanes of M_2 and M_3 respectively we have, $P(M_2;\lambda) = (\lambda-1)(\lambda-4)(\lambda-5)$ and $P(M_3;\lambda) = (\lambda-1)(\lambda-3)(\lambda-5)$ and therefore both M_2 and M_3 are tangential 1-blocks over GF(5). Note that $P(M_1;\lambda) = (\lambda-1)(\lambda-4)(\lambda-5)$ and M_1 and M_2 are therefore nonisomorphic tangential 1-blocks over GF(5) with identical characteristic polynomials.

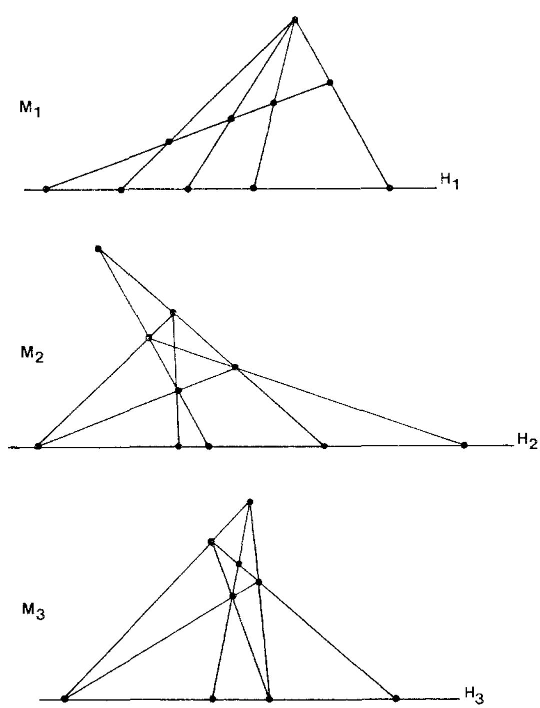

FIGURE 1

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