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A Characterisation of the Matroids Representable over $GF(3)$ and the Rationals

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It follows from a fundamental (1958) result of Tutte that a binary matroid is representable over the rationals if and only if it can be represented by a totally unimodular matrix, that is, by a matrix over the rationals with the property that all subdeterminants belong to $\{0, 1, -1\}$. For an arbitrary field $\mathbf{F}$, it is of interest to ask for a matrix characterisation of those matroids representable over $\mathbf{F}$ and the rationals. In this paper this question is answered when $\mathbf{F}$ is $GF(3)$. It is shown that a ternary matroid is representable over the rationals if and only if it can be represented over the rationals by a matrix A with the property that all subdeterminants of A belong to the set $\{0, \pm 2^i: i \text{ an integer}\}$. While ternary matroids are uniquely representable over $GF(3)$, this is not generally the case for representations of ternary matroids over other fields. A characterisation is given of the class of ternary matroids that are uniquely representable over the rationals. © 1995 Academic Press, Inc.

1. INTRODUCTION

It follows from a celebrated (1958) theorem of Tutte [22] that a binary matroid is representable over the rationals if and only if it can be represented over the rationals by a totally unimodular matrix. A matrix U over the rationals is *totally unimodular* if all subdeterminants of U belong to the set $\{0, 1, -1\}$. It is natural to ask if an analogous result holds for other fields. In particular, for a field $\mathbf{F}$, one can ask for a matrix characterisation of the matroids representable over $\mathbf{F}$ and the rationals. In this paper we solve this problem when $\mathbf{F}$ is $GF(3)$, that is, we give a matrix characterisation for the class of matroids that are representable over $GF(3)$ and the rationals. A matrix A over the rationals is *dyadic* if all non-zero subdeterminants of A belong to the set $\{\pm 2^i: i \in \mathbf{Z}\}$. It is easily seen that the matroid represented by the columns of a dyadic matrix is ternary. The main theorem of this paper proves the converse. We show that if M is representable over $GF(3)$ and the rationals, then $M = M[A]$ for some dyadic matrix A.

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A major problem that presents itself in matroid representation theory is the existence of inequivalent representations. Strong results are only known to exist in cases where matroids are uniquely representable. All known proofs of the excluded-minor characterisations of the classes of binary and ternary matroids use the uniqueness of representations of these matroids in an essential way [2, 6, 7, 19, 22]. Also regular matroids are uniquely representable over any field. In contrast to this, a ternary matroid may have inequivalent representations over the rationals. For example, it is easily seen that the whirl $\mathcal{W}^3$ has inequivalent representations over the rationals. Indeed, of the infinite number of equivalence classes of representations of $\mathcal{W}^3$, only three contain dyadic matrices. This is strikingly different from the case of regular matroids. Any representation of a regular matroid can be transformed to a totally unimodular representation by standard matrix operations.

Loosely speaking, the reason why inequivalent representations cause difficulties is that one no longer has leverage for induction. In trying to show that a matroid M has a representation of a certain type, it is natural to try to show that this property is inherited from a representation of a minor of that type. If we do not have unique representability, then not all representations of minors need extend to representations of M. This means that we cannot guarantee that a representation of the desired type extends to a representation of M.

Most of this paper is devoted to overcoming difficulties caused by the existence of inequivalent representations. Essentially the technique used is to show that when a 3-connected ternary matroid has inequivalent representations over the rationals, the inequivalent representations correspond to a single equivalence class of representations over an appropriate transcendental extension field of the rationals. I believe that it is of great interest to know whether the techniques of this paper can be generalised to assist in solving any other of the many outstanding problems in matroid representation theory.

The paper is structured as follows. Section 2 outlines known results that are used throughout the paper. Section 3 presents a result for 3-connected matroids. This result is essentially a technical lemma. The proof is an unfortunately long case analysis using standard techniques. It is recommended that the reader skip the proof on a first reading. It would be nice to see an elegant proof of this theorem. Weak maps are standard matroid constructions that are used frequently in this paper. Section 4 outlines some basic facts on weak maps. A homomorphism of an integral domain induces a map on representations of matroids over that integral domain. The image of a representation of a matroid under such a map is a representation of a weak-map image of the original matroid. The two main sections of the paper are Sections 5 and 6. Section 5 introduces a certain class of matroids that turn out to be representable over all fields except possibly $GF(2)$.

These matroids have natural representations over a transcendental extension field of the rationals. These are the so-called "near-regular" matroids. It is shown that members of this class, while not in general uniquely representable over the rationals, have well-behaved canonical representations over this transcendental extension field. It turns out to be easy to show that a near-regular matroid can be represented over the rationals by a dyadic matrix, but while near-regular matroids are representable over $GF(3)$ and the rationals the converse is not generally true. The main task of Section 6 is to deal with the transition to ternary matroids representable over the rationals that are not near-regular. Section 7 presents the main results—the proofs essentially summarise information from previous sections. As well as showing that all matroids that are representable over $GF(3)$ and the rationals can be represented by dyadic matrices, it is also shown that a 3-connected ternary matroid has inequivalent representations over the rationals if and only if it is non-binary and near-regular.

Finally we note that in [26] the techniques of this paper are used to give matrix characterisations of those matroids that are representable over $GF(3)$ and $\mathbf{F}$ for any given field $\mathbf{F}$. This is acheived by finding appropriate generalisations of Theorems 6.6 and 7.1.

2. Preliminaries

Familiarity is assumed with the elements of matroid theory. Terminology follows Oxley [17] with a single exception noted in the following paragraph. One aspect of matroid theory that we assume particular familiarity with is the theory of matroid representations. Essentially, it is assumed that the reader is familiar with the substance of [17, Chapter 6].

Connectivity

For a good discussion of the theory of matroid connectivity we again refer the reader to [17]. Recall that the *simplification* of a matroid M is obtained by deleting all the loops of M and all but one element of each parallel class of M. Dually, the *cosimplification* of M is obtained by contracting all the coloops of M and all but one element of each series class of M. We denote the simplification and cosimplification of M by $\mathrm{si}(M)$ and $\mathrm{co}(M)$ respectively. This notation differs from [17]. Note that, as defined here, $\mathrm{si}(M)$ and $\mathrm{co}(M)$ are minors of M, which turns out to be convenient in our arguments.

Recall that a matroid is not 3-connected if and only if it has a 2-separation. A 2-separation of M is a partition $\{X, Y\}$ of $E(M)$ with the property that $|X|, |Y| \geqslant 2$, and $r(X) + r(Y) \leqslant r(M) + 1$. We are frequently interested in cases where M may not be 3-connected but where either $\mathrm{si}(M)$ or $\mathrm{co}(M)$

is 3-connected. The following elementary facts on 2-separations are used in Section 3.

(2.1) If $\{X, Y\}$ is a 2-separation of a connected matroid M, then either $\{\mathrm{cl}(X), Y-\mathrm{cl}(X)\}$ is a 2-separation of M or Y is contained in a non-trivial parallel class of M.

(2.2) ([20, (5.1)]) If M is connected and $\mathrm{si}(M)$ (respectively $\mathrm{co}(M)$) is 3-connected, then any 2-separation $\{X, Y\}$ of M has the property that either X or Y is contained in a parallel (respectively series) class.

(2.3) If M is a simple matroid with $r(M)>3$, and $\mathrm{co}(M)$ is not 3-connected, then there exists a 2 separation $\{X, Y\}$ of M with $|X|, |Y|>2$.

3-connected, non-binary matroids

Recall that, for $r>2$, the *whirl* $\mathcal{W}^r$ is the matroid defined as follows. Let $P=\{p_1, p_2, ..., p_r\}$ be the vertices of an r-simplex. Then $\mathcal{W}^r$ is obtained by placing a point freely on each of the lines $\{p_1, p_2\}$, $\{p_2, p_3\}, ...,$ $\{p_{r-1}, p_r\}$ and $\{p_r, p_1\}$. If $r=2$, then $\mathcal{W}^2=U_{2,4}$. The following results are straightforward consequences of Seymour's Splitter Theorem [20]. For a discussion of this theorem and its consequences see [17, Chapter 11].

(2.4) Let M and N be 3-connected matroids with the property that N is a non-binary minor of M, $|E(N)|\geqslant 4$, and if N is a whirl, then M has no larger whirl as a minor. Then there is a sequence $M_0, M_1, ..., M_n$ of 3-connected matroids such that $M_0\cong N$, $M_n\cong M$, and, for all i in $\{0, 1, ..., n-1\}$, M_i is a single-element deletion or a single-element contraction of M_{i+1}.

In particular we have

(2.5) Let M be a non-binary, 3-connected matroid. If M is not a whirl, then there exists $x\in E$ such that either $M\backslash x$ or M/x is non-binary and 3-connected.

The following result is proved in [25]. It is a straightforward consequence of (2.5).

(2.6) Let M be a 3-connected, non-binary matroid. If $r(M)\geqslant 3$, then there is an element $x\in E(M)$ with the property that $\mathrm{si}(M/x)$ is 3-connected and non-binary.

A result that is crucial in many arguments on non-binary, 3-connected matroids is the following theorem of Seymour [21]. A matroid N *uses* a set X if $X\subseteq E(N)$.

(2.7) If x_1 and x_2 are elements of a 3-connected, non-binary matroid M, then M has a $U_{2,4}$-minor using $\{x_1, x_2\}$.

While, in general, a 3-connected, ternary, non-binary matroid may have inequivalent representations over a field **F**, there are sharp constraints on these representations. The following result is a routine strengthening of [25, Lemma 2.6].

(2.8) Let M be a connected, ternary matroid representable over a field **F** with an element x such that $M\backslash x$ is connected and si$(M\backslash x)$ is 3-connected and non-binary. Let A be a matrix representation of $M\backslash x$ over **F** that extends to a representation of M. If **x** and **y** are vectors such that $[A \mid \mathbf{x}]$ and $[A \mid \mathbf{y}]$ both represent M over **F**, then **x** is a scalar multiple of **y**.

3. A 3-Connectivity Theorem

Let M be a 3-connected, non-binary matroid such that the rank of its dual M^* is at least four. The triple (a, b, c) of distinct elements of M is a *distinguished triple* if it is coindependent and it has the property that co$(M\backslash a)$, co$(M\backslash b)$, co$(M\backslash c)$, co$(M\backslash a, b)$ and co$(M\backslash a, c)$ are all 3-connected and nonbinary.

The purpose of this section is to prove

(3.1) Theorem. *A 3-connected, non-binary matroid with $r(M^*) \geqslant 4$ has a distinguished triple.*

In fact it is not Theorem 3.1 that is used in this paper but its dual, stated as Corollary 3.8. This corollary is used in the proofs of Theorems 5.9 and 6.6.

Proof of Theorem 3.1. Throughout M denotes a 3-connected, non-binary matroid with $r(M^*) \geqslant 4$ and ground set E. We proceed by induction on the rank of M. If $r(M) = 2$, the result clearly holds, so assume that $r(M) \geqslant 3$. We first show that the theorem holds when $r(M) = 3$. In fact in this case it turns out to be easier to establish a somewhat stronger conclusion which is certainly not generally true for higher ranks.

(3.2) If $r(M) = 3$, then M has a distinguished triple (a, b, c) with the property that $M\backslash a$, $M\backslash b$ and $M\backslash c$ are all 3-connected.

Proof. The proof of 3.2 is by induction on $|E|$. Certainly $|E| \geqslant 7$. We first prove

(3.2.1) If $|E| = 7$ then 3.2 holds.

Proof. We have $r(M)=3$ and $|E|=7$. Assume that M has a four-point line. It is easily checked that, up to isomorphism, there are just four such matroids; these are illustrated in Fig. 3.1.

Routine checking shows that, in each matroid, (a, b, c) is a distinguished triple where a, b, and c are the points labelled in the diagrams. It is also easily checked that $M\backslash a$, $M\backslash b$, and $M\backslash c$ are all 3-connected.

Now say M has no 4-point lines. Assume that M is ternary. In this case M is an extension of $\mathcal{W}^3$ and it follows from [14, Lemma 2.2] that M is either the non-Fano matroid F_7^- or the matroid P_7 illustrated in Fig. 3.2. Again it is routine to check that in each matroid (a, b, c) is a distinguished triple where a, b, and c are the points labelled in the diagrams. It is also easily checked that $M\backslash a$, $M\backslash b$, and $M\backslash c$ are all 3-connected.

Assume that M is not ternary. Then M has either a $U_{2,5}$- or a $U_{3,5}$-minor. It follows by [15, Theorem 1.6] that M has a $U_{3,5}$-minor. It routinely follows that there exists $a \in E$ such that $M\backslash a$ is $U_{3,6}$ or one of the matroids P_6 or Q_6 illustrated in Fig. 3.3. If $M\backslash a$ is $U_{3,6}$, then it is clear that M has a distinguished triple (a, b, c) with the property that $M\backslash a$, $M\backslash b$, and $M\backslash c$ are all 3-connected, so we may assume that $M\backslash a$ is either P_6 or Q_6.

It is easily seen that if $M\backslash a$ is P_6, then (a, b, c) is a distinguished triple of M where b and c are as labelled in Figure 3.3. It is also clear that $M\backslash a$, $M\backslash b$ and $M\backslash c$ are all 3-connected. Say $M\backslash a$ is Q_6, and consider the labelling indicated in Figure 3.3. Evidently, in M, either $\{a, c_1, x\}$ or $\{a, c_2, x\}$ is not collinear. Assume without loss of generality that $\{a, c_2, x\}$ is not a circuit. Then one readily checks that (a, b, c_1) is a distinguished triple of M with $M\backslash a$, $M\backslash b$ and $M\backslash c_1$ all 3-connected. ▮

We now complete the proof of (3.2). By (3.2.1), (3.2) holds if $|E|=7$. Assume for induction that $|E|>7$ and that the conclusion of (3.2) holds

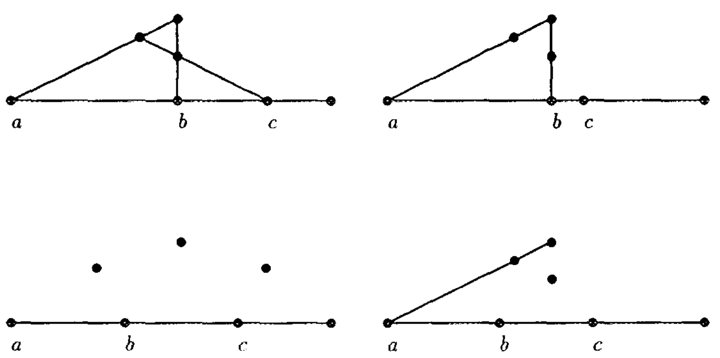

FIG. 3.1. Distinguished triples in matroids with a four point line.

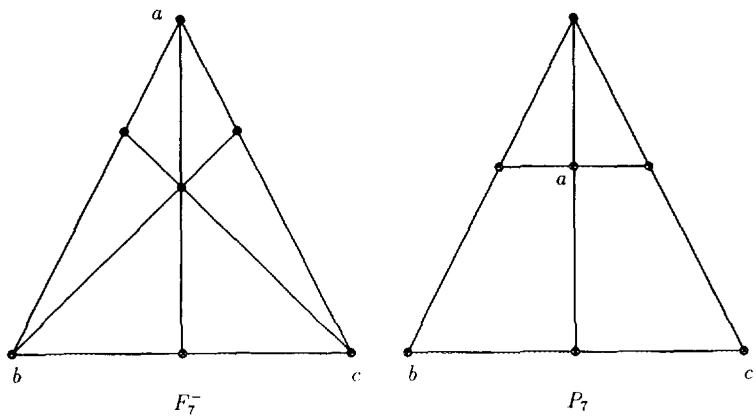

FIG. 3.2. Distinguished triples in F_7^- and P_7.

for all matroids satisfying the conditions of (3.2) whose ground sets have cardinality less than $|E|$.

Certainly there is a point $p \in E$ such that $M\backslash p$ is non-binary. If $M\backslash p$ is not 3-connected, then $E-\{p\}$ is the union of two lines l_1 and l_2. If $l_1 \cap l_2 \neq \emptyset$, then it is easily seen that $M\backslash(l_1 \cap l_2)$ is non-binary and 3-connected, so assume that $l_1 \cap l_2 = \emptyset$. If either l_1 or l_2 has more than four points, then deleting a point from the line with more than four points clearly leaves a non-binary, 3-connected matroid. Assume then, that neither l_1 nor l_2 has more than four points. Without loss of generality we may assume that $|l_1| = 4$ and $|l_2| \in \{3, 4\}$. Let q be a point of l_2. Clearly $M\backslash q$ is non-binary and 3-connected. We conclude that there exists a point $a \in E$ such that $M\backslash a$ is 3-connected and non-binary. By the inductive hypothesis there exist points b and c in $E-a$ such that $M\backslash a, b$ and $M\backslash a, c$ are non-binary and 3-connected. Clearly $M\backslash b$ and $M\backslash c$ are also non-binary and 3-connected. This establishes 3.2. ∎

Now assume that $r(M) \geqslant 4$, and, for induction, that any matroid satisfying the hypotheses of Theorem 3.1 whose rank is less than $r(M)$ has a

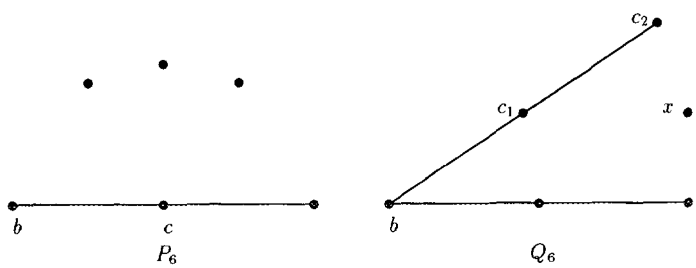

FIG. 3.3. The matroids P_6 and Q_6.

distinguished triple. By (2.6) there exists an element $x \in E$ with the property that $\mathrm{si}(M/x)$ is non-binary and 3-connected. We first show that the theorem holds when M/x has no non-trivial parallel classes.

(3.3) If M/x is non-binary and 3-connected, then M has a distinguished triple.

Proof. It follows from the inductive hypothesis that there exists a coindependent triple (a, b, c) of distinct elements of M/x for which $\mathrm{co}(M/x\backslash a)$, $\mathrm{co}(M/x\backslash b)$, $\mathrm{co}(M/x\backslash c)$, $\mathrm{co}(M/x\backslash a, b)$, and $\mathrm{co}(M/x\backslash a, c)$ are all 3-connected and non-binary. We first prove

(3.3.1) If N is a loopless matroid and $y \in E(N)$ has the property that $\mathrm{co}(N/y)$ is 3-connected, then $\mathrm{co}(N)$ is 3-connected.

Proof. We prove the dual which seems more obvious intuitively. Now N^* is coloop free, $\mathrm{si}(N^*\backslash y)$ is 3-connected and N^* is an extension of $N^*\backslash y$. If y is in a parallel class or is a loop, then it is clear that $\mathrm{si}(N^*\backslash y) = \mathrm{si}(N^*)$, and hence $\mathrm{si}(N^*)$ is 3-connected. Otherwise, $\mathrm{si}(N^*)$ can be regarded as an extension of $\mathrm{si}(N^*\backslash y)$ in which y is neither a loop, coloop, or parallel to an element of $\mathrm{si}(N^*\backslash y)$. Again it follows that $\mathrm{si}(N^*)$ is 3-connected. ▮

By (3.3.1), $\mathrm{co}(M\backslash a)$, $\mathrm{co}(M\backslash b)$, $\mathrm{co}(M\backslash c)$, $\mathrm{co}(M\backslash a, b)$, and $\mathrm{co}(M\backslash a, c)$ are all 3-connected. Moreover these matroids are all non-binary since they contain non-binary minors. Since a, b, c is coindependent in M/x, this set is certainly coindependent in M. It follows that (a, b, c) is a distinguished triple of M, and (3.3) is established. ▮

We may now assume that M/x has at least one non-trivial parallel class, that is, that M has at least one non-trivial line containing x. We first consider the case where there is such a line with at least four points.

(3.4) If M has a line l containing x with $|l| \geqslant 4$, then M has a distinguished triple.

Proof Say $l = \{x, p_1, p_2, ..., p_n\}$, where $n \geqslant 3$. First note that if $\{i, j\} \subset \{1, 2, ..., n\}$, then $M\backslash p_i, p_j$ clearly contains a minor isomorphic to $\mathrm{si}(M/x)$ so that $M\backslash p_i, p_j$ is non-binary. Also, if i, j and k are distinct elements of $\{1, 2, ..., n\}$, then $\{p_i, p_j, p_k\}$ is coindependent in M, otherwise $\{p_i, p_j, p_k\}$ would be a triangle and a triad of M, a situation that cannot occur in a 3-connected matroid with at least five elements. It follows that to show that (p_i, p_j, p_k) is a distinguished triple, it only has to be shown that the cosimplifications of the appropriate deletions are 3-connected. It is well known, and easily seen, that if the element z of the 3-connected matroid N

is on a line with at least four points, then $M\backslash z$ is 3-connected. It follows that $M\backslash p_1$, $M\backslash p_2$, and $M\backslash p_3$ are all 3-connected.

Assume that $|l| \geqslant 5$, that is, $n \geqslant 4$. Then $M\backslash p_1, p_2$ and $M\backslash p_1, p_3$ are also 3-connected, and (p_1, p_2, p_3) is a distinguished triple.

We may therefore assume that $n = 3$, so that $l = \{x, p_1, p_2, p_3\}$. The two following preliminary results will be used frequently throughout the rest of the proof.

(3.4.1) Let N be a connected matroid with an element z having the property that N/z is connected. Say $\{Z, Y\}$ is a 2-separation of N with $z \in Z$, and $|Z| > 2$. Then,

(i) $\{Z - z, Y\}$ is a 2-separation of N/z, and

(ii) $(N/z) \mid Y = N \mid Y$.

Proof. $r_{N/z}(Z - z) = r_N(Z) - 1$, and $r_{N/z}(Y) \leqslant r_N(Y)$. It follows routinely from this that $\{Z - z, Y\}$ is a 1-separation of N/z or a 2-separation of N/z. Since N/z is connected, the former case does not occur, so $\{Z - z, Y\}$, is an exact 2-separation of N/z. But this case only occurs if $r_{N/z}(Y) = r_N(Y)$. We conclude that this is the case. Thus $r(Y \cup z) = r(Y) + 1$, so z is a coloop of $N|(Y \cup z)$ and $(N/z)|Y = N|Y$. ▮

(3.4.2) Let N be a 3-connected matroid with rank at least three, and having an element z with the property that $\mathrm{si}(N/z)$ is 3-connected. If $\{z, p, q\}$ is a circuit of N, then either $\mathrm{co}(N\backslash p)$ or $\mathrm{co}(N\backslash q)$ is 3-connected.

Proof. Assume that neither $\mathrm{co}(N\backslash p)$ nor $\mathrm{co}(N\backslash q)$ is 3-connected. Then neither $N\backslash p$ nor $N\backslash q$ is 3-connected. Thus by Tutte's Triangle Lemma ([24], see also [17, Lemma 8.4.9]) N has a triad T_p containing exactly one of z and q. If T_p contains q, then the 3-connected matroid $N/z\backslash p$ has a 2-element cocircuit; a contradiction. Thus T_p contains p and z but not q. Similarly, N has a triad containing q and z but not p. Now N/p has $\{q, z\}$ as a circuit that is in a cocircuit. Thus $\mathrm{si}(N/p)$ has a 2-element cocircuit and so is not 3-connected. Thus, by a result of Bixby ([3], see also [17, Proposition 8.4.8]), $\mathrm{co}(N\backslash p)$ is 3-connected. ▮

Now return to the proof of (3.4). We have a line $l = \{x, p_1, p_2, p_3\}$ of M. Moreover, for $i \in \{1, 2, 3\}$, $M\backslash p_i$ is 3-connected. Consider $M\backslash p_1$. By (3.4.2), either $\mathrm{co}(M\backslash p_1, p_2)$ or $\mathrm{co}(M\backslash p_1, p_3)$ is 3-connected. If both are 3-connected, then (p_1, p_2, p_3) is a distinguished triple of M. Say not; assume without loss of generality that $\mathrm{co}(M\backslash p_1, p_2)$ is not 3-connected. Consider $M\backslash p_2$. By (3.4.2), either $\mathrm{co}(M\backslash p_2, p_1)$ or $\mathrm{co}(M\backslash p_2, p_3)$ is 3-connected. We conclude that the latter matroid is 3-connected, and that (p_3, p_2, p_1) is a distinguished triple M. ▮

For the remainder of the proof we assume that all non-trivial lines of M that contain x have at most three points.

(3.5) If there is exactly one non-trivial line of M containing x, then M has a distinguished triple.

Proof. Let $l=\{x, p, q\}$ be the non-trivial line of M containing x. Note that $\mathrm{si}(M/x) \cong M\backslash p/x \cong M\backslash q/x$. We use this fact frequently. We obtain a distinguished triple of M that contains $\{p, q\}$.

(3.5.1) $\mathrm{co}(M\backslash p)$ and $\mathrm{co}(M\backslash q)$ are 3-connected and non-binary.

Proof. It clearly suffices to show that $\mathrm{co}(M\backslash p)$ is 3-connected and non-binary. Now $\mathrm{si}(M/x) \cong M\backslash p/x$, so $M\backslash p$ is certainly non-binary. It also follows that $M\backslash p/x$ is 3-connected. Therefore $M\backslash p$ is 3-connected unless it has x in a 2-element cocircuit. In either case $\mathrm{co}(M\backslash p)$ is 3-connected. ▮

Identify $\mathrm{si}(M/x)$ with $M\backslash p/x$. We now show

(3.5.2) There exists a 3-element, coindependent subset $\{a, b, c\}$ of $M\backslash p/x$ with the property that $\mathrm{co}(M\backslash p/x\backslash a)$, $\mathrm{co}(M\backslash p/x\backslash b)$, and $\mathrm{co}(M\backslash p/x\backslash c)$ are all non-binary and 3-connected.

Proof. If M has corank greater than four, then $M\backslash p/x$ has corank greater that three. In this case it follows from the inductive hypothesis that $M\backslash p/x$ has a distinguished triple (a, b, c) and $\{a, b, c\}$ is the required set.
Assume that M has corank four. Then $M\backslash p/x$ has corank three. Now $(M\backslash p/x)^*$ is a 3-connected, non-binary, rank-3 matroid. It now follows by [16, Theorem 3.1], that $(M\backslash p/x)^*$ has a restriction isomorphic to one of $U_{3,6}$, P_6, Q_6, or $\mathcal{W}^3$. (Recall that the matroids P_6 and Q_6 are illustrated in Figure 3.3.) One readily checks that each of these matroids has an independent 3-element set $\{a, b, c\}$ such that the simplification of the contraction of any one of these is 3-connected and non-binary. It follows that $(M\backslash p/x)^*$ has such a set, and by duality, that $M\backslash p/x$ has a 3-element set with the required properties. ▮

Let $\{a, b, c\}$ be a 3-element, coindependent set of $M\backslash p/x$ satisfying (3.5.2). We would like to guarantee that for some $e \in \{a, b, c\} - \{q\}$, $\mathrm{co}(M\backslash e)$ is 3-connected. We first establish some preliminary facts.

(3.5.3) Let s be an element of the set $E-\{x, p, q\}$ with the property that $\mathrm{co}(M/x\backslash p, s)$ is 3-connected but $\mathrm{co}(M\backslash s)$ is not 3-connected. Then $\{l, E-(l \cup \{s\})\}$ is a 2-separation of $M\backslash s$, and $l \cup \{s\}$ contains a cocircuit C^* with $s \in C^*$.

Proof. Since $\mathrm{co}(M\backslash s)$ is not 3-connected, there is a 2-separation $\{X, Y\}$ of $M\backslash s$ with the property that neither X nor Y is contained in a series class of $M\backslash s$. Recall that $l=\{x, p, q\}$. One of X or Y contains at least two points of l; say X does. One readily checks that $\{X\cup l, Y-l\}$ is also a 2-separation of $M\backslash s$ with the property that neither $X\cup l$ nor $Y-l$ is a series class of $M\backslash s$. Hence we may assume without loss of generality that $\{X, Y\}$ is a 2-separation of $M\backslash s$ with $l\subseteq X$.

By (3.4.1), $\{X-\{x\}, Y\}$ is a 2-separation of $M\backslash s/x$. It may be that this 2-separation is not exact, that is, the 2-separation may also be a 1-separation. Set $A=X-l$. Say $A=\emptyset$. Then $\{l, Y\}$ is a 2-separation of $M\backslash s$. We now show that $\{l, E-(l\cup\{s\})\}$ is also a 2-separation of $M\backslash s$ in the case that $A\neq\emptyset$.

Assume then, that $A\neq\emptyset$. Now $\{s, p, q\}$ is not a triad of M, otherwise the 3-connected matroid $M\backslash p/x$ has a 2-element cocircuit. Hence

$$r_{M/x\backslash s,p}(Y\cup A)=r(M\backslash s)-1.$$

In the following equations, (2) follows from the submodularity of the rank function, (3) follows from (3.4.1), and (4) follows from the above equation.

$$\begin{aligned} &r_{M\backslash s}(l)+r_{M\backslash s}(Y\cup A) && (1)\\ &\leqslant 2+r_{M\backslash s}(Y)+|A| && (2)\\ &=2+r_{M/x\backslash s,p}(Y)+|A| && (3)\\ &=2+r(M\backslash s)-1 && (4)\\ &=r(M\backslash s)+1. && (5)\end{aligned}$$

But $Y\cup A=E-(l\cup\{s\})$, so it follows that $\{l, E-(l\cup\{s\})\}$ is a 2-separation of $M\backslash s$. Since $r_{M\backslash s}(l)=2$, $E-(l\cup\{s\})$ spans a hyperplane H of M. Moreover, $s\notin H$, otherwise $\{l, E-l\}$ is a 2-separation of M. Hence $l\cup\{s\}$ contains a cocircuit C^* that contains s. ∎

(3.5.4) Let S be the set of all elements of $E-\{x, p, q\}$ with the property that for $s\in S$, $\mathrm{co}(M/x\backslash p, s)$ is 3-connected, but $\mathrm{co}(M\backslash s)$ is not 3-connected. Then the corank of S in $M/x\backslash p$ is less than or equal to 2.

Proof. Order the elements of S, say $S=\{s_1, s_2, ..., s_n\}$. As before, set $l=\{x, p, q\}$. Say $i\in\{1, 2, ..., n\}$ and consider s_i. By (3.5.3), $l\cup\{s_i\}$ contains a cocircuit C_i^* that contains s_i. Set $H_i=E-C_i^*$. We now show that

$$r_M\left(\bigcap_{i=1}^{n} H_i\right)=r(M)-n.$$

Since l cannot contain a cocircuit, $r(M\backslash l) = r(M)$. Moreover, each of s_1, $s_2, ..., s_n$ is a coloop of $M\backslash l$. The assertion follows unless, for all i and some fixed a and b, C_i^* is $\{s_i, a, b\}$. In the exceptional case, $n \leqslant 2$ or $\{s_1, s_2, a\}$ is a cocircuit of M meeting l in a single element. We conclude that it is indeed the case that

$$r_M\left(\bigcap_{i=1}^{n} H_i\right) = r(M) - n.$$

Now consider $M/x\backslash p$. By (3.4.1)

$$r_{M/x\backslash p}\left(\bigcap_{i=1}^{n} H_i\right) = r_M\left(\bigcap_{i=1}^{n} H_i\right).$$

Hence

$$r_{M/x\backslash p}\left(\bigcap_{i=1}^{n} H_i\right) = r(M/x\backslash p) - n + 1.$$

But

$$E(M/x\backslash p) - \bigcap_{i=1}^{n} H_i = S \cup \{q\},$$

so, using the formula for the rank function of the dual matroid, we have

$$\begin{aligned} r_{(M/x\backslash p)^*}(S \cup \{q\}) &= |S \cup \{q\}| - r(M/x\backslash p) + r_{M/x\backslash p}\left(\bigcap_{i=1}^{n} H_i\right) \\ &= n + 1 - r(M/x\backslash p) + r(M/x\backslash p) - n + 1 \\ &= 2. \end{aligned}$$

We conclude that $r_{(M/x\backslash p)^*}(S) \leqslant 2$. ∎

Return to the proof of (3.5). We first show that there exists an element $e \in E - \{x, p, q\}$ such that $\text{co}(M\backslash e)$ is 3-connected and non-binary. Recall that $\{a, b, c\}$ is a coindependent, 3-element set of elements of $M\backslash p/x$ with the property that $\text{co}(M\backslash p/x\backslash a)$, $\text{co}(M\backslash p/x\backslash b)$, and $\text{co}(M\backslash p/x\backslash c)$ are all non-binary and 3-connected. We consider two cases. For the first assume that $q \notin \{a, b, c\}$. Since $\{a, b, c\}$ is coindependent, by (3.5.4), there exists an element $e \in \{a, b, c\}$ such that $\text{co}(M\backslash a)$ is 3-connected. Consider the second case. Here we may assume that $q = c$. Assume that neither $\text{co}(M\backslash a)$ nor $\text{co}(M\backslash b)$ is 3-connected. Then, by (3.5.3), $\{x, p, q, a\}$ and $\{x, p, q, b\}$ contain cocircuits that contain a and b respectively. We deduce that $\{p, q, a, b\}$ contains a cocircuit of M/x. It follows easily that $\{q, a, b\}$

is a triad of $M/x\backslash p$. This contradicts the assumption that $\{a, b, q\}$ is coindependent in $M/x\backslash p$. It follows that, for some $e \in \{a, b\}$, co$(M\backslash e)$ is 3-connected. In both of the above cases it is clear that $M\backslash e$ is non-binary.

Arguing as in (3.5.1) we see that co(co$(M\backslash e)\backslash p$) is 3-connected. It is easily seen that co(co$(M\backslash e)\backslash p$) = co$(M\backslash e, p)$, so co$(M\backslash e, p)$ is 3-connected. But $M/x\backslash e, q \cong M/x\backslash e, p$, so again arguing as in (3.5.1), we see that co$(M\backslash e, q)$ is 3-connected. Evidently both $M\backslash e, p$ and $M\backslash e, q$ are non-binary. To establish that (e, p, q) is a distinguished triple, all that remains is to show that $\{e, p, q\}$ is coindependent. Now, in M/x, $\{e, p, q\}$ is a parallel extension of $\{e, p\}$, and this set is certainly coindependent. But a parallel extension of a coindependent set is coindependent. Therefore $\{e, p, q\}$ is coindependent in M/x and consequently in M. ▮

We may now assume that M has at least two non-trivial three point lines containing x.

(3.6) If M has exactly two non-trivial lines containing x, then M has a distinguished triple.

Proof. Say $\{x, p_1, p_2\}$ and $\{x, q_1, q_2\}$ are the non-trivial lines containing x. Assume that co$(M\backslash p_1)$ and co$(M\backslash p_2)$ are 3-connected. By (3.4.2), either co$(M\backslash q_1)$ or co$(M\backslash q_2)$ is 3-connected. Assume that co$(M\backslash q_1)$ is 3-connected. Say $i \in \{1, 2\}$, and consider $M\backslash q_1, p_i$. Evidently $M\backslash q_1, p_i/x \cong$ si(M/x), a 3-connected matroid. It may be that x is a coloop of $M\backslash q_1, p_i$. In this case co$(M\backslash q_1, p_i) \cong$ si(M/x), a 3-connected matroid. If x is not a coloop, then, since $M\backslash q_1, p_i/x$ is 3-connected, $M\backslash q_1, p_i$ is certainly connected. Say $\{X, Y\}$ is a 2-separation of $M\backslash q_1, p_i$, where $x \in X$. Assume without loss of generality that X is a flat of $M\backslash q_1, p_i$. Evidently, either $\{X - x, Y\}$ is a 2-separation of $M\backslash q_1, p_i/x$, or $|X| = 2$. We conclude that $|X| = 2$. But X is also a flat. Clearly a trivial line which is part of a 2-separation is a series pair. We conclude that co$(M\backslash q_1, p_i)$ is 3-connected. It is clear that $M\backslash q_1, p_i$ is non-binary. Moreover an argument identical to that at the end of (3.5) shows that $\{q_1, p_1, p_2\}$ is coindependent in M. Therefore (q_1, p_1, p_2) is a distinguished triple of M.

It now follows that we may assume without loss of generality that neither co$(M\backslash p_1)$ nor co$(M\backslash q_1)$ is 3-connected. Since co$(M\backslash p_1)$ is not 3-connected, there exists a 2-separation $\{X, Y\}$ of $M\backslash p_1$ with $x \in X$, and $|X| > 2$. By (3.4.1), $\{X - x, Y\}$ is a 2-separation of $M\backslash p_1/x$. But, apart from the parallel pair $\{q_1, q_2\}$, this matroid is isomorphic to si(M/x). We conclude that $X = \{q_1, q_2, x\}$, and that $\{x, q_1, q_2, p_1\}$ contains a cocircuit of M. Similarly, $\{x, p_1, p_2, q_1\}$ contains a cocircuit. Assume that these cocircuits are distinct. Then $r(E - \{x, p_1, p_2, q_1, q_2\}) = r(M) - 2$. But $r(\{x, p_1, p_2, q_1, q_2\}) = 3$, and it follows that M is not 3-connected. It follows that the cocircuits are not distinct. We conclude that $\{x, p_1, q_1\}$ is a triad of M.

We seek a distinguished triple of M. By (3.4.2), both $\mathrm{co}(M\backslash p_2)$ and $\mathrm{co}(M\backslash q_2)$ are 3-connected. Also they are certainly non-binary. Let $H = E - \{x, p_1, q_1\}$. It is evident that $M|H \cong \mathrm{si}(M/x)$. We first consider the case when M has corank four.

(3.6.1) If $r^*(M) = 4$, then M has a distinguished triple.

Proof. Since $M \mid H \cong \mathrm{si}(M/x)$, $M \mid H$ is 3-connected and non-binary. Since $r^*(M) = 4$, $r^*(M \mid H) = 2$. It follows that $M \mid H$ is uniform; indeed, $M \mid H \cong U_{r-1, r+1}$. But $r(M) > 3$, so $r(M|H) > 2$, and therefore $M \mid H$ has no 3-circuits.

We now show that $\mathrm{co}(M\backslash x)$ is 3-connected and non-binary. Since $M \mid H$ has no 3-circuits, there is no other point on the line of M spanned by $\{p_2, q_2\}$. This means that $\mathrm{cl}_M(\{p_1, q_1\}) \cap H = \varnothing$. But $M \mid H$ only differs from $M\backslash x$ in the series pair $\{p_1, q_1\}$. Hence $\mathrm{co}(M\backslash x)$ is an extension of $M \mid H$ in which the point of extension is neither a loop, coloop, nor parallel to any element of $M \mid H$. It follows that $\mathrm{co}(M\backslash x)$ is indeed 3-connected and non-binary.

We now show that there exists a pair $\{b, c\}$ of elements of $H - \{p_2, q_2\}$ with the property that (x, b, c) is a distinguished triple of M. We begin by showing that there exists a unique circuit of M containing $\{p_1, q_1\}$ and otherwise only elements of $H - \{p_2, q_2\}$. An easy argument shows that there is at least one circuit with this property, say C is such a circuit. Now $\{p_1, q_1, p_2, q_2\}$ is a circuit of M. Hence $(C \cup \{p_1, q_1, p_2, q_2\}) - \{p_1\}$ contains a circuit C'. But q_1 is a coloop of $M \mid ((C \cup \{p_1, q_1, p_2, q_2\}) - \{p_1\})$, so C' is contained in H. Since $M \mid H \cong U_{r-1, r+1}$, this circuit contains all but one element of H. It follows that there is at most one element of $H - \{p_2, q_2\}$ that is not in C. If C contains all of $H - \{p_2, q_2\}$, it is clear that C is unique. Say C contains all but one element of $H - \{p_2, q_2\}$. In this case C spans a hyperplane of M. Now assume that C'' is another circuit containing $\{p_1, q_1\}$ and otherwise only elements of $H - \{p_2, q_2\}$. Then $|C''| = |C|$, and $|C'' \cap C| = |C| - 1$. So both C and C'' are spanned by $C \cap C''$. It follows that $C \cup C''$ is contained in a hyperplane of M. But it is easily seen that $C \cup C''$ contains a basis of M. This contradiction establishes that C is unique.

Let t be an element of $C \cap H$, and consider $\mathrm{co}(M\backslash t)$. It is routinely seen that $H - \{p_2, q_2, t\}$ is a series class of $M\backslash t$, and that $\mathrm{co}(M\backslash t)$ is obtained by contracting all but one element (say s) of this series class. In $\mathrm{co}(M\backslash t)$, s is on the line spanned by $\{p_2, q_2\}$. Neither $\{s, p_2\}$, nor $\{s, q_2\}$ is a parallel pair, otherwise $M \mid H$ would have an $(r-1)$-element circuit, contradicting the fact that $M \mid H \cong U_{r-1, r+1}$. It is now straightforward to see that

$\mathrm{co}(M\backslash t)$ is either isomorphic to $M(K_4)$ or $\mathcal{W}^3$, the latter case occuring if $\{p_1, q_1, s\}$ is independent in $\mathrm{co}(M\backslash t)$. But $(H-\{p_2, q_2, t\}) \cup \{p_1, q_1\}$ does not contain C, and since C is unique it follows that $(H-\{p_2, q_2, t\}) \cup \{p_1, q_1\}$ is independent in M. Hence $\{p_1, q_1, s\}$ is independent in $\mathrm{co}(M\backslash t)$. We conclude that $\mathrm{co}(M\backslash t) \cong \mathcal{W}^3$ and that $\mathrm{co}(M\backslash t)$ is 3-connected and non-binary. Furthermore, it is easily seen that x is a spoke element of $\mathrm{co}(M\backslash t)$, so that $\mathrm{co}(\mathrm{co}(M\backslash t)\backslash x) \cong U_{2,4}$. But $\mathrm{co}(\mathrm{co}(M\backslash t)\backslash x) = \mathrm{co}(M\backslash t, x)$, so $\mathrm{co}(M\backslash t, x)$ is 3-connected and non-binary.

Clearly $|C \cap H| \geqslant 2$. It now follows that if b and c are any distinct elements of $C \cap H$, then (x, b, c) is a distinguished triple of M. ∎

For the remainder of the proof of (3.6) we assume that $r^*(M) \geqslant 5$. The proof of (3.6.2) below is just the same as that for (3.5.2).

(3.6.2) There exists a 3-element, coindependent set $\{a, b, c\}$ of $M \mid H$ with the property that $\mathrm{co}((M \mid H)\backslash a)$, $\mathrm{co}((M \mid H)\backslash b)$, and $\mathrm{co}((M \mid H)\backslash c)$ are all 3-connected and non-binary. ∎

Let $\{a, b, c\}$ be a triple satisfying (3.6.2). We now show, that for some $z \in \{a, b, c\}$, (z, p_2, q_2) is a distinguished triple of M. We first prove

(3.6.3) There is an element $z \in (\{a, b, c\} - \{p_2, q_2\})$ such that $\{z, p_2, q_2\}$ is coindependent in $M \mid H$.

Proof. Evidently a 3-element subset of $M \mid H$ is coindependent if and only if it is not a triad. Assume without loss of generality that $a \notin \{p_2, q_2\}$. If $\{a, p_2, q_2\}$ is not a triad we are done, so assume that this set is a triad. Since $\{a, b, c\}$ is not a triad, it follows that $\{p_2, q_2\} \neq \{b, c\}$. Say $b \notin \{p_2, q_2\}$. If $\{b, p_2, q_2\}$ is not a triad we are done, so assume that this set is also a triad. Now, in $(M \mid H)^*$, both $\{a, p_2, q_2\}$ and $\{b, p_2, q_2\}$ are triangles. An easy argument shows that either $\{a, b, p_2, q_2\}$ is a 4-point line, or p_2 and q_2 are parallel. The latter cannot happen in a 3-connected matroid, so $\{a, b, p_2, q_2\}$ is a 4-point line. Hence every 3-element subset of this set is a triangle. It follows that $\{a, b, p_2\}$ and $\{a, b, q_2\}$ are both triads of $M|H$. Hence $c \notin \{p_2, q_2\}$. Again we are either done, or we may assume that $\{c, p_2, q_2\}$ is a triad. In this case, arguing as before, we conclude that $\{a, b, c, p_2, q_2\}$ is a 5-point line of $(M \mid H)^*$. But then, $\{a, b, c\}$ is a triad of $M \mid H$. This contradiction establishes the result. ∎

Consider (z, p_2, q_2), where $z \in \{a, b, c\}$ and $\{z, p_2, q_2\}$ is not a triad of $M \mid H$. Evidently this set is not a triad of M. Certainly $M\backslash z$, $M\backslash p_2$, and $M\backslash q_2$ are non-binary. It is easily seen that $M\backslash z, p_2/x$ has a minor isomorphic to $(M \mid H)\backslash z$, a non-binary matroid. Hence $M\backslash z, p_2$ is non-binary. Similarly $M\backslash z, q_2$ is non-binary. We now show that $\mathrm{co}(M\backslash z)$ is 3-connected.

Assume that $\text{co}(M\backslash z)$ is not 3-connected. Then there is a 2-separation $\{X, Y\}$ of $M\backslash z$ such that $x \in X$ and neither X nor Y is contained in a series class of $M\backslash z$. We may also assume that X is a flat of $M\backslash z$. It follows that $|X|, |Y| > 2$. Now assume that $\{x, p_1, q_1\} \not\subseteq X$. By submodularity $r(X \cap H) < r(X)$ and $r(Y \cap H) < r(Y)$. Also both $X \cap H$ and $Y \cap H$ are non-empty. But this implies that $\{X \cap H, Y \cap H\}$ is a 1-separation of $(M \mid H)\backslash z$, a connected matroid. Therefore, $\{x, p_1, q_1\} \subseteq X$, and, since X is a flat, $\{p_2, q_2\} \subseteq X$. It now follows routinely that $\{X \cap H, Y\}$ is a 2-separation of $(M \mid H)\backslash z$. But $\text{co}((M \mid H)\backslash z)$ is 3-connected. Hence either $X \cap H$ or Y is contained in a series class of $(M \mid H)\backslash z$. But $\{z, p_2, q_2\}$ is not a triad of $M \mid H$ so $\{p_2, q_2\}$ is not a series pair of $(M \mid H)\backslash z$. Therefore $X \cap H$ is not in a series class. We conclude that Y is contained in a series class of $(M \mid H)\backslash z$. But the members of Y are also in series in $M/x\backslash z$, since this matroid is obtained from $(M \mid H)\backslash z$ by adding p_1 and q_1 in parallel to p_2 and q_2 respectively. But then the members of Y must be in series in $M\backslash z$. From this contradiction we conclude that $\text{co}(M\backslash z)$ is 3-connected.

We now show that $\text{co}(M\backslash z, p_2)$ is 3-connected. Consider $\text{co}(M\backslash z)$. This matroid can be obtained from $M\backslash z$ by contracting all but one member of each series class. It is clear that no pair of the set $\{x, p_1, p_2, q_1, q_2\}$ belongs to a series class, so it follows that we can assume without loss of generality that this set belongs to the ground set of $\text{co}(M\backslash z)$. In this matroid, $\{x, p_1, p_2\}$ and $\{x, q_1, q_2\}$ are lines, and furthermore $\{x, p_1, q_1\}$ is a triad. But then, $\text{co}(\text{co}(M\backslash z)\backslash p_1)$ is not 3-connected, so by (3.4.2), $\text{co}(\text{co}(M\backslash z)\backslash p_2)$ is 3-connected. It is routine to show that this matroid is equal to $\text{co}(M\backslash z, p_2)$. An identical argument shows that $\text{co}(M\backslash z, q_2)$ is also 3-connected. We conclude that (z, p_2, q_2) is a distinguished triple of M. ∎

Finally we prove

(3.7) If M has at least three non-trivial lines containing x, then M has a distinguished triple.

Proof. Say $\{x, p_1, p_2\}$ is a line of M. Assume that $M\backslash p_1$ is not 3-connected Then there is a 2-separation $\{X, Y\}$ of $M\backslash p_1$ with $x \in X$. If $|X| = 2$, then X is a 2-cocircuit of M, and $X \cup \{p_1\}$ is a triad of M. But a cocircuit of M containing z must contain at least one other point of each non-trivial line containing x. Since there are at least three such lines, there are no triads of M containing x. Therefore $|X| > 2$. It follows by (3.4.1) that $\{X - x, Y\}$ is a 2-separation of $M\backslash p_1/x$ where Y contains no parallel pairs of M/x. Since $\text{si}(M\backslash p_1/x)$ is 3-connected, $X - x$ must be contained in a single parallel class of $M\backslash p_1/x$. But this matroid has at least two distinct parallel classes. We conclude that $M\backslash p_1$ is 3-connected.

Now say that $\{x, q_1, q_2\}$ and $\{x, r_1, r_2\}$ are also lines of M. By the above argument, if $z \in \{q_1, q_2, r_1, r_2\}$, then $M \backslash z$ is 3-connected. Now consider $M \backslash p_1$. By (3.4.2), either $\text{co}(M \backslash p_1, q_1)$ or $\text{co}(M \backslash p_1, q_2)$ is 3-connected. Assume the former is 3-connected. Similarly, we may also assume that $\text{co}(M \backslash p_1, r_1)$ is 3-connected. It is easily seen that these matroids are nonbinary.

To show that (p_1, q_1, r_1) is a distinguished triple of M all that remains is to show that $\{p_1, q_1, r_1\}$ is not a triad of M. But this is clear since any hyperplane of M that does not meet $\{p_1, q_1, r_1\}$ cannot contain all of $\{x, p_2, q_2, r_2\}$, so $\{p_1, q_1, r_1\}$ is a proper subset of a cocircuit. ▮

Since (3.3)–(3.7) cover all cases, (3.1) follows. ▮

By invoking duality, we immediately obtain

(3.8.) COROLLARY. *Let M be a 3-connected, non-binary matroid with $r(M) \geqslant 4$. Then there exists an independent triple (a, b, c) with the property that* $\text{si}(M/a)$, $\text{si}(M/B)$, $\text{si}(M/c)$, $\text{si}(M/a, b)$ *and* $\text{si}(M/a, c)$ *are all 3-connected and non-binary.* ▮

4. WEAK MAPS AND HOMOMORPHISMS

Let M and N be matroids on a common ground set E. The identity map on E is a *weak map* from M to N if every independent set in N is also independent in M. In this case, N is a *weak-map image* of M. If $M \neq N$, then the weak map is *proper*. If, moreover, M and N have the same rank, N is a *rank-preserving* weak-map image of M. A good survey of the theory of weak maps is given in Kung and Nguyen [9].

Weak maps are very general constructions, and it is not surprising that there are few strong results describing their behaviour. A striking exception is Lucas's [13] characterisation of weak maps of binary matroids. For ternary matroids, weak maps are not as well behaved, but something can still be said. In [18], Oxley and Whittle prove

(4.1) ([18, Theorem 1.1]) Let M and N be ternary matroids such that N is a rank-preserving weak-map image of M. If M is 3-connected, and N is connected and non-binary, then $M = N$.

We use 4.1 frequently in later sections.

It is possible to determine whether one matroid is a weak-map image of another by comparing representations of the two matroids. Let A and B be matrices of the same size, so that their rows and columns are indexed by the same sets. Submatrices A' and B' of A and B respectively are *corresponding submatrices* if their rows and columns are indexed by the same subsets of the index sets of the rows and columns of A and B respectively. Lucas [13] proves

(4.2) Let M_1 and M_2 be matroids on a common ground set E represented over fields $\mathbf{F}_1$ and $\mathbf{F}_2$ by the $r \times n$ matrices $[I \mid A_1]$ and $[I \mid A_2]$, respectively, where corresponding columns of $[I \mid A_1]$ and $[I \mid A_2]$ represent the same elements of E. Then M_2 is a weak-map image of M_1 if and only if the following property holds. If D is a square submatrix of $[I \mid A_1]$ with $|D| = 0$, and D' is the corresponding submatrix of $[I \mid A_2]$, then $|D'| = 0$.

In particular we have

(4.3) $M_1 = M_2$ if and only if the following property holds. For each square submatrix D of $[I \mid A_1]$ and corresponding submatrix D' of $[I \mid A_2]$, we have $|D| = 0$ if and only if $|D'| = 0$.

It is usual to discuss matroid representation in terms of fields, although, in fact matroids are often represented over integral domains. This causes no difficulty; every integral domain is a subring of its field of quotients, so a representation of a matroid over an integral domain is effectively a representation over a field. In the following discussion we are interested in homomorphisms. In this case we do need to consider integral domains, since a non-trivial homomorphism defined on an integral domain need not extend to its field of quotients.

It has often been noted that while the well-understood class of strong maps between matroids is geometric in nature—strong maps generalise linear transformations—weak maps seem more algebraic. The following result is in accord with this perspective. Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be integral domains and consider a function $\varphi\colon \mathbf{I}_1 \to \mathbf{I}_2$. If A is a matrix over $\mathbf{I}_2$, then $\varphi(A)$ denotes the matrix over $\mathbf{I}_2$ whose (i, j)-th entry is $\varphi(a_{ij})$. In this case, if D is a submatrix of A, then $\varphi(D)$ denotes the corresponding submatrix of $\varphi(A)$.

(4.4) ([9, Exercise 9.2]) Let M_1 be represented over the integral domain $\mathbf{I}_1$ by the matrix $[I \mid A]$, let $\varphi\colon \mathbf{I}_1 \to \mathbf{I}_2$ be a homomorphism from $\mathbf{I}_1$ into $\mathbf{I}_2$, and let M_2 denote the matroid represented over $\mathbf{I}_2$ by $\varphi([I \mid A_1])$. Then M_2 is a weak-map image of M_1.

Proof. Let D be a square submatrix of $[I \mid A]$. It follows immediately from the definitions of determinants and homomorphisms that $|\varphi(D)| = \varphi(|D|)$. Therefore, if $|D| = 0$, then also $|\varphi(D)| = 0$. The result now follows by (4.2). ∎

Certain homomorphisms will be of particular interest. We first recall some terminology. Let $\mathbf{F}$ be a field, and α be a transcendental over $\mathbf{F}$. Then $\mathbf{F}[\alpha]$ denotes the integral domain of polynomials in α over $\mathbf{F}$, and $\mathbf{F}(\alpha)$ denotes the extension field obtained by extending $\mathbf{F}$ by the transcendental α. Recall also that $\mathbf{F}(\alpha)$ is the field of quotients of $\mathbf{F}[\alpha]$, that is, the elements of $\mathbf{F}(\alpha)$ are rational functions in α with coefficients in $\mathbf{F}$. If $a \in \mathbf{F}$, then the function φ: $\mathbf{F}[\alpha] \to \mathbf{F}$ defined by evaluating the elements of $\mathbf{F}[\alpha]$ at a is known to be a homomorphism. Because we apply this homomorphism frequently in the following sections we standardise some terminology. If A is a matrix over $\mathbf{F}[\alpha]$, then $A(a)$ denotes the matrix obtained by evaluating each entry of A at a. If D is a submatrix of A, then $D(a)$ denotes the corresponding submatrix of $A(a)$, and of course, if the polynomial p is an entry of A, then $p(a)$ denotes the corresponding entry of $A(a)$.

5. Near-Regular Matroids

Let $\mathbf{Q}$ denote the field of rational numbers, and let α be a transcendental over $\mathbf{Q}$. We consider matrices over the transcendental extension field $\mathbf{Q}(\alpha)$. Let $\mathscr{A}$ denote the set $\{\pm\alpha^i(\alpha-1)^j: i, j \in \mathbf{Z}\}$. A matrix A over $\mathbf{Q}(\alpha)$ is *near-unimodular in* α if all non-zero subdeterminants of A are in $\mathscr{A}$. If the particular transcendental is clear from the context we just say that A is *near-unimodular*. A matroid M is *near-regular* if $M = M[A]$ for some near-unimodular matrix A. In this case we say that A is a *near-unimodular* representation of M.

It could be argued that near-unimodular matrices should be called "totally near-unimodular" keeping a parallel with totally unimodular matrices. The reason for not doing this is that the terminology is somewhat clumsy. Also, our interest is always in the case where the condition that subdeterminants belong to $\{\pm\alpha^i(\alpha-1)^j: i, j \in \mathbf{Z}\}$ applies to all subdeterminants of a matrix A. Indeed I know of no situation where it is of interest that just $|A|$ satisfies the condition.

Let A be a near-unimodular matrix. If B is obtained from A by multiplying each entry of a given row or column by a fixed element of $\mathscr{A}$, then B is obtained from A by *a proper scaling* of A.

(5.1) Proposition. *Let A be a near-unimodular matrix, and B be a matrix over* $\mathbf{Q}[\alpha]$.

(i) *If B is obtained from A by a sequence of proper scalings, then B is near-unimodular.*

(ii) *If B is obtained from A by a sequence of pivots, then B is near-unimodular.*

Proof. It is evident that (i) holds. To show that (ii) holds we may assume that B is obtained from A by a single pivot. Say B is obtained from A by pivoting on the non-zero entry x_{st} of the latter. Let A' and B' be corresponding submatrices of A and B respectively, each having their rows indexed by the sets J_R and J_C, respectively. We want to show that $|B'| \in \mathscr{A}$. If $s \in J_R$, then it follows from elementary facts of determinants that $|B'| = x_{st}^{-1}|A'|$. But $|A'| \in \mathscr{A}$ and $x_{st} \in \mathscr{A}$ so it follows that $|B'| \in \mathscr{A}$. Hence we may assume that $s \notin J_R$. In this case, if $t \in J_C$, then B' has a zero column, so $|B'| = 0$. Thus we may also assume that $t \notin J_C$. Now let A'' and B'' be the submatrices of A and B whose rows and columns are indexed by $J_R \cup \{s\}$ and $J_C \cup \{t\}$. As the only non-zero entry in the column of Y'' indexed by t is a 1, $|B''| = \pm|B'|$. But as above, $|B''| = x\,|A''|$ for some $x \in \mathscr{A}$. Again it follows that $|B'| \in \mathscr{A}$ and the proposition is proved. ∎

With very minor modifications, the above proof is an unashamed lift of the proof of the analogous result for totally unimodular matrices given by Oxley [17, Theorem 2.2.20]. A proof has been give here because Proposition 5.1 is vital in what follows. The next proposition is a routine consequence of Proposition 5.1.

(5.2) PROPOSITION. *The class of near-regular matroids is minor closed and is closed under duality.*

A property of near-regular matroids that is straightforward to prove is

(5.3) PROPOSITION. *Direct sums and 2-sums of near-regular matroids are near-regular.*

It is a consequence of Proposition 5.3 that, in dealing with the class of near-regular matroids, one can focus on 3-connected members of this class. Since whirls are basic building blocks for non-binary, 3-connected matroids it is of interest to examine near-unimodular matrices that represent whirls. Two near-unimodular matrices are *equivalent* if they are equivalent representations of the same matroid.

(5.4) PROPOSITION. *Up to equivalence, all near-unimodular representations of $\mathcal{W}^r$, $r \geq 2$, are of the form $[I \mid A]$, where*

$$A = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ 1 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & (-1)^r x \end{bmatrix},$$

and $x \in \{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$.

Proof. Say the near-unimodular matrix B represents M. Then B can be transformed by a sequence of pivots and proper scalings into a matrix $[I \mid C]$, where the columns of I correspond to the spokes of $\mathcal{W}^r$. Moreover, by reordering the columns of C if necessary, we can assume that an entry c_{ij} of C is non-zero if and only it the corresponding entry a_{ij} of A is non-zero. Furthermore, we can also assume that the leading entry of each row and column of C is non-zero. Hence, for $\{i, j\} \subseteq \{1, 2, ..., r\}$, and $(i, j) \neq (r, r)$, $a_{ij} = c_{ij}$. In other words, any near-unimodular matrix that represents $\mathcal{W}^r$ is equivalent to one of the form $[I \mid A]$ for some choice of x.

We now examine possible values for x. The only subdeterminant of A that is not in $\{0, \pm 1, \pm x\}$ is $|A|$, and $|A| = \pm(x-1)$. Now $[I \mid A]$ is a representation of $\mathcal{W}^r$ if and only if $x \notin \{0, 1\}$. Thus $[I \mid A]$ is a near-unimodular representation of $\mathcal{W}^r$ if and only if both x and $x-1$ are in $\mathcal{A} - \{0, 1\}$. Routine checking shows that this only happens when x belongs to the set specified in the statement of the proposition. ▌

(5.5) Proposition. *Let the matrix A be near-unimodular in α, and say $\beta \in \{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$. Then A is near-unimodular in β.*

Proof. Note that, regarded as functions, each member of $\{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$ is equal to its inverse. It follows that if β belongs to this set, then α is in $\{\beta, -(\beta-1), \beta/(\beta-1), -1(\beta-1), 1/\beta, (\beta-1)/\beta\}$. If any one of these values is substituted into an element of $Q[\alpha]$ of the form $\alpha^i(\alpha-1)^j$ is clear that one obtains an element of $Q[\beta]$ of the form $\beta^{i'}(\beta-1)^{j'}$. This proves the proposition. ▌

We now consider the representability of near-regular matroids. Let $\mathbf{F}$ be a field, b be an element of $\mathbf{F} - \{0, 1\}$, and A be a near-unimodular matrix. We extend the notation defined in Section 4 and define $A(b)$ to be the matrix over $\mathbf{F}$ obtained by letting $\alpha = b$. Although A is a matrix over $\mathbf{Q}(\alpha)$ it is easily seen that $A(b)$ is well-defined for any field $\mathbf{F}$. This definition holds if A is 1×1. In particular, if A' is a submatrix of A, then $|A'|$ is a 1×1 near-unimodular matrix. Hence $|A'|(b)$ is obtained by evaluating $|A'|$ at b.

(5.6) Lemma. *Let A be a near-unimodular matrix, $\mathbf{F}$ be a field, and $b \in \mathbf{F} - \{0, 1\}$. If A' is a square submatrix of A, then $|A'(b)| = |A'|(b)$.*

Proof. Consider the subdomain $\mathbf{I} = \langle \alpha^i(\alpha-1)^j : i, j \in \mathbf{Z} \rangle$ of $\mathbf{Q}(\alpha)$. Members of $\mathbf{I}$ have the form $\sum_{k=1}^{n} a_k \alpha^{i_k}(\alpha-1)^{j_k}$, where, for $1 \leqslant k \leqslant n$, a_k, i_k and j_k are integers. Define the function $\phi: \mathbf{I} \to \mathbf{F}$ by

$$\varphi\left(\sum_{k=1}^{n} a_k \alpha^{i_k}(\alpha-1)^{j_k}\right) = \sum_{k=1}^{n} a_k b^{i_k}(b-1)^{j_k}.$$

Here, of course, as an element of **F**, a_k is interpreted to mean

$$\pm(\underbrace{1+1+\cdots+1}_{|a_k|\text{ terms}})$$

depending on whether a_k is positive or negative. It is routinely checked that φ is a homomorphism. Since A is near-unimodular, A is a matrix over **I**. Moreover, $A(b)=\varphi(A)$. It now follows from the fact that φ is a homomorphism that, for any submatrix A' of A, $|\varphi(A')|=\varphi\,|A'|$. Hence $|A'(b)|=|A'|(b)$. ∎

(5.7) COROLLARY. *Let $[I \mid A]$ be a near-unimodular matrix,* **F** *be a field, and b be an element of* $\mathbf{F}-\{0,1\}$. *Then* $M[I \mid A]=M[[I \mid A](b)]$.

Proof. Since $b^r(b-1)^s=0$ if and only if $b\in\{0,1\}$, it follows from Lemma 5.6 that a subdeterminant of A is non-zero if and only if the corresponding subdeterminant of $A(b)$ is non-zero. But then, by (4.3), we have $M[I \mid A]=M[[I \mid A](b)]$. ∎

(5.8) COROLLARY. *If M is a near-regular matroid then M is representable over every field except possibly $GF(2)$.*

Proof. Let A be a near-unimodular matrix that represents M, and **F** be a field other than $GF(2)$. Then $\mathbf{F}-\{0,1\}\neq\varnothing$, so by Corollary 5.7, there exists a matrix B over **F** with the property that $M[B]=M[A]$, in other words, $M[B]=M$. ∎

Of course, it follows immediately from Corollary 5.8 that near-regular matroids are representable over both $GF(3)$ and **Q**. However the converse is not true, for example it will soon be seen that the non-Fano matroid is not near-regular. It will transpire that the 3-connected, non-binary near-regular matroids are exactly the 3-connected ternary matroids that are not uniquely representable over **Q**. Note also that it is shown in [26] that near-regular matroids are exactly the class of matroids representable over all fields except possibly $GF(2)$. The importance of the remaining results in this section is that they show that near-unimodular representations are well behaved in the sense that one can always extend a near-unimodular representation of a 3-connected matroid M to a near-unimodular representation of any near-regular, single-element extension of M.

(5.9) THEOREM. *Let M be a ternary, non-binary, 3-connected matroid with an element $x\in E(M)$ with the property that $M\backslash x$ is non-binary, 3-connected and near-regular. If a near-unimodular representation of $M\backslash x$ extends to a representation of M over* $\mathbf{Q}[\alpha]$, *then, up to a scaling of the* *vector that represents* x, *that representation is near-unimodular, and* M *is near-regular.*

Proof. Consider a representation of M over $\mathbf{Q}[\alpha]$ that is obtained by extending a near-unimodular representation of $M\backslash x$. Using pivoting, proper scaling and Proposition 5.2, we may assume, without loss of generality, that M is represented by $B=[I \mid A \mid \mathbf{x}]$, where $[I \mid A]$ is a near-unimodular representation of $M\backslash x$.

The proof of the theorem is in two parts. In the first part we show that all non-zero subdeterminants of B are of the form $k\alpha^i(\alpha-1)^j$ for some $k\in\mathbf{Q}$. In the second we show that if $\mathbf{x}$ is appropriately scaled, then $k\in\{1-1\}$.

(5.9.1) Each entry of B is of the form $k\alpha^i(\alpha-1)^j$ for some $k\in\mathbf{Q}$.

Proof. Let $\mathbf{C}$ denote the field of complex numbers. Certainly B is well defined as a matrix over $\mathbf{C}(\alpha)$, that is, as a matrix whose entries are rational functions in α over the complex numbers. Since the complex numbers are algebraically closed, any numerator or denominator of a non-zero entry of B that has degree greater than 0 splits into linear factors over $\mathbf{C}$. (Of course, for entries in $[I \mid A]$ the only possible factors are $\alpha-1$ and α, but for entries in $\mathbf{x}$ it is conceivable that other factors occur.) Multiply each column of B by the lowest common denominator of the entries in that column. For the columns of $[I \mid A]$, this is a proper scaling. Finally divide the last column of the resulting matrix by the greatest common divisor of the entries in that column.

The upshot of the above discussion is that we may assume without loss of generality that M is represented over $\mathbf{C}(\alpha)$ by $B=[I \mid A \mid \mathbf{x}]$, where $[I \mid A]$ is near-unimodular as a matrix over $\mathbf{Q}(\alpha)$, all the entries of B are polynomials, and the greatest common divisor of the entries of $\mathbf{x}$ has degree 0.

Assume that some subdeterminant of B is not of the form $k\alpha^i(\alpha-1)^j$. Since the complex numbers are algebraicaly closed, this subdeterminant has a root $c\in\mathbf{C}$, where $c\notin\{0,1\}$. Recall from Section 4 that $B(c)$ is the matrix obtained by evaluating each entry of B at c. Consider $B(c)$. By (4.4), $M[B(c)]$ is a weak-map image of M. Moreover, by (4.3), $M[B]\neq M[B(c)]$ so this weak map is proper. Now $M[B(c)]\backslash x=M[[I \mid A](c)]$. But $[I \mid A]$ is near-unimodular, and $c\notin\{0,1\}$, so by Corollary 5.7, $M[[I \mid A](c)]=M\backslash x$, and this is a 3-connected, non-binary matroid whose rank is equal to $r(M)$. Moreover, the greatest common divisor of the entries in $\mathbf{x}$ has degree 0, so $\alpha-c$ is not a factor of all entries of $\mathbf{x}$. It follows that $\mathbf{x}(c)$ has at least one non-zero entry. This shows that x is not a loop of $M[B(c)]$, so this matroid is connected. In other words, $M[B(c)]$ is a connected, non-binary matroid that is a rank-preserving, weak-map image of M. An easy argument shows that if M has no $U_{2,5}$- or $U_{3,5}$-minor, then no rank-preserving, weak-map image of M has a $U_{2,5}$- or $U_{3,5}$-minor. Hence $M[B(c)]$ has no $U_{2,5}$- or $U_{3,5}$-minor. Also, since $M[B(c)]$ is representable over $\mathbf{C}$, it has no F_7- or F_7^*-minor. It now follows from the excluded-minor characterisation of ternary matroids [19] that $M[B(c)]$ is ternary. But then, by (4.1), $M[B(c)]=M$, a contradiction. Therefore the only roots over $\mathbf{C}$ of any subdeterminant of B are in $\{0, 1\}$, and it follows that each subdeterminant is of the form $k\alpha^i(\alpha-1)^j$. ∎

Now assume that x is scaled so that the leading coefficients of the polynomials that are non-zero entries of $\mathbf{x}$ are integers, and the greatest common divisor of these coefficients is one. It follows that the leading coefficients of all the polynomials corresponding to non-zero subdeterminants of B are integers. We complete the proof of the theorem by showing that these integers are all in $\{1, -1\}$. The proof is by induction on the rank of M. The result certainly holds if M has rank 2. Assume that M has rank 3. We first note

(5.9.2) No near-unimodular representation of $\mathcal{W}^3$ extends to a representation of F_7^- over $\mathbf{Q}(\alpha)$. Moreover, F_7^- is not near-regular.

Proof. Consider the following near-unimodular representation of $\mathcal{W}^3$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -\alpha \end{bmatrix}.$$

It follows from Propositions 5.4 and 5.5 that if this representation of $\mathcal{W}^3$ does not extend to a representation of F_7^-, then no near-unimodular representation of $\mathcal{W}^3$ extends to a representation of F_7^-. The unique extension of the above representation of $\mathcal{W}^3$ that is isomorphic to F_7^- occurs when the extension element $\mathbf{y}$ is on each of the lines spanned by $\{(1, 0, 0)', (0, 1, 1)'\}$, $\{(0, 0, 1)', (1, 1, 0)'\}$ and $\{(0, 1, 0)', (1, 0, -\alpha)'\}$. If $\mathbf{y}$ is on the first two of these lines, then, up to scaling, $\mathbf{y}=(1, 1, 1)'$. But

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -\alpha & 1 \end{vmatrix} = \alpha + 1,$$

and $\alpha+1$ is certainly not zero as an element of $\mathbf{Q}(\alpha)$. It follows that $\mathbf{y}$ is not on the line spanned by $\{(0, 1, 0)', (1, 0, -\alpha)'\}$. We conclude that no near-unimodular representation of $F_7^- \backslash x$ extends to a representation of F_7^-. It also follows that F_7^- has no near-unimodular representation, so F_7^- is not near-regular. ∎

With (5.9.2) in hand it can be checked that every 3-connected, near-regular, rank-3 matroid is a restriction of the matroid M_3 illustrated in Figure 5.1. We omit the details of this check. It is quite straightforward if one uses facts on rank-3, 3-connected, ternary matroids established in, for example, [8, 10, 14] and the fact that the matroid $AG(2, 3)\backslash p$, obtained by deleting a point from the ternary affine plane, is not near regular. This latter fact follows from [4, Exercise 24.14] or [17, Exercise 6.4.9].

It is easily checked that the matrix

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1-\alpha \\ 0 & 0 & 1 & 0 & 1 & -\alpha & -1 & -\alpha \end{bmatrix}$$

is a near-unimodular representation of M_3, so that M_3 is near-regular. Any counterexample to the theorem must then come from a 3-connected, non-binary minor N_3 of M_3 with the property that a near-unimodular representation of N_3 extends to a $\mathbf{Q}(\alpha)$ – representation of a matroid N'_3 that is not a minor of M_3. We show that no such matroid exists. First note that $\mathcal{W}^3$ extends to the non-Fano matroid F_7^-.

Another routine check shows that every 3-connected, ternary, non-binary, rank-3 matroid that is also representable over $\mathbf{Q}(\alpha)$ is either a minor of M_3 or has an F_7^--minor. An easy argument then shows that the theorem holds if M has rank 3.

Therefore we may assume that $r(M) \geqslant 4$, and, for induction, assume that the result holds for all matroids satisfying the hypotheses of the theorem whose rank is less that $r(M)$. By Corollary 3.8, there exists an independent triple $\{a, b, c\}$ of distinct elements of $E(M\backslash x)$ with the property that

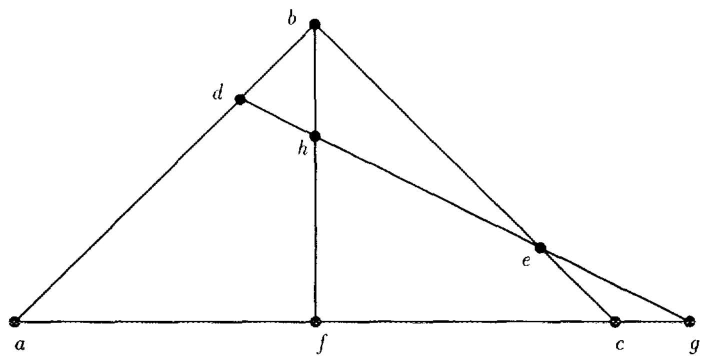

FIG. 5.1. The maximum rank-3 near-regular matroid.

si$(M\backslash x/a)$, si$(M\backslash x/b)$, si$(M\backslash x/c)$, si$(M\backslash x/a, b)$, and si$(M\backslash x/a, c)$ are all 3-connected and non-binary. It is clear that if $\{a, c, x\}$ is collinear, then $\{a, b, x\}$ is not. Hence we may assume without loss of generality that $\{a, b, x\}$ is not collinear. By pivoting if necessary, we may also assume that in the representation $[I \mid A \mid \mathbf{x}]$, a and b are represented by the last two columns of I. By dividing each entry of $\mathbf{x}$ by its leading non-zero entry if necessary, we may also assume that the leading non-zero entry of $\mathbf{x}$ is 1. Since $\{a, b, x\}$ is not collinear, this leading non-zero entry is not in the second-to-last or last row of $[I \mid A \mid \mathbf{x}]$. Say $\mathbf{x} = (x_1, x_2, ..., x_r)$, and set $\mathbf{x_a} = (x_1, x_2, ..., x_{r-1})$. Then $\mathbf{x_a}$ is a vector that extends a representation of $M\backslash x/a$ to a representation of M/a. But si$(M\backslash x/a)$ is 3-connected and non-binary. Therefore, by (2.8), the coordinates of $\mathbf{x_a}$ are uniquely determined given that the leading non-zero entry of $\mathbf{x_a}$ is a 1. It now follows by induction that the leading coefficients of all of the non-zero entries of $\mathbf{x_a}$ are in $\{1, -1\}$. The same argument applied to M/b shows that the leading coefficients of all non-zero entries of $\mathbf{x}$ are all in $\{1, -1\}$.

Now assume that some square submatrix of $[I \mid A \mid \mathbf{x}]$ has a non-zero determinant that does not have a leading coefficient in $\{1, -1\}$. Let D be such a submatrix having minimum size. It follows from the above that D must be at least 2×2. In fact, since the representations of M/a and M/b obtained by deleting the last and second-to-last rows of $[I \mid A \mid \mathbf{x}]$ are certainly near-unimodular, D must meet both of these rows. Say D is $n \times n$ where $n > 2$. An elementary matrix-theoretic argument shows that the representation of M obtained after pivoting on a non-zero element d of D has an $(n-1) \times (n-1)$ submatrix D' with the property that $|D'| = |D|/d$. Evidently, the leading coefficient of $|D'| \notin \{1, -1\}$ if and only if the leading coefficient of $|D| \notin \{1, -1\}$. Moreover, if d is not in the last or second-to-last row of $[I \mid A \mid \mathbf{x}]$, then all entries of D' correspond to entries of a matrix obtained by a pivot on a near-unimodular matrix. It follows that all entries of D' are of the form $\pm\alpha^i(\alpha-1)^j$. We deduce that, by performing a sequence of such pivots on entries not in the second or second-to-last row, we obtain a 2×2 matrix D'' with the property that the leading coefficient of $|D''| \notin \{1, -1\}$, and the property that all of its entries are of the form $\pm\alpha^i(\alpha-1)^j$. It is clear that no entry of D'' is zero. It is also clear that, after performing proper scalings if necessary, we can assume without loss of generality that

$$D'' = \begin{bmatrix} 1 & 1 \\ 1 & \pm\alpha^r(\alpha-1)^s \end{bmatrix}$$

for some integers r and s. But we know that $|D''| = k\alpha^i(\alpha-1)^j$ for some integer $k \notin \{0, 1, -1\}$. A routine computation shows that the only solution to

$$\pm\alpha^r(\alpha-1)^s - 1 = k\alpha^i(\alpha-1)^j$$

occurs when $(r, s) = (0, 0)$, that is when

$$D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The upshot of the above discussion is that a sequence of pivots and proper scalings can be performed on $[I \mid A \mid x]$ to obtain a matrix $[I \mid C]$ representing M over $\mathbf{Q}(\alpha)$ with the property that each entry of $[I \mid C]$ is in $\pm\alpha^r(\alpha-1)^s$, and the property that C contains a 2×2 submatrix equal to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

By performing an appropriate permutation of the rows and columns of C, we may assume that

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Say that C is $r \times n$. For $3 \leqslant i \leqslant n$ multiply column i of C by the lowest common denominator of the entries in that column, and for $3 \leqslant i \leqslant r$ multiply row i of the resulting matrix by the greatest common divisor of the entries in that row. After these scalings we obtain a matrix that has the abovementioned properties of C, but has the additional property that each entry is a polynomial. In other words, we may assume without loss of generality that each entry of C is a polynomial. This means that $C(0)$ and $C(1)$, the matrices obtained by evaluating each entry of C at 0 and 1 respectively are both well-defined. Certainly both $M[C(0)]$ and $M[C(1)]$ are rank-preserving, weak-map images of M, but it may be that be that at least one of $M[C(0)]$ and $M[C(1)]$ is not connected, so that we cannot apply (4.1). We remedy this situation now. For a matrix Z, there is a natural bipartite graph associated with Z. The vertices are the index sets of the rows and columns of Z, and $\{r_i, c_j\}$ is an edge if and only if $z_{ij} \neq 0$. It is known [5, Proposition 2.4] that the matroid $M[I \mid Z]$ is connected if and only if the bipartite graph associated with Z is connected. We now scale C to obtain a matrix C' with the property that the bipartite graphs associated with $C'(0)$ and $C'(1)$ are both connected.

Let C_c denote the following set of columns of C. The i-th column of C is in C_c if and only if $i > 2$, and at least one of c_{1i} and c_{2i} is non-zero. Similarly, C_r denotes the following set of rows of C. The i-th row of C is in C_r if and only if $i > 2$, and at least one c_{i1} and c_{i2} is non-zero. At least one of C_c and C_r is non-empty, otherwise M is not connected. Assume without loss of generality that the set of columns is non-empty. Take a column in the set. Consider the non-zero entries in the first two coordinates of this column. Divide each entry of the column by the greatest common divisor of these entries. It is easily seen that in the first two entries of the resulting column there is an element of the form $\pm\alpha^i$, for some non-negative integer i, and an element of the form $\pm(\alpha-1)^j$, for some non-negative integer j. (Note that ± 1 has both forms.) Repeat the process for each column in the set. Not all entries of the resulting matrix are guaranteed to be polynomials. Fix this situation by an appropriate scaling of all but the first two rows. Interchange columns so that the columns that have a non-zero entry in one of the first two coordinates form the first s columns. Let P denote the matrix we now have. Consider the submatrix

$$P' = \begin{bmatrix} p_{11} & \cdots & p_{1s} \\ p_{21} & \cdots & p_{2s} \end{bmatrix}.$$

It is evident that the bipartite graphs associated with both $P'(0)$ and $P'(1)$ are connected.

Now consider the following set P_r of rows of P. The first two rows of P are not in P_r. Otherwise a row is in the set if and only if it has a non-zero entry in one of its first s coordinates. Thus P_r cannot be empty, for otherwise M would not be connected. It is clear that we can repeat the above process on this set of rows to obtain a rescaling of P. It now follows by an obvious inductive process that C can be scaled to produce the desired matrix C' with the properties that the bipartite graphs associated with $C'(0)$ and $C'(1)$ are both connected. Moreover, at no stage have either the first two rows or columns of any matrix been scaled. Therefore

$$\begin{bmatrix} c'_{11} & c'_{12} \\ c'_{21} & c'_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

One of the columns of C' represents x. The matrix obtained by deleting this column is, when the identity is adjoined, a near-unimodular representation of $M\backslash x$, a non-binary matroid. If all entries of this matrix were in $\{0, 1, -1\}$, then the matrix would be unimodular, and would never represent a non-binary matroid. It follows that at least one entry of this matrix has either α or $\alpha-1$ as a factor. Hence we may assume that one of $M[C'(0)]$ and $M[C'(1)]$ is a proper weak-map image of M. Assume without loss of generality that $M[C'(0)]$ is. But $M[C'(0)]$ is connected, so by (4.1) this matroid is binary. Moreover, $C'(0)$ is a matrix over $\mathbf{Q}$, so $M[C'(0)]$ is regular. Now all entries of $M[C'(0)]$, are in $\{0, 1, -1\}$, and this matrix represents a regular matroid. It follows by the fact that binary matroids are uniquely representable over $\mathbf{Q}$, that $C'(0)$ is unimodular. But $C'(0)$ has a subdeterminant equal to -2. This contradiction shows that the leading coefficients of all non-zero determinants of submatrices of $[I \mid A \mid \mathbf{x}]$ are indeed in $\{1, -1\}$, and the theorem is proved. ▌

(5.10) COROLLARY. *Let M be a 3-connected, ternary, non-binary matroid with largest whirl minor $\mathcal{W}^r$. If a near-unimodular representation of $\mathcal{W}^r$ extends to a representation of M over $\mathbf{Q}(\alpha)$, then M is near-regular.*

Proof. By (2.4) there exists a sequence $\mathcal{W}^r \cong M_0, M_1, ..., M_n = M$ of 3-connected matroids with the property that for $1 \leqslant i \leqslant n$, M_i is a single-element extension or coextension of M_{i-1}. The corollary now follows from this fact, Theorem 5.9, and an easy duality argument. ▌

Say M is ternary, and $M \backslash x$ is non-binary, 3-connected and near-regular. Theorem 5.9 tells us that if a near-unimodular representation of $M \backslash x$ does extend to a representation of M, then M is near-regular. We now show that if M is near-regular, then a near-unimodular representation of $M \backslash x$ will always extend to a representation of M.

(5.11) THEOREM. *Let M be a 3-connected, non-binary, near-regular matroid, and x be an element of $E(M)$ with the property that $M \backslash x$ is 3-connected and non-binary. Let $[I \mid A]$ be a near-unimodular representation of $M \backslash x$. Then there exists a vector $\mathbf{x}$ with the property that $[I \mid A \mid \mathbf{x}]$ represents M.*

Proof. Let $[I \mid B \mid \mathbf{x}']$ be a near-unimodular representation of M. Call an operation on a near-unimodular matrix *good* if the matrix that results from the operation is both near-unimodular and represents the same matroid. We prove by induction that there exists a sequence of good operations that transforms $[I \mid B \mid \mathbf{x}']$ into a matrix that is, apart from the last column, equal to $[I \mid A]$. This last column is the desired $\mathbf{x}$. It is certainly the case that row and column permutations, pivots and scalings are good operations. Moreover, these operations are invertible so we may assume without loss of generality that a column of $[I \mid A]$ represents the same element of the ground set of $M \backslash x$ as the corresponding column of $[I \mid B]$. Assume that $M \backslash x$ is a whirl. It follows by Proposition 5.4 that—again after appropriate scalings, permutations and pivots—we can assume that

$$A = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ 1 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & (-1)^r u \end{bmatrix},$$

where $u \in \{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$, and that

$$B = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ 1 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & (-1)^r v \end{bmatrix},$$

where $v \in \{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$. Now make the substitution $\beta = u$ in each entry of $[I \mid B \mid \mathbf{x}']$. By Proposition 5.5, the resulting matrix is near-unimodular as a matrix over $\mathbf{Q}(\beta)$. This matrix certainly represents M. Next, make the substitution $v = \beta$ for all the entries of this near-unimodular matrix over $\mathbf{Q}(\beta)$. Again by Proposition 5.5, we obtain a near-unimodular matrix, and one that represents M. But this matrix is, apart from the last column, equal to $[I \mid A]$.

Assume that $M \backslash x$ is not a whirl, and assume, for induction, that the theorem holds for all matroids satisfying the hypotheses whose ground sets have cardinality less than $|E(M)|$. By (2.5), there exists an element $y \in E(M \backslash x)$ with the property that either $M \backslash x, y$ or $M \backslash x/y$ is 3-connected and non-binary. Assume the former. By pivoting if necessary, we may assume that $[I \mid A] = [I \mid A' \mid \mathbf{y}]$. By the induction assumption, one can perform a sequence of good operations on $[I \mid B \mid \mathbf{x}]$ to obtain the matrix $[I \mid A' \mid \mathbf{y}' \mid \mathbf{x}'']$. But $M[I \mid A']$ is 3-connected and non-binary, and $M \backslash x$ is ternary. So, by (2.8) there is, up to scaling, a unique vector that can be added to $[I \mid A']$ to obtain a representation of $M \backslash x$. In other words $\mathbf{y}'$ is a scalar multiple of $\mathbf{y}$ and we conclude that $[I \mid A \mid \mathbf{x}'']$ represents M. Moreover, this matrix is obtained from $[I \mid B \mid \mathbf{x}']$ by a sequence of good operations. A similar argument holds in the case when $M \backslash x/y$ is 3-connected. ∎

6. Dyadic Matroids

Recall that a matrix A over $\mathbf{Q}$ is *dyadic* if every non-zero subdeterminant of A is a signed integral power of 2. A matroid M is *dyadic* if $M = M[A]$ for some dyadic matrix A. If the matrix B is obtained from the dyadic matrix A by multiplying each entry of a given row or column of A by a fixed integral power of 2, then B is obtained from A by a *proper scaling* of A.

The term "dyadic matrix" was introduced by Zaslavsky (see [11, 12]). I have not completely followed Zaslavsky's terminology: the class of matrices that we call "dyadic", Zaslavsky calls "totally dyadic", and the class of matroids that we call "dyadic", Zaslavsky calls "subregular". The reason for changing the latter terminology is because of the danger of confusing "subregular" with "near-regular".

Dyadic matrices are special cases of matrices studied by Lee [11, 12], although Lee's interest is more in problems arising from linear programming than matroid representation theory. The following three propositions are routine. They also follow from more general results in [11, 12].

(6.1) PROPOSITION. *Let A be a dyadic matrix, and B be a matrix over* $\mathbf{Q}$.

(i) *If B is obtained from A by a sequence of proper scalings, then B is a dyadic matrix.*

(ii) *If B is obtained from A by a sequence of pivots, then B is also a dyadic matrix.*

(6.2) PROPOSITION. *The class of dyadic matroids is minor closed and is closed under duality.*

(6.3) PROPOSITION. *Direct sums and 2-sums of dyadic matroids are also dyadic matroids.*

We aim to show that a matroid is representable over $GF(3)$ and $\mathbf{Q}$ if and only if it is dyadic. In one direction this is very easy. The following proposition follows from [12, Proposition 3.1]. Nonetheless a proof is given here.

(6.4) PROPOSITION. *If M is a dyadic matroid and* $\mathbf{F}$ *is a field whose characteristic is not 2, then M is representable over* $\mathbf{F}$.

Proof. Say $M = M[A]$ where A is a dyadic matrix. By scaling if necessary we may assume without loss of generality that the entries in A are integers. Let $\varphi\colon \mathbf{Z} \to \mathbf{F}$ be the natural homomorphism, that is, $\varphi(a) = \pm(\underbrace{1 + 1 + \cdots + 1}_{|a| \text{ terms}})$ depending on whether a is positive or negative. It follows by (4.2) that $M[\varphi(A)]$ is a weak-map image of M. Say B is a square submatrix of A, and $|B| \neq 0$. Then $|B| \in \{\pm 2^i : i \in \mathbf{Z}\}$. But $|\varphi(B)| = \varphi(|B|)$, and hence $|\varphi(B)| \neq 0$. Therefore M is a weak-map image of $M[\varphi(A)]$. We conclude that $M = M[\varphi(A)]$, and hence that M is representable over $\mathbf{F}$. ∎

In particular, it follows from Proposition 6.4 that a dyadic matroid is representable over both $GF(3)$ and $\mathbf{Q}$. We now work towards the converse of this fact. We first note

(6.5) PROPOSITION. *Near-regular matroids are dyadic matroids.*

Proof. If M is near-regular, then M has a near-unimodular representation $[I \mid A]$. Consider $[I \mid A](2)$, the matrix obtained by making the substitution $\alpha = 2$ in $[I \mid A]$. This matrix is clearly a dyadic matrix. It now follows from Corollary 5.7 that $M[[I \mid A](2)] = M$. ∎

(6.6) THEOREM. *Let M be a 3-connected matroid that is representable over both $GF(3)$ and $\mathbf{Q}$. Assume that M is not near-regular, but all 3-connected minors of M are near-regular. Then M is a dyadic matroid, and M is uniquely representable over $\mathbf{Q}$.*

The method of proof is as follows. We begin by showing that the theorem holds if $r(M) = 3$. If $r(M) \geq 4$ we proceed by first constructing a matrix $[A \mid \mathbf{x}]$ over $\mathbf{Q}(\alpha)$ that purports to represent M. We then show that there exists a unique evaluation of the entries of this matrix having the property that the resulting matrix over $\mathbf{Q}$ is in fact a representation of M. It is then shown that this matrix is dyadic.

Proof of Theorem 6.6. Clearly M is non-binary and $r(M) > 2$. Assume that M has rank 3. It is a straightforward exercise to check that the only minor-minimal rank-3 matroid that is both $GF(3)$- and $\mathbf{Q}$-representable and is not near-unimodular is the non-Fano matroid F_7^-. It follows from the proof of (5.9.2) that the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

uniquely represents F_7^- over $\mathbf{Q}$. One routinely checks that A is a dyadic matrix. This fact is also essentially well known.

Assume that M has rank r, where $r > 3$. If $r(M^*) = 3$, then by the above, M^* is a dyadic matroid. It then follows from Proposition 6.2 that M is also dyadic. Therefore we may assume that both M and M^* have rank at least 4. Since whirls are near-regular, M is not a whirl. By (2.5) there exists an element x in the ground set E of M with the property that either $M \backslash x$ or M/x is 3-connected and non-binary. It is routinely seen that under the current assumptions no generality is lost in assuming that $M \backslash x$ is 3-connected and non-binary. By Corollary 3.8, there exists an independent triple $\{a, b, c\}$ of distinct elements of $E - \{x\}$ with the property that $\mathrm{si}(M \backslash x/a)$, $\mathrm{si}(M \backslash x/b)$, $\mathrm{si}(M \backslash x/c)$, $\mathrm{si}(M \backslash x/a, b)$, and $\mathrm{si}(M \backslash x/a, c)$ are all non-binary and 3-connected. If $\{x, a, c\}$ is collinear, then clearly $\{x, a, b\}$ is not collinear. Assume without loss of generality that $\{x, a, b\}$ is not collinear.

We now focus on a particular representation of $M\backslash x$. Since $M\backslash x$ is a 3-connected minor of M, it is near-regular and therefore has a near-unimodular representation. Since one can scale and pivot on a near-unimodular matrix it follows that $M\backslash x$ can be represented by a near-unimodular matrix $[I \mid A]$ where the last two columns of I represent a and b respectively. Following standard practice we say that $M\backslash x$ is represented by A, the identity matrix being implicit. Let A_a, A_b, and A_{ab} denote the matrices obtained by deleting the second-last, the last, and the last two rows of A respectively. Under the current convention, A_a, A_b and A_{ab} represent $M\backslash x/a$, $M\backslash x/b$, and $M\backslash x/a, b$ respectively. Say $s \in \{a, b, \{a, b\}\}$. Certainly x is not a loop of M/s. We now show that a near-unimodular representation of $M\backslash x/s$ extends uniquely to a near-unimodular representation of M/s where the vector representing x is chosen to have leading non-zero coefficient 1. If x is in a non-trivial parallel class of M/s this is clear, so assume that x is not in such a parallel class. Then $\text{si}(M/s)$ is a 3-connected extension of $\text{si}(M\backslash x/s)$. By Theorem 5.11, any near-unimodular representation of $\text{si}(M\backslash x/s)$ does extend to a representation of $\text{si}(M/s)$, and by (2.8) this extension is unique. The fact that a near-unimodular representation of $M\backslash x/s$, extends uniquely to a near-unimodular representation of M/s now follows routinely. It follows that a unique column can be added to each of A_a, A_b, and A_{ab} to obtain representations of M/a, M/b and $M/a, b$ respectively. Clearly the first $r-2$ entries of these column vectors agree.

Let $\mathbf{x} = (x_1, x_2, ..., x_r)$ be defined as follows: $(x_1, x_2, ..., x_{r-2})$ is the vector that can be added to A_{ab} to represent $M/a, b$, while x_{r-1} and x_r are the last entries of the vectors that can be added to A_b and A_a to represent M/b and M/a respectively. Let M' be the matroid on $E(M)$ that is represented by the matrix $[A \mid \mathbf{x}]$, where, of course, $\mathbf{x}$ represents x. It now follows that M' is a $\mathbf{Q}(\alpha)$-representable matroid on $E(M)$ with the property that $M'\backslash x = M\backslash x$, $M'/a = M/a$ and $M'/b = M/b$. (Note that this conclusion would not hold if x was a loop of $M/a, b$, that is, if $\{x, a, b\}$ was collinear in M.) Certainly $M \neq M'$, for otherwise, by Theorem 5.9, M would be near-regular. We now show that, for some $q \in \mathbf{Q}$, the matrix obtained by evaluating each entry of $[A \mid \mathbf{x}]$ at q represents M. We first prove a lemma.

(6.7) LEMMA. *Let N be a 3-connected, near-regular matroid that is represented by the near-unimodular matrix $[I \mid B]$. If the matrix $[I \mid C]$ over $\mathbf{Q}$ also represents N, then there exists $q \in \mathbf{Q}$ with the property that $[I \mid C]$ is equivalent to a matrix obtained by evaluating the entries of $[I \mid B]$ at q.*

Proof. If N is binary, then the result certainly holds, so assume that N is not binary. Assume that N is a whirl. Once more we note that after appropriate scaling, pivoting and column permutations we can assume that

$$B = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ 1 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & (-1)^r u \end{bmatrix},$$

where $u \in \{\alpha, -(\alpha-1), \alpha/(\alpha-1), -1/(\alpha-1), 1/\alpha, (\alpha-1)/\alpha\}$, and that

$$C = \begin{bmatrix} 1 & 0 & & 0 & 1 \\ 1 & 1 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 1 & (-1)^r p \end{bmatrix},$$

for some $p \in \mathbf{Q} - \{0, 1\}$. But, regarded as functions, each of the possible values of u is a permutation of $\mathbf{Q} - \{0, 1\}$. It follows that in each case there exist $q \in \mathbf{Q}$ with the property that $u(q) = p$.

We may therefore assume that N is not a whirl. Say N has a largest whirl minor $\mathcal{W}^r$. Then by (2.4), there exists a, sequence $\mathcal{W}^r \cong N_0, N_1, ..., N_k = N$ of 3-connected matroids with the property that for $1 \leqslant i \leqslant k$, N_i is a single-element extension or coextension of its predecessor. We can assume that appropriate pivots have been performed so that both B and C have corresponding submatrices B' and C' that represent $\mathcal{W}^r$. We have shown that there exists $q \in \mathbf{Q}$ with the property that C' is equivalent to the matrix obtained from B' by evaluating its entries at q. The extensions and coextensions used to build N from $\mathcal{W}^r$ correspond to building B and C from B' and C' by adding columns and rows respectively. But by (2.8) these columns and rows are unique up to scalar multiples. It follows routinely that C is equivalent to the matrix obtained from B by evaluating each of its entries at q. ▌

It follows from Lemma 6.7 that every $\mathbf{Q}$-representation of $M\backslash x$ is obtained from A by evaluating its entries at an appropriate rational number. But some $\mathbf{Q}$-representation of $M\backslash x$ extends to a $\mathbf{Q}$-representation of M. Therefore there exists a rational number q with the property that the representation $A(q)$ of $M\backslash x$ obtained by evaluating the entries of A at q extends to a representation of M. Say $[A(q) \mid \mathbf{q}]$ represents M where $\mathbf{q}=(q_1, q_2, ..., q_r)$ and the leading non-zero coefficient of $\mathbf{q}$ is 1. Evidently

$$(q_1, q_2, ..., q_{r-1}) = (x_1(q), x_2(q), ..., x_{r-1}(q)),$$

and

$$(q_1, q_2, ..., q_{r-2}, q_r) = (x_1(q), x_2(q), ..., x_{r-2}(q), x_r(q)).$$

It follows that $\mathbf{q}=x(q)$. In other words, the matrix $[A \mid \mathbf{x}](q)$ obtained by evaluating each entry of $[A \mid \mathbf{x}]$ at q represents M.

If q is not a zero of any non-zero subdeterminant of $[A \mid \mathbf{x}]$, then it is easily seen that $M[[A \mid \mathbf{x}](q)] = M[A \mid \mathbf{x}]$. It follows that there exists a submatrix D of $[A \mid \mathbf{x}]$ with the property that $|D|$ is non-zero over $\mathbf{Q}(\alpha)$, and $|D|$ has q as a factor. Certainly D meets the rows indexed by a and b, so D is at least 2×2. Say D is $n\times n$ where $n>2$. Consider the entries of D that are in neither of the rows indexed by a or b, nor in the column $\mathbf{x}$. If these entries are all zero, then it follows from elementary facts on matrices that D is 3×3, and, up to a factor of $\pm\alpha^i(\alpha-1)^j$, $|D|$ is equal to $|D'|$ for some 2×2 submatrix D' of D. Assume that at least one entry is non-zero. It is easily seen that the matrix obtained by pivoting on this entry has all the desired features of $[A \mid \mathbf{x}]$. Moreover it follows, again by elementary matrix theory, that this matrix has an $(n-1)\times(n-1)$ submatrix D' with the property that $|D'| = \pm\alpha^i(\alpha-1)^j\,|D|$ for some integers i and j. A consequence of this discussion is that we may assume without loss of generality that D is 2×2.

Consider the entries of D. If an entry of D is in the row indexed by a, then that entry is an entry of the near-unimodular matrix that represents M/b. If it is not in that row, then it is an entry of the near-unimodular matrix that represents M/a. It follows that each entry is of the form $\pm\alpha^i(\alpha-1)^j$. This imposes constraints on the possibilities for $|D|$, and hence q. We now show

(6.6.1) q is the only factor of $|D|$ in $\mathbf{Q}-\{0,1\}$; moreover $q\in\{-1, 1/2, 2\}$.

Proof. It is clear that we can properly scale D to ensure that all entries of D are polynomials. Then the only factor of $|D|$ that is not a power of α or $\alpha-1$ is one that has either the form

$$(\alpha-1)^i \pm \alpha^j$$

or the form

$$(\alpha-1)^i\alpha^j\pm 1$$

for some non-negative integers i and j. We examine the possible rational roots of these polynomials. Assume that $q=m/n$ is a root where m and n are relatively prime.

Certainly $(i,j)\neq(0,0)$. Assume that $j=0$. Then we have either $q^i+1=0$ or $q^i-1=0$. A routine check shows that in cases where either of these polynomials has a root in $\mathbf{Q}-\{0,1\}$, then that root is unique. It is also readily checked that for some cases $q=2$ and for the remainder $q=-1$. The case $i=0$ is identical. Therefore we may assume that $i,j\geqslant 1$. We examine the two polynomials in turn. If q is a root of the first we have

$$(q-1)^i\pm q^j=\left(\frac{m-n}{n}\right)^i\pm\left(\frac{m}{n}\right)^j=0.$$

If $i=j$, then $q=1/2$ is the unique root in $\mathbf{Q}-\{0,1\}$. Assume that $i\neq j$. Then $(\alpha-1)^i\pm\alpha^j$ is, up to sign, monic. It is well known (see for example [1, Proposition V.3.8]) that any rational root of such a polynomial must be an integer, that is, we may assume that $n=1$. We then have

$$(m-1)^i\pm m^j=0.$$

An obvious parity argument shows that this case does not occur.

A similar argument shows that for $i,j>1$, the polynomial $(\alpha-1)^i\alpha^j\pm 1$ has no rational roots and the lemma is proved. ∎

By (6.6.1), if p is any rational number in $\mathbf{Q}-\{0,1,q\}$ then the matrix obtained by evaluating the entries of $[A\mid \mathbf{x}]$ at p does not represent M, for the submatrix corresponding to D has a non-zero determinant. But, by Lemma 6.7, every representation of M over $\mathbf{Q}$ can be obtained, up to equivalence, by such an evaluation. We conclude that M is uniquely representable over $\mathbf{Q}$. It remains to show that $[A\mid \mathbf{x}](q)$ is a dyadic matrix.

If $q\in\{-1,1/2,2\}$, then any rational number of the form $q^i(q-1)^j$ is certainly a signed integral power of 2. It follows that every entry of $[A\mid \mathbf{x}](q)$ is a signed integral power of q. To check that $[A\mid \mathbf{x}](q)$ is a dyadic matrix we need only show that every subdeterminant is a signed power of 2. Assume not. By scaling if necessary we may assume that every non-zero entry of $[A\mid \mathbf{x}](q)$ is a signed non-negative power of 2. Since we are assuming that $[A\mid \mathbf{x}](q)$ is not a dyadic matrix, there exists a subdeterminant that has an odd prime p as a factor. Consider the matrix $A(GF(p))$ obtained by interpreting the entries of A as elements of $GF(p)$. By arguments that must by now be very familiar we deduce that $M[A(GF(p)]$

is a connected, ternary, non-binary matroid that is a proper, rank-preserving, weak-map image of the 3-connected, ternary matroid M. By (4.1) this cannot occur. It follows that $[A \mid \mathbf{x}](q)$ is indeed a dyadic matrix, so that M is a dyadic matroid and the theorem is proved. ▌

(6.8) COROLLARY. *Let M be a 3-connected, ternary matroid that is not near-regular. If M is representable over $\mathbf{Q}$, then M is uniquely representable over $\mathbf{Q}$.*

Proof. The proof is by induction on the number of 3-connected minors that are not near-regular. If all 3-connected minors of M are near-regular, then the corollary follows by Theorem 6.7. Hence we may assume that M has at least one 3-connected minor that is not near-regular. By the obvious induction assumption we may also assume that all 3-connected minors of M that are not near-regular are uniquely representable over $\mathbf{Q}$. By (2.4), there exists an element x in $E(M)$ with the property that $M \backslash x$ or M/x is 3-connected and is not near-regular. Assume without loss of generality that $M \backslash x$ is 3-connected and is not near-regular. Then, $M \backslash x$ has a unique $\mathbf{Q}$-representation. Evidently this $\mathbf{Q}$-representation of $M \backslash x$ extends to a representation of M. But by (2.8) a representation of $M \backslash x$ that extends to a representation of M does so uniquely. We conclude that M is uniquely representable over $\mathbf{Q}$. ▌

7. MAIN RESULTS

At last we are able to prove

(7.1) THEOREM. *A matroid M is representable over both $GF(3)$ and $\mathbf{Q}$ if and only if it is a dyadic matroid.*

Proof. If M is a dyadic matroid, then it follows from Proposition 6.4 that M is representable over $GF(3)$ and $\mathbf{Q}$. Consider the converse.

Assume that M is 3-connected. If M is binary or a whirl, the result certainly holds, so assume that M is neither binary nor a whirl. Assume, for induction, that all proper 3-connected minors of M are dyadic matroids. Consider M. If every 3-connected minor of M is near-regular then M is dyadic by Theorem 6.6. Otherwise M has a 3-connected minor that is not near-regular. Arguing as in Corollary 6.8, we can now assume without loss of generality that there exists an element $x \in E(M)$ with the property that $M \backslash x$ is 3-connected and not near-regular. By Corollary 6.8, M is uniquely representable over $\mathbf{Q}$. Say the matrix $[I \mid A \mid \mathbf{x}]$ represents M, where $\mathbf{x}$ represents x. It follows from the induction assumption that we may assume (after pivoting and scaling $[I \mid A \mid \mathbf{x}]$ if necessary) that A is a dyadic matrix. We may further assume (after a sequence of proper scalings if necessary) that all entries of A are integers. We may finally assume (in this case by appropriately scaling $\mathbf{x}$ if necessary) that the entries of $\mathbf{x}$ are integers and that the greatest common divisor of the non-zero entries of $\mathbf{x}$ is 1.

We now invoke the standard weak-map argument. Assume that $[I \mid A \mid \mathbf{x}]$ is not a dyadic matrix. Then it has a subdeterminant that is divisible by an odd prime p. Let $P = [I \mid A \mid \mathbf{x}]_p$, that is, the matrix over $GF(p)$ whose entries are the entries of $[I \mid A \mid \mathbf{x}]$ treated as integers modulo p. Certainly $M[P]$ is a weak-map image of M. But, arguing as in Proposition 6.4, one deduces that $M[P]\backslash x = M\backslash x$. Therefore $M[P]$ is a non-binary, rank-preserving, weak-map image of M. But at least one entry of $\mathbf{x}$ is not divisible by p. Hence x is not a loop of $M[P]$ and it follows that $M[P]$ is connected. It is also clear that $M[P]$ is ternary. It now follows by (4.1) that $M[P] = M$. Since $[I \mid A \mid \mathbf{x}]$ has a subdeterminant divisible by p, there exists a submatrix of $[I \mid A \mid \mathbf{x}]$ with a non-zero determinant and the property that the determinant of the corresponding submatrix of P is zero. It follows by (4.3) that $M \neq M[P]$. This contradiction establishes that $[A \mid \mathbf{x}]$ is indeed a dyadic matrix and we conclude that M is a dyadic matroid.

Assume that M is not 3-connected. Then M can be obtained by taking 2-sums and direct sums of 3-connected matroids that are representable over $GF(3)$ and $\mathbf{Q}$, that is by taking 2-sums and direct sums of dyadic matroids. It follows by Proposition 6.3 that M is a dyadic matroid. ∎

We are now also in a position to characterise when a 3-connected matroid uniquely representable over $\mathbf{Q}$.

(7.2) THEOREM. *Let M be a 3-connected matroid that is ternary, representable over $\mathbf{Q}$, and has rank greater than 2. Then M is not uniquely representable over $\mathbf{Q}$ if and only if it is a non-binary near-reqular matroid.*

Proof. If M is not near-regular, then M is uniquely representable over $\mathbf{Q}$ by Corollary 6.8. If M is binary, then M is certainly uniquely representable over $\mathbf{Q}$. Assume that M is non-binary and near-regular. Let A be a near-unimodular matrix that represents M. Assume that A is scaled so that its entries are polynomials and the greatest common divisor of any row or column has degree zero. It is easily seen that in this case, A represents a non-binary matroid if and only if at least one entry of A has degree greater that 0. By Lemma 5.6, if $\{q_1, q_2\} \subset \mathbf{Q} - \{0, 1\}$, then $M[A(q_1)] = M[A(q_2)] = M$. A routine argument now shows that if $q_1 \neq q_2$, then $A(q_1)$ and $A(q_2)$ are inequivalent representations of M. ∎

The reason for insisting that M have rank at least three in Theorem 7.2 is to exclude $U_{2,4}$. This matroid is non-binary and near-regular and is often regarded as being uniquely representable over **Q** since, regarded as a matroid, the automorphism group of $PG(1, \mathbf{Q})$ is the symmetric group. It is not clear to me that this is the correct notion of equivalence. Indeed there certainly exist representations of $U_{2,4}$ in $PG(2, \mathbf{Q})$ with the property that no automorphism of this projective space takes one to the other. In what follows we adopt the convention that $U_{2,4}$ is not uniquely **Q**-representable.

One routinely checks that a 2-sum or a direct sum of **Q**-representable matroids is uniquely **Q**-representable if and only if both the summands are. The following corollary follows straightforwardly from this fact and Theorem 7.2.

(7.3) COROLLARY. *Let M be a connected ternary matroid representable over* **Q**. *Then M is uniquely representable over* **Q** *if and only if whenever M is decomposed as a 2-sum of 3-connected matroids, none of the summands is a non-binary, near-regular matroid.*

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