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Characteristic Polynomials of Weighted Lattices

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1. INTRODUCTION

One of the most interesting and useful matroid invariants is the characteristic, or chromatic, polynomial. (For a good survey of its applications see [5].) Now characteristic polynomials can be defined for more general structures. For example, the definition is easily extended to graded lattices and they can also be defined for polymatroids. Moreover, much of the combinatorial significance of characteristic polynomials is retained in some of these more general settings. For example, the characteristic polynomial of a polynmatroid representable over a finite field can be used to evaluate its critical exponent [21], generalizing results of Crapo and Rota [8]. Also, characteristic polynomials of certain polymatroids derived from hypergraphs can be used to enumerate hypergraph colourings in the same way that characteristic polynomials of graphic matroids can be used to enumerate graph colourings [11, 20]. This suggests that it is worth while to study characteristic polynomials in a more general setting.

In this paper we define characteristic polynomials for weighted lattices, that is, for lattices endowed with order respecting functions into the integers. The umbrella of weighted lattices includes all the structures considered above—matroids, geometric lattices, polymatroids, and graded lattices. The general plan of the paper is, in each section, to develop some theory for weighted lattices, and then to use the theory to obtain a non-trivial result for characteristic polynomials. In particular, the paper is structured as follows.

In Section 3, the concept of a respectful closure operator on a weighted lattice is introduced—essentially these are closure operators which respect function values. It is shown that, excluding weighted lattices with trivial

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characteristic polynomials, the characteristic polynomial of a weighted lattice is equal to that of its quotient relative to a respectful closure operator. Section 4 considers minors of weighted lattices. A deletion-contraction recursion for characteristic polynomials of weighted lattices is given. In Section 5, a notion of modularity for weighted lattices is introduced. The main theorem of this section, and possibly of the whole paper, is an extension of Stanley's modular factorisation theorem [14, Theorem 2] to weighted lattices. Sections 6 and 7 develop further structure theory for weighted lattices endowed with modular elements. Section 8 considers weighted lattice separation.

2. Preliminaries

We assume that the reader is familiar with the basic concepts of matroid theory as set forth in [17–19]. Familiarity is also assumed with the elements of lattice theory, and with the theory of Möbius functions of partially ordered sets. Attractive presentations of relevant order-theoretic concepts are given in Stanley [16, Chap. 3] and Aigner [1, particularly Chap. 3, Sect. 4]. Terminology for partially ordered sets and lattices follows [16]. In any unexplained context the symbols $\hat{0}$ and $\hat{1}$ are used for the least and greatest elements of a lattice, respectively.

Let S be a set, then an integer valued set function *on* S is a function $f: 2^S \to Z$, and we say that S is the *ground set* of f. In this paper we only consider set functions on finite sets. The function f is *normalised* if $f(\emptyset)=0$, is *increasing* if $f(A) \geqslant f(B)$ whenever A and B are subsets of S with $A \supseteq B$, and is *submodular* if $f(A)+f(B) \geqslant f(A \cup B)+f(A \cap B)$ for all subsets A and B of S. If f is a normalised, increasing, submodular function, then f is a *polymatroid on* S.

With a normalised, increasing function f on S, we associate a canonical closure operator. A subset F of S is a *flat* of f if $f(F \cup a) > f(F)$ for all a in $S-F$. A subset is a *closed set* of f if it is an intersection of flats. If A is a subset of S, then $\mathrm{cl}_f(A)$ is the minimal closed set of f containing A. Clearly cl_f is a closure operator, the *closure operator* of f. Just as clearly, the closed sets of f form a lattice, the *lattice of closed sets* of f. A particularly well-behaved class of functions consists of those functions whose flats and closed sets coincide. Note that polymatroids belong to this class, so that one may refer to the *lattice of flats* of a polymatroid.

A *weighted lattice* is an ordered pair (L, f), where L is a finite lattice and f is a function from L into the non-negative integers with the properties that $f(\hat{0})=0$ and that, if a and b are members of L with $a \leqslant b$, then $f(a) \leqslant f(b)$. If (L, f) is a weighted lattice, then L and f are the *underlying lattice* and *underlying function* of (L, f), respectively. To simplify notation we generally use the symbol L_f to denote a weighted lattice. In this case L_f also denotes the underlying unweighted lattice of L_f: the underlying function of L_f is f.

With a normalised, increasing function f on S we naturally associate two weighted lattices. The first has, as its underlying lattice, the lattice of all subsets of S ordered by inclusion and has f as its underlying function. This is the *Boolean weighted lattice* of f, and is denoted by B_f. The other has, as its underlying lattice, the lattice of closed sets of f and has, as its underlying function, the restriction of f to these sets. This is the *weighted lattice of closed sets* of f, and is denoted by L_f. In the case that f is a polymatroid we may refer to the *weighted lattice of flats* of f.

To appreciate the unifying role played by weighted lattices it is worth observing that if M is a matroid with rank function r, then both M and its associated geometric lattice have natural interpretations as weighted lattices, viz. B_r and L_r.

In a sense weighted lattices generalize graded lattices, but of course a graded lattice may be weighted by a function which bears no relationship to its rank function. On the other hand, nice examples of weighted lattices are provided by graded lattices with their rank functions, and in particular, by geometric lattices with their rank functions. In any unexplained context a geometric lattice will always be assumed to be weighted by its rank function.

The weighted lattice L_f is *submodular* if, for all elements a and b of L_f, $f(a)+f(b) \geqslant f(a \wedge b)+f(a \vee b)$. A graded lattice is *upper semimodular* if it is submodular when weighted by its rank function. Note that the underlying lattice of a submodular weighted lattice may not be upper semimodular, and a weighted lattice whose underlying lattice is upper semimodular may not be submodular.

The *characteristic polynomial* of the weighted lattice L_f, denoted $P(L_f; \lambda)$, is defined by

$$P(L_f; \lambda) = \sum_{x \in L_f} \mu(\hat{0}, x)\, \lambda^{f(\hat{1})-f(x)}.$$

Here μ represents the Möbius function of the underlying lattice of L_f. The *characteristic polynomial* of the normalised, increasing function f on S is equal to that of its Boolean weighted lattice, that is, $P(f; \lambda) = P(B_f; \lambda)$. It is then easily seen that

$$P(f; \lambda) = \sum_{A \subseteq S} (-1)^{|A|}\, \lambda^{f(S)-f(A)}.$$

We often use Hasse diagrams to depict weighted lattices. Vertices of such diagrams are labelled by ordered pairs of the form $(a, f(a))$, where a denotes the element of the lattice corresponding to the given vertex.

3. Respectful Closure Operators

Recall that a *closure operator* on a lattice L is a function $\mathrm{cl}: L \to L$ with the following properties:

(i) $\mathrm{cl}(x) \geqslant x$, for all x in L;

(ii) if x and y are in L and $x \leqslant y$, then $\mathrm{cl}(x) \leqslant \mathrm{cl}(y)$;

(iii) $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$, for all x in L.

An element x of L is *closed* if $\mathrm{cl}(x) = x$. It is well known, and easily verified, that the closed elements of L with the induced order form a lattice: this is the *quotient* of L relative to the closure operator cl.

Now let L_f be a weighted lattice. A *respectful* closure operator on L_f is a closure operator cl on the underlying lattice of L_f with the property that $f(x) = f(\mathrm{cl}(x))$ for every element x of L_f.

We generalize the definition of flat to weighted lattices by saying that an element x of L_f is a *flat* if $f(y) > f(x)$ for all elements y of L_f with $y > x$. Equivalently x is a flat if $f(y) > f(x)$ for all elements y of L_f which cover x. An alternative characterisation of respectful closure operators is then given by

Proposition 3.1. *Let* cl *be a closure operator on the weighted lattice* L_f. *Then* cl *is respectful if and only if every flat of* L_f *is closed.*

Proof. It is clear that if cl is respectful, then every flat of L_f is closed. For the converse, assume that every flat of L_f is closed, and let x be an element of L_f. It is easily seen that there exists at least one flat z of L_f with $z \geqslant x$ and $f(z) = f(x)$. Since cl is a closure operator, $\mathrm{cl}(z) \geqslant \mathrm{cl}(x)$. But $\mathrm{cl}(z) = z$ so $z \geqslant \mathrm{cl}(x)$. Therefore $f(z) \geqslant f(\mathrm{cl}(x))$, and the result follows since $\mathrm{cl}(x) \geqslant x$. ∎

Assume that cl is a respectful closure operator on L_f. By restricting f to the closed elements of L_f, the quotient of L_f relative to cl becomes a weighted lattice: this is the *weighted quotient* of L_f relative to cl. (Of course, this construction can be performed for any closure operator on L_f, but it is only respectful ones that are of interest to us.)

A canonical respectful closure operator is obtained by setting $\mathrm{cl}(x)$ to be $\bigwedge \{ y : y \text{ a flat of } L_f \text{ and } y \geqslant x \}$ for all elements x of L_f. It is easily checked that cl is, indeed, a respectful closure operator, and that if cl_1 is any other respectful closure operator on L_f, then cl_1 is a refinement of cl in the sense that the elements which are closed with respect to cl_1 form a superset of the closed elements of cl. This closure operator is the one of most interest to us; we call it the *principal closure operator* of L_f.

Now let f be a normalised, increasing function on a finite set. Then it is immediate that the weighted quotient of B_f (the Boolean weighted lattice of f) relative to the principal closure operator of B_f is precisely L_f (the weighted lattice of closed sets of f). To reinforce the point—perhaps unnecessarily—observe that if r is the rank function of a matroid M, then the principal closure operator of B_r is just the usual closure operator of M. The weighted quotient of B_r relative to this closure operator is just the geometric lattice one associates with M weighted by its rank function.

For the remainder of this section we assume that L_f is a weighted lattice, that cl is a respectful closure operator on L_f, and that L'_f is the weighted quotient of L_f relative to cl. The value of taking a weighted quotient of a weighted lattice is that one frequently obtains a structure which is simpler than the original weighted lattice, yet which retains sufficient information to solve problems associated with it. This claim is justified by

THEOREM 3.2. *If* $\mathrm{cl}(\hat{0}) = \hat{0}$, *then* $P(L_f, \lambda) = P(L'_f; \lambda)$.

Theorem 3.2 will follow straightforwardly from the following fundamental result of Rota [13]. The version cited here follows [7, Theorem 1] (see also [1, Theorem 4.2.7]).

PROPOSITION 3.3. *Let P be a locally finite poset, let* cl *be a closure operator on P, and let $\bar{P}$ denote the quotient of P relative to* cl. *Then, for all elements x and y of P,*

$$\sum_{z \in P,\ \mathrm{cl}(z) = \mathrm{cl}(y)} \mu_P(x, z) = \begin{cases} \mu_{\bar{P}}(\mathrm{cl}(x), \mathrm{cl}(y)), & \text{if } x = \mathrm{cl}(x) \\ 0, & \text{if } x < \mathrm{cl}(x). \end{cases}$$

Proof of Theorem 3.2. Let μ and μ' denote the Möbius functions of L_f and L'_f, respectively. Then,

$$\begin{aligned} P(L_f; \lambda) &= \sum_{x \in L_f} \mu(\hat{0}, x)\, \lambda^{f(\hat{1}) - f(x)} \\ &= \sum_{x \in L'_f} \left(\sum_{y \in L_f:\, \mathrm{cl}(y) = x} \mu(\hat{0}, y)\, \lambda^{f(\hat{1}) - f(y)} \right). \end{aligned}$$

By Proposition 3.1, if $\mathrm{cl}(y) = x$, then $f(y) = f(x)$. Hence

$$P(L_f; \lambda) = \sum_{x \in L'_f} \left(\sum_{y \in L_f:\, \mathrm{cl}(y) = x} \mu(\hat{0}, y) \right) \lambda^{f(\hat{1}) - f(x)}.$$

Since $\mathrm{cl}(\hat{0}) = \hat{0}$, it follows from Proposition 3.3 that $\sum_{y \in L_f:\, \mathrm{cl}(y) = x} \mu(\hat{0}, y) = \mu'(\hat{0}, x)$. Hence

$$P(L_f; \lambda) = \sum_{x \in L'_f} \mu'(\hat{0}, x)\, \lambda^{f(\hat{1}) - f(x)} = P(L'_f; \lambda). \quad \blacksquare$$

A condition of Theorem 3.2 is that $\mathrm{cl}(\hat{0}) = \hat{0}$. Consider the case where this does not hold. An atom e of the weighted lattice L_f is a *loop* of L_f if $e \leqslant \mathrm{cl}(\hat{0})$ where cl denotes the principal closure operator of L_f. Equivalently, e is a loop of L_f, if, for every flat x of L_f, $e \leqslant x$. Clearly, $f(e) = 0$ if e is a loop, but the converse is not, in general, true. The following result follows routinely from Proposition 3.3.

PROPOSITION 3.4. *If L_f has a loop, then $P(L_f; \lambda) = 0$.*

4. MINORS OF WEIGHTED LATTICES

The proof of the following preliminary lemma is routine and is omitted.

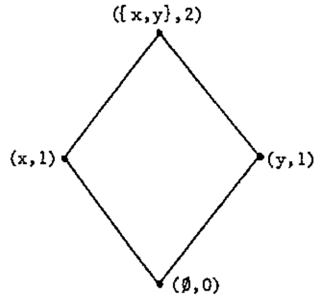
LEMMA 4.1. *Let L be a lattice, and let x be an element of L. Then the set $P = \{y : y \in L, y \nleqslant x\} \cup \hat{0}$, with the induced order, is a lattice.*

Let L_f be a weighted lattice and let x be an element of L_f. The *contraction of L_f by x*, denoted L_f/x, is the weighted lattice whose underlying lattice is the interval $[x, \hat{1}]$ of L_f and whose underlying function, denoted f/x, is defined, for all y in $[x, \hat{1}]$, by $f/x(y) = f(y) - f(x)$. The *deletion of L_f by x*, denoted $L_f\backslash x$, is the weighted lattice whose underlying lattice is the set $P = \{y : y \in L, y \nleqslant x\} \cup \hat{0}$ with the induced order and whose underlying function is the restriction of f to P. It follows from Lemma 4.1 that the deletion of f by x is well-defined.

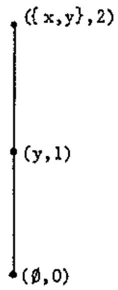
Let f be a normalised, increasing function on S, and let A be a subset of S. Then it is natural to define the *deletion* of A from f, denoted $f\backslash A$, by restricting f to the power set of $S - A$; that is, $f\backslash A(B) = f(B)$ for all subsets B of $S - A$. The *contraction* of A from f, denoted f/A, is the function on $S - A$ defined, for all subsets B of $S - A$, by $f/A(B) = f(A \cup B) - f(A)$. In the case that f is the rank function of a matroid, these operations reduce to the usual matroid-theoretic ones.

To throw some light on the operations of deletion and contraction for weighted lattices it is worth while comparing them with the synonomous operations on normalised, increasing functions. Assume that L_f is the weighted lattice of closed sets of the normalised, increasing function f on S, and let A be a closed set of f. It is easily seen that L_f/A is the weighted lattice of closed sets of f/A; that is, $L_f/A = L_{f/A}$. Of course, for matroids this is very well known. These facts show that it is reasonable to denote the underlying function of the weighted lattice L_f/x by f/x.

No such correspondence exists, in general, for deletion. For example, let r be the rank function of the free matroid M on the set $\{x, y\}$. Then Fig. 1(a) illustrates both B_r and the weighted lattice of closed sets of M (in



(a) B_r



(b) $B_r\backslash x$

FIGURE 1

this case the two are identical) and Fig. 1(b) illustrates $B_r\backslash x$. Now $B_r\backslash x$ is not a weighted lattice which can be associated in any reasonable way with $M\backslash x$, or indeed, with any matroid.

Nonethless the terminology is not unreasonable. Let f be the function on $\{x, y, z\}$ defined by $f(\varnothing)=0$, $f(x)=f(y)=1$, and $f(A)=2$ for all other subsets of $\{x, y, z\}$. It is easily checked that the weighted lattices of flats of f and $f\backslash x$ are isomorphic to B_r and $B_r\backslash x$, respectively. Informally, we can say that whether or not deletion in the weighted lattice corresponds to deletion in the function depends on the choice of function. A more formal description of the situation follows.

Let L_f be a strictly weighted lattice (that is, $f(x)<f(y)$ for all x and y in L_f with $x<y$). Let g be the function whose ground set is the set of elements of L_f defined, for all subsets A of L_f, by $g(A)=f(\bigvee(A))$. It is not hard to see that the weighted lattice of closed sets of g, denoted L_g, is isomorphic to L_f. The reasonably routine verification of the following proposition is left to the reader.

PROPOSITION 4.2. *If A is an element of L_g (that is, A is a closed set of g) then $L_g\backslash A$ is isomorphic to the weighted lattice of closed sets of $g\backslash A$.*

As a somewhat concrete illustration consider the content of Proposition 4.2 for matroids. Let M be a matroid with rank function r. Let f be the function whose ground set is the set $\mathscr{F}$ of flats of M, and which is defined, for all subsets A of $\mathscr{F}$, by $f(A)=r(\bigcup\{F: F\in A\})$. Then f is a *most* natural submodular function to associate with a matroid. Now L_f, the weighted lattice of closed sets of f, is isomorphic to the weighted lattice of flats of M. Proposition 4.2 then says that deletion in L_f corresponds to deletion by flats in f. The following theorem justifies the existence of this section.

THEOREM 4.3. *Let L_f be a weighted lattice with at least three elements, and let x be an atom of L_f. Then,*

$$P(L_f; \lambda) = P(L_f \backslash x; \lambda) - P(L_f/x; \lambda).$$

We first prove a lemma. Let μ and $\mu \backslash x$ denote the Möbius functions of L_f and $L_f \backslash x$, respectively.

LEMMA 4.4. *Let x be an atom of L_f, and let y be an element of $L_f \backslash x$.*

(i) *If $y \not\geqslant x$, then $\mu(\hat{0}, y) = \mu \backslash x(\hat{0}, y)$.*

(ii) *If $y > x$, then $\mu(\hat{0}, y) = \mu \backslash x(\hat{0}, y) - \mu(x, y)$.*

Proof. Part (i) is clear. Consider (ii). First assume that y covers x. Then

$$\mu \backslash x(\hat{0}, y) = - \sum_{\hat{0} \leqslant z < y; z \neq x} \mu \backslash x(\hat{0}, z).$$

But, since y covers x, if a is in $\{z : \hat{0} \leqslant z < y, z \neq x\}$, then $a \not\geqslant x$ and, by (i), $\mu \backslash x(\hat{0}, a) = \mu(\hat{0}, a)$. Therefore

$$\mu \backslash x(\hat{0}, y) = - \sum_{\hat{0} \leqslant z < y; z \neq x} \mu(\hat{0}, z).$$

Now

$$\mu(\hat{0}, y) = - \sum_{\hat{0} \leqslant z < y; z \neq x} \mu(\hat{0}, z) - \mu(\hat{0}, x).$$

But x covers $\hat{0}$, and y covers x, so $\mu(\hat{0}, x) = \mu(x, y) = -1$. Therefore $\mu(\hat{0}, y) = \mu \backslash x(\hat{0}, y) - \mu(x, y)$.

Now assume that $y > x$, and that the result holds for all z in L_f such that $x < z < y$. Clearly,

$$\mu(\hat{0}, y) = - \sum_{x < z < y} \mu(\hat{0}, z) - \sum_{\hat{0} \leqslant z < y; z \not\geqslant x} \mu(\hat{0}, z) - \mu(\hat{0}, x).$$

Now $\mu(\hat{0}, x) = -1 = -\mu(x, x)$; so by (i) and the induction assumption, it follows that

$$\begin{aligned} \mu(\hat{0}, y) &= - \sum_{x < z < y} \mu \backslash x(\hat{0}, z) + \sum_{x < z < y} \mu(x, z) - \sum_{0 \leqslant z < y; z \neq x} \mu \backslash x(\hat{0}, z) + \mu(x, x) \\ &= \mu \backslash x(\hat{0}, y) - \mu(x, y). \quad \blacksquare \end{aligned}$$

Proof of Theorem 4.3. Since L_f has at least three elements and x is an atom of L_f, it follows that $x \neq \hat{1}$, so that $\hat{1}$ is the maximum element of $L_f \backslash x$. Therefore,

$$P(L_f \backslash x; \lambda) = \sum_{y \in L_f; y \neq x} \mu \backslash x(\hat{0}, y) \, \lambda^{f(\hat{1}) - f(y)}.$$

Consider L_f/x. We have

$$P(L_f/x;\lambda)=\sum_{y\in L_f;\, y\geqslant x}\mu(x,y)\,\lambda^{f/x(\hat{1})-f/x(y)}.$$

But $f/x(\hat{1})-f/x(y)=f(\hat{1})-f(y)$. Therefore,

$$P(L_f/x;\lambda)=\sum_{y\in L_f;\, y\geqslant x}\mu(x,y)\,\lambda^{f(\hat{1})-f(y)}.$$

Using these facts and Lemma 4.4, it is now easily seen that the following chain of equalities holds:

$$\begin{aligned}
&P(L_f\backslash x,\lambda)-P(L_f/x;\lambda)\\
&\quad=\sum_{y\neq x}\mu\backslash x(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}-\sum_{y\geqslant x}\mu(x,y)\,\lambda^{f(\hat{1})-f(y)}\\
&\quad=\sum_{y\not\geqslant x}\mu\backslash x(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}+\sum_{y>x}\mu\backslash x(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}\\
&\qquad-\sum_{y>x}\mu(x,y)\,\lambda^{f(\hat{1})-f(y)}-\mu(x,x)\,\lambda^{f(\hat{1})-f(x)}\\
&\quad=\sum_{y\not\geqslant x}\mu(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}+\sum_{y>x}\mu(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}+\mu(\hat{0},x)\,\lambda^{f(\hat{1})-f(x)}\\
&\quad=\sum_{y\in L_f}\mu(\hat{0},y)\,\lambda^{f(\hat{1})-f(y)}=P(L_f;\lambda).\quad\blacksquare
\end{aligned}$$

Theorem 4.3 does not hold for 2-element weighted lattices. The characteristic polynomials of these lattices are easily computed directly: if L_f is a 2-element weighted lattice, then $P(L_f;\lambda)=\lambda^{f(\hat{1})}-1$.

Note that Theorem 4.3 can be used to compute the characteristic polynomial of a loopless matroid (or, indeed, any normalised, increasing function) in two ways; either from the Boolean weighted lattice of the matroid or from the weighted lattice of flats of the matroid.

For a concrete example to illustrate the content of Theorem 4.3 consider the weighted lattice L_f illustrated in Fig. 2(a). Figures 2(b) and 2(c) illustrate $L_f\backslash q$ and L_f/q, respectively. It is easily verified by direct computation that $P(L_f;\lambda)=\lambda^6-3\lambda^4+2\lambda$, that $P(L_f\backslash q;\lambda)=\lambda^6-2\lambda^4+1$, and that $P(L_f/q;\lambda)=\lambda^4-2\lambda+1$. Hence, $P(L_f;\lambda)=P(L_f\backslash q;\lambda)-P(L_f/q;\lambda)$.

Consider a further class of examples. Let L_f be a weighted lattice with set of atoms $\{x_1,\ldots,x_n\}$. We define the *lower truncation* of L_f, denoted $d(L_f)$, by $d(L_f)=(\ldots(L_f\backslash x_1)\ldots\backslash x_n)$. It is easily seen that the order of deleting the atoms is irrelevant, so that $d(L_f)$ is well-defined.

A number of interesting weighted lattices arise as lower truncations of geometric lattices. Note that if L_f is a geometric lattice, the atoms of $d(L_f)$ all have weight equal to 2 so that $d(L_f)$ is a 2-polymatroid. We calculate

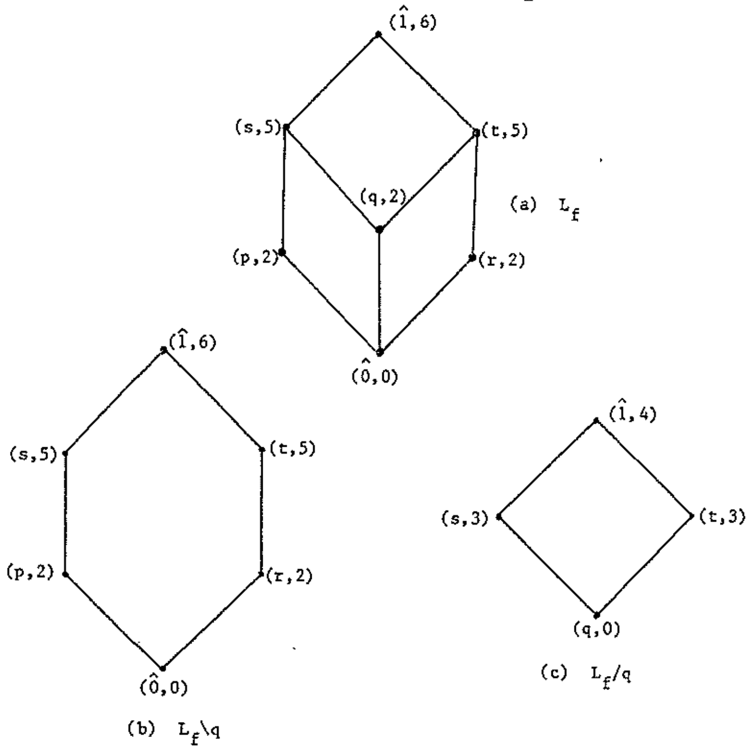


FIGURE 2

the characteristic polynomial of some of these lattices. The technique we use is summarised in the following proposition.

PROPOSITION 4.5. *Let L_f be a weighted lattice with set of atoms $\{x_1, ..., x_n\}$, and assume that $L_f/x_1 \cong L_f/x_2 \cong \cdots \cong L_f/x_n$. Then*

$$P(d(L_f); \lambda) = P(L_f; \lambda) + n(P(L_f/x_1; \lambda)).$$

Proof. The result follows routinely from Theorem 4.3 after observing that if $1 < i \leqslant n$, then

$$(\dots(L_f\backslash x_1)\backslash x_2 \dots\backslash x_{i-1})/x_i = L_f/x_i. \quad \blacksquare$$

Consider the rank-r vector space over the finite field $GF(q)$. Denote the weighted lattice of flats of this vector space by $V(r, q)$. If x is an atom of $V(r, q)$, then $V(r, q)/x \cong V(r-1, q)$. Also, $V(r, q)$ has $(q^r-1)/(q-1)$

atoms, and $P(V(r,q);\lambda)=(\lambda-1)(\lambda-q)\cdots(\lambda-q^r)$. It then follows from Proposition 4.5, that

$$P(d(V(r,q));\lambda)=(\lambda-1)(\lambda-q)\cdots(\lambda-q^{r-1})(\lambda-q^r+(q^r-1)/(q-1)).$$

Let P_r and V_r denote the rank-r partition and Boolean lattice, respectively (weighted, of course, by their rank functions). Then a similar calculation shows that

$$P(d(P_r);\lambda)=(\lambda-1)(\lambda-2)\cdots(\lambda-(r-1))\left(\lambda-r+\binom{r}{2}\right),$$

and that

$$P(d(V_r);\lambda)=(\lambda-1)^{r-1}(\lambda-1+r).$$

Note that $d(V(r,q))$ is naturally associated with the Grassmanian of rank 2 flats of $V(r,q)$. Also, $(d(P_r);\lambda)$ enumerates the number of λ-colourings of the complete 3-hypergraph on $r+1$ vertices.

5. Modular Elements of Weighted Lattices

The ordered pair (x, y) of elements of a lattice is a *modular pair* (written xMy) if, for all $z \leqslant y$, we have $z \vee (x \wedge y)=(z \vee x) \wedge y$. Wilcox [22] showed that the relation of being a modular pair is symmetric if and only if the lattice is upper semimodular. Stanley [15] defines an element x of a lattice L to be *modular* if xMy and yMx for all elements y of L. This is a natural definition in that a lattice is modular if and only if every element of the lattice is modular in the above sense.

It follows immediately from the results of [2, Chap. IV, Sect. 2] that the modular elements of an upper semimodular lattice with height function h are the elements x of L which have the property that for all elements y of L,

$$h(x)+h(y)=h(x \wedge y)+h(x \vee y).$$

Indeed, for such lattices—especially geometric lattices—this property is often given as the definition of a modular element. For arbitrary weighted lattices, no nice correspondence like the above holds: for a useful concept of modularity we need a definition which incorporates both aspects.

Let L_f be a weighted lattice. Then the element x of L_f is a *modular element* of L_f if, for all elements y of L_f, the following properties hold:

(i) yMx and xMy (in other words x is modular in the underlying lattice of L_f)

(ii) $f(x)+f(y)=f(x \vee y)+f(x \wedge y)$.

The main purpose of this section is to prove the following theorem. It is a direct generalization of Stanley [14, Theorem 2].

THEOREM 5.1. *If x is a modular element of the weighted lattice L_f, then*

$$P(L_f; \lambda)=P(L_f \mid x; \lambda)\left[\sum_{a: a \wedge x=\hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1})-f(x)-f(a)}\right].$$

We need some preliminary results; some of these are of sufficient independent interest to be entitled propositions. In the following sequence it will be assumed that x is a modular element of the weighted lattice L_f. Also, if a is an element of L_f, then α_a: $[a \wedge x, x] \to [a, a \vee x]$ is defined by $\alpha_a(y)=a \vee y$, and β_a: $[a, a \vee x] \to [a \wedge x, x]$ is defined by $\beta_a(y)=x \wedge y$.

The *restriction* of L_f to a, denoted $L_f \mid a$, is obtained by restricting L_f to the interval $[0, a]$. If a and b are elements of L_f and $a \leqslant b$, then the underlying lattice of $(L_f/a) \mid b$ is, of course, the interval $[a, b]$ of the underlying lattice of L_f: the underlying function of $(L_f/a) \mid b$ is the restriction of the underlying function of L_f/a to this interval. Consistent with earlier usage in this paper we denote this function by f/a also.

PROPOSITION 5.2. *If a is an element of L_f, then α_a is a weighted lattice isomorphism between $(L_f/(a \wedge x)) \mid x$ and $(L_f/a) \mid (a \vee x)$. The inverse of α_a is β_a.*

Proof. It follows from [14, Lemma 1] that α_a is a lattice isomorphism with inverse β_a. To show that α_a is a weighted lattice isomorphism we must show that if y is in $[a \wedge x, x]$, then $f/a(\alpha_a(y))=f/(a \wedge x)(y)$. Now $f/a(\alpha_a(y)) = f(a \vee y) - f(a)$, and $f/(a \wedge x)(y) = f(y) - f(a \wedge x)$. Therefore we must show that

$$f(a \vee y)-f(a)=f(y)-f(a \wedge x). \tag{1}$$

From the modularity of x, we see that

$$f(x)+f(a)=f(x \vee a)+f(x \wedge a), \tag{2}$$

and that

$$f(x)+f(a \vee y)=f(x \vee a \vee y)+f(x \wedge (a \vee y)).$$

But y is in $[a \wedge x, x]$ so $x \vee a \vee y=x \vee a$. Also, $x \wedge (a \vee y)=\beta_a\alpha_a(y)=y$. Therefore,

$$f(x)+f(a \vee y)=f(x \vee a)+f(y). \tag{3}$$

Equation (1) is now established by subtracting (2) from (3). ∎

PROPOSITION 5.3. *Let a be an element of L_f. Then,*

(i) *$x \vee a$ is modular in L_f/a,*

(ii) *$x \wedge a$ is modular in $L_f \mid a$,*

(iii) *if $x \wedge a = \hat{0}$, then x is modular in $L_f \backslash a$.*

Proof. Consider (i). Assume that y and z are elements of $[a, \hat{1}]$. If $z \leqslant y$, then, using the modularity of x in L_f, we see that

$$\begin{aligned} z \vee ((a \vee x) \wedge y) &= z \vee (a \vee (x \wedge y)) \\ &= (z \vee a) \vee (x \wedge y) = (z \vee (a \vee x)) \wedge y. \end{aligned}$$

Now assume that $z \leqslant a \vee x$. Since $z \in [a, a \vee x]$, it follows from Proposition 5.2 that $z = \alpha_a(x') = a \vee x'$ for some $x' \leqslant x$. It then follows that

$$\begin{aligned} z \vee (y \wedge (a \vee x)) &= x' \vee a \vee (a \vee (x \wedge y)) \\ &= a \vee ((x' \vee y) \wedge x) = (x' \vee y) \wedge (a \vee x) \\ &= (z \vee y) \wedge (a \vee x). \end{aligned}$$

We conclude that $x \vee a$ is modular in the underlying lattice of L_f/a. To complete the proof we need to show that if y is in $[a, \hat{1}]$, then

$$f/a(y) + f/a(x \vee a) = f/a(y \vee x \vee a) + f/a(y \wedge (x \vee a)).$$

Now, since x is modular, $y \wedge (x \vee a) = a \vee (x \wedge y)$. It then follows from Proposition 5.1 that

$$f/a(a \vee (x \wedge y)) = f/(a \wedge x)(\beta_a(a \vee (x \wedge y))) = f(x \wedge y) - f(x \wedge a).$$

Also $y \vee x \vee a = y \vee x$, so $f/a(y \vee x) = f(y \vee x) - f(a)$. Therefore

$$\begin{aligned} &f/a(y \vee x \vee a) + f/a(y \wedge (x \vee a)) \\ &\quad = f(x \wedge y) - f(x \wedge a) + f(x \vee y) - f(a) \\ &\quad = f(x) + f(y) - f(x) - f(a) + f(x \vee a) - f(a) \\ &\quad = f/a(y) + f/a(x \vee a), \end{aligned}$$

and (i) is proved.

The proof of (ii) is similar to (i) and is omitted. The proof of (iii) is almost trivial. ∎

LEMMA 5.4. *Let z be an element of a lattice L with the property that if $y \in L$ and $y > \hat{0}$, then $z \wedge y > \hat{0}$, and let a be an element of L which is not in $[\hat{0}, z]$. Then $\mu(\hat{0}, a) = 0$.*

Proof. Consider $z \wedge a$ in $[\hat{0}, a]$. If y is in $[\hat{0}, a]$, then $z \wedge a \wedge y = z \wedge y$, so that $z \wedge a$ has no complement in $[\hat{0}, a]$. It is then a consequence of Crapo's complementation theorem [6, Theorem 3] that $\mu(\hat{0}, a) = 0$. ∎

We are now in a position to prove Theorem 5.1. A proof could also be given which mimics the technique of Stanley [14, Theorem 2], but I cannot resist the opportunity of applying Theorem 4.3.

Proof of Theorem 5.1. It is clear that the result holds for weighted lattices with at most two elements. Assume that L_f has at least three elements, and, for induction, that the result holds for all weighted lattices with less elements than L_f. If the only element which meets x at $\hat{0}$ is $\hat{0}$, then by Lemma 5.4,

$$\begin{aligned} P(L_f; \lambda) &= \sum_{a \leqslant x} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(a)} \\ &= P(L_f \mid x; \lambda)[\mu(\hat{0}, \hat{0})\, \lambda^{f(\hat{1}) - f(x) - f(\hat{0})}] \\ &= P(L_f \mid x; \lambda)\left[\sum_{a : x \wedge a = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)}\right]. \end{aligned}$$

Assume then that there exists at least one element $q > \hat{0}$ such that $q \wedge x = \hat{0}$. This clearly implies the existence of an atom with this property. Let p be an atom of L_f such that $p \wedge x = \hat{0}$. By Theorem 4.3,

$$P(L_f; \lambda) = P(L_f \backslash p; \lambda) - P(L_f / p; \lambda).$$

Let $\mu \backslash p$ denote the Möbius function of $L_f \backslash p$. By Proposition 5.3(iii), x is modular in $L_f \backslash p$. Clearly $(L_f \backslash p) \mid x \cong L_f \mid x$. Hence,

$$P(L_f \backslash p; \lambda) = P(L_f \mid x; \lambda)\left[\sum_{a \neq p : a \wedge x = \hat{0}} \mu \backslash p(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)}\right].$$

By Proposition 5.2, $(L_f/p) \mid (p \vee x) \cong L_f \mid x$, and by Proposition 5.3(i), $p \vee x$ is modular in L_f/p. Hence

$$P(L_f | p; \lambda) = P(L_f \mid x; \lambda)\left[\sum_{a \geqslant p : (p \vee x) \wedge a = p} \mu(p, a)\, \lambda^{f/p(\hat{1}) - f/p(x \vee p) - f/p(a)}\right].$$

Clearly $f/p(\hat{1}) - f/p(x \vee p) - f/p(a) = f(\hat{1}) - f(x) - f(a)$. Also, since x is modular, $(p \vee x) \wedge a = p \vee (x \wedge a)$, so that if $a \geqslant p$, then $(p \vee x) \wedge a = p$ if and only if $x \wedge a = \hat{0}$. Therefore,

$$P(L_f/p; \lambda) = P(L_f \mid x; \lambda)\left[\sum_{a \geqslant p : a \wedge x = \hat{0}} \mu(p, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)}\right],$$

and it follows that

$$P(L_f;\lambda)=P(L_f \mid x;\lambda)\left[\sum_{a\neq p:a\wedge x=\hat{0}} \mu\backslash p(\hat{0},a)\,\lambda^{f(\hat{1})-f(x)-f(a)}\right.$$
$$\left.-\sum_{a\geqslant p:a\wedge x=\hat{0}} \mu(p,a)\,\lambda^{f(\hat{1})-f(x)-f(a)}\right].$$

It follows from a straightforward application of Lemma 4.4 that

$$\sum_{a\neq p,a\wedge x=\hat{0}} \mu\backslash p(\hat{0},a)\,\lambda^{f(\hat{1})-f(x)-f(a)}-\sum_{a\geqslant p:a\wedge x=\hat{0}} \mu(p,a)\,\lambda^{f(\hat{1})-f(x)-f(a)}$$
$$=\sum_{a:x\wedge a=\hat{0}} \mu(\hat{0},a)\,\lambda^{f(\hat{1})-f(x)-f(a)}$$

and the theorem is proved. ∎

It is natural to ask if the conclusion of Theorem 5.1 holds for any weaker hypotheses than those given. Now, for x to be modular in L_f, we must have, for all elements y of L_f: (a) yMx, (b) xMy, and (c) $f(x)+f(y)=f(x\wedge y)+f(x\vee y)$. Can any of these conditions be dropped? The element x of the weighted lattice L_f illustrated in Fig. 3(a) certainly has properties (a) and (b), but $P(L_f;\lambda)=\lambda^4-\lambda^3-\lambda^2+1$ and $P(L_f \mid x;\lambda)=\lambda^2-1$. Therefore (c) cannot be dropped. Also, the element x of the weighted lattice L_g illustrated in Fig. 3(b) has properties (b) and (c), but $P(L_g;\lambda)=\lambda^3-2\lambda^2+1$ and $P(L_g \mid x;\lambda)=\lambda^2-\lambda$, so that (a) cannot be dropped. However, I can find no counter example to the conjecture that if x is an

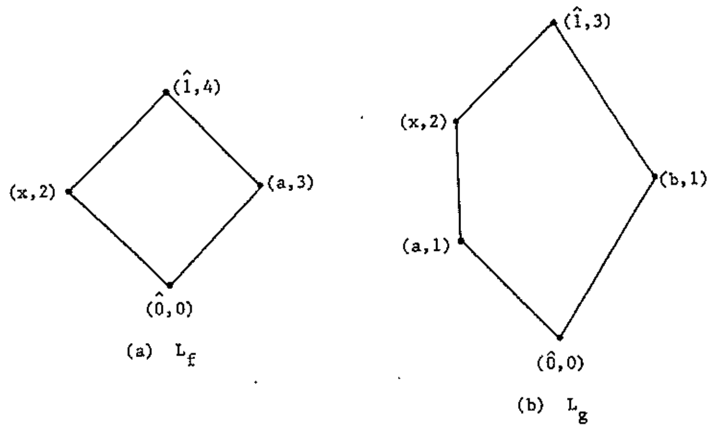


FIGURE 3

element of the weighted lattice L_f, and x has properties (a) and (c), then $P(L_f \mid x; \lambda)$ divides $P(L_f; \lambda)$: nor, unfortunately, can I prove it.

Note that without (b), conditions (a) and (c) are quite weak. To see this consider an example. Let L be any finite lattice. Let L' be the lattice obtained from L by adding the element x in such a way that in L', x covers $\hat{0}$ and $\hat{1}$ covers x. Define $f: L' \to Z$ by $f(\hat{0})=0$, $f(\hat{1})=2$, and $f(a)=1$ for all other elements a of L'. It is readily verified that, for all elements y of L', yMx and $f(y)+f(x)=f(y \vee x)+f(y \wedge x)$.

In any event the whole question is perhaps academic. For classes of weighted lattices where characteristic polynomials are known to be of combinatorial significance the problem does not really arise.

THEOREM 5.5. *Let L_f be a weighted lattice which is strictly increasing and submodular, and let x be an element of L_f which has the property that, for all y in L_f, $f(x)+f(y)=f(x \wedge y)+f(x \vee y)$. Then x is a modular element of L_f.*

Proof. Let a be an element of L_f. Now let y be an element of L_f such that $y \leqslant x$. Then $f(x)+f(a \vee y) = f(x \vee a \vee y)+f((y \vee a) \wedge x)$. But $x \vee a \vee y = x \vee a$ so that

$$f((y \vee a) \wedge x)=f(y \vee a)+f(x)-f(x \vee a). \tag{1}$$

Now,

$$f(a)+f(y \vee (a \wedge x)) \geqslant f(a \vee y \vee (a \wedge x))+f(a \wedge (y \vee (a \wedge x))).$$

Since $a \wedge x \leqslant a$, $a \vee y \vee (a \wedge x) = a \vee y$, and since $y \in [a \wedge x, x]$, $a \wedge (y \vee (a \wedge x))=a \wedge x$. Therefore,

$$f(y \vee (a \wedge x)) \geqslant f(y \vee a)+f(a \wedge x)-f(a). \tag{2}$$

But $f(x)-f(x \vee a)=f(a \wedge x)-f(a)$ and it follows, upon comparing (1) with (2), that $f(y \vee (a \wedge x)) \geqslant f((y \vee a) \wedge x)$. In any lattice $y \vee (a \wedge x) \leqslant (y \vee a) \wedge x$, and it now follows from the fact that f is strictly increasing that $y \vee (a \wedge x)=(y \vee a) \wedge x$: in other words, aMx.

Now assume that $y \leqslant a$. Then,

$$f(y \vee (x \wedge a))=f(x \vee y \vee (x \wedge a))+f(x \wedge (y \vee (x \wedge a)))-f(x).$$

But $x \vee y \vee (x \wedge a)=x \vee y$ and $x \wedge (y \vee (x \wedge a))=x \wedge a$, so

$$f(y \vee (x \wedge a))=f(x \vee y)+f(x \wedge a)-f(x). \tag{3}$$

Now $x \vee y \vee a=x \vee a$, so

$$f((y \vee x) \wedge a) \leqslant f(y \vee x)+f(a)-f(a \vee x). \tag{4}$$

Comparing (3) with (4), we see that $f(y \vee (x \wedge a)) \geqslant f((y \vee x) \wedge a)$. But $y \vee (x \wedge a) \leqslant (y \vee x) \wedge a$, and f is strictly increasing, so $y \vee (x \wedge a) = (y \vee x) \wedge a$, and we conclude that xMa. Since aMx and xMa for all a in L_f, the theorem is proved. ∎

Before discussing the significance of Theorem 5.5, we first note the following immediate corollary of Theorems 5.1 and 3.2.

COROLLARY 5.6. *If x is an element of the weighted lattice L_f, and x is modular in some quotient of L_f relative to a respectful closure operator, then* $P(L_f \mid x; \lambda)$ *divides* $P(L_f; \lambda)$.

Let f be a normalised, increasing set function, and let C be a closed set of f. By Corollary 5.6, if C is modular in the weighted lattice of closed sets of f, then $P(f \mid C; \lambda)$ divides $P(f; \lambda)$. For polymatroids the situation is even simpler since we may also invoke Theorem 5.5.

COROLLARY 5.7. *Let f be a polymatroid and let F be a flat of f. If, for all flats F' of F,*

$$f(F) + f(F') = f(F \cup F') + f(F \cap F'),$$

then $P(f \mid F; \lambda)$ *divides* $P(f; \lambda)$.

Proof. Let L_f denote the weighted lattice of flats of f. Then, in L_f, $f(F \vee F') = f(F \cup F')$ and $F \wedge F' = F \cap F'$. Hence, by Theorem 5.5, F is modular in L_f. Therefore, by Corollary 5.6, $P(f \mid F; \lambda)$ divides $P(f; \lambda)$. ∎

While modularity is satisfactorily defined for weighted lattices, the definition cannot be extended unambiguously to set functions. Let f be a normalised increasing function on E, and let A be a subset of E. The phrase "A is modular in f" has two plausible meanings: it could mean that A is modular in the Boolean weighted lattice of f, or it could mean that A is modular in the weighted lattice of closed sets of f. In matroid theory, it is almost always the latter meaning that is intended, but the former meaning does get occasional use in the theory of submodular functions (see, for example, [12, Sect. 1]). We do not define "modular" for set functions as all situations in which the term is useful can be dealt with by reference to an appropriate weighted lattice.

The following result generalizes Brylawski [3, Proposition 3.5].

PROPOSITION 5.8. *If x is modular in L_f and y is modular in $L_f \mid x$, then y is modular in L_f.*

Proof. Assume that z and a are elements of L_f. If $a \leqslant y$, then, using the hypotheses of the proposition, we see that

$$\begin{aligned} a \vee (z \wedge y) &= a \vee ((z \wedge x) \wedge y) = (a \vee (z \wedge x)) \wedge y \\ &= ((a \vee z) \wedge x) \wedge y = (a \vee z) \wedge y, \end{aligned}$$

and it follows that zMy.

Now assume that $a \leqslant z$. Then

$$(a \wedge x) \vee (y \wedge (z \wedge x)) = (a \wedge x) \vee (z \wedge y) = ((z \wedge y) \vee a) \wedge x.$$

Also,

$$((a \wedge x) \vee y) \wedge (z \wedge x) = ((y \vee a) \wedge x) \wedge (z \wedge x) = ((y \vee a) \wedge z) \wedge x.$$

It follows from the fact that y is modular in $L_f \mid x$ that $(a \wedge x) \vee (y \wedge (z \wedge x)) = ((a \wedge x) \vee y) \wedge (z \wedge x)$. Therefore, $(a \vee (y \wedge z)) \wedge x = ((a \vee y) \wedge z) \wedge x$. But, both $a \vee (y \wedge z)$ and $(a \vee y) \wedge z$ are in $[a, a \vee x]$. It then follows from Proposition 5.2 that $a \vee (y \wedge z) = (a \vee y) \wedge z$, so that yMz.

Further use of the modularity of x in L_f and the modularity of y in $L_f \mid x$ shows that

$$\begin{aligned} f(z) + f(y) &= f(z \wedge x) + f(z \vee x) - f(x) + f(y) \\ &= f(z \wedge x \wedge y) + f((z \wedge x) \vee y) + f(z \vee x) - f(x) \\ &= f(z \wedge y) + f((y \vee z) \wedge x) + f(z \vee x) - f(x) \\ &= f(z \wedge y) + f(y \vee z) + f(x) - f(y \vee z \vee x) + f(z \vee x) - f(x) \\ &= f(z \wedge y) + f(z \vee y). \quad \blacksquare \end{aligned}$$

COROLLARY 5.9. *If x and y are modular elements of L_f, then $x \wedge y$ is a modular element of L_f.*

Proof. By Proposition 5.3(ii), $x \wedge y$ is modular in $L_f \mid x$. Hence, by Proposition 5.8, $x \wedge y$ is modular in L_f.

6. SUPERSOLVABLE WEIGHTED LATTICES

A *maximal chain* of a lattice is a chain $\hat{0} = x_0 < x_1 < \cdots < x_{n-1} < x_n = \hat{1}$ of the lattice with the property that, for $1 \leqslant i \leqslant n$, x_i covers x_{i-1}. Let L_f be a weighted lattice. We say that L_f is *supersolvable* if L_f contains a maximal chain of modular elements. The theory of supersolvable (unweighted) lattices was developed by Stanley [15]. Note that a non-supersolvable weighted lattice may have a supersolvable underlying lattice (in the sense of [15]). More importantly, note that it follows from [15, Proposition 2.1] that if a weighted lattice is supersolvable in the sense of this paper, then its underlying unweighted lattice is supersolvable in Stanley's sense. Note particularly that it follows routinely from [15, Corollary 2.1], that a geometric lattice is (Stanley) supersolvable if and only if, when weighted by its rank function, it is supersolvable as a weighted lattice. The following theorem generalizes [15, Theorem 4.1].

THEOREM 6.1. *Let* $\hat{0}=x_0<x_1<\cdots<x_{n-1}<x_n=\hat{1}$ *be a maximal chain of modular elements of the supersolvable weighted lattice* L_f. *Then,*

$$P(L_f;\lambda)=\prod_{i=1}^{n}(\lambda^{a_i}-b_i),$$

where, for $1\leqslant i\leqslant n$, $a_i=f(x_i)-f(x_{i-1})$, *and* b_i *denotes the number of atoms of* $[\hat{0},x_i]$ *which do not belong to* $[\hat{0},x_{i-1}]$.

Proof. Consider x_{n-1}. Since x_{n-1} is modular in L_f, it follows from Theorem 5.1, that

$$P(L_f;\lambda)=P(L_f\mid x_{n-1};\lambda)\sum_{a:a\wedge x_{n-1}=\hat{0}}\mu(\hat{0},a)\,\lambda^{f(\hat{1})-f(x_{n-1})-f(a)}.$$

Say z is an element of L_f such that $z\wedge x_{n-1}=\hat{0}$ and $z\neq\hat{0}$. If $z'<z$ and $z'\neq\hat{0}$, then, since x_{n-1} is a copoint of L_f, $z'\vee x_{n-1}=z\vee x_n=\hat{1}$. Therefore, $z'\vee(x_{n-1}\wedge z)=z'$ and $(z'\vee x_{n-1})\wedge z=z$, which contradicts the modularity of x_{n-1}. We may conclude that the elements a of L_f with the property that $a\wedge x_{n-1}=\hat{0}$ are exactly the atoms of L_f not belonging to $[\hat{0},x_{n-1}]$. If a is such an atom, then, since x_{n-1} is modular, $f(a)=f(\hat{1})-f(x_{n-1})$. Of course, $\mu(\hat{0},a)=-1$. Therefore

$$\begin{aligned}\sum_{a:a\wedge x_{n-1}=\hat{0}}\mu(\hat{0},a)\,\lambda^{f(\hat{1})-f(x_{n-1})-f(a)}&=\mu(\hat{0},\hat{0})\,\lambda^{f(\hat{1})-f(x_{n-1})}-b_n\\&=\lambda^{a_n}-b_n.\end{aligned}$$

Thus, $P(L_f;\lambda)=P(L_f\mid x_{n-1};\lambda)(\lambda^{a_n}-b_n)$. It follows from Proposition 5.3(iii) that, for $1\leqslant i\leqslant n$, x_{i-1} is a modular element of $L_f\mid x_i$, so the above process can be repeated and the result obtained. ∎

In the class of upper semimodular lattices weighted by their rank functions, Theorem 6.1 is Stanley's theorem.

The following result is the weighted lattice theoretic version of a useful result in matroid theory [14, Sect. 3]. It follows easily from the proof of Theorem 6.1.

COROLLARY 6.2. *Let x be a modular copoint of the weighted lattice L_f, and let k denote the number of atoms of L_f which are not comparable with x. Then,*

$$P(L_f; \lambda) = P(L_f \mid x; \lambda)(\lambda^{f(\hat{1}) - f(x)} - k).$$

It is time for some examples. Let $[0, n]$ denote the set $\{0, 1, ..., n\}$ with the natural order. Let n be a positive integer and let f: $[0, n] \to Z$ be an increasing function for which $f(0) = 0$. This defines a weighted lattice whose underlying lattice is $[0, n]$ and whose underlying function is f. It follows from Theorem 6.1, that

$$\begin{aligned} P(L_f; \lambda) &= (\lambda^{f(n) - f(n-1)} - 0)(\lambda^{f(n-1) - f(n-2)} - 0) \cdots (\lambda^{f(2) - f(1)} - 0)(\lambda^{f(1)} - 1) \\ &= \lambda^{f(n) - f(1)}(\lambda^{f(1)} - 1). \end{aligned}$$

This is easily checked to be correct by direct computation, since if i is in $[0, n]$, then $\mu(0, i) \neq 0$ if and only if $i \in \{0, 1\}$.

We now consider a less trivial example. Let $(a_1, ..., a_n)$ be a sequence of positive integers such that, if $1 < i \leqslant n$, then $a_1 \leqslant a_i$; and let $(b_1, ..., b_n)$ be a sequence of non-negative integers with $b_1 = 1$. Let V be a vector space over a large enough field—the reals will always do—and assume that $r(V)$, the rank of V, is greater than or equal to $a_1 + \cdots + a_n$. Now let $W = \{w_1, ..., w_n\}$ be a collection of subspaces of V with the following properties:

(i) $r(w_1) = a_1$, and for $1 < i \leqslant n$, $r(w_i) = a_1 + a_i$;

(ii) if $1 \leqslant i \leqslant n$, then $w_i \supseteq w_1$; furthermore, if i and j are distinct members of $\{1, ..., n\}$, then $w_i \cap w_j = w_1$.

It is clear that it is always possible to find such a collection of subspaces. For $1 < i \leqslant n$, let $X_i = \{x_{ij}: 1 \leqslant j \leqslant b_i\}$ be a collection of subspaces of w_i with the following properties:

(i) the rank of each member of X_i is a_i;

(ii) each member of X_i is skew to w_1 (that is, if $x \in X_i$, then $x_{ij} \cap w_1 = 0$);

(iii) if X_i' is a subset of X_i and $|X_i'| \geqslant 2$, then $r(\bigcup \{x : x \in X_i'\}) = r(w_i)$.

It is clear that conditions (i) and (ii) can be made to hold. For condition (iii) to hold we first need a field which is large enough, which we have. We also need $r(w_i)$ to be less than or equal to $2a_i$: we have this too, since $a_1 \leqslant a_i$ and $r(w_i) = a_1 + a_i$. Note that conditions (ii) and (iii) above amount to saying that the members of X_i are *freely placed* on w_i. Of course, if $b_i = 0$, then X_i is empty. Let $S = W \cup X_1 \cup \cdots \cup X_n$, and let f be the function defined, for all subsets A of S, by $f(A) = r(\bigcup \{a : a \in A\})$. It is clear

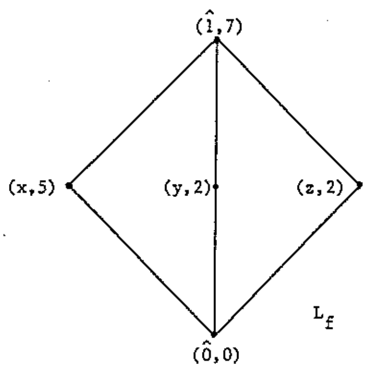


FIGURE 4

that f is a polymatroid. For $0 \leqslant k \leqslant n$, let $F_k = \{w_1, ..., w_k\} \cup \{x_{ij} : i \leqslant k\}$. It is routinely verified that, for $0 \leqslant k \leqslant n$, F_k is a flat of f, and that $\varnothing = F_0 < F_1 < \cdots < F_{n-1} < F_n = S$ is a maximal chain of the weighted lattice L_f of flats of f. Moreover, it is straightforward to check, using Theorem 5.5, that if $0 \leqslant k \leqslant n-1$, then F_k is modular in $L_f \mid F_{k+1}$. It then follows from Proposition 5.8 that each of these flats is modular in L_f, so that L_f is supersolvable. It is then evident that $P(L_f; \lambda) = \prod_{i=1}^{n} (\lambda^{a_i} - b_i)$.

This example embarrasses me somewhat. The underlying geometric idea is simple in the extreme, but the simplicity seemed to get lost in the writing down. It is really just a natural generalisation of [15, Example 2.10]. The point of the example is to establish

PROPOSITION 6.3. *Let $(a_1, ..., a_n)$ be a sequence of positive integers such that, for $1 < i \leqslant n$, $a_1 \leqslant a_i$, and let $(b_1, ..., b_n)$ be a sequence of non-negative integers with $b_1 = 1$. Then there exists a supersolvable, submodular weighted lattice L_f with maximal chain of modular elements $\hat{0} = x_0 < x_1 < \cdots < x_{n-1} < x_n = \hat{1}$ such that for $1 \leqslant i \leqslant n$, $a_i = f(x_i) - f(x_{i-1})$, and b_i denotes the number of atoms of $[\hat{0}, x_i]$ which do not belong to $[\hat{0}, x_{i-1}]$.*

Of course, $P(L_f; \lambda) = \prod_{i=1}^{n} (\lambda^{a_i} - b_i)$. It is trivial to check that the condition that $b_1 = 1$ cannot be dropped in Proposition 6.3, but the condition that $a_1 \leqslant a_i$ for $1 < i \leqslant n$ is not so obviously necessary. I have not checked the matter thoroughly but it seems that this condition can be dropped if one is prepared to forgo submodularity. Consider the weighted lattice L_f illustrated in Fig. 4. Here $\hat{0} < x < \hat{1}$ is a maximal chain of modular elements. In the notation of Proposition 6.3, we have $a_1 = 5$, $a_2 = 2$, $b_1 = 1$, and $b_2 = 2$, and $P(L_f; \lambda) = (\lambda^5 - 1)(\lambda^2 - 2)$. Of course, L_f is not submodular.

7. Complete Principal Truncations

A useful construction for matroids and geometric lattices is the complete principal truncation of [4, Sect. 4]. Of particular interest is the fact, observed by Brylawski [3, Corollary 7.4], that if F is a modular flat of a matroid M, then

$$P(M;\lambda)=P(M\mid F;\lambda)\,P(\bar{T}_F(M);\lambda)/(\lambda-1).$$

Here $\bar{T}_F(M)$ denotes the complete principal truncation of M at F.

Let L_f be a weighted lattice, and x be an element of L_f for which $f(x)\geqslant 1$. Then the *complete principal truncation* of L_f at x, denoted $\bar{T}_x(L_f)$, consists of a partially ordered set (also denoted $\bar{T}_x(L_f)$) and a function $\bar{T}_x(f)\colon \bar{T}_x(L_f)\to Z$. These are defined as follows:

(i) $\bar{T}_x(L_f)$ is equal to $\{a\in L_f : a\wedge x=\hat{0}$ or $a\wedge x=x\}$ with the induced order;

(ii) if a is in $\bar{T}_x(L_f)$, then $\bar{T}_x(f)(a)=f(a)$, if $a\wedge x=\hat{0}$, and $\bar{T}_x(f)(a)=f(a)-f(x)+1$, if $a\wedge x=x$.

Proposition 7.1. *If x is a modular element of L_f, and $f(x)\geqslant 1$, then $\bar{T}_x(L_f)$ is a weighted lattice.*

Proof. It is a trivial exercise to check that $\bar{T}_x(L_f)$ is a lattice regardless of whether or not x is modular. Now assume that a and b are elements of $\bar{T}_x(L_f)$, and that $a<b$. If both $a\wedge x=\hat{0}$, and $b\wedge x=\hat{0}$ or if both $a\wedge x=x$ and $b\wedge x=x$, then it is clear that $\bar{T}_x(f)(a)\leqslant\bar{T}_x(f)(b)$. The only other possibility is that $a\wedge x=\hat{0}$ and $b\wedge x=x$. In this case we see, from the modularity of x, that $f(a)=f(a\vee x)-f(x)$. But $a\vee x\leqslant b\vee x=b$, so that $f(a)\leqslant f(b\vee x)-f(x)$. Now $\bar{T}_x(f)(a)=f(a)$, and $\bar{T}_x(f)(b)=f(b)-f(x)+1$. Therefore $\bar{T}_x(f)(a)\leqslant\bar{T}_x(f)(b)$, and we conclude that $\bar{T}_x(f)$ is increasing. The result then follows after finally observing that $\bar{T}_x(L_f)(\hat{0})=\hat{0}$. ∎

For submodular weighted lattices we can be stronger.

Proposition 7.2. *If L_f is a submodular weighted lattice, and x is any element of L_f with $f(x)\geqslant 1$, then $\bar{T}_x(L_f)$ is a submodular weighted lattice.*

Proof. The proof that $\bar{T}_x(L_f)$ is a weighted lattice is almost identical to the proof of Proposition 7.1. We now show that $\bar{T}_x(L_f)$ is submodular. Let a and b be elements of $\bar{T}_x(L_f)$. The only non-trivial case is covered by assuming that $a\wedge x=\hat{0}$ and $b\wedge x=x$. In this case, $(a\wedge b)\wedge x=\hat{0}$ and $(a\vee b)\wedge x=x$. Therefore $\bar{T}_x(f)(a)+\bar{T}_x(f)(b)=f(a)+f(b)-f(x)+1$, and $\bar{T}_x(f)(a\wedge b)+\bar{T}_x(f)(a\vee b)=f(a\wedge b)+f(a\vee b)-f(x)+1$. The submodularity of $\bar{T}_x(f)$ then follows from the submodularity of f. ∎

Of course, the complete principal truncation as described here coincides with the usual definition in the class of geometric lattices weighted by their rank functions.

THEOREM 7.3. *If x is a modular element of the weighted lattice L_f and $f(x) \geqslant 1$, then $P(L_f;\lambda) = P(L_f \mid x; \lambda)\, P(\bar{T}_x(L_f); \lambda)/(\lambda - 1)$.*

Proof. First observe that x is a modular element of $\bar{T}_x(L_f)$. (Note that while this is easily seen to be so in this case, there exist highly structured weighted lattices whose atoms are not modular; as the example at the end of Section 4 shows.) Let μ and μ' denote the Möbius function of L_f and $\bar{T}_x(L_f)$, respectively, and, for simplicity, set $f' = \bar{T}_x(f)$. Using the modularity of x in $\bar{T}_x(L_f)$, and Crapo's complementation theorem [6, Theorem 2], we see that if z is an element of $\bar{T}_x(L_f)$, and $z \geqslant x$, then

$$\mu'(\hat{0}, z) = \sum_{a\,:\,a \wedge x = \hat{0},\, a \vee x = z} \mu'(\hat{0}, a)\, \mu'(a, z).$$

But, by Proposition 5.2, if $a \wedge x = \hat{0}$ and $a \vee x = z$, then $\bar{T}_x(L_f) \mid x \cong (\bar{T}_x(L_f) \mid z)/a$, so that $\mu'(a, z) = \mu'(\hat{0}, x) = -1$. Therefore,

$$\mu'(\hat{0}, z) = -\sum_{a\,:\,a \wedge x = \hat{0},\, a \vee x = z} \mu(\hat{0}, a).$$

It also follows that if $a \wedge x = \hat{0}$ and $a \vee x = z$, then $f'(z) = 1 + f'(a)$. But $f'(a) = f(a)$, so $f'(z) = 1 + f(a)$. Therefore $f'(\hat{1}) - f'(z) = f(\hat{1}) - f(x) - f(a) + 1$. Hence,

$$\begin{aligned}
&\sum_{z \in \bar{T}_x(L_f)\,:\,z \geqslant x} \mu'(\hat{0}, z)\, \lambda^{f'(1) - f'(z)} \\
&\quad = -\sum_{a \in \bar{T}_x(L_f)\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a) + 1} \\
&\quad = -\lambda \sum_{a \in L_f\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)}.
\end{aligned}$$

We now see that

$$\begin{aligned}
P(\bar{T}_x(L_f)); \lambda) &= \sum_{a \in L_f\,:\,a \wedge x = \hat{0}} \mu'(\hat{0}, a)\, \lambda^{f'(\hat{1}) - f'(a)} \\
&\quad + \sum_{z \in L_f\,:\,z \wedge x = x} \mu'(\hat{0}, z)\, \lambda^{f'(1) - f'(z)} \\
&= \sum_{a \in L_f\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)} \\
&\quad + \lambda \sum_{a \in L_f\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)} \\
&= (\lambda - 1) \sum_{a \in L_f\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)},
\end{aligned}$$

and the result follows by Theorem 5.1. ∎

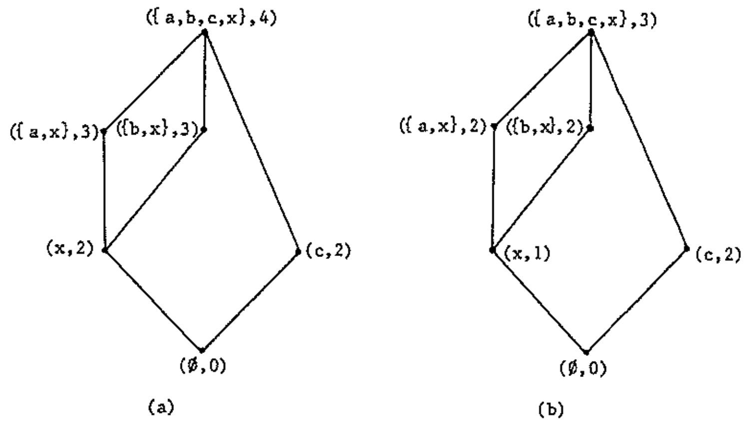


FIGURE 5

So far, the complete principal truncation appears as a somewhat contrived construction tailor made for Theorem 7.3. The point is that this construction has a natural and appealing geometric interpretation for matroids. If F is a flat of the matroid M, then $\bar{T}_F(M)$ is obtained by first putting a set X of $r(F)-1$ points freely on the flat F and then contracting X from the extended matroid. It is straightforward to verify that this interpretation extends, at least, to submodular weighted lattices. For illustration, consider the following example.

Let a and b denote planes in rank-4 space meeting at the line x, and let c be a line skew to a, b, and x. There is a natural polymatroid f which can be associated with this situation. The weighted lattice of flats, L_f, of f is illustrated in Fig. 5(a). Clearly x is a modular element of L_f. Consider $\bar{T}_x(L_f)$. Geometrically, this corresponds to placing a point freely on x and contracting it from the extended configuration. The result is that x becomes a point, a and b become lines meeting at x, and c becomes a line skew to a, b, and x in rank-3 space. It is easily seen that the weighted lattice $\bar{T}_x(L_f)$, illustrated in Fig. 5(b) corresponds to this configuration. It is readily verified that $P(L_f; \lambda) = (\lambda^2-1)^2$, $P(L_f \mid x, \lambda) = \lambda^2 - 1$, and $P(\bar{T}_x(L_f); \lambda) = (\lambda-1)(\lambda^2-1)$, so that all is as it should be according to Theorems 5.1 and 7.3.

8. SEPARATORS

Recall that the *direct product* $A \times B$ of lattices A and B is the set $\{(a, b) : a \in A, b \in B\}$ of ordered pairs together with the order relation $(a, b) \leqslant (c, d)$ in $A \times B$ if and only if $a \leqslant c$ and $b \leqslant d$ in A and B, respectively. For the set-function-theoretic counterpart of this construction we have that if f_1 and f_2 are normalised integer-valued set functions on the disjoint sets S_1 and S_2, respectively, then the *direct sum* f_1+f_2 of f_1 and f_2 is the function defined for all subsets X of $S_1 \cup S_2$ by $(f_1+f_2)(X)=f_1(X\cap S_1)+f_2(X\cap S_2)$. This definition accords with the usual one for matroids when f_1 and f_2 are rank functions. It also agrees with the implied definition of direct sum for submodular functions in [9]. As usual, for weighted lattices we combine both approaches. The *direct sum* L_f+L_g of the weighted lattices L_f and L_g has as its underlying lattice $L_f \times L_g$: the underlying function $f+g$ of $L_f \times L_g$ is defined for all elements (a,b) of $L_f \times L_g$, by $(f+g)(a,b)=f(a)+g(b)$. The element x of the weighted lattice L_f is a *separator* of L_f if there exists an element x' of L_f with the property that $L_f \cong L_f \mid x + L_f \mid x'$.

We seek to characterise separators of weighted lattices. To this end we define an element x of L_f to be *neutral* if, for all a and b in L_f, the following properties hold:

(i) $(x \wedge a) \vee (a \wedge b) \vee (b \wedge x)=(x \vee a) \wedge (a \vee b) \wedge (b \vee x)$;

(ii) $f(x)+f(a)=f(x \vee a)+f(x \wedge a)$.

This definition extends that of Grätzer [10, Chap. III, Sect. 2] to weighted lattices. A *distributive* element x of a lattice L is one with the property that, for all a and b in L, $a \vee (x \wedge y)=(a \vee x) \wedge (a \vee y)$. In general, neutrality is a stronger condition than distributivity, but in relatively complemented lattices—in particular, in geometric lattices—the two concepts coincide (Grätzer [10, Chap. III, Sect. 2, Theorem 6]; also an easy consequence of results in Crapo and Rota [8, Chap. 12]). Consequently no distinction need be made between the two types of element in these lattices—the term "distributive" being frequently used. We need the distinction, and we need the stronger condition of neutrality.

It is also worth noting [10, Chap. III, Sect. 2, Theorem 5] that an element x of a lattice L is neutral if and only if, for all a and b in L, $x \wedge (a \vee b)=(x \wedge a) \vee (x \wedge b)$ and $a \wedge (x \vee b)=(a \wedge x) \vee (a \wedge b)$, and dually.

THEOREM 8.1. *Let x be an element of the weighted lattice L_f. Then x is a separator of L_f if and only if x is a complemented neutral element of L_f.*

Before proving the theorem we first prove a lemma. Evidently a neutral element has at most one complement.

LEMMA 8.2. *If x is a neutral element of L_f with complement x', then for all a in L_f, $f(a)=f(a \wedge x)+f(a \wedge x')$.*

Proof. Since x is neutral in L_f, $f(a \wedge x) = f(a) + f(x) - f(a \vee x)$, and $f(a \wedge x') + f(x) = f(a \wedge x' \wedge x) + f((a \wedge x') \vee x) = f(a \wedge x' \wedge x) + f((a \vee x) \wedge (x' \vee x))$. But $x' \wedge x = \hat{0}$, and $x' \vee x = \hat{1}$. Therefore $f(a \wedge x') = f(a \vee x) - f(x)$, so that $f(a \wedge x) + f(a \vee x') = f(a)$. ∎

Proof of Theorem 8.1. It is easily seen that any separator of L_f is a complemented neutral element of L_f. For the converse, assume that x' is a complement of the neutral element x. It is a consequence of the discussion following [10, Chap. II, Sect. 4, Theorem 1] that the function $\phi: L_f \mid x \times L_f \mid x' \to L_f$ defined, for all $a \leqslant x$ and $b \leqslant x'$, by $\phi(a, b) = a \vee b$, is a lattice isomorphism between the underlying lattice of $L_f \mid x \times L_f \mid x'$ and the underlying lattice of L_f. Now if a is in L_f, then it is easily seen that $\phi^{-1}(a) = (a \wedge x, a \wedge x')$. Hence, by Lemma 8.2, $f(\phi^{-1}(a)) = f(a \wedge x) + f(a \wedge x') = f(a)$ and we conclude that ϕ is a weighted lattice isomorphism. ∎

Since every element of a Boolean latice is neutral, Theorem 8.1 implies that the separators of a normalised, increasing set function are just the subsets which are modular in the Boolean weighted lattice of the function—essentially a well-known fact.

A weighted lattice is *indecomposable* if it has no separators apart from $\hat{0}$ and $\hat{1}$. The proof of the following theorem is straightforward and is omitted.

THEOREM 8.3. *Let L_f be a weighted lattice. Then the set of separators of L_f with the induced order is a sublattice of L_f. If $\{a_1, ..., a_n\}$ is the set of atoms of this sublattice, then $\prod_{i=1}^{n} L_f/a_i$ is the unique decomposition of L_f into indecomposable sublattices.*

For characteristic polynomials we have:

THEOREM 8.4. *If x is a separator of the weighted lattice L_f, and the complement of x is x', then $P(L_f; \lambda) = P(L_f \mid x; \lambda)\, P(L_f \mid x'; \lambda)$.*

Proof. Obviously, separators are modular, so by Theorem 5.1,

$$P(L_f; \lambda) = P(L_f \mid x; \lambda) \left[\sum_{a\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)} \right].$$

Assume that $a \wedge x = \hat{0}$. Consider $a \vee x'$. Then $x \wedge (a \vee x') = (x \wedge a) \vee (x \wedge x') = \hat{0}$, and $x \vee (a \vee x') = \hat{1}$, so that $a \vee x'$ is a complement of x. We conclude that $a \wedge x = \hat{0}$ if and only if $a \leqslant x'$. Therefore,

$$\sum_{a\,:\,a \wedge x = \hat{0}} \mu(\hat{0}, a)\, \lambda^{f(\hat{1}) - f(x) - f(a)} = \sum_{a \leqslant x'} \mu(\hat{0}, a)\, \lambda^{f(x') - f(a)} = P(L_f \mid x'; \lambda),$$

and the theorem is proved. ∎

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