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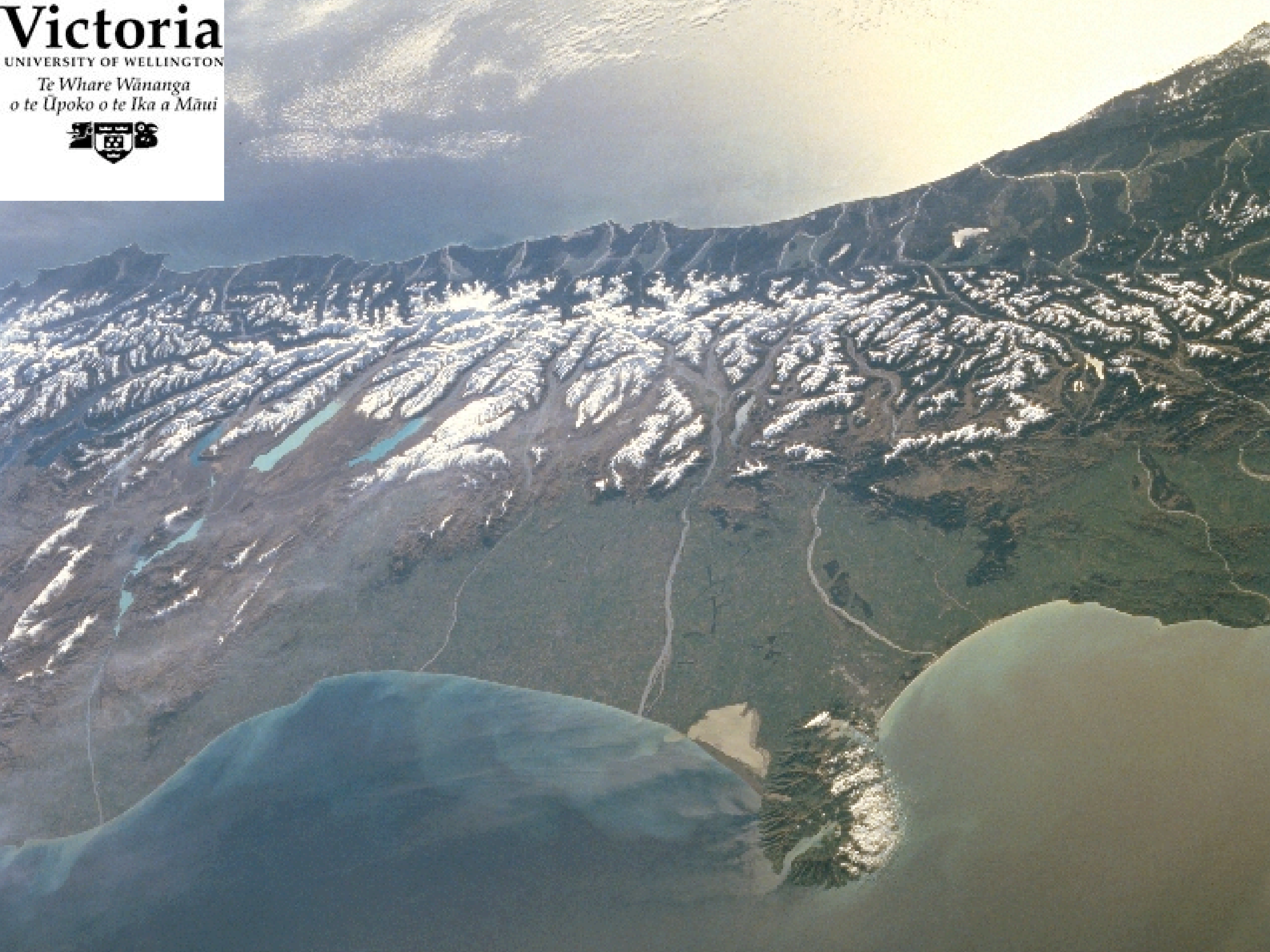

Multiple fields:

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Some of the features we encountered in looking at sound in a vortex flow are really generic features of the fact that vorticity is a specific example of dealing with multiple interacting fields.

Suppose we have many interacting fields $\phi^A(t,\vec{x})$.

Lagrangian: $$\mathcal{L}(\partial_\mu \phi^A, \phi^A).$$

Action: $$S[\phi^A] = \int \mathrm{d}^{d+1}x \, \mathcal{L}(\partial_\mu \phi^A, \phi^A).$$

Linearize in the (by now) usual fashion...

Multiple fields:

$$\phi^A(t,\vec{x}) = \phi_0^A(t,\vec{x}) + \epsilon\phi_1^A(t,\vec{x}) + \frac{\epsilon^2}{2}\phi_2^A(t,\vec{x}) + O(\epsilon^3).$$

Action:

$$S[\phi^A] = S[\phi_0^A]$$

$$+\frac{\epsilon^2}{2}\int \mathrm{d}^{d+1}x \left[\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial(\partial_\nu\phi^B)} \right\} \partial_\mu\phi_1^A\,\partial_\nu\phi_1^B \right.$$

$$+2\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial\phi^B} \right\} \partial_\mu\phi_1^A\,\phi_1^B$$

$$\left. +\left\{ \frac{\partial^2 \mathcal{L}}{\partial\phi^A\,\partial\phi^B} \right\} \phi_1^A\,\phi_1^B \right]$$

$$+O(\epsilon^3).$$

(linear term vanishes by background EOM)

Multiple fields:

Linearized EOM (fluctuations):

$$\partial_\mu \left(\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\} \partial_\nu \phi_1^B \right)$$

(formally self-adjoint)

$$+\partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B}\, \phi_1^B \right)$$

(2nd-order linear PDE)

$$-\partial_\mu \phi_1^B\, \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial \phi^A}$$

$$- \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A\, \partial \phi^B} \right) \phi_1^B = 0.$$

(multiple interacting field-theory normal modes...)

Massage this a little to make it more palatable...

Multiple fields:

First, generalize $f^{\mu\nu}$:

$$f^{\mu\nu}{}_{AB} \equiv \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial(\partial_\nu \phi^B)} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\,\partial(\partial_\mu \phi^B)}\right).$$

Symmetric in $(\mu\nu)$ and (AB).

[space-time and field space...]

Define:

$$\Gamma^{\mu}{}_{AB} \equiv +\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial \phi^B} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\,\partial \phi^A}$$
$$+\frac{1}{2}\partial_\nu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\,\partial(\partial_\mu \phi^B)} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial(\partial_\nu \phi^B)}\right).$$

Antisymmetric in field space [AB]...

Multiple fields:

Finally define:

$$K_{AB} = -\frac{\partial^2 \mathcal{L}}{\partial\phi^A\,\partial\phi^B} + \frac{1}{2}\partial_\mu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial\phi^B}\right) + \frac{1}{2}\partial_\mu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi^B)\,\partial\phi^A}\right).$$

This "potential" or "mass matrix" is symmetric in (AB).

Assemble:

$$\partial_\mu\left(f^{\mu\nu}{}_{AB}\,\partial_\nu\phi_1^B\right) + \frac{1}{2}\left[\Gamma^\mu_{AB}\,\partial_\mu\phi_1^B + \partial_\mu(\Gamma^\mu_{AB}\,\phi_1^B)\right] + K_{AB}\,\phi_1^B = 0.$$

Multiple fields:

Note the presence (depending on the particular field of interest) of combinations of 2nd order, 1st order, and 0th order terms.

(This is exactly the behaviour we encountered for sound interacting with a vortex flow...)

The general situation may correspond to multiple metrics, and the causal structure is best investigated using the theory of characteristics...

This leads to uncharted territory well beyond the scope of this workshop....

Multiple fields:

Symbol of the PDE...

Normal cone:

$$\mathcal{N}(q) \equiv \{p_\mu \mid \det\left(f^{\mu\nu}{}_{AB}\; p_\mu\, p_\mu\right) = 0\}\,.$$

Locus of the normals to the characteristic surfaces.

It may be remarked that the present state of the theory of algebraic surfaces does not permit entirely satisfactory applications to the questions of reality of geometric structures which confront us here...

--- Courant and Hilbert

Multiple fields:

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Monge cone: (ray cone, characteristic cone, null cone)

Define $Q(q,p)$ on the co-tangent bundle

$$Q(q,p) \equiv \det\left(f^{\mu\nu}{}_{AB}(q)\, p_\mu\, p_\mu\right).$$

$$\mathcal{M}(q) = \left\{ t^\mu = \frac{\partial Q(q,p)}{\partial p_\mu} \,\middle|\, p_\mu \in \mathcal{N}(q) \right\}.$$

Envelope of the set of characteristic surfaces through "q".

The "Monge cone" is dual to the "normal cone".

Even if the normal cone is relatively simple,
the Monge cone can be absolutely foul.

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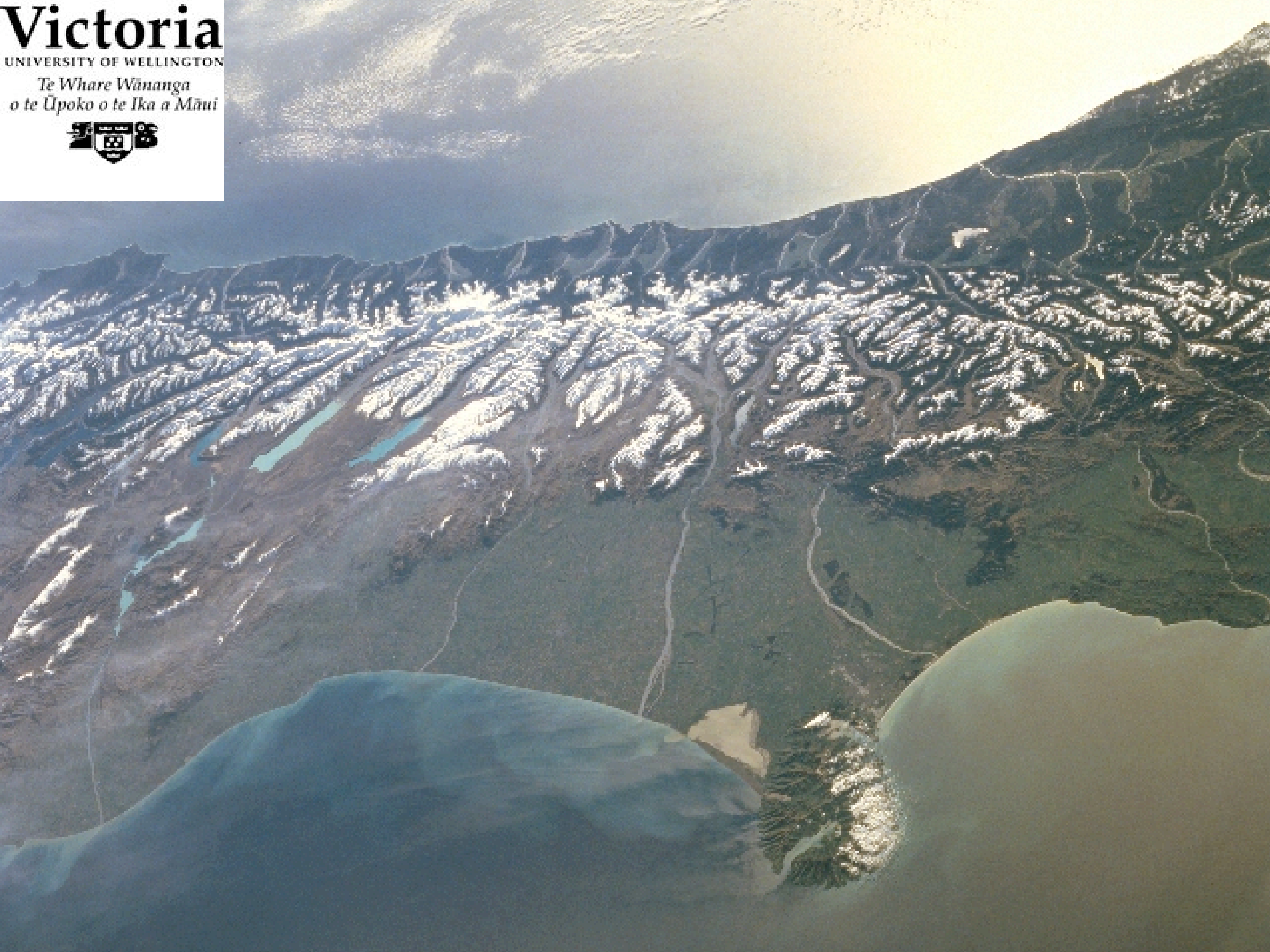


Physics examples:

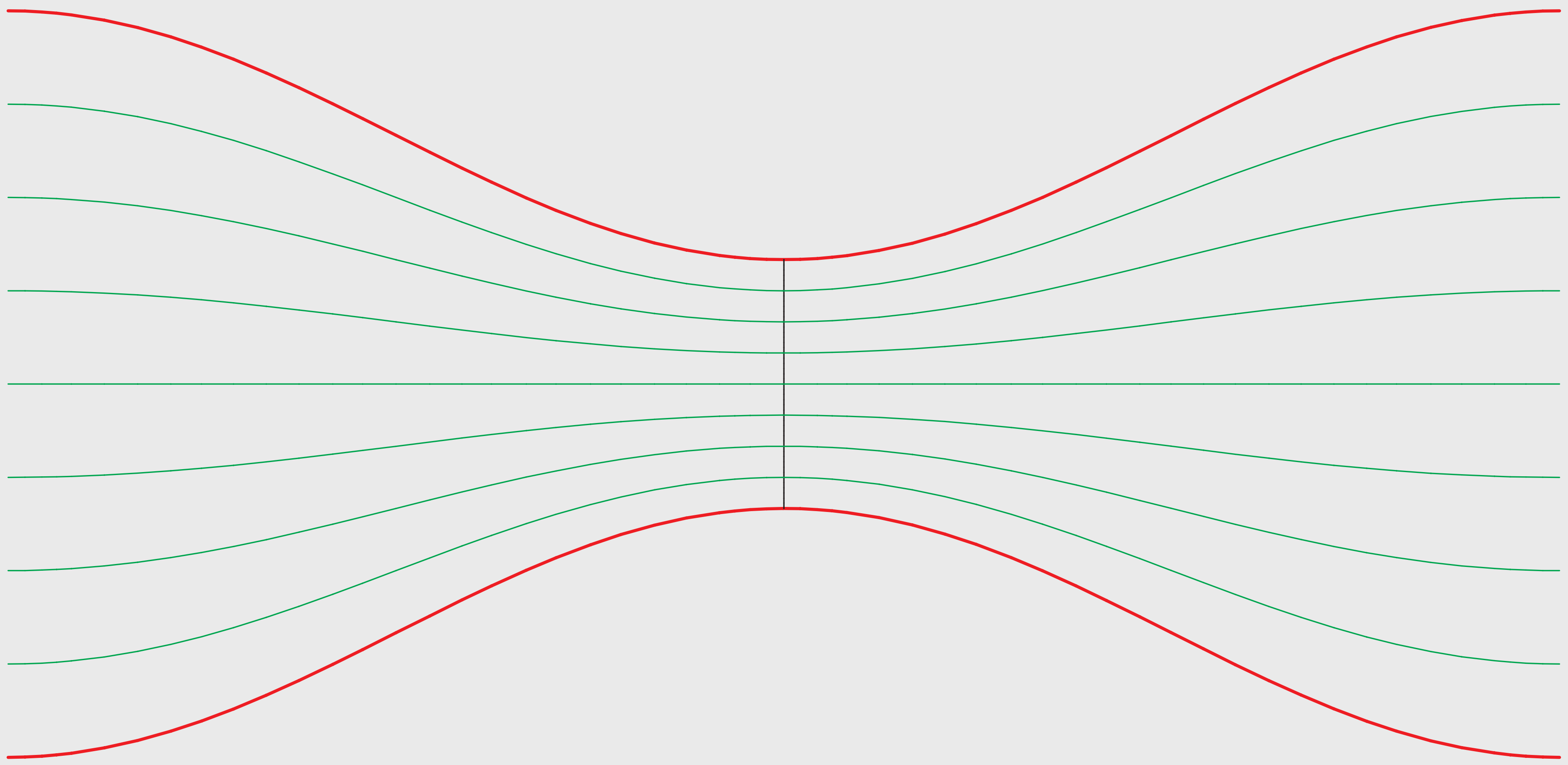
There are numerous physical examples where we have direct experimental/observational evidence for acoustic metrics, up to and including acoustic horizons (dumb holes).

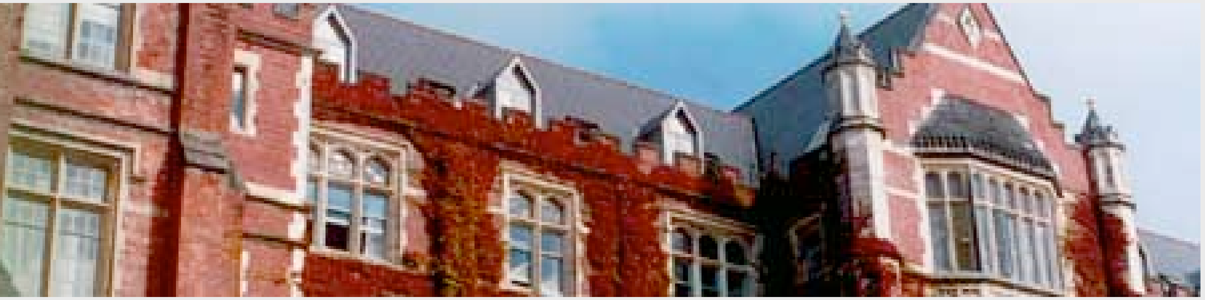
NB: "dumb" = "mute" (silent).

Main examples:

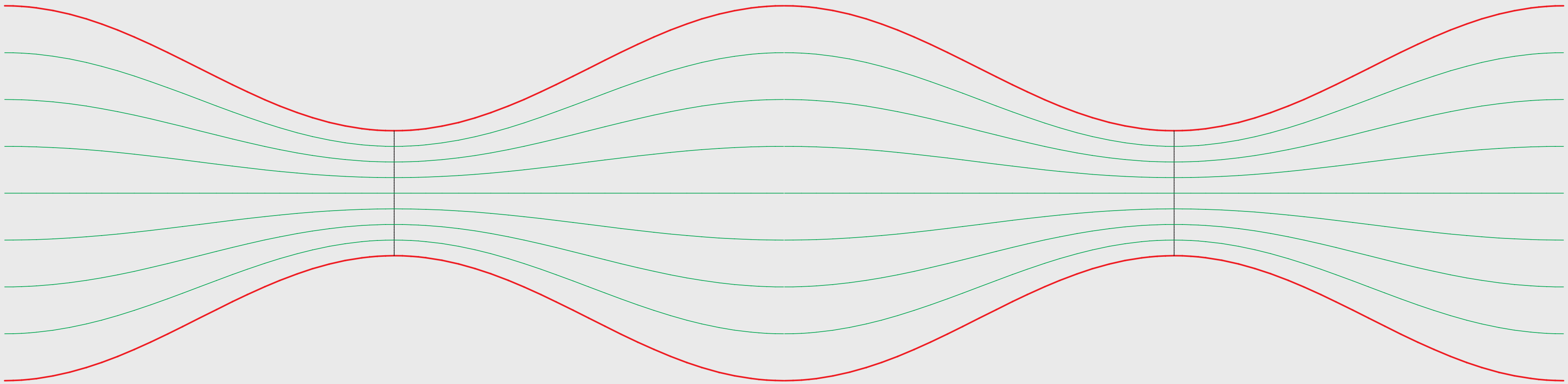
- --- draining bathtub (acoustics and/or surface waves).
- --- supersonic wind tunnels (Laval nozzles).
- --- oscillating bubbles (acoustic apparent horizons).
- --- Parker wind (stellar coronal outflow).
- --- Bondi accretion.

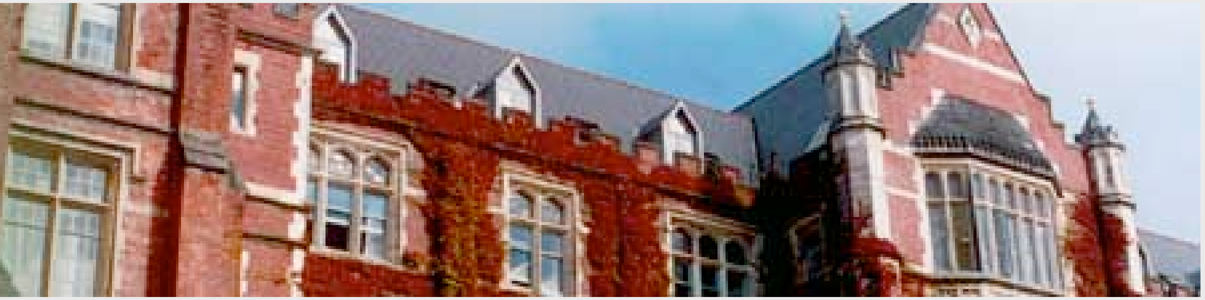
Laval nozzle:



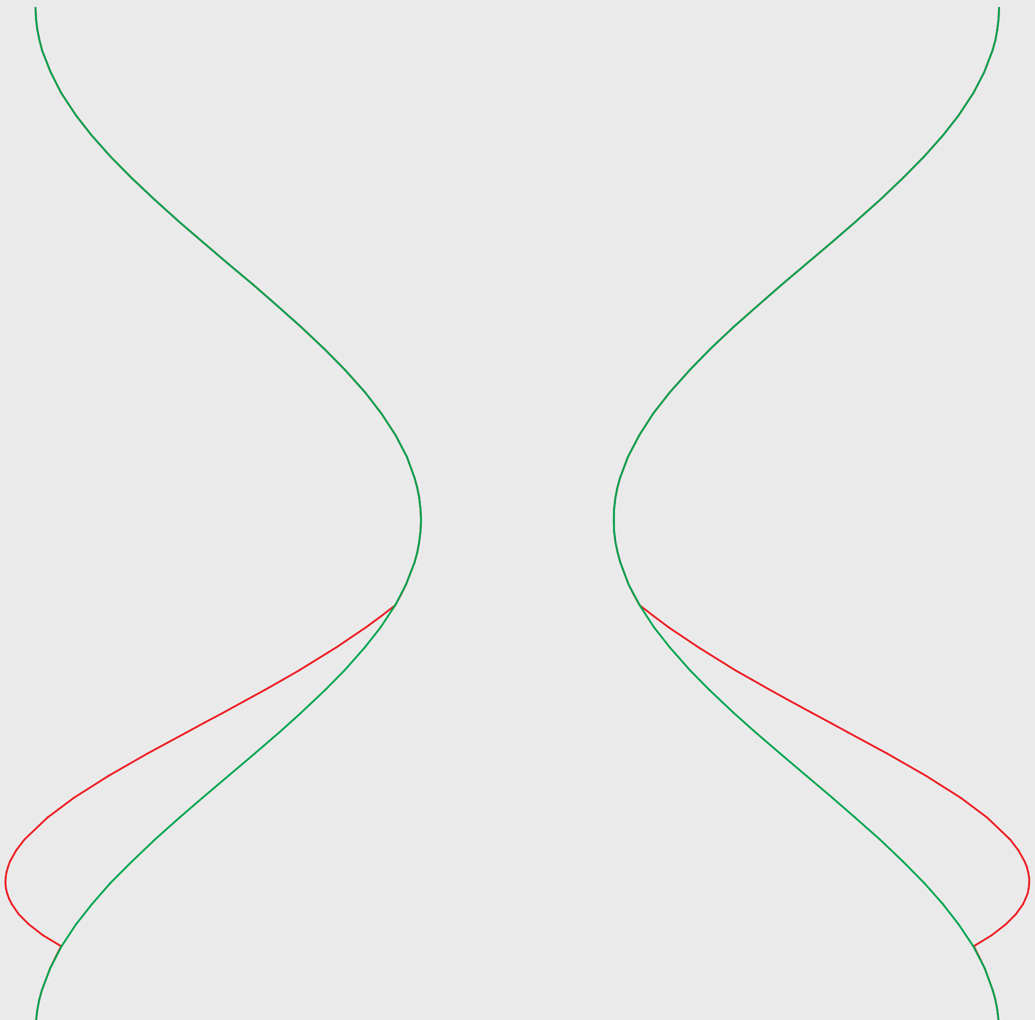


Paired Laval nozzles:





Oscillating bubbles:



Bubble experiments often achieve supersonic collapse --- Mach 5+

(bubble surface)

(apparent horizon)

Apparent horizon lasts for less than a sound crossing time...

Oscillating bubbles:

Spherically symmetric flow constant density fluid.

R(t) is the bubble radius:

$$v = \dot{R}(t)\, \frac{R(t)^2}{r^2}.$$

The acoustic metric is:

$$\mathrm{d}s^2 = -c_s^2 \mathrm{d}t^2 + \left(\mathrm{d}r - \dot{R}(r)\, \frac{R(t)^2}{r^2}\, \mathrm{d}t \right)^2 + r^2 (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2).$$

There is a nasty approximation hiding here,
but qualitative physics is OK...

Parker wind (coronal outflow):

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The solar wind is subsonic as it emerges from the surface.

As it moves out, the plasma density drops,
and so does the speed of sound.

The solar wind goes supersonic in the upper reaches
of the photosphere.

[acoustic black hole horizon, inverted]

The solar wind then remains supersonic out to the
heliopause --- in the outer solar system.

[acoustic white hole horizon, inverted]

Bondi--Hoyle accretion:

Gas cloud with $p = \rho\, c_s^2$.

Symmetric free fall (approximate), onto a central object of mass M:

$$v^2 = \frac{2G\,M}{r}$$

Ignore (for simplicity) corrections due to back-pressure.

Infall velocity exceeds speed of sound when:

$$r < \frac{2G\,M}{c_s^2} = \frac{2G\,M}{c^2}\,\frac{c^2}{c_s^2} = R_S\,\frac{c^2}{c_s^2}$$

Note "acoustic Schwarzschild radius"!

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Bondi--Hoyle accretion:

Ignoring back-pressure, the geometrical acoustics metric for Bondi--Hoyle accretion is:

$$\mathrm{d}s^2 = -c_s^2\, \mathrm{d}t^2 + \left(\mathrm{d}r + \sqrt{\frac{2GM}{r}}\mathrm{d}t\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right).$$

These "acoustic horizons" (aka "sonic points") are of direct observational interest in astrophysics...

Replacing sound speed by light speed, the metric above is in fact identical to the Schwarzschild solution of general relativity in Painleve--Gullstrand coordinates.

(This appears to be an accident, not fundamental!)

(Micro) Summary:

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In short, there are no end of physical situations where we know that "acoustic metrics" are useful.

Futhermore there are many situations in which we have direct observational evidence of the existence of "acoustic horizons".

More generally, we should talk of "analogue metrics" for optical and other analogue systems.

And now for something completely different:
More on why the GR community is so interested....

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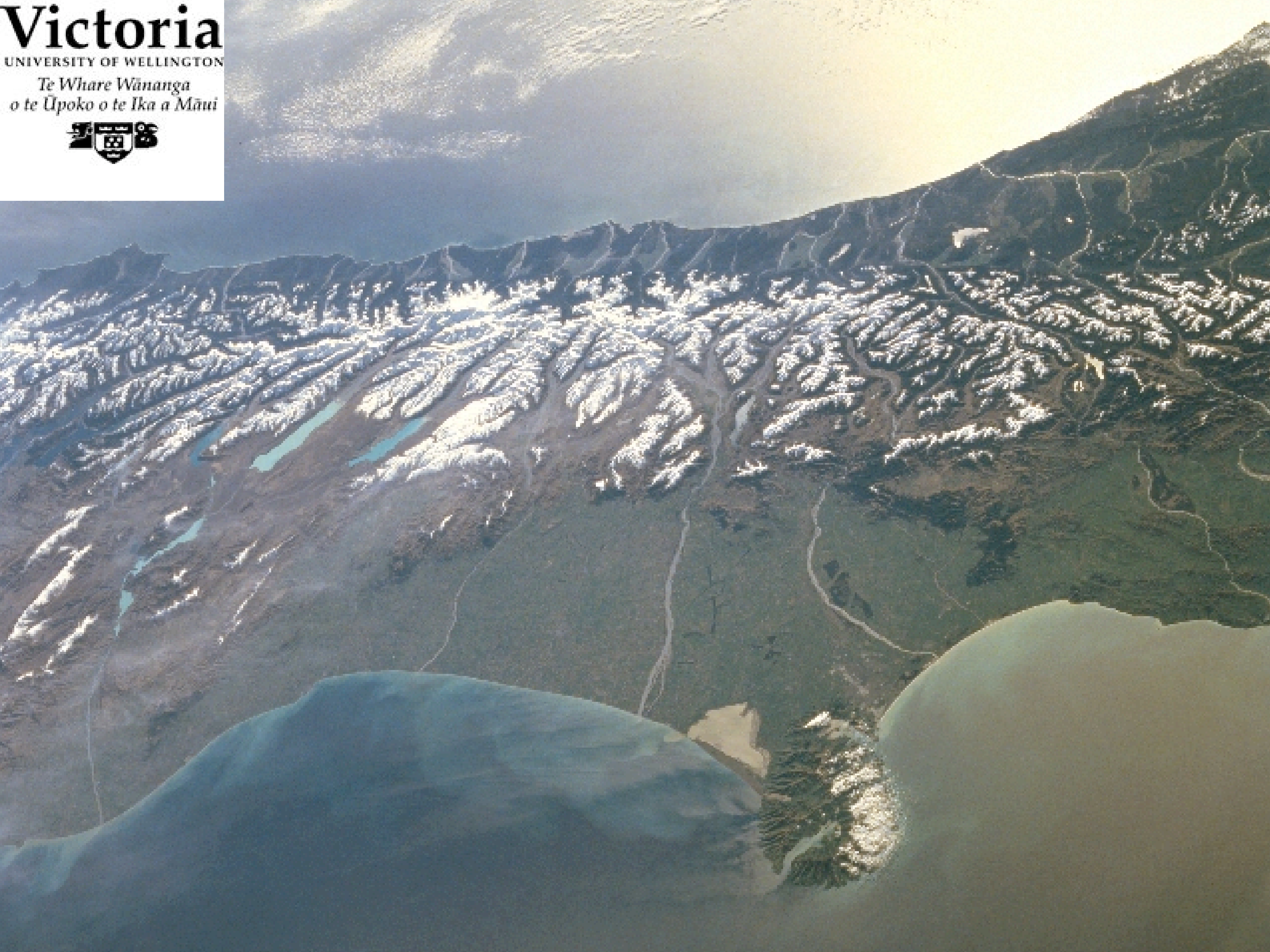


Black holes are not (completely) black:

There is a small quantum mechanical leakage from the horizon --- Hawking radiation.

$$k\, T_H = \frac{\hbar\, g_H}{2\pi\, c_s}.$$

Hawking temperature depends on "surface gravity" and "signal speed" (sound/ light/ whatever...)

Need an apparent horizon that is "long lived" compared to

$$\tau_H = \frac{c_s}{g_H}$$

the timescale determined by surface gravity.

Black holes are not (completely) black:

Hawking radiation is not specifically a GR effect.

Hawking radiation has to do with QFT in the presence of a horizon --- and does not care about the dynamics that set up the horizon.

Hawking radiation is pure kinematics:
How do quantum modes react to
the presence of a horizon?

Very roughly speaking:
negative energy quantum fluctuations fall in,
positive energy quantum fluctuations escape...

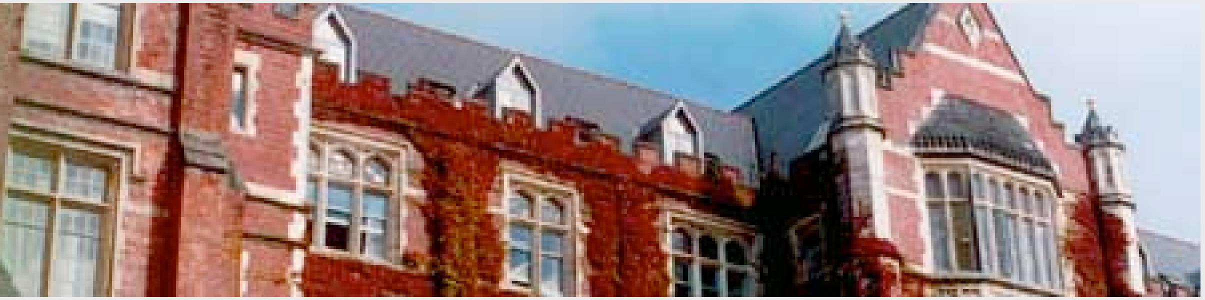
Black holes are not (completely) black:

Despite thousands of theory papers, there are no experimental/ observational tests for the existence of Hawking radiation!

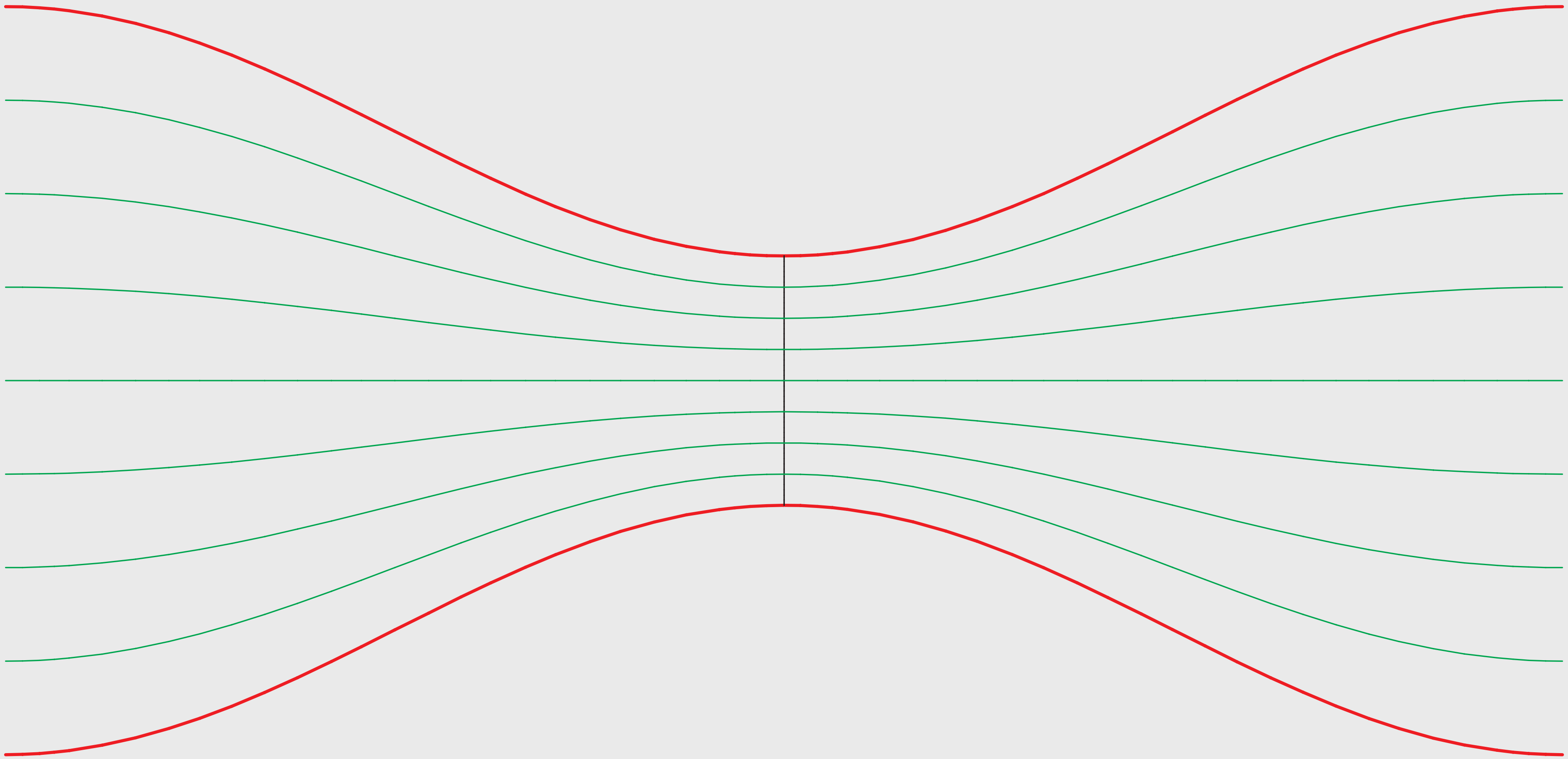
Astrophysical black holes are too heavy (implies low Hawking temperature).

Primordial black holes might be suitable, if (when?) we can find any....

Instead, can we look for analogue Hawking radiation in analogue black holes?



Laval nozzle:



BEC dumb holes:

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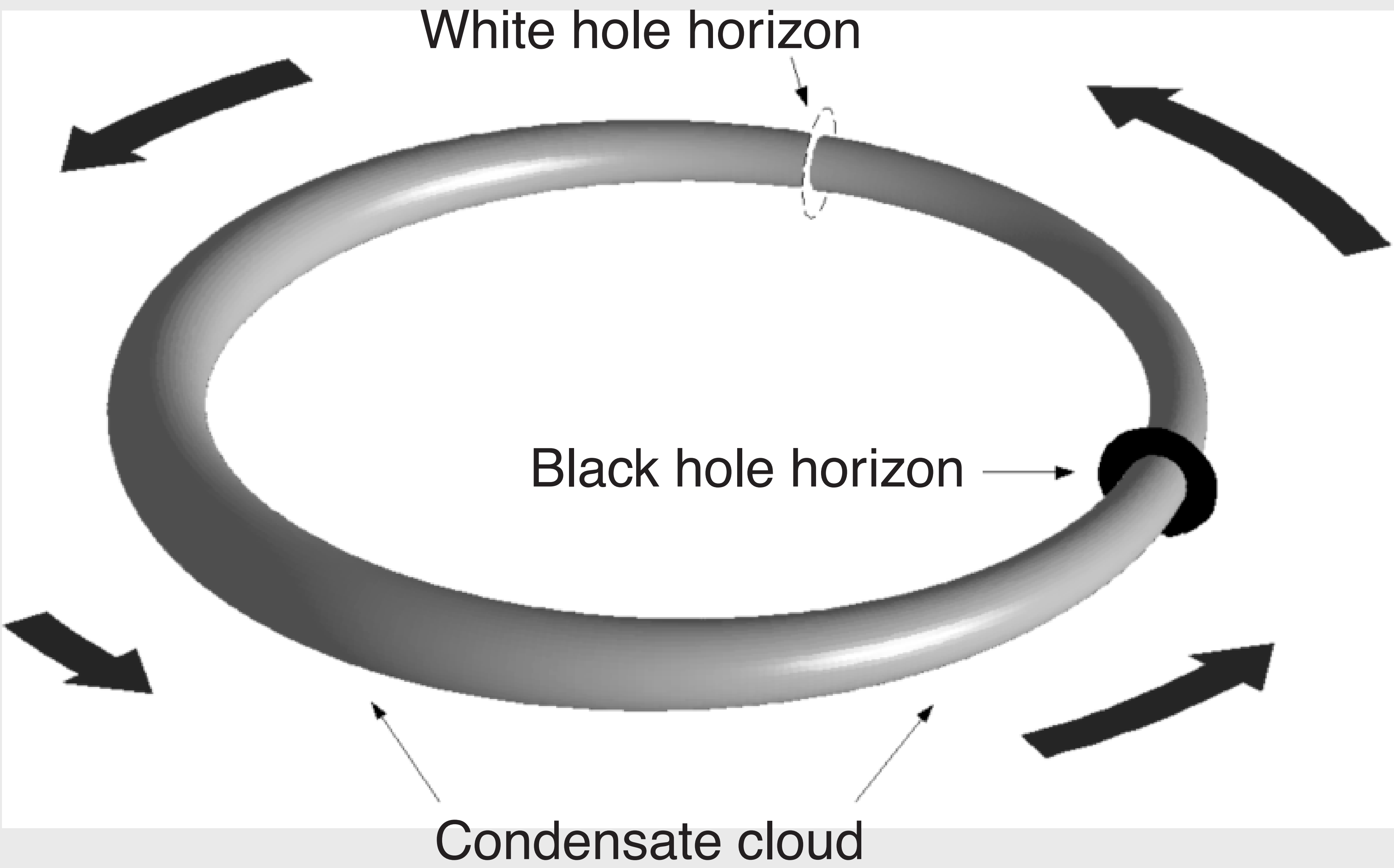
Acoustic black holes based on BECs are technologically interesting for two specific reasons:

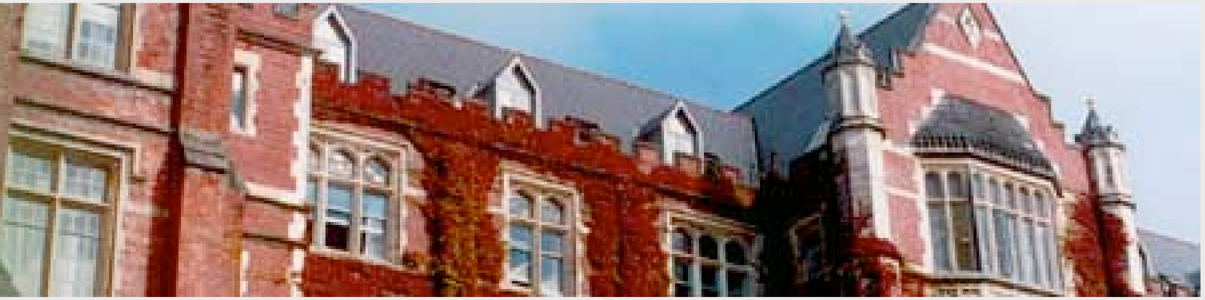
--- the speed of sound is low (mm/sec),
(implying supersonic flow is "easy").

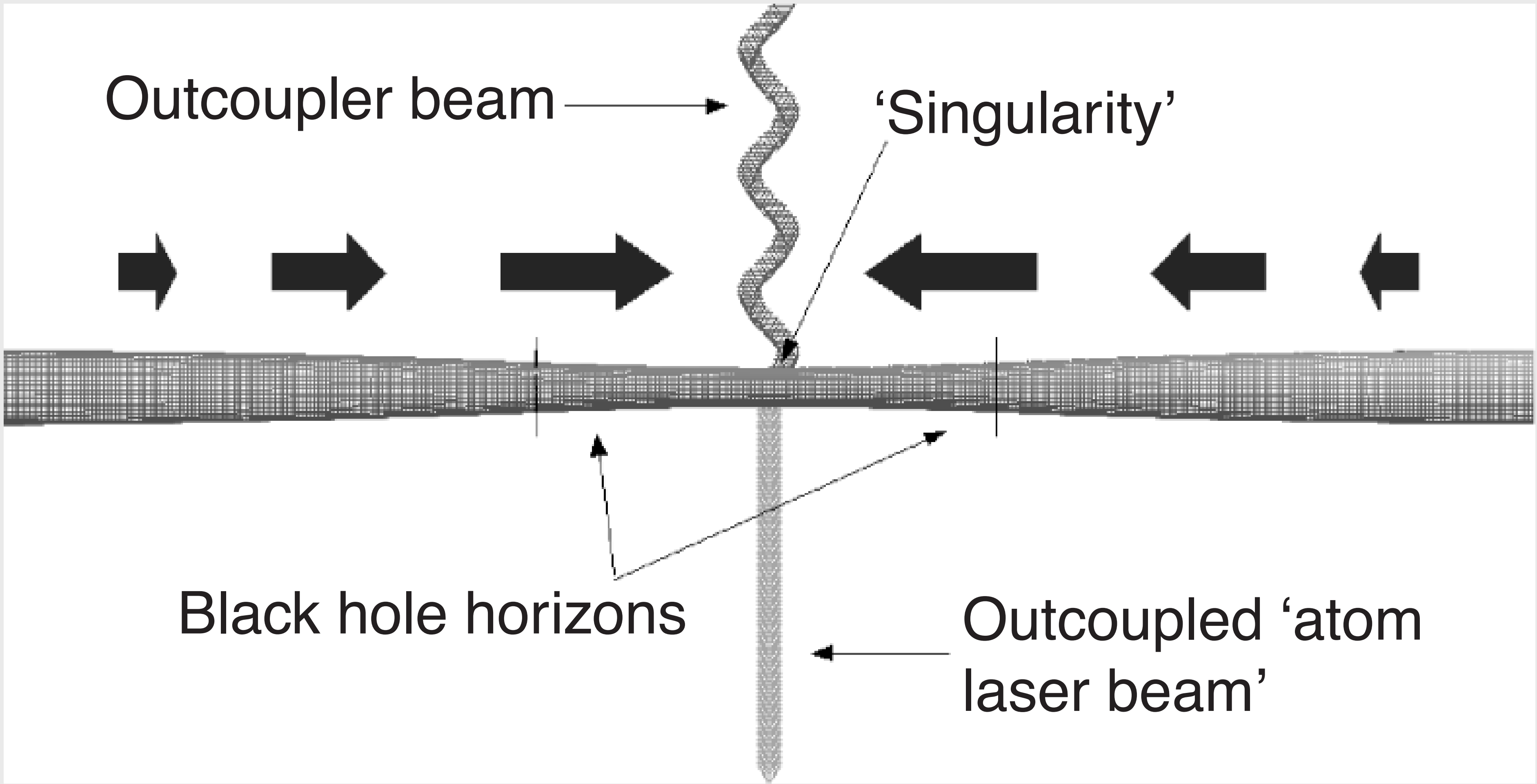
--- the condensate temperature is extremely low,
as low as one nano-Kelvin,
(implying little background).

The interest in BEC dumb holes is not "fundamental".



White hole horizon
Black hole horizon
Condensate cloud




Outcoupler beam
'Singularity'
Black hole horizons
Outcoupled 'atom
laser beam'

BEC dumb holes:

Best (most favourable) estimates:

Under favourable conditions might get

$$T_H \approx 70 \text{ nK}$$

Compare to

$$T_{condensation} \approx 90 \text{ nK}$$

and

$$T_{condensate} \approx 1 \text{ nK}$$

Some experimental BEC groups are now interested...

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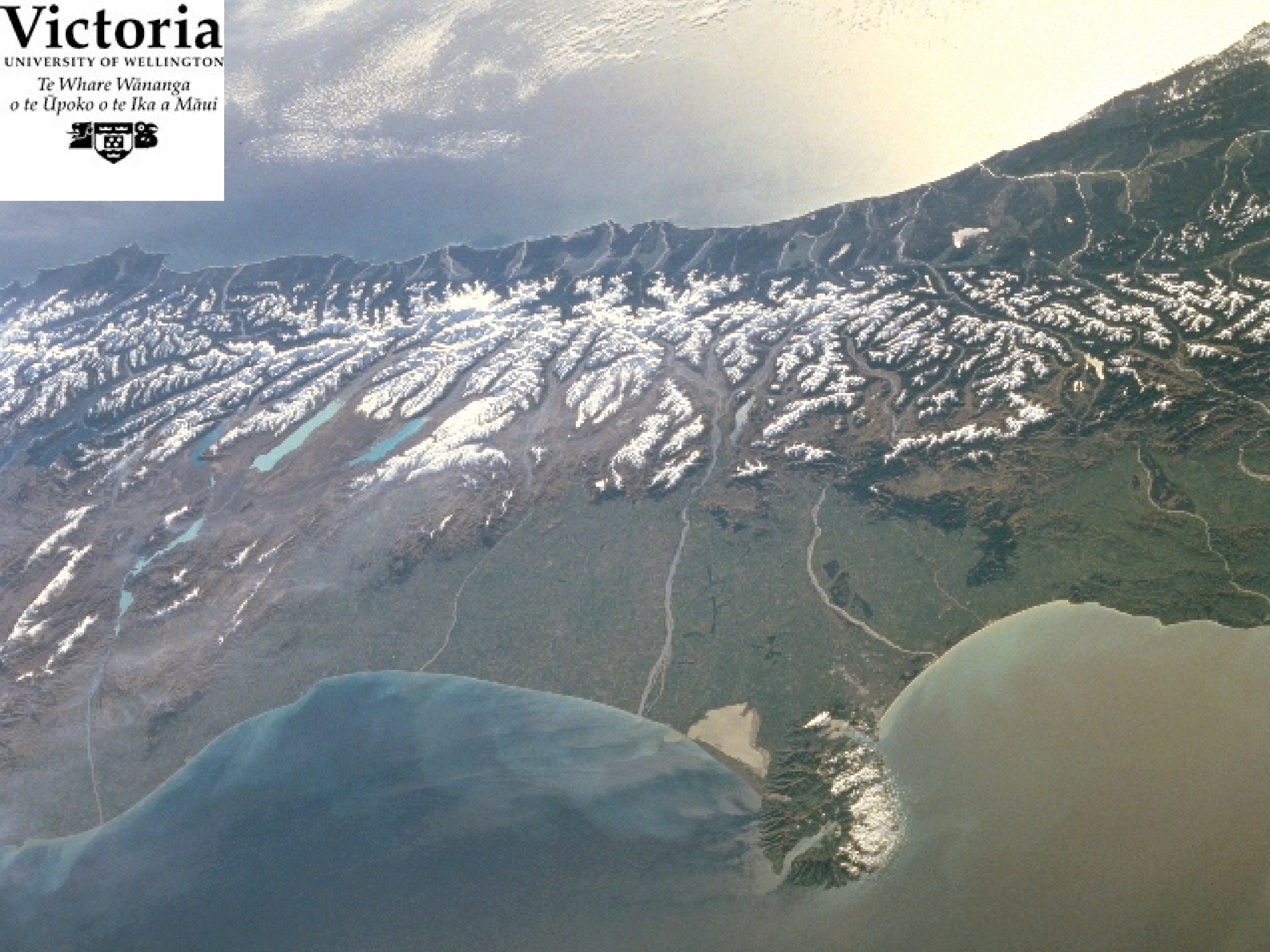

Kerr spacetime:

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"Analogue models" for curved spacetime can be very useful for guiding physical intuition in general relativity.

The "acoustic metric" describing sound in a flowing fluid is perhaps the simplest of the "analogue models".

A "draining bathtub" vortex can be set up to exhibit both a horizon and an ergo-surface.

How close can we get to modelling the actual geometry of the Kerr spacetime using a fluid vortex?

Mimicking Kerr spacetime:

Can we construct an acoustic geometry that mimics Kerr spacetime in detail?

There is a fundamental geometrical obstruction:

For simple fluids the spatial slices of the acoustic geometry are always conformally flat.

The spatial slices of Kerr are never conformally flat.

The best you can hope for is to consider the equatorial slice of the Kerr spacetime.

Zero radial flow:

$$\vec{v}(r) = v_{\hat{\theta}}(r)\ \hat{\theta}.$$

$$\vec{a} = (\vec{v}\cdot\nabla)\vec{v} = -\frac{v_{\hat{\theta}}(r)^2}{r}\ \hat{r}.$$

$$\vec{f} = f_{\hat{r}}\ \hat{r} = \left\{ -\rho(r)\ \frac{v_{\hat{\theta}}(r)^2}{r} + c^2\ \partial_r \rho(r) \right\} \hat{r}\ .$$

This is the external force required to maintain the vortex.

For zero external force: $$\frac{v_\theta(r)^2}{c(r)^2} = -\ r\ \partial_r \ln \rho(r)$$

General radial flow:

$$\vec{v} = v_{\hat{r}}(r)\ \hat{r} + v_{\hat{\theta}}(r)\ \hat{\theta}.$$

Continuity implies:

$$\oint \rho(r)\ \vec{v}(r) \cdot \hat{r}\ \mathrm{d}s = 2\pi\ \rho(r)\ v_{\hat{r}}(r)\ r = 2\pi\ k_1.$$

$$\rho(r) = \frac{k_1}{r\ v_{\hat{r}}(r)}.$$

$$\vec{f} = \frac{k_1}{r v_{\hat{r}}}\ (\vec{v} \cdot \nabla)\vec{v} + c_s^2\ \partial_r \left(\frac{k_1}{r v_{\hat{r}}} \right)\ \hat{r},$$

General radial flow:

Calculating external force and decomposing into angular and radial pieces:

$$f_{\hat{r}} = \vec{f} \cdot \hat{r} = k_1 \left\{ \frac{1}{r v_{\hat{r}}} \; \left[\frac{1}{2} \partial_r [v_{\hat{r}}(r)^2] - \frac{v_{\hat{\theta}}(r)^2}{r} \right] + c_s^2 \, \partial_r \left(\frac{1}{r v_{\hat{r}}} \right) \right\},$$

$$f_{\hat{\theta}} = \vec{f} \cdot \hat{\theta} = k_1 \left\{ \frac{1}{r^2} \partial_r [r \; v_{\hat{\theta}}(r)] \right\}.$$

With enough effort you can mimic any velocity profile.

The Kerr equator:

In Boyer--Lindquist coordinates:

$$(\mathrm{d}s^2)_{(2+1)} = -\mathrm{d}t^2 + \frac{2m}{r}(\mathrm{d}t - a\;\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{1 - 2m/r + a^2/r^2} + (r^2 + a^2)\;\mathrm{d}\phi^2.$$

In the r-ϕ plane:

$$(\mathrm{d}s^2)_{(2)} = \frac{\mathrm{d}r^2}{1 - 2m/r + a^2/r^2} + \left(r^2 + a^2 + \frac{2ma^2}{r}\right)\mathrm{d}\phi^2.$$

This is conformally flat, but not obviously so.

Adopt new coordinates:

$$\frac{\mathrm{d}r^2}{1 - 2m/r + a^2/r^2} + \left(r^2 + a^2 + \frac{2ma^2}{r}\right)\mathrm{d}\phi^2 = \Omega(\tilde{r})^2\;[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\;\mathrm{d}\phi^2].$$

The Kerr equator:

This gives two equations:

$$\left. \begin{aligned} \left(r^2 + a^2 + \frac{2ma^2}{r}\right) &= \Omega(\tilde{r})^2\ \tilde{r}^2, \\ \frac{\mathrm{d}r^2}{1 - 2m/r + a^2/r^2} &= \Omega(\tilde{r})^2\ \mathrm{d}\tilde{r}^2 \end{aligned} \right\} \quad \text{Eliminate } \Omega(\tilde{r})$$

Differential equation:

$$\frac{1}{\tilde{r}(r)}\ \frac{\mathrm{d}\tilde{r}(r)}{\mathrm{d}r} = \frac{1}{\sqrt{1 - 2m/r + a^2/r^2}\sqrt{r^2 + a^2 + 2ma^2/r}},$$

$$\tilde{r}(r) = \exp\left\{\int \frac{\mathrm{d}r}{\sqrt{1 - 2m/r + a^2/r^2}\sqrt{r^2 + a^2 + 2ma^2/r}}\right\}.$$

The Kerr equator:

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Fix boundary conditions:

$$\tilde{r}(r) = r \exp\left[-\int_r^\infty \left\{ \frac{1}{\sqrt{1-2m/\bar{r}+a^2/\bar{r}^2}\sqrt{\bar{r}^2+a^2+2ma^2/\bar{r}}} - \frac{1}{\bar{r}} \right\} \mathrm{d}\bar{r} \right]$$

Define:
$$\begin{cases} \tilde{r} = r\; F(r), \text{ with } \lim_{r\to\infty} F(r) = 1, \\ r = \tilde{r}\; H(\tilde{r}) \text{ with } \lim_{\tilde{r}\to\infty} H(\tilde{r}) = 1. \end{cases}$$

$$\Omega(\tilde{r})^2 = \frac{r^2+a^2+2ma^2/r}{\tilde{r}^2} = H(\tilde{r})^2 \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right),$$

$$(\mathrm{d}s^2)_{(2+1)} = -\mathrm{d}t^2 + \frac{2m}{r}(\mathrm{d}t^2 - 2a\,\mathrm{d}\phi\,\mathrm{d}t) + \Omega(\tilde{r})^2\,[\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\phi^2].$$

The Kerr equator:

This is now in “acoustic form”:

$$(\mathrm{d}s^2)_{(2+1)} \;=\; \Omega(\tilde{r})^2 \left\{ -\Omega(\tilde{r})^{-2} \left[1 - \frac{2m}{r}\right] \mathrm{d}t^2 - \Omega(\tilde{r})^{-2} \frac{4am}{r}\, \mathrm{d}\phi\, \mathrm{d}t + [\mathrm{d}\tilde{r}^2 + \tilde{r}^2\, \mathrm{d}\phi^2] \right\}.$$

Pick off the coefficients:

$$\frac{\rho}{c} = \Omega(\tilde{r})^2 = H^2(\tilde{r}) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right).$$

$$v_\phi = \Omega(\tilde{r})^{-2}\, \frac{2am}{r} = -\frac{2am}{r}\, H^{-2}(\tilde{r}) \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)^{-1}.$$

$$c^2 = \Omega(\tilde{r})^{-4}\, H^2(\tilde{r}) \left\{1 - \frac{2m}{r} + \frac{a^2}{r^2}\right\}$$

$$\rho = H(\tilde{r}) \sqrt{1 - \frac{2m}{r} + \frac{a^2}{r^2}}$$

This is the “equivalent vortex”!

The Kerr equator:

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$$\frac{\rho}{c} = \Omega^2(r) = F^{-2}(r)\ \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right).$$

$$v_\phi = -\frac{2am}{r}\ F^2(r)\ \left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)^{-1}.$$

$$c^2 = F^2(r)\ \left\{1 - \frac{2m}{r} + \frac{a^2}{r^2}\right\}\left(1 + \frac{a^2}{r^2} + \frac{2ma^2}{r^3}\right)^{-2}.$$

$$\rho(r) = F^{-1}(r)\ \sqrt{1 - \frac{2m}{r} + \frac{a^2}{r^2}}$$

$$F(r) = \exp\left[-\int_r^\infty \left\{\frac{1}{\sqrt{1 - 2m/\bar{r} + a^2/\bar{r}^2}\sqrt{\bar{r}^2 + a^2 + 2ma^2/\bar{r}}} - \frac{1}{\bar{r}}\right\} \mathrm{d}\bar{r}\right].$$

This has the advantage of being completely explicit, albeit a trifle messy!

The Kerr equator:

Ergo-surface: $$r_E = 2m.$$

Horizon: $$r_H = m + \sqrt{m^2 - a^2} < r_E.$$

The Kerr equator can [in principle] be exactly simulated by a very specific vortex.

This needs a very specific external force, and a very specific equation of state.

This is not likely to be experimentally feasible.

Somewhat disappointing!

The Kerr equator:

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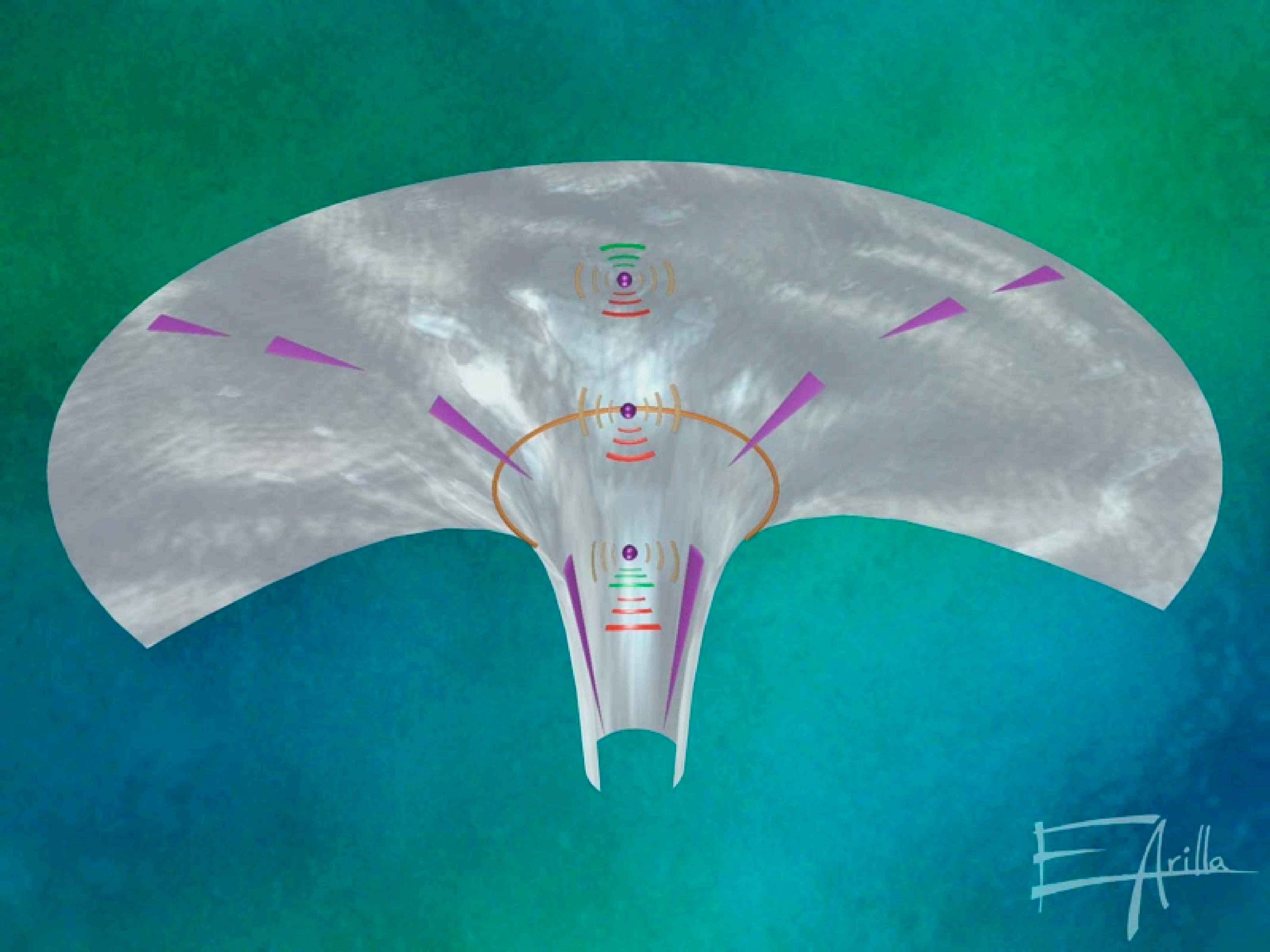
Technical surprise:

The Doran coordinates were not useful?

(Doran coordinates are the natural extension of Painleve--Gullstrand coordinates, which are very useful for the "acoustic Schwarzschild" geometry.

The problem lies with the off-diagonal parts of the space metric...

For the future: simple "analogue models" that generate fully general geometries...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Conclusions:

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I hope by now you are convinced that analogue models are multi-directionally useful...

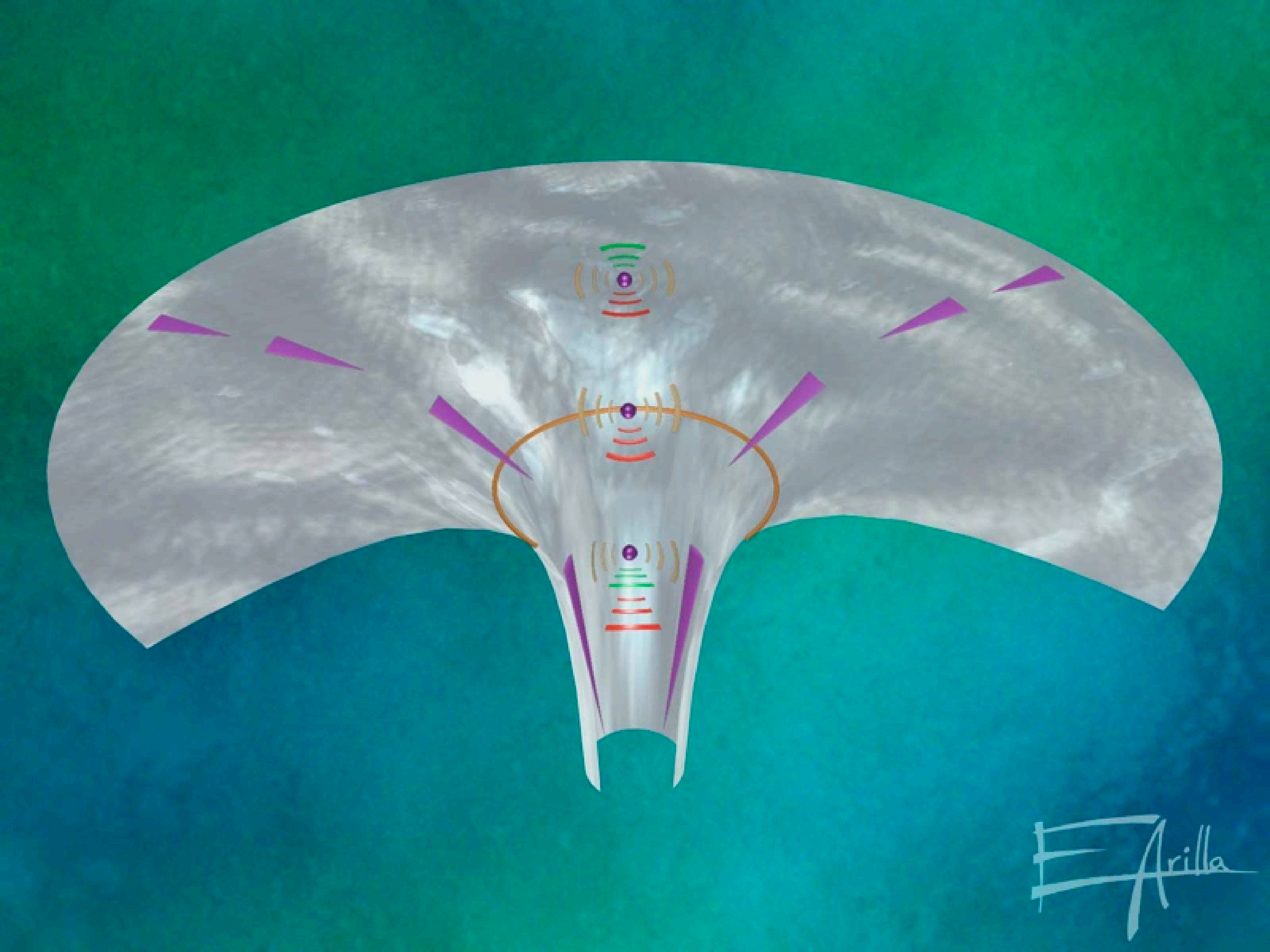
They are useful for both theorists and experimentalists, and serve to build strong cross-connections between otherwise disparate fields.

In this talk I have focussed on the "acoustic geometries", but there are additionally "optical geometries", "surface wave geometries", the "mechano-optical analogy"...

And I cherish more than anything else the Analogies, my most trustworthy masters.

They know all the secrets of Nature, and they ought least to be neglected in Geometry.

--- Johannes Kepler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