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Effective refractive index tensor for weak-field gravity

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Abstract:

Gravitational lensing in a weak but otherwise arbitrary gravitational field can to linearized order be described in terms of an analogy that uses a 3 x 3 tensor to characterize an "effective refractive index".

If the sources generating the gravitational field all have small internal fluxes, stresses, and pressures, then the 3 x 3 tensor is automatically isotropic and the "effective refractive index" is simply a scalar that can be determined in terms of a classic result involving the Newtonian gravitational potential.

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Abstract:

In contrast if anisotropic stresses are ever important then the gravitational field acts similarly to an anisotropic crystal.

We derive simple formulae for the refractive index tensor, and indicate some situations in which this will be important.

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Weak-field gravity in Einstein's general relativity is actually more general than straightforward Newtonian gravity.

While the approximate validity of Newtonian gravity is certainly limited to the weak-field regime, Newtonian gravity makes significant additional assumptions as to the smallness of effects that depend on the internal stresses, pressures, and energy fluxes in the massive bodies that act as source for the gravitational field.

While there is no significant doubt that for planets, and indeed most stars, internal stresses can safely be neglected, the situation for neutron star crusts (or indeed the “dark matter” that makes up approximately 90% of most spiral galaxies) is much more uncertain.

In view of this we have developed a formalism that makes no assumptions about the relative smallness of internal stresses (and pressures and fluxes), to see how gravitational lensing is affected.

Null curve:

$$g_{ab}\,\mathrm{d}X^a(\lambda)\,\mathrm{d}X^b(\lambda) = 0\,.$$

Weak field (quasi-Cartesian coordinates):

$$\eta_{ab}\,\mathrm{d}X^a(\lambda)\,\mathrm{d}X^b(\lambda) + h_{ab}\,\mathrm{d}X^a(\lambda)\,\mathrm{d}X^b(\lambda) = 0\,.$$

Static weak-field:

$$g_{ab}\,\frac{\mathrm{d}X^a}{\mathrm{d}\lambda}\,\frac{\mathrm{d}X^b}{\mathrm{d}\lambda} = (-1 + h_{tt})\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + (\delta_{ij} + h_{ij})\,\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = 0\,.$$

Choose: $\lambda = t$

$$g_{ab}\,\frac{\mathrm{d}X^a}{\mathrm{d}t}\,\frac{\mathrm{d}X^b}{\mathrm{d}t} = (-1 + h_{tt}) + (\delta_{ij} + h_{ij})\,\dot{x}^i\,\dot{x}^j = 0\,.$$

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Now introduce the "coordinate speed of light" by defining:

$$\dot{x}^i = ||\dot{x}^i|| \, \hat{k}^i; \qquad ||\hat{k}^i|| = 1 = \sqrt{\delta_{ij} \, \hat{k}^i \, \hat{k}^j};$$

$$||\dot{x}^i|| = \sqrt{\frac{1 - h_{tt}}{1 + h_{ij} \, \hat{k}^i \, \hat{k}^j}} \approx 1 - \frac{1}{2} h_{tt} - \frac{1}{2} h_{ij} \, \hat{k}^i \, \hat{k}^j + \mathcal{O}(h_{ab}^2)$$

Refractive index:

$$n(\hat{k}) = \frac{1}{||\dot{x}^i||} \approx 1 + \frac{1}{2} h_{tt} + \frac{1}{2} h_{ij} \, \hat{k}^i \, \hat{k}^j + \mathcal{O}(h_{ab}^2).$$

$$n_{ij} \equiv \left(1 + \frac{1}{2} h_{tt}\right) \delta_{ij} + \frac{1}{2} h_{ij} \,,$$

To connect the refractive index tensor to the presence of stress-energy, define:

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$$\begin{aligned} \nabla^2\Phi &= 4\pi\, G_N\, \rho, \\ \nabla^2\Psi_{ij} &= 4\pi\, G_N\, T_{ij}, \end{aligned}$$

a Newtonian scalar potential,
and post-Newtonian tensor potential.

Because the spacetime is (for now) assumed static:

$$T_{ab} = \begin{bmatrix} \rho & 0 \\ 0 & T_{ij} \end{bmatrix}.$$

Weak-field Einstein:

$$\nabla^2 h_{ab} = -16\pi\, G_N\, \left(T_{ab} - \frac{1}{2}\, T\, \eta_{ab} \right) + \mathcal{O}(h^2),$$

In Einstein-Fock-de Donder gauge, and with suitable boundary conditions:

$$\nabla^2 h_{ij} = -16\pi G_N \left(T_{ij} - \delta_{ij}\, \frac{1}{2} \left[-\rho + \delta^{kl}\; T_{kl} \right] \right).$$

$$\nabla^2 h_{tt} = -8\pi\, G_N\, (\rho + \delta^{kl} T_{kl})$$

In terms of the density scalar potential and stress-pressure tensor potential

$$h_{tt} = -2(\Phi + \delta^{kl}\; \Psi_{kl});$$

$$h_{ij} = -2(2\Psi_{ij} + \delta_{ij}[\Phi - \delta^{kl}\; \Psi_{kl}]).$$

Refractive index tensor:

$$n_{ij} = (1 - 2\Phi)\,\delta_{ij} - 2\Psi_{ij}\,.$$

Internal stresses isotropic (perfect fluid):

$$T_{ij} \to p\,\delta_{ij} \text{ and } \Psi_{ij} \to \Psi_0\,\delta_{ij},$$

$$\nabla^2\Phi \;= 4\pi\,G_N\,\rho, \qquad\qquad n_{ij} = (1 - 2\Phi - 2\Psi_0)\,\delta_{ij}\,.$$
$$\nabla^2\Psi_0 = 4\pi\,G_N\,p,$$

Internal stresses negligible:

$$h_{tt} = -2\Phi; \qquad h_{ij} = -2\Phi\,\delta_{ij}; \qquad n_{ij} = (1 - 2\Phi)\,\delta_{ij}\,.$$

Anisotropic stress implies anisotropic refractive index.

Stationary non-static geometries:

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$$h_{ab} = \begin{bmatrix} h_{tt} & h_{tj} \\ h_{it} & h_{ij} \end{bmatrix}$$

Photon trajectory:

$$g_{ab} \frac{\mathrm{d}X^a}{\mathrm{d}\lambda} \frac{\mathrm{d}X^b}{\mathrm{d}\lambda} = (-1 + h_{tt}) \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2h_{tj} \frac{\mathrm{d}t}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} + (\delta_{ij} + h_{ij}) \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j}{\mathrm{d}\lambda} = 0\,.$$

Choose: $\lambda = t$

Coordinate speed of light:

$$||\dot{x}^i|| = \frac{1}{(\delta_{ij} + h_{ij})\,\hat{k}^i\,\hat{k}^j} \left(-\,h_{tj}\,\hat{k}^j + \sqrt{(h_{tj}\,\hat{k}^j)^2 - (-1 + h_{tt})(\delta_{ij} + h_{ij})\,\hat{k}^i\,\hat{k}^j}\right).$$

Refractive index:

$$n(\hat{k}) = 1 + \frac{1}{2}\left(h_{tt} + 2h_{tj}\,\hat{k}^j + h_{ij}\,\hat{k}^i\,\hat{k}^j\right) + \mathcal{O}(h^2)$$

$$n(\hat{k}) = n_{ij}\,\hat{k}^i\,\hat{k}^j + h_{tj}\,\hat{k}^j + \mathcal{O}(h^2)$$

Effective medium moving with velocity $-h_{tj}$

Define the "flux potential":

$$\nabla^2\Pi_j = 4\pi\,G_N\,T_{tj}$$

Einstein equations:

$$\nabla^2 h_{tj} = -16\pi\,G_N\,T_{tj}. \qquad h_{tj} = -4\,\Pi_j,$$

Stationary case:

$$n(\hat{k}) = n_{ij}(\Phi, \Psi)\ \hat{k}^i\, \hat{k}^j\ - 4\, \Pi_j\, \hat{k}^j + \mathcal{O}(\Phi^2, \Psi^2, \Pi^2)$$

$$n_{ij} = (1 - 2\Phi)\ \delta_{ij} - 2\Psi_{ij}\ .$$

Potentials:

$$\nabla^2\Phi \quad = 4\pi\ G_N\ \rho,$$

$$\nabla^2\Psi_{ij} = 4\pi\ G_N\ T_{ij},$$

$$\nabla^2\Pi_j = 4\pi\ G_N\ T_{tj}$$

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Charges:

$$m = \int T_{tt}\, \mathrm{d}^3x = \int \rho\, \mathrm{d}^3x;$$

$$p_i = \int T_{ti}\, \mathrm{d}^3x = \int j_i\, \mathrm{d}^3x;$$

$$\mu_{ij} = \int T_{ij}\, \mathrm{d}^3x;$$

$$\Sigma_{ab} = \left[\begin{array}{c|c} m & p_i \\ \hline p_j & \mu_{ij} \end{array}\right] = \int T_{ab}\, \mathrm{d}^3x.$$

Dominant multi-poles:

$$\Phi = -G_N\, m/r; \qquad \Pi_i = -G_N\, p_i/r; \qquad \Psi_{ij} = -G_N\, \mu_{ij}/r.$$

$$n(\hat{k}) = 1 + \frac{2G_N\, m}{r} + \frac{4G_N\, p_i\, \hat{k}^i}{r} + \frac{2G_N\, \mu_{ij}\, \hat{k}^i\, \hat{k}^j}{r} = 1 + \frac{2G_N\, \Sigma_{ab}\, k^a\, k^b}{r},$$

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The "null energy condition" [NEC] then guarantees the refractive index is always greater than unity, and the coordinate speed of light is always less than unity.

This ties the discussion back to "superluminal censorship", in that physically sensible sources always lead to Shapiro time delay (not a time advance).

Central messages:

1) Pressure and stress can affect gravity lensing.
2) Anisotropic stress-energy implies anisotropic "refractive index" implies anisotropic propagation of light.