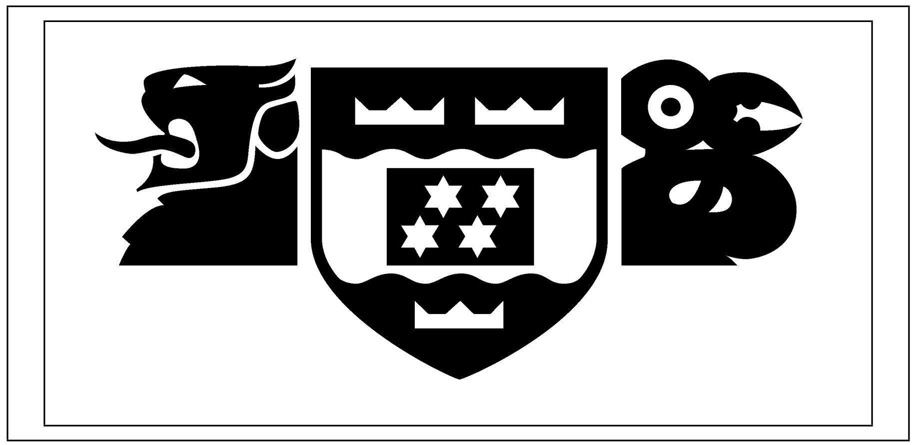

Victoria University of Wellington
Te Whare Wānanga o te Ūpoko o te Ika a Maui

Jerk, snap, and the cosmological EOS

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Taylor expand the cosmological equation of state around the current epoch:

$$p = p_0 + \kappa_0\,(\rho - \rho_0) + \frac{1}{2}\,\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2}\bigg|_0 (\rho - \rho_0)^2 + O[(\rho - \rho_0)^3].$$

This is the simplest model one can consider that does not make any *a priori* restrictions on the nature of the cosmological fluid.

What can we say about the coefficients?

Determining the first three Taylor coefficients of the EOS at the current epoch requires a measurement of the deceleration, jerk, and snap — the *second*, *third*, and *fourth* derivatives of the scale factor with respect to time.

Model building:

Assume you know or can determine $a(t)$. Use the Einstein equations in reverse to calculate the energy density $\rho(t)$ and pressure $p(t)$ via

$$8\pi G_N \; \rho(t) = 3c^2 \left[\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right];$$

$$8\pi G_N \; p(t) = -c^2 \left[\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} + 2\frac{\ddot{a}}{a}\right].$$

Use this to calculate $w = p/\rho$, dp/dρ, and $H(z)$...

Deceleration, jerk, and snap:

- Standard terminology in mechanics: The first four time derivatives of position are velocity, acceleration, jerk, and snap.
- Jerk is also sometimes referred to as jolt.
- Less common alternative terminologies for jerk are: pulse, impulse, bounce, surge, shock, and super-acceleration.
- Snap is also sometimes called jounce.
- The fifth and sixth time derivatives are sometimes somewhat facetiously referred to as crackle and pop.

Deceleration, jerk, and snap:

- In a cosmological setting this makes it appropriate to define Hubble, deceleration, jerk, and snap parameters as

$$H(t) = +\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t};$$

$$q(t) = -\frac{1}{a}\,\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\,\left[\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right]^{-2};$$

$$j(t) = +\frac{1}{a}\,\frac{\mathrm{d}^3 a}{\mathrm{d}t^3}\,\left[\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right]^{-3};$$

$$s(t) = +\frac{1}{a}\,\frac{\mathrm{d}^4 a}{\mathrm{d}t^4}\,\left[\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right]^{-4}.$$

Deceleration, jerk, and snap:

- In particular, at arbitrary time t

$$w(t) = \frac{p}{\rho} = -\frac{H^2(1-2q) + kc^2/a^2}{3(H^2 + kc^2/a^2)} = -\frac{(1-2q) + kc^2/(H^2a^2)}{3[1 + kc^2/(H^2a^2)]}.$$

- Inflation implies $H_0 a_0/c \gg 1$.

$$\rho_0 \approx \frac{3}{8\pi G_N}\, H_0^2 > 0; \qquad w_0 \approx -\frac{(1-2q_0)}{3}.$$

- But determining w_0 is not the same as extracting the equation of state.

Linearized EOS:

Linearize the cosmological EOS around the present epoch as

$$p = p_0 + \kappa_0\,(\rho - \rho_0) + O[(\rho - \rho_0)^2].$$

To calculate κ_0 use

$$\kappa_0 = \frac{\mathrm{d}p/\mathrm{d}t|_0}{\mathrm{d}\rho/\mathrm{d}t|_0}.$$

From the definition of deceleration and jerk parameters:

$$8\pi G_N\,\frac{\mathrm{d}\rho}{\mathrm{d}t} = -6c^2 H\left[(1+q)H^2 + \frac{kc^2}{a^2}\right],$$

$$8\pi G_N\,\frac{\mathrm{d}p}{\mathrm{d}t} = 2c^2 H\left[(1-j)H^2 + \frac{kc^2}{a^2}\right].$$

Linearized EOS:

This leads to

$$\kappa_0 = -\frac{1}{3} \left[\frac{1 - j_0 + kc^2/(H_0^2 a_0^2)}{1 + q_0 + kc^2/(H_0^2 a_0^2)} \right],$$

which approximates (using $H_0 a_0/c \gg 1$) to

$$\kappa_0 = -\frac{1}{3} \left[\frac{1 - j_0}{1 + q_0} \right].$$

The key observation here is that to obtain the linearized equation of state you need significantly more information than the deceleration parameter q_0; you also need to measure the jerk parameter j_0.

Taylor expanded EOS:

Taylor expand the scale factor:

$$a(t) = a_0 \left\{ 1 + H_0\,(t-t_0) - \frac{1}{2}\,q_0\,H_0^2\,(t-t_0)^2 + \frac{1}{3!}\,j_0\,H_0^3\,(t-t_0)^3 + \frac{1}{4!}\,s_0\,H_0^4\,(t-t_0)^4 + O([t-t_0]^5) \right\}.$$

Take explicit time derivatives and so verify that:

$$\left.\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2}\right|_0 = -\frac{(1 + kc^2/[H_0^2 a_0^2])}{6\rho_0(1 + q_0 + kc^2/[H_0^2 a_0^2])^3} \times \left\{ s_0(1+q_0) + j_0(1 + j_0 + 4q_0 + q_0^2) + q_0(1+2q_0) + (s_0 + j_0 + q_0 + q_0 j_0)\frac{kc^2}{H_0^2 a_0^2} \right\}.$$

In the approximation $H_0 a_0 / c \gg 1$ this reduces to:

$$\left.\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2}\right|_0 = -\frac{s_0(1+q_0) + j_0(1 + j_0 + 4q_0 + q_0^2) + q_0(1+2q_0)}{6\rho_0(1+q_0)^3}.$$

As expected, this second derivative depends linearly on the snap.

Hubble law:

[Cosmographic: Independent of the Einstein equations]

The physical distance travelled by a photon that is emitted at time t_* and absorbed at the current epoch t_0 is

$$D = c \int \mathrm{d}t = c\,(t_0 - t_*).$$

In terms of this physical distance the Hubble law is *exact*

$$1 + z = \frac{a(t_0)}{a(t_*)} = \frac{a(t_0)}{a(t_0 - D/c)},$$

but impractical.

Hubble law:

A more useful result is obtained by performing a fourth-order Taylor series expansion,

$$\frac{a(t_0)}{a(t_0 - D/c)} = 1 + \frac{H_0 D}{c} + \frac{2+q_0}{2}\,\frac{H_0^2 D^2}{c^2} + \frac{6(1+q_0)+j_0}{6}\,\frac{H_0^3 D^3}{c^3}$$
$$+\frac{24 - s_0 + 8j_0 + 36q_0 + 6q_0^2}{24}\,\frac{H_0^4 D^4}{c^4} + O\left[\left(\frac{H_0 D}{c}\right)^5\right],$$

followed by reversion of the resulting series $z(D) \to D(z)$ to obtain:

$$D(z) = \frac{c\,z}{H_0}\Big\{1 - \left[1 + \frac{q_0}{2}\right] z + \left[1 + q_0 + \frac{q_0^2}{2} - \frac{j_0}{6}\right] z^2$$
$$-\left[1 + \frac{3}{2}q_0(1+q_0) + \frac{5}{8}q_0^3 - \frac{1}{2}j_0 - \frac{5}{12}q_0 j_0 - \frac{s_0}{24}\right] z^3$$
$$+O(z^4)\Big\}.$$

Hubble law:

The observational Hubble law is given in terns of "luminosity distance":

$$\text{(energy flux)} = \frac{L}{4\pi\, d_L^2}.$$

Let the photon be emitted at $r = 0$ at time t_*, and absorbed at $r = r_0$ at time t_0.

Then it is a purely geometrical result that

$$d_L = a(t_0)^2\, \frac{r_0}{a(t_*)} = \frac{a_0}{a(t_0 - D/c)}\, (a_0\, r_0).$$

To calculate $d_L(D)$ we need $r_0(D)$.

[Brief agony suppressed.]

Hubble law:

After some calculation, the luminosity distance as a function of D, the physical distance travelled is:

$$
\begin{aligned}
d_L(D) \;=\; & D\Big\{1 + \frac{3}{2}\left(\frac{H_0 D}{c}\right) + \frac{1}{6}\left[11 + 4q_0 - \frac{kc^2}{H_0^2 a_0^2}\right]\left(\frac{H_0 D}{c}\right)^2 \\
& + \frac{1}{24}\left[50 + 40q_0 + 5j_0 - 10\frac{kc^2}{H_0^2 a_0^2}\right]\left(\frac{H_0 D}{c}\right)^3 \\
& + O\left[\left(\frac{H_0 D}{c}\right)^4\right]\Big\}.
\end{aligned}
$$

Now using the series expansion for for $D(z)$ we finally derive the luminosity-distance version of the Hubble law.

Hubble law:

$$
\begin{aligned}
d_L(z) \;=\; & \frac{c\,z}{H_0}\Big\{1+\frac{1}{2}\left[1-q_0\right]z-\frac{1}{6}\left[1-q_0-3q_0^2+j_0+\frac{kc^2}{H_0^2a_0^2}\right]z^2 \\
& +\frac{1}{24}\Big[2-2q_0-15q_0^2-15q_0^3+5j_0 \\
& \quad +10q_0j_0+s_0+\frac{2kc^2(1+3q_0)}{H_0^2a_0^2}\Big]z^3+O(z^4)\Big\}.
\end{aligned}
$$

The first two terms above are Weinberg's version of the Hubble law. His equation (14.6.8).

The third term is equivalent to Chiba and Nakamura.

The fourth order term appears to be new, and (as expected) depends linearly on the snap.

Hubble law:

It is important to realise that this Hubble law, and indeed the entire discussion of this section, is completely model-independent.

The argument assumes only that the geometry of the universe is well approximated by a FRW cosmology but does not invoke the Einstein field equations [Friedmann equation] or any particular matter model.

The jerk j_0 first shows up in the Hubble law at *third* order (order z^3); but this was one of the parameters we needed to make the *lowest-order* estimate for the slope of the EOS.

Most important formulae:

Cosmographic (Hubble law):

$$
\begin{aligned}
D(z) &= \frac{c\,z}{H_0}\Big\{1 - \left[1 + \frac{q_0}{2}\right] z + \left[1 + q_0 + \frac{q_0^2}{2} - \frac{j_0}{6}\right] z^2 \\
&\quad - \Big[1 + \frac{3}{2} q_0 (1 + q_0) + \frac{5}{8} q_0^3 - \frac{1}{2} j_0 - \frac{5}{12} q_0 j_0 - \frac{s_0}{24}\Big] z^3 \\
&\qquad + O(z^4)\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
d_L(z) &= \frac{c\,z}{H_0}\Big\{1 + \frac{1}{2}\left[1 - q_0\right] z - \frac{1}{6}\left[1 - q_0 - 3q_0^2 + j_0 + \frac{kc^2}{H_0^2 a_0^2}\right] z^2 \\
&\quad + \frac{1}{24}\Big[2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 \\
&\qquad + 10 q_0 j_0 + s_0 + \frac{2kc^2(1 + 3q_0)}{H_0^2 a_0^2}\Big] z^3 + O(z^4)\Big\}.
\end{aligned}
$$

Most important formulae:

Cosmodynamic (EOS):

$$w_0 = \frac{p_0}{\rho_0} = -\frac{(1-2q_0) + kc^2/(H_0^2 a_0^2)}{3[1 + kc^2/(H_0^2 a_0)]}.$$

$$\kappa_0 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_0 = -\frac{1}{3}\;\left[\frac{1 - j_0 + kc^2/(H_0^2 a_0^2)}{1 + q_0 + kc^2/(H_0^2 a_0^2)}\right],$$

$$\left.\frac{\mathrm{d}^2 p}{\mathrm{d}\rho^2}\right|_0 = -\frac{(1 + kc^2/[H_0^2 a_0^2])}{6\rho_0(1 + q_0 + kc^2/[H_0^2 a_0^2])^3}$$
$$\times\left\{s_0(1+q_0) + j_0(1 + j_0 + 4q_0 + q_0^2) + q_0(1+2q_0) + (s_0 + j_0 + q_0 + q_0 j_0)\frac{kc^2}{H_0^2 a_0^2}\right\}.$$

Conclusions:

There are currently many different models for the cosmological fluid under active consideration.

Though these models often make dramatically differing predictions in the distant past (*e.g.*, a "bounce") or future (*e.g.*, a "big rip") there is considerable degeneracy among the models in that many physically quite different models are compatible with present day observations.

To understand the origin of this degeneracy I have chosen to rephrase the question in terms of a phenomenological approach where cosmological observations are used to construct an "observed" equation of state.

Conclusions:

The key result is that even at the linearized level, determining the slope of the EOS requires information coming from the third order term in the Hubble law.

Despite the fact that some parameters in cosmology are now known to high accuracy, other parameters can still only be crudely bounded.

The jerk is one of these parameters, and as a consequence direct observational constraints on the cosmological EOS are likely to remain poor for the foreseeable future.