Emergent spacetimes, rainbow spacetimes, and pseudo-Finsler spacetimes.

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Abstract:

The theoretical physics community is increasingly pushing at the boundaries of classical differential geometry (Riemannian and Lorentzian manifolds), and seeking new mathematical tools to investigate various extensions of Einstein gravity.

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Among the as yet mathematically imprecise concepts being mooted are the notions of emergent spacetime (where the manifold picture breaks down at short distances), rainbow spacetimes (where the "metric" somehow depends on energy and momentum), and particular unexplored sub-classes of pseudo-Finsler spacetime.

Abstract:

I will outline why these ideas are considered interesting, (at least by the physicists), and indicate some of the foundational mathematical issues that remain open.

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Mathematics and physics:

Divided by a common language...

The usual warnings:

Everybody knows:

Einstein's theory of gravity, the general relativity, is based on Riemannian geometry...

Everybody knows wrong:

Einstein's theory of gravity, the general relativity, is based on pseudo-Riemannian geometry... aka Lorentzian geometry...

The distinction is important...

The usual warnings:

The spacetime metric isn't...

...isn't a metric that is...

Mathematical definition of a metric:

- $d : X \times X \to I\!R$
- $d(x, y) \geq 0$
- $d(x, y) = 0 \quad \Longrightarrow \quad x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Care to guess how many of these axioms the physicist's spacetime metric satisfies?

The usual warnings:

- $d : X \times X \to \mathbb{R}$ XXX
- $d(x, y) \geq 0$ XXX
- $d(x, y) = 0 \implies x = y$ XXX
- $d(x, y) = d(y, x)$ $\surd\surd\surd$
- $d(x, z) \leq d(x, y) + d(y, z)$ XXX

The usual warnings:

Working in flat (Minkowski) spacetime for simplicity:

- $d^2 : X \times X \to \mathbb{R}$
- $d^2(x,y) \begin{cases} > 0 & \text{spacelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{timelike} \end{cases}$ (GR conventions)
- $d^2(x,y) = d^2(y,x)$
- If 3 events are spacelike separated: $d(x,z) \leq d(x,y) + d(y,z)$
- If 3 events are timelike separated: $|d(x,z)| \geq |d(x,y)| + |d(y,z)|$

Note anti-triangle inequality for timelike separated events.

(Equivalent to the "twin pseudo-paradox".)

The usual warnings:

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I emphasize this is utterly standard and uncontroversial elementary (special relativity) physics...

Side effect of the minus sign in the spacetime interval:

$$d^2(X_1, X_2) = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

And the minus sign is there because you are using the spacetime metric to encode the “light cones”.

Lorentzian signature: $(-, +, +, +)$

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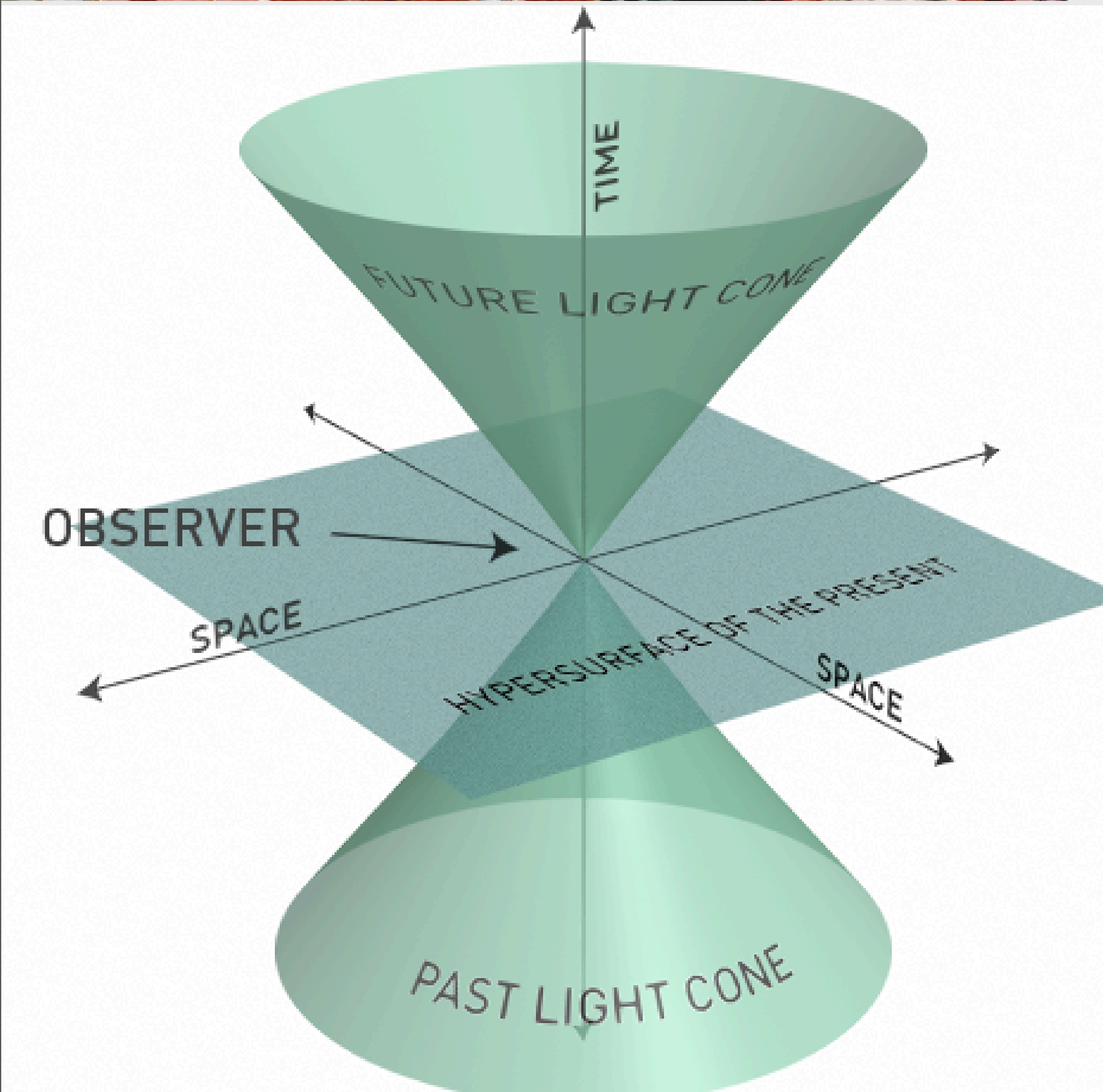

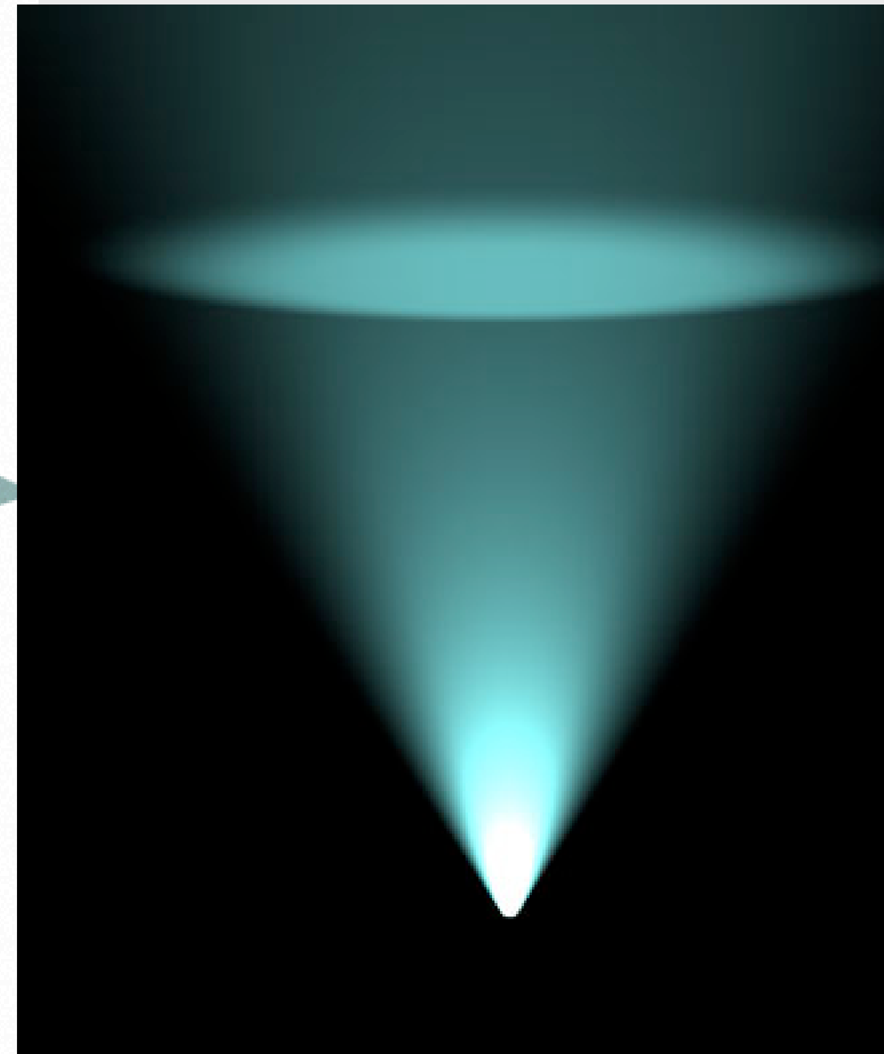

The usual warnings:

The difference between the mathematician's metric
and the physicist's spacetime metric
is not an issue of "right" or "wrong"...

... it's just that the meaning of key words shifts
as you cross discipline boundaries...

...and every now and then the semantic shift
can really blindside you...

The usual warnings:

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For instance: Roughly 90% of research in Riemannian geometry is of doubtful relevance to physics, simply because of the differences between Riemannian geometry and pseudo-Riemannian spacetime (Lorentzian geometry)...

The number of math texts specifically aimed at Lorentzian geometry is *very* small...

The needs of the physics community do not always coincide with the most straightforward line of mathematical development...

Outline:

With that cautionary tale out of the way,
let me now talk a little about:

Emergent spacetimes;
Rainbow spacetimes;
pseudo-Finsler spacetimes.

Emergence:

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The word "emergence" is being tossed around an awful lot lately.....

But what does it really mean?

--- "More is different"? [Anderson]

--- The sum is greater than its parts?

--- Universality?

--- Mean field?

Short distance physics is often radically different from long distance physics...

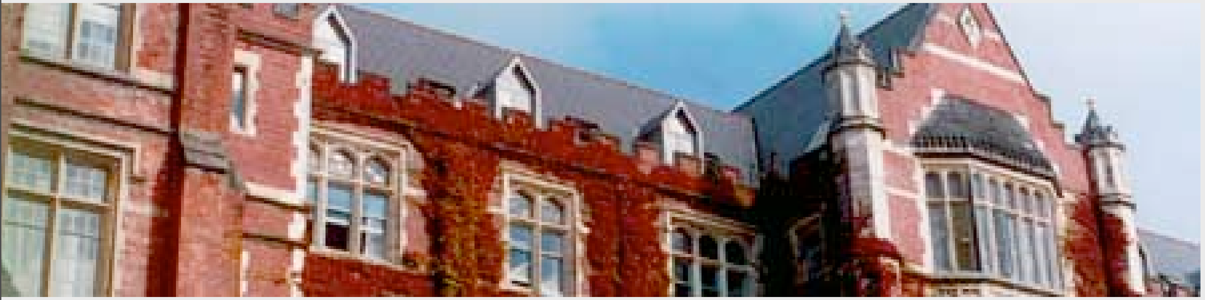
Emergence:

Prime example: Fluid dynamics

Long distance physics:	Euler equation	(generic)
	Continuity equation	(generic)
	Equation of state	(specific)

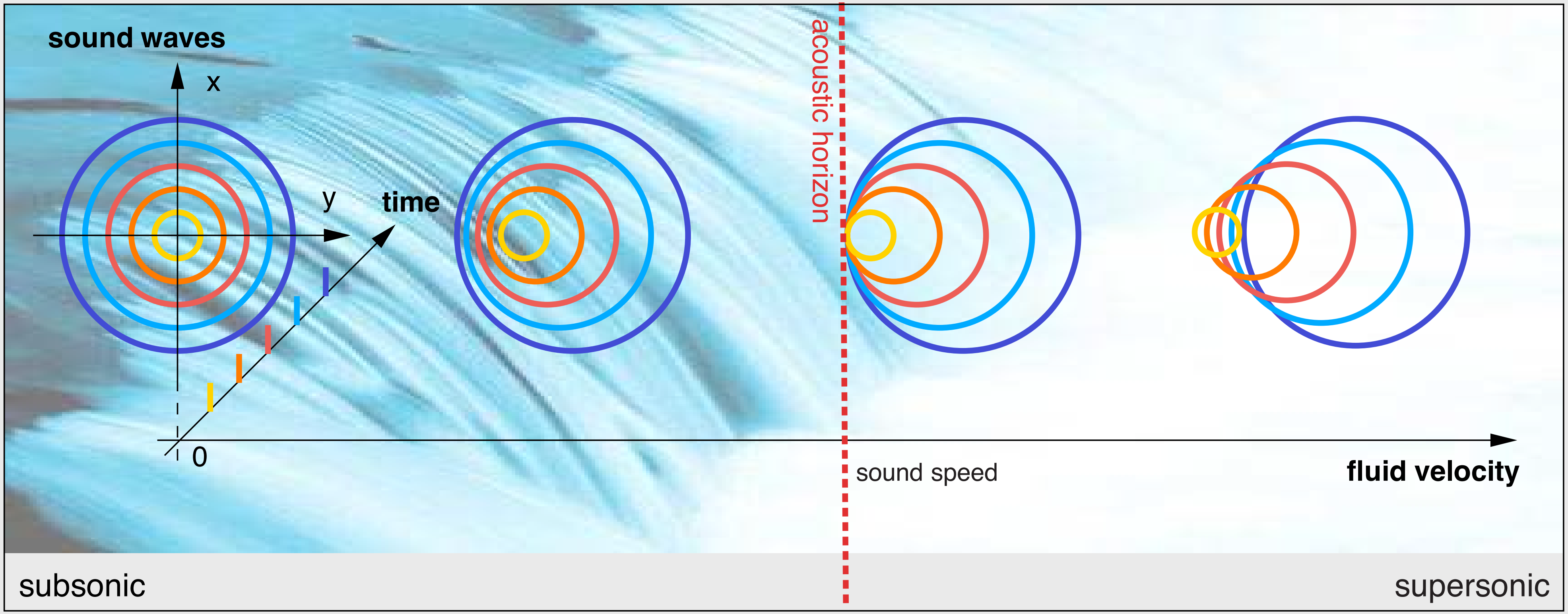
Short distance physics: Quantum molecular dynamics

Starting from fluid dynamics, it is possible to define an "emergent" notion of geometry, "acoustic spacetime"...



Acoustic spacetime:

The simplest "emergent spacetimes" are the "acoustic spacetimes"...



Consider sound waves in a moving fluid... [Unruh 81]

Acoustic spacetime:

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Theorem: Consider an irrotational, inviscid, barotropic perfect fluid, governed by the Euler equation, continuity equation, and an equation of state.

The dynamics of the linearized perturbations (sound, phonons) is governed by a D'Alembertian equation

$$\Delta_g \Phi = \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g}\; g^{ab} \partial_b\, \Phi \right) = 0$$

involving an "acoustic metric".

[Algebraic function of the background fields.]

Acoustic spacetime:

Theorem: (3+1 dimensions)

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{\rho_0 c}\left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\,\delta^{ij} - v_0^i\, v_0^j) \end{array}\right].$$

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho_0}{c}\left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{array}\right].$$

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\rho_0}{c}\left[-c^2\,\mathrm{d}t^2 + (\mathrm{d}x^i - v_0^i\,\mathrm{d}t)\,\delta_{ij}\,(\mathrm{d}x^j - v_0^j\,\mathrm{d}t)\right].$$

Acoustic spacetime:

There is by now a quite sizable literature on acoustic, and other more general emergent/ analogue spacetimes.

Unruh: Experimental black hole evaporation? Phys Rev Lett 46 (1981) 1351-1353.

Barcelo, Liberati, Visser: Analogue gravity, Living Reviews in Relativity, 8 (2005) 12.

Main message: Finding an effective low-energy spacetime metric is not all that difficult....

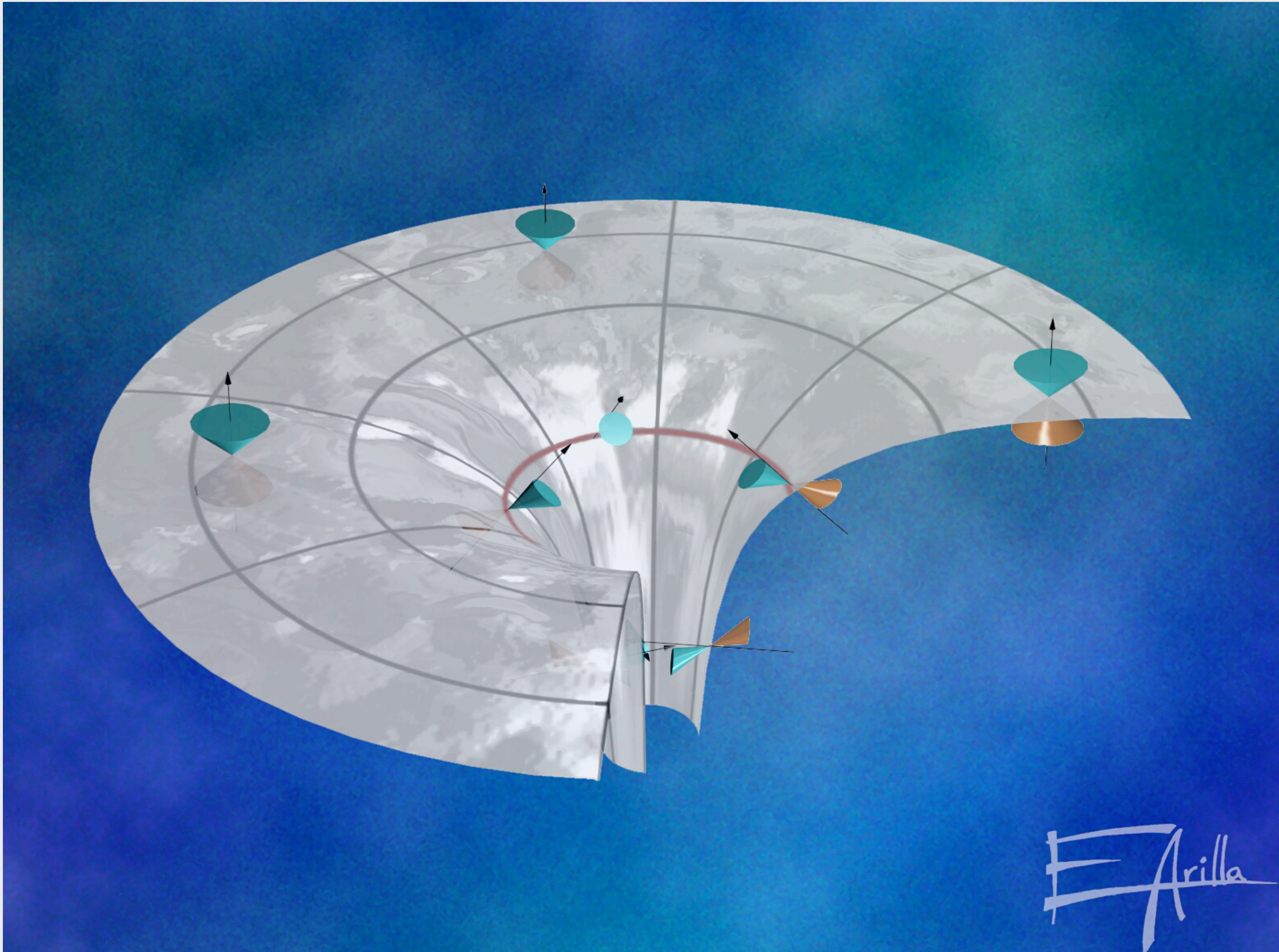
Acoustic spacetime:

Forces us to think about:

"Discretium" ==> Continuum?

What does it mean to have on object that is "manifold-like" at large distances, and yet fundamentally discrete at short distances...

(And as we have just seen, objects like this certainly exist in physical reality.)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Rainbow spacetime:

There is no general widely accepted precise mathematical definition of what is meant by a "rainbow geometry"...

The physicist's definition is rather imprecise:
"energy dependent metric"?
"momentum dependent metric"?
"4-momentum dependent metric"?

Q: 4-momentum of what? The observer? The object being observed?

[A: It depends...]

Rainbow spacetime:

To capture the notion of "energy-momentum dependence" need a metric that depends on the tangent vector...

Consider a fluid at rest, in very many cases the dispersion relation can be written in the form:

$$\omega^2 = F(k)$$

for some possibly nonlinear function F(k)...

(2nd-order in time; arbitrary order in space...)

[Unruh, Jacobson]

Rainbow spacetime:

Phase velocity: $$c_k^2 = \frac{\omega^2}{k^2} = \frac{F(k)}{k^2}$$

Dispersion relation: $$\omega^2 = c_k^2 \, k^2$$

Fluid in motion: Doppler shift the frequency...

$$\omega \longrightarrow \omega - \vec{v} \cdot \vec{k}$$

$$\left(\omega - \vec{v} \cdot \vec{k}\right)^2 - c_k^2 \, k^2 = 0$$

[non-relativistic]

Rainbow spacetime:

Rewrite as: $$g_k^{ab}\; k_a\, k_b = 0.$$ [dispersion relation]

Pick off components:

$$g_k^{ab} \propto \left[\begin{array}{c|c} -\,1 & -v^j \\ \hline -\,v^i & c_k^2\; \delta^{ij} - v^i v^j \end{array} \right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array} \right].$$

Momentum dependent metric depending on phase velocity.

Rainbow spacetime:

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Dispersion relation approach is physically transparent...

Only weakness: Conformal factor left unspecified...

(This is a standard side-effect of the geometrical quasi-particle approximation,
cf geometrical acoustics,
cf geometrical optics.)

[PDE is better]
[Weinfurtner]

The momentum in question is now the momentum of an individual "mode" of the field ---
hence phase velocity + dispersion relation.

Rainbow spacetime:

Similar (but distinct) steps can be taken to develop a different rainbow metric based on group velocity.

Consider a wave packet centered on momentum k.

That packet will propagate with the group velocity.

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t)^2 = c_k^2\,\mathrm{d}t^2$$

Group velocity.

Rainbow spacetime:

Rewrite as: $$\mathrm{d}s^2 = 0 = g_{ab}\, dx^a\, \mathrm{d}x^b$$ [propagation]

Pick off components:

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\, \delta^{ij} - v^i v^j \end{array}\right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array}\right].$$

Momentum dependent metric depending on group velocity.

Rainbow spacetime:

Thus there are at least two distinct very different notions of "rainbow metric" in an analogue setting.

They answer different questions:

* What is the dispersion relation of a pure mode?
* How do wave packets propagate?

If you are lucky there is a "hydrodynamic" limit:

$$\lim_{k\to 0} c^2_{\text{phase}}(k) = c^2_{\text{hydrodynamic}} = \lim_{k\to 0} c^2_{\text{group}}(k)$$

$$\neq 0!$$

Rainbow spacetime:

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But in general: Rainbow ==> multi-metric

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\,\delta^{ij} - v^i v^j \end{array} \right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array} \right].$$

With: $$c_k \to \left\{ \begin{array}{l} c_{\mathrm{phase}} \\ c_{\mathrm{group}} \\ c_{\mathrm{hydrodynamic}} \end{array} \right. .$$

Signal speed?
c ==> infinity?
causal structure

Can one now come up with a clean and useful generalization of these ides to write down a "generic" energy-momentum dependent notion of manifold?

Other hints: Renormalization group flows (compare with Ricci-Hamilton flows and Perelman flows);
also,
Maupertuis' constant-energy variational principle.

Rainbow spacetime:

Bogoliubov dispersion relation (eg, BECs):

$$\omega^2 = c_0^2 \, k^2 + \left(\frac{\hbar}{2m}\right)^2 k^4$$

$$c^2 = c_0^2 + \left(\frac{\hbar}{2m}\right)^2 k^2$$

(supersonic)

Controlled breaking of Lorentz invariance...

See "quantum gravity phenomenology"... [Liberati...]

See "cosmological particle production" [Weinfurtner]

Rainbow spacetime:

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Surface waves in finite depth of liquid:

[Lamb]
[Hydrodynamics]

$$\omega^2 = g\, k\, \tanh(k\, d) = c_0^2\, k^2\, \frac{\tanh(k\, d)}{k\, d}$$

$$c_0^2 = g\; d.$$

$$c^2 = c_0^2\, \cancel{k^2}\, \frac{\tanh(k\, d)}{k\, d}$$

(subsonic)

$$\omega^2 = c_0^2\, k^2\, \left\{ 1 - \frac{(k\, d)^2}{3} + \frac{2(k\, d)^2}{15} + \dots \right\}$$

So analogue models provide concrete examples for both supersonic an subsonic dispersion, and more...

Rainbow spacetime:

Surface waves in infinite depth of liquid:

$$\omega = \sqrt{g\,k}; \qquad c_{\text{phase}} = \sqrt{g/k}.$$

$$c_{\text{group}} = \frac{\partial \omega}{\partial k} = \frac{\sqrt{g/k}}{2} = \frac{c_{\text{phase}}}{2}.$$

No hydrodynamic limit...

No well-defined low-momentum spacetime...

[You could argue that this is an unphysical limit...]

Rainbow spacetime:

Surface waves in finite depth of liquid + surface tension:

$$\omega^2 = c_0^2\, k^2\, \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

$$c^2 = c_0^2\, \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

$$c_0^2 = g\, d.$$

Asymptotically supersonic, though it can be adjusted to have a subsonic dip.

Water: $$\epsilon = \frac{\sigma}{\rho c_0^2 d} = \frac{\sigma}{\rho g d^2} = \frac{(0.27\ \mathrm{cm})^2}{d^2}.$$

Rainbow spacetime:

$$c^2 = c_0^2 \, \left\{1 + \epsilon \, (kd)^2\right\} \, \frac{\tanh(kd)}{kd}.$$

$$c^2 = c_0^2 \, \left\{1 + \frac{3\epsilon - 1}{3} \, (kd)^2 - \frac{5\epsilon - 2}{15} \, (kd)^4 + \mathcal{O}[(kd)^6]\right\}.$$

Can tune away the lowest order Lorentz violation...

(Water at 0.47 cm depth)

These are just some examples of the types of dispersion relation you can arrange to set up...

Finsler spacetime:

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1854: $$ds = \sqrt[4]{g_{abcd}\; dx^a dx^b dx^c dx^d}$$

Riemann's inaugural lecture at Goettingen

But Riemann never developed the idea...

Left to Paul Finsler in early 20'th century...

But physicists need pseudo-Finsler spacetime, not Finsler space...

Finsler spacetime:

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Physical model: Birefringent crystal [Born+Wolf]

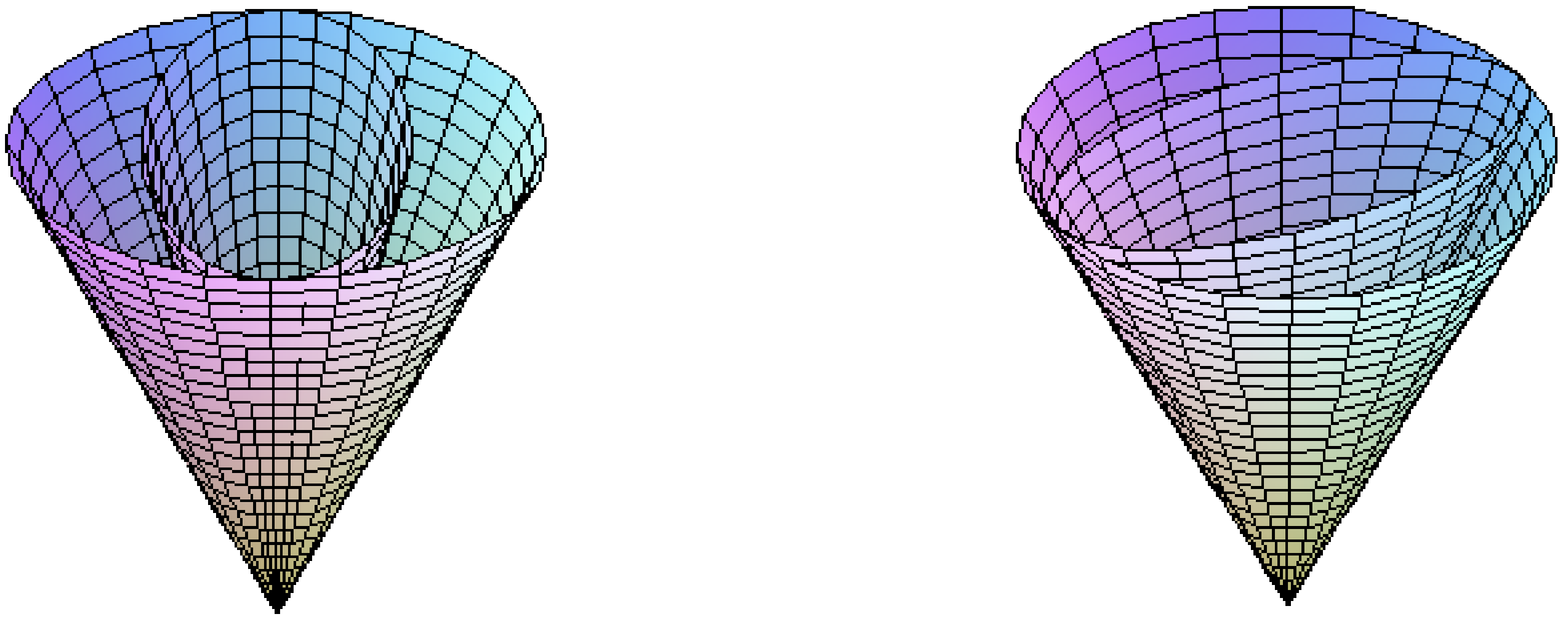
Maxwell ==> $$f^{ab}_{\mathrm{AB}}\; p_a p_b \;\epsilon^B = 0$$

Fresnel equation $$\det[f^{ab}_{\mathrm{AB}}\; p_a p_b] = 0.$$

Expand determinant:

$$\det[f^{ab}_{\mathrm{AB}}\; p_a p_b] = Q^{abcd\dots}\; p_a p_b p_c p_d \dots$$

(OK, technically co-Finsler rather than Finsler)



Light cones in a uni-axial birefringent crystal

Finsler spacetime:

Remember: In special relativity ----

$$d_\gamma(x,y) = \int_x^y \sqrt{g_{ab}(dx^a/d\tau)(dx^b/d\tau)}\,d\tau,$$

- $d_\gamma(x,y) \in \mathbb{R}^+$ for spacelike paths;
- $d_\gamma(x,y) = 0$ for null paths;
- $d_\gamma(x,y) \in \mathbb{I}^+$ for timelike paths;

Even in SR and GR, "distances" do not have to be real numbers...

Finsler spacetime:

Generalize this to a Finsler structure:

Start with the simple multi-metric case:

$$Q(x,p) = \Pi_{i=1}^{n} (g_i^{ab} p_a p_b),$$

$$G(x,p) = \sqrt[2n]{\Pi_{i=1}^{n} (g_i^{ab} p_a p_b)},$$

$$G(x,p) \in \exp\left(\frac{i\pi\ell}{2n}\right) \mathbb{R}^+,$$

- $\ell = 0 \to G(x,p) \in \mathbb{R}^+ \to$ outside all n signal cones;
- $\ell = n \to G(x,p) \in \mathbb{I}^+ \to$ inside all n signal cones.

Finsler spacetime:

That is:

- Spacelike $\leftrightarrow$ outside all n signal cones $\leftrightarrow$ G real;
- Null $\leftrightarrow$ on any one of the n signal cones $\leftrightarrow$ G zero;
- Timelike $\leftrightarrow$ inside all n signal cones $\leftrightarrow$ G imaginary;
- plus the various "intermediate" cases:

$$\text{"intermediate"} \leftrightarrow \text{inside } \ell \text{ of } n \text{ signal cones} \leftrightarrow G \in i^{\ell/n} \times \mathbb{R}^{+}.$$

This basic idea survives even if we go beyond the multi-metric special case...

Finsler spacetime:

$Q(x,p) = 0$ defines a polynomial of degree "2n"...

... and therefore defines "n" nested "conoids"...

This is Courant-Hilbert's "Monge cone"...

$$
\begin{aligned}
Q(x,p) = 0 \quad &\Leftrightarrow \quad Q(x,(E,\vec{p})) = 0; \\
&\Leftrightarrow \quad \text{polynomial of degree } 2n \text{ in } E \text{ for any fixed } \vec{p}; \\
&\Leftrightarrow \quad \text{in each direction } \exists\ 2n \text{ roots in } E; \\
&\Leftrightarrow \quad \text{corresponds to } n \text{ [topological] cones.}
\end{aligned}
$$

Finsler spacetime:

In short:

[Liberati *et al*]

- pseudo-co–Finsler functions arise naturally from the leading symbol of hyperbolic systems of PDEs;
- pseudo-co–Finsler geometries provide the natural “geometric” interpretation of a multi-component PDE before fine tuning;
- In particular the natural geometric interpretation of 2-BEC models (in the hydrodynamic limit, and before fine tuning) is as a 4-smooth pseudo-co–Finsler geometry.

Despite their somewhat abstract mathematical character, Finsler spacetimes are of direct physical interest...

Finsler spacetime:

OK, technically, make that pseudo-co-Finsler spacetimes...

The "indicatrix" is now more subtle...

"indicatrix" <==> "mass shell"

Lots of interesting stuff here...

But most of the standard mathematical literature is "not relevant" ...

... to a large extent, the wrong questions have been asked ...

Conclusion:

Many interesting extensions and modifications of the general relativity notion of spacetime have concrete and well controlled models within the "emergent spacetime" framework.

This tells us which rocks to start looking under...

“It is important to keep an open mind; just not so open that your brains fall out”

--- Albert Einstein

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