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Analogue models for Finsler spacetime and Rainbow spacetime

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Abstract:

We are by now very used to seeing how analogue Lorentzian spacetimes automatically emerge from small fluctuations in many different physical and mathematical systems.

But sometimes life gets a little more difficult: if the physical or mathematical model is a little too complicated one might have to move beyond Lorentzian spacetime.

Abstract:

I will explain two (relatively) simple things that can happen:

--- Finsler spacetimes ---

--- Rainbow spacetimes ---

Both of these extensions of Lorentzian geometry seem (in their own way) to be physically relevant.

Collaboration: (this project)

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Finsler spacetimes: gr-qc/0605121, and in preparation...

Rainbow spacetimes: in preparation...

Introduction:

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If a physical system is governed by second-order PDEs then the linearized fluctuations can very often be proved to be governed by a PDE of the form:

$$\partial_a \left(f^{ab} \, \partial_b \theta \right) = 0.$$

This happens, for instance for:

Barotropic irrotational inviscid fluid mechanics,
Generic superfluids, Bose-Einstein condensates,
Euler-Lagrange equations, etc...

Introduction:

But as soon as you see the PDE:

$$\partial_a \left(f^{ab} \, \partial_b \theta \right) = 0.$$

Then you extract a Lorentzian spacetime by defining:

$$\sqrt{-g} \; g^{ab} = f^{ab}.$$

Lorentzian <=> Hyperbolic PDE

Riemannian <=> Elliptic PDE

That's all...

Introduction:

This is all very nice, but sometimes the universe makes life a little more difficult for us...

Two relatively simple things that can happen:

- Multiple interacting fields;
- Higher-order PDEs.

Introduction:

These generalizations (complications) actually do show up in several physically interesting systems, for example:

- Multiple interacting fields:
  - Optics (birefringent media).
  - Interacting BECs.
  - Complex superfluids.
  - Seismic waves (s and p waves).

Introduction:

- Higher-order PDEs:
  - BECs beyond the hydrodynamic limit.
  - Superfluids beyond the phonon limit.
  - Some models with Lorentz violation at high energy.
  - Generic curvature-squared models.

Introduction:

In the case of multiple fields one typically derives:

$$\partial_a \left(\mathbf{f}^{ab} \, \partial_b \vec{\theta} \right) = 0,$$

Here $\vec{\theta}$ is now a vector of fields,

while $\mathbf{f}^{ab}$ is a tensor density that is matrix-valued in field space.

This naturally leads to the concept of Finsler spacetime... (details below...)

Introduction:

In the case of higher derivatives one typically derives:

$$\partial_a \left(\widehat{f}^{ab} \, \partial_b \theta \right) = 0,$$

where $\widehat{f}^{ab}$ is a tensor density whose components are themselves differential operators...

I'll soon show how to get a "rainbow spacetime" out of this...

Of course, you could do both, or even worse, but one thing at a time...

Example: BECs

For 2 coupled BECs in the hydrodynamic limit, perturbations are governed by:

$$\partial_a \left(\mathbf{f}^{ab} \; \partial_b \vec{\theta} \right) + \frac{1}{2} \left\{ \mathbf{\Gamma}^a \; \partial_a \vec{\theta} + \partial_a \left(\mathbf{\Gamma}^a \vec{\theta} \right) \right\} + \mathbf{K} \; \vec{\theta} = 0.$$

The matrices $\mathbf{f}^{ab}$, $\mathbf{\Gamma}^a$, $\mathbf{K}$, are calculable functions of the BEC background...

More details than you could possibly want:

cond-mat/0409639, gr-qc/0506029, gr-qc/0510125,
gr-qc/0511105, gr-qc/0512127, gr-qc/0512139,
gr-qc/0605121 ...

Example: BECs

This is a self-adjoint system of 2nd-order linear PDEs.

The spacetime geometry is encoded in the "leading symbol" of the PDEs.

Courant and Hilbert: The "leading symbol" determines the characteristics, the "signal speed", and so the causal structure.

Eikonal approximation: Causal structure completely determined by leading term in the Fresnel equation:

$$\det[\mathbf{f}^{ab} k_a k_b] = 0,$$

Example: BECs

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Determinant is in “field space”.

For a 2-BEC system:

$$(f_{11}^{ab} k_a k_b)(f_{22}^{cd} k_c k_d) - (f_{12}^{ab} k_a k_b)(f_{21}^{cd} k_c k_d) = 0.$$

Define:

$$Q^{abcd} \equiv f_{11}^{(ab} f_{22}^{cd)} - f_{12}^{(ab} f_{21}^{cd)},$$

then the determinant condition is equivalent to:

$$Q^{abcd} k_a k_b k_c k_d = 0,$$

which defines a co-Finsler structure.

Example: BECs

Co-Finsler:
$$Q^{abcd} k_a k_b k_c k_d = 0,$$

Compare to:

Lorentzian:
$$g^{ab}\ k_a k_b = 0$$

Finsler distance defined by Legendre transform:

$$ds^4 = g_{abcd}\ dx^a dx^b dx^c dx^d.$$

History:

1854: $$ds = \sqrt[4]{g_{abcd}\; dx^a dx^b dx^c dx^d}$$

Riemann's inaugural lecture at Goettingen

But Riemann never developed the idea...

Left to Paul Finsler in early 20'th century...

But physicists need pseudo-Finsler spacetime,
not Finsler space...

Generalities:

Finsler function: $\quad F(x,t): \quad F(x,\lambda t) = \lambda\, F(x,t),$

Defined on the tangent bundle (minus the zero vector).

Finsler distance:

$$d_\gamma(x,y) = \int_x^y F(x(\tau), dx/d\tau)\; d\tau; \qquad \tau = \text{ arbitrary parameter.}$$

Finsler metric: $$g_{ij}(x,t) = \frac{1}{2}\frac{\partial^2[F^2(x,t)]}{\partial t^i\;\partial t^j}.$$

Generalities:

The Finsler metric depends on the direction of the tangent vector, but not its magnitude...

Because the metric does not depend on the magnitude of the tangent vector, Finsler spacetimes are not considered to be "rainbow spacetimes".

Given a metric, you can define curvature...

Even the mathematicians say: Curvature computations "quickly become mind-numbingly complex".

Generalities:

co–Finsler function:

cotangent bundle

Legendre transformation

$$G^2(x,p) = t^j(p)\, p_j - F^2(x, t(p))$$

$$\frac{\partial [F^2]}{\partial t^j}(x,t) = p_j.$$

$$\frac{\partial p_j}{\partial t^k} = \frac{\partial [F^2]}{\partial t^j\ \partial t^k} = 2 g_{jk}(x,t)$$

metric
non-singular!

Generalities:

- F^2 is homogeneous of degree 2.
- g_{ij} is homogeneous of degree 0.
- $\partial[F^2]/\partial t$ is homogeneous of degree 1.
- Therefore $p(t)$ is homogeneous of degree 1 *and* $t(p)$ is homogeneous of degree 1.
- Therefore $t(p)p - F^2(t(p))$ is homogeneous of degree 2.
- Therefore $G(p)$ is homogeneous of degree 1.

Finsler <=> co-Finsler.

PDE analysis:

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For a second-order system of PDEs:

$$\partial_a \left(f^{ab}{}_{\mathrm{AB}}\, \partial_b \theta^B \right) + \text{lower order terms} = 0.$$

Equivalently:

$$\partial_a \left(\mathbf{f}^{ab}\, \partial_b \vec{\theta}\, \right) + \text{lower order terms} = 0.$$

Eikonal approximation:

$$f^{ab}_{\mathrm{AB}}\, p_a p_b\, \epsilon^B + \text{lower-order terms} = 0$$

PDE analysis:

Fresnel equation (leading term):

$$\det[f_{\mathrm{AB}}^{ab}\; p_a p_b] = 0.$$

Expand determinant:

$$\det[f_{\mathrm{AB}}^{ab}\; p_a p_b] = Q^{abcd\dots}\; p_a p_b p_c p_d \dots$$

Define:

$$Q(x,p) = Q^{abcd\dots}\; p_a p_b p_c p_d \dots$$

PDE analysis:

Co-Finsler function:

$$G(x,p) = \sqrt[2n]{Q(x,p)} = [Q(x,p)]^{1/(2n)}$$

(Generic to any system of 2nd order PDEs.)

- $Q(x,p)$ is homogeneous of degree $2n$.
- $G(x,p)$ is homogeneous of degree 1, and hence is a co–Finsler function.
- We can now Legendre transform $G \to F$, providing a chain

$$Q(x,p) \to G(x,p) \to F(x,t).$$

Can this route be reversed?

PDE analysis:

Step 1: We can always reverse $F(x,t) \to G(x,p)$ by Legendre transformation.
Step 2: We can always define

$$g^{ab}(x,p) = \frac{1}{2}\frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}[G(x,p)^2],$$

this is homogeneous of degree 0, but is generically not smooth at $p=0$.

Now *if* $g^{ab}(x,p)$ is smooth at $p=0$ then there exits a limit

$$g^{ab}(x, p \to 0) = \bar{g}^{ab}(x),$$

but then

$$g^{ab}(x,p) = \bar{g}^{ab}(x) \qquad [\forall p],$$

and so: smooth => (Finsler ==> Riemann).

PDE analysis:

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Generalize this:

Definition: A co–Finsler function $G(x,p)$ is $2n$-smooth iff

$$\frac{1}{(2n)!} \lim_{p\to 0} \left(\frac{\partial}{\partial p}\right)^{2n} G(x,p)^{2n} = \bar{Q}^{abcd\dots}$$

Lemma: If $G(x,p)$ is $2n$-smooth then

$$G(x,p)^{2n} = \bar{Q}^{abcd\dots}\; p_a p_b p_c p_d \dots$$

$$G(x,p) = \sqrt[2n]{\bar{Q}^{abcd\dots}\; p_a p_b p_c p_d \dots}.$$

So 2n-smooth ==> (F <=> G <=> Q)

PDE analysis:

- For those co–Finsler functions that are $2n$ smooth we recover the tensor $Q^{abcd\dots}$.
- Not all co–Finsler functions are $2n$ smooth, and for those functions we *cannot* extract $Q^{abcd\dots}$ in any meaningful way.
- But those specific co–Finsler functions that arise from the leading symbol of a 2nd-order system of PDEs are naturally $2n$-smooth, and so for the specific co–Finsler structures we are physically interested in

$$Q(x,p) \leftrightarrow G(x,p) \leftrightarrow F(x,t).$$

- Therefore, in the physically interesting case the Finsler function $F(x,t)$ encodes all the information present in $Q^{abcd\dots}$.

Lorentzian signature:

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Remember: In special relativity ----

$$d_\gamma(x,y) = \int_x^y \sqrt{g_{ab}(dx^a/d\tau)(dx^b/d\tau)}\,d\tau,$$

- $d_\gamma(x,y) \in \mathbb{R}^+$ for spacelike paths;
- $d_\gamma(x,y) = 0$ for null paths;
- $d_\gamma(x,y) \in \mathbb{I}^+$ for timelike paths;

Even in SR and GR, "distances" do not have to be real numbers...

Lorentzian signature:

Generalize this to a Finsler structure:

Start with the simple multi-metric case:

$$Q(x,p) = \Pi_{i=1}^{n}(g_i^{ab}p_a p_b),$$

$$G(x,p) = \sqrt[2n]{\Pi_{i=1}^{n}(g_i^{ab}p_a p_b)},$$

$$G(x,p) \in \exp\left(\frac{i\pi\ell}{2n}\right) \; I\!R^+,$$

- $\ell = 0 \to G(x,p) \in I\!R^+ \to$ outside all n signal cones;
- $\ell = n \to G(x,p) \in I\!I^+ \to$ inside all n signal cones.

Lorentzian signature:

That is:

- Spacelike $\leftrightarrow$ outside all n signal cones $\leftrightarrow$ G real;
- Null $\leftrightarrow$ on any one of the n signal cones $\leftrightarrow$ G zero;
- Timelike $\leftrightarrow$ inside all n signal cones $\leftrightarrow$ G imaginary;
- plus the various "intermediate" cases:

$$\text{“intermediate”} \leftrightarrow \text{inside } \ell \text{ of } n \text{ signal cones} \leftrightarrow G \in i^{\ell/n} \times I\!R^{+}.$$

This basic idea survives even if we go beyond the multi-metric special case...

Lorentzian signature:

$Q(x,p) = 0$ defines a polynomial of degree "2n"...

... and therefore defines "n" nested "conoids"...

This is Courant-Hilbert's "Monge cone"...

$$
\begin{aligned}
Q(x,p) = 0 \quad &\Leftrightarrow \quad Q(x,(E,\vec{p})) = 0; \\
&\Leftrightarrow \quad \text{polynomial of degree } 2n \text{ in } E \text{ for any fixed } \vec{p}; \\
&\Leftrightarrow \quad \text{in each direction } \exists\ 2n \text{ roots in } E; \\
&\Leftrightarrow \quad \text{corresponds to } n \text{ [topological] cones.}
\end{aligned}
$$

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In short:

- pseudo-co–Finsler functions arise naturally from the leading symbol of hyperbolic systems of PDEs;
- pseudo-co–Finsler geometries provide the natural "geometric" interpretation of a multi-component PDE before fine tuning;
- In particular the natural geometric interpretation of 2-BEC models (in the hydrodynamic limit, and before fine tuning) is as a 4-smooth pseudo-co–Finsler geometry.

Despite their somewhat abstract mathematical character, Finsler spacetimes are of direct physical interest...

Rainbow geometries:

There is no general widely accepted precise mathematical definition of what is meant by a "rainbow geometry"...

The physicist's definition is rather imprecise:
"energy dependent metric"?
"momentum dependent metric"?
"4-momentum dependent metric"?

Q: 4-momentum of what? The observer? The object being observed?

Rainbow geometries:

Finsler geometries, with a direction-dependent metric that is independent of the magnitude of the tangent vector, are not “rainbow”...

To capture the essence of “energy dependence” need a metric that depends also on the magnitude of the tangent vector....

Rainbow geometries:

Provisional mathematical characterization:

Rainbow metric <=> metric on the co-tangent bundle that factorizes over the base manifold:

$$g_{AB}(x,p) = \left[\begin{array}{c|c} g_{ab}(x,p) & 0 \\ \hline 0 & g_{ab}(x,p) \end{array}\right].$$

(Base manifold "n" dimensions;
co-tangent bundle "2n" dimensions.)

Physical interpretation left ambiguous for now....

Rainbow geometries:

Example: BEC beyond the hydrodynamic limit...

...keeping the "quantum pressure" term...

Linearized GP:

$$\partial_t n + \nabla\left[\left(\frac{n_0\hbar}{m}\nabla\theta\right) + (n\cdot\mathbf{v})\right] = 0;$$

$$\partial_t\theta + \mathbf{v}\cdot\nabla\theta + \frac{\widehat{U}}{\hbar}\,n = 0.$$

Differential operator:

$$\widehat{U}\;n = \left(U - \frac{\hbar^2}{2m}\,\widehat{D}_2\right)\,n,$$

Quantum pressure:

$$\widehat{D}_2 = \frac{1}{2}\left\{\frac{(\nabla n_0)^2 - (\nabla^2 n_0) n_0}{n_0^3} - \frac{\nabla n_0}{n_0^2}\nabla + \frac{1}{n_0}\nabla^2\right\}.$$

This 2nd-order differential operator comes from linearizing the so-called "quantum pressure"...

In the EOM it is suppressed by 2 powers of hbar...

Hydrodynamic/acoustic limit <=> ignore this term.

Eikonal approximation: $$\widehat{D}_2 \to -\frac{k^2}{2n_0};$$

Rainbow geometries:

General situation: Solve for “n”...

$$n = \hbar \, \widehat{U}^{-1} \, [\partial_t \theta + \mathbf{v} \cdot \nabla \theta] = \hbar \, \widehat{U}^{-1} \, \frac{D\theta}{Dt},$$

Insert into other EOM:

$$\partial_t n + \nabla \left[\left(\frac{n_0 \hbar}{m} \nabla \theta \right) + (n \cdot \mathbf{v}) \right] = 0;$$

Obtain:

$$\partial_a \left(\widehat{f}^{ab} \, \partial_b \theta \right) = 0$$

Tensor density with differential-operator components...

Rainbow geometries:

$$\hat{f}^{ab} = \left[\begin{array}{c|c} -\,\widehat{U}^{-1} & -\widehat{U}^{-1} v^j \\ \hline -\,v^i \widehat{U}^{-1} & \frac{n_0}{m}\delta^{ij} - v^i \widehat{U}^{-1} v^j \end{array} \right].$$

$$\partial_a \left(\widehat{f}^{ab} \, \partial_b \theta \right) = 0$$

2nd-order in time...

Infinite order in space...

How do we interpret this physically?

Only known way: Adopt the eikonal approximation...

$$\hat{U} \to U_k(t,x) = U(t,x) + \frac{\hbar^2 k^2}{4 m n_0},$$

Eikonal approximation:

$$\widehat{f}^{ab} \to f^{ab}_k = U_k^{-1} \left[\begin{array}{c|c} -1 & -v^j \\ \hline -\, v^i & \frac{n_0 U_k}{m}\; \delta^{ij} - v^i v^j \end{array} \right].$$

Phase velocity: $$c_k^2 = \frac{n_0 U_k}{m}$$

$$\widehat{f}^{ab} \to f^{ab}_k = U_k^{-1} \left[\begin{array}{c|c} -1 & -v^j \\ \hline -\, v^i & c_k^2\; \delta^{ij} - v^i v^j \end{array} \right].$$

Metric: $$f^{ab}_k \equiv \sqrt{-g_k}\;\; g_k^{ab}$$

Eikonal approximation:

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$$g_{ab} \equiv \left(\frac{n_0}{c_k}\right)^{\frac{2}{d-1}} \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array}\right]$$

EOM: $$\frac{1}{\sqrt{-g_k}}\, \partial_a \left(\sqrt{-g_k}\, g_k^{ab}\, \partial_b \theta \right) = 0.$$

This analysis requires a "separation of scales" between the wave-vector of the perturbation and the scale on which the background varies....

Dispersion relation:

Another approach is to start straight from the dispersion relations....

Consider a fluid at rest, in very many cases the dispersion relation can be written in the form:

$$\omega^2 = F(k)$$

for some possibly nonlinear function F(k)...

(2nd-order in time; arbitrary order in space...)

Eg: BECs (acoustic and post-acoustic), ripplons, gravity waves (fluid mechanics), etc, etc ...

Dispersion relation:

Phase velocity: $$c_k^2 = \frac{\omega^2}{k^2} = \frac{F(k)}{k^2}$$

Dispersion relation: $$\omega^2 = c_k^2\, k^2$$

Fluid in motion: Doppler shift the frequency...

$$\omega \longrightarrow \omega - \vec{v} \cdot \vec{k}$$

$$\left(\omega - \vec{v} \cdot \vec{k}\right)^2 - c_k^2\, k^2 = 0$$

Dispersion relation:

Rewrite as: $$g_k^{ab}\; k_a\, k_b = 0.$$

Pick off components:

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\, \delta^{ij} - v^i v^j \end{array}\right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array}\right].$$

Momentum dependent metric depending on phase velocity.

Dispersion relation:

Dispersion relation approach is physically transparent...

Only weakness: Conformal factor left unspecified...

(This is a standard side-effect of the geometrical quasi-particle approximation,
cf geometrical acoustics,
cf geometrical optics.)

Rainbow geometries:

In analogue spacetimes it is now clear that the momentum in the "rainbow metric" should be the momentum of the quasiparticle under investigation...

In other contexts, eg DSR
--- distorted special relativity ----
it might be useful to consider a metric that depends on the 4-momentum of the observer..

Conclusions:

Analogue spacetimes are a good way of providing clean and interesting physical models for both Finsler spaces and “rainbow spacetimes”...

The extremely abstract mathematics of Finsler geometry now has some direct relevance to physics....

We also now have a well-controlled physical model that can help us sharpen our notion of what exactly a “rainbow spacetime” can and should be...

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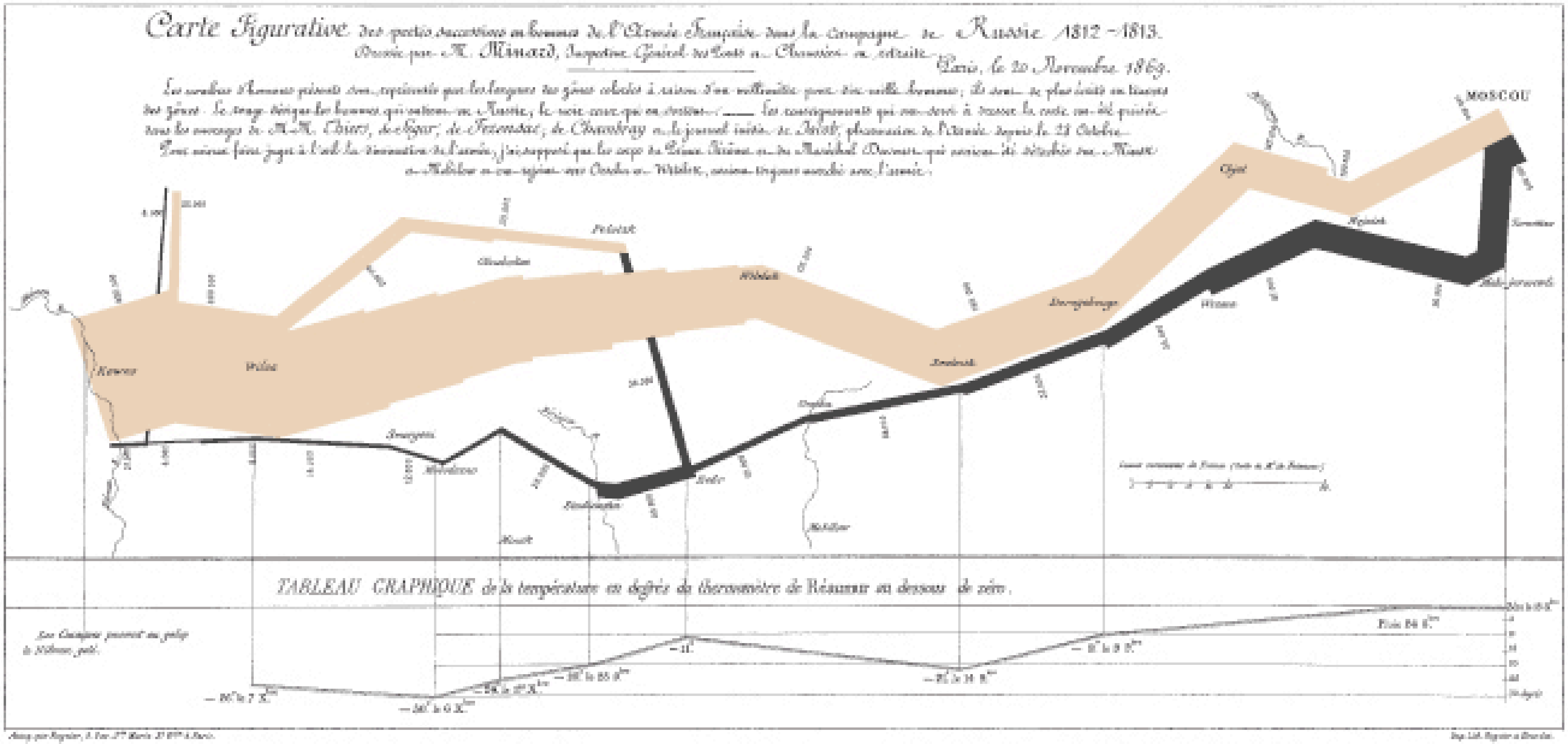

Napoleon's March to Moscow   The War of 1812

Charles Joseph Minard

This classic of Charles Joseph Minard (1781–1870), the French engineer, shows the terrible fate of Napoleon's army in Russia. Described by E. J. Marey as seeming to defy the pen of the historian by its brutal eloquence, this combination of data map and time-series, drawn in 1869, portrays the devastating losses suffered in Napoleon's Russian campaign of 1812. Beginning at the left on the Polish-Russian border near the Niemen River, the thick band shows the size of the army (422,000 men) as it invaded Russia in June 1812. The width of the band indicates the size of the army at each place on the map. In September, the army reached Moscow, which was by then sacked and deserted, with 100,000 men. The path of Napoleon's retreat from Moscow is depicted by the darker, lower band, which is linked to a temperature scale and dates at the bottom of the chart. It was a bitterly cold winter, and many froze on the march out of Russia. As the graphic shows, the crossing of the Berezina River was a disaster, and the army finally struggled back into Poland with only 10,000 men remaining. Also shown are the movements of auxiliary troops, as they sought to protect the rear and the flank of the advancing army. Minard's graphic tells a rich, coherent story with its multivariate data, far more enlightening than just a single number bouncing along over time. Six variables are plotted: the size of the army, its location on a two-dimensional surface, direction of the army's movement, and temperature on various dates during the retreat from Moscow. It may well be the best statistical graphic ever drawn.

Edward R. Tufte, *The Visual Display of Quantitative Information*   Graphics Press, Box 430, Cheshire, Connecticut 06410