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Analogue rainbow geometries

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Abstract:

We develop several models for "rainbow geometries" based on "analogue spacetimes".

These geometries are useful as concrete physical examples of how to construct physically well-motivated rainbow geometries, which may then be of interest as guideposts when considering possible energy-dependent modifications of general relativity.

One class of models is based on generalizing the acoustic spacetimes of classical fluid mechanics by inserting the momentum-dependent group velocity and phase velocity into the spacetime metric --- this leads to two distinct "rainbow metrics", which describe distinct aspects of the physics, and which converge on the ordinary acoustic metric in the hydrodynamic limit.

Abstract:

A second class of energy-dependent geometries can be built by using the Maupertuis form of the least action principle to rewrite Newton's second law in terms of geodesic equations on an energy-dependent manifold.

[no time to discuss this]

Abstract:

While these particular models are not themselves of direct relevance to quantum gravity, they do provide mathematically and physically well-defined examples of what a "rainbow spacetime" should be.

Acoustic spacetime:

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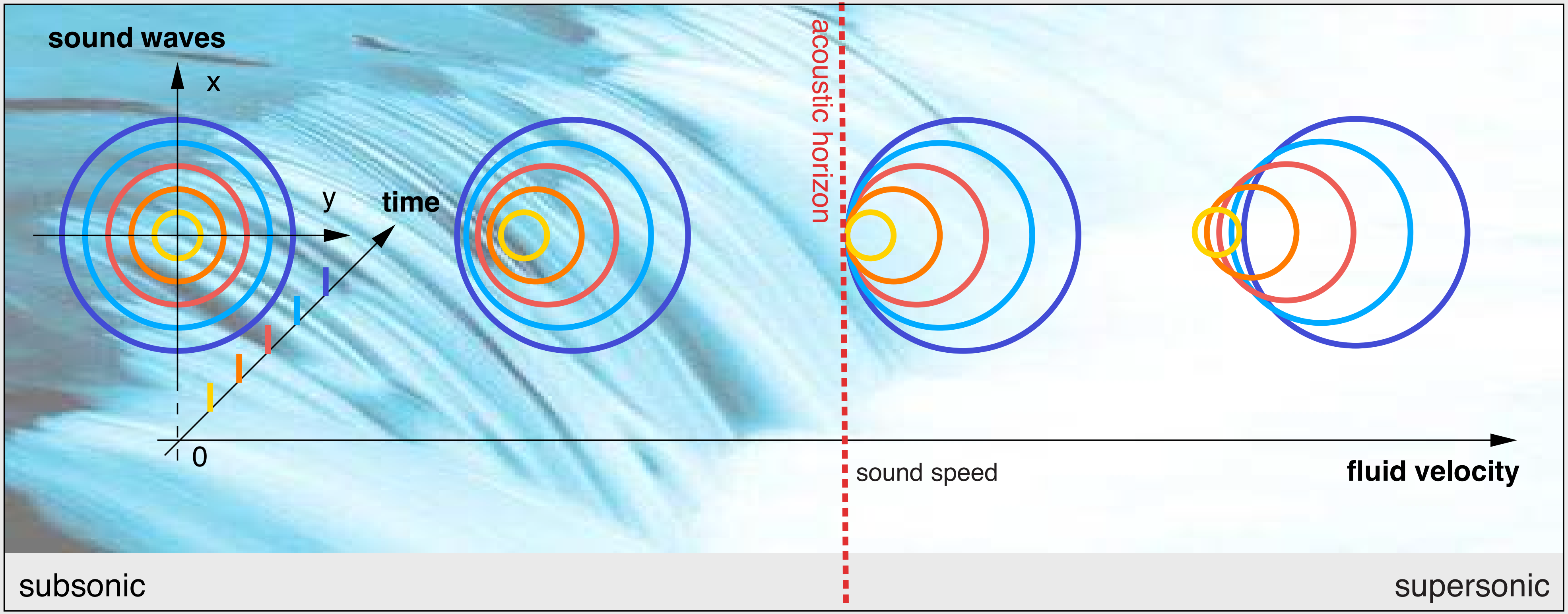
The simplest "analogue spacetimes" are the "acoustic spacetimes"...



Consider sound waves in a moving fluid... [Unruh]

Rainbow spacetime:

There is no general widely accepted precise mathematical definition of what is meant by a "rainbow geometry"...

The physicist's definition is rather imprecise:
"energy dependent metric"?
"momentum dependent metric"?
"4-momentum dependent metric"?

Q: 4-momentum of what? The observer? The object being observed?

There is a physics reason for this vagueness...

Rainbow spacetime:

To capture the essence of "energy dependence", need a metric that depends on some tangent or cotangent vector...

Consider a fluid at rest, in very many cases the dispersion relation can be written in the form:

$$\omega^2 = F(k)$$

for some possibly nonlinear function F(k)...

(2nd-order in time; arbitrary order in space...)

[Unruh, Jacobson]

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Rainbow spacetime:

Phase velocity:

$$c_k^2 = \frac{\omega^2}{k^2} = \frac{F(k)}{k^2}$$

Dispersion relation:

$$\omega^2 = c_k^2 \, k^2$$

Fluid in motion: Doppler shift the frequency...

$$\omega \longrightarrow \omega - \vec{v} \cdot \vec{k}$$

$$\left(\omega - \vec{v} \cdot \vec{k}\right)^2 - c_k^2 \, k^2 = 0$$

Rainbow spacetime:

Rewrite as: $$g_k^{ab}\; k_a\, k_b = 0.$$

Pick off components:

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\,\delta^{ij} - v^i v^j \end{array}\right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array}\right].$$

Momentum dependent metric depending on phase velocity.

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Dispersion relation approach is physically transparent...

Only weakness: Conformal factor left unspecified...

(This is a standard side-effect of the geometrical quasi-particle approximation, cf geometrical acoustics, cf geometrical optics.)

[PDE is better] [Weinfurtner]

The momentum in question is now clearly the momentum of an individual "mode" of the field that is: phase velocity <==> dispersion relation.

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Similar (but distinct) steps can be taken to develop a rainbow metric based on group velocity.

Consider a wave packet centered on momentum k.

That packet will propagate with the group velocity.

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t)^2 = c_k^2\,\mathrm{d}t^2$$

Group velocity.

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Rewrite as: $$\mathrm{d}s^2 = 0 = g_{ab}\, dx^a\, \mathrm{d}x^b$$

Pick off components:

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\, \delta^{ij} - v^i v^j \end{array}\right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array}\right].$$

Momentum dependent metric depending on group velocity.

Rainbow spacetime:

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There are at least two distinct very different notions of "Rainbow metric" in an analogue setting.

They answer different questions:

* What is the dispersion relation of a pure mode?

* How do wave packets propagate?

If you are lucky there is a "hydrodynamic" limit:

$$\lim_{k \to 0} c^2_{\text{phase}}(k) = c^2_{\text{hydrodynamic}} = \lim_{k \to 0} c^2_{\text{group}}(k)$$

$$\neq 0!$$

Rainbow spacetime:

In general: Rainbow ==> multi-metric

$$g_k^{ab} \propto \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & c_k^2\, \delta^{ij} - v^i v^j \end{array} \right].$$

$$g^k_{ab} \propto \left[\begin{array}{c|c} -(c_k^2 - v^2) & -v^j \\ \hline -v^i & \delta^{ij} \end{array} \right].$$

With: $c_k \to \begin{cases} c_{\text{phase}} \\ c_{\text{group}} \\ c_{\text{hydrodynamic}} \end{cases}.$

Signal velocity?

Front velocity?

c ==> infinity?

Rainbow spacetime:

Bogoliubov dispersion relation (eg, BECs):

$$\omega^2 = c_0^2\, k^2 + \left(\frac{\hbar}{2m}\right)^2 k^4$$

$$c^2 = c_0^2 + \left(\frac{\hbar}{2m}\right)^2 k^2 \qquad \text{(supersonic)}$$

Controlled breaking of Lorentz invariance...

See "quantum gravity phenomenology"... [Liberati...]

See "cosmological particle production" [Weinfurtner]

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Surface waves in finite depth of liquid: [Lamb]

$$\omega^2 = g\,k\,\tanh(k\,d) = c_0^2\,k^2\,\frac{\tanh(k\,d)}{k\,d} \qquad c_0^2 = g\,d.$$

$$c^2 = c_0^2\,\cancel{k^2}\,\frac{\tanh(k\,d)}{k\,d}$$

(subsonic)

$$\omega^2 = c_0^2\,k^2\left\{1 - \frac{(k\,d)^2}{3} + \frac{2(k\,d)^2}{15} + \dots\right\}$$

So analogue models provide concrete examples for both supersonic and subsonic dispersion, and more...

Rainbow spacetime:

Surface waves in infinite depth of liquid:

$$\omega = \sqrt{g\,k}; \qquad c_{\text{phase}} = \sqrt{g/k}.$$

$$c_{\text{group}} = \frac{\partial\omega}{\partial k} = \frac{\sqrt{g/k}}{2} = \frac{c_{\text{phase}}}{2}.$$

No hydrodynamic limit...

No well-defined low-momentum spacetime...

[You could argue that this is an unphysical limit...]

Rainbow spacetime:

Surface waves in finite depth of liquid + surface tension:

$$\omega^2 = c_0^2\, k^2 \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

$$c^2 = c_0^2 \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

$$c_0^2 = g\, d.$$

Asymptotically supersonic, though it can be adjusted to have a subsonic dip.

Water: $\epsilon = \frac{\sigma}{\rho c_0^2 d} = \frac{\sigma}{\rho g d^2} = \frac{(0.27\ \mathrm{cm})^2}{d^2}.$

Rainbow spacetime:

$$c^2 = c_0^2 \; \left\{1 + \epsilon \,(kd)^2\right\} \; \frac{\tanh(kd)}{kd}.$$

$$c^2 = c_0^2 \; \left\{1 + \frac{3\epsilon - 1}{3}\,(kd)^2 - \frac{5\epsilon - 2}{15}\,(kd)^4 + \mathcal{O}[(kd)^6]\right\}.$$

Can tune away the lowest order Lorentz violation...

(Water at 0.47 cm depth)

These are just some examples of the types of dispersion relation you can arrange...

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Can also arrange for particle masses:

$$\omega^2 = \omega_0^2 + c_0^2 \; k^2 + \frac{k^4}{K^2} + \mathcal{O}[(k)^6].$$

[2 interacting BECs: Weinfurtner *et al*...]

Basic message: Lots of physically well behaved and well controlled toy models for many different types of "beyond the standard model" physics...

Conclusion:

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Many interesting extensions and modifications of the general relativity notion of spacetime have concrete and well controlled models within the "analogue spacetime" framework.

This tells us which rocks to start looking under...

"It is important to keep an open mind; just not so open that your brains fall out"

--- Albert Einstein