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Te Whare Wānanga
o te Ūpoko o te Ika a Māui

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Emergent dispersion relations --- lessons for quantum gravity.

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Experimental search for quantum gravity
Perimeter Institute
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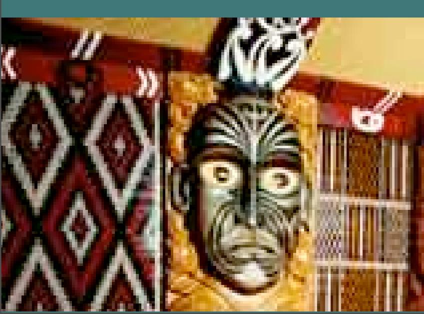

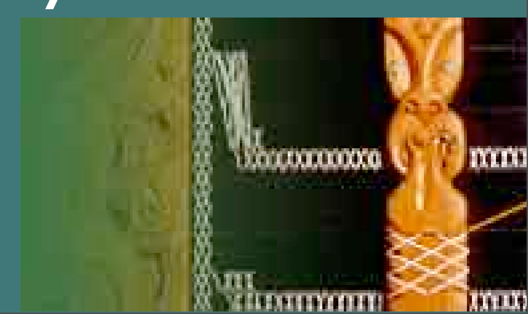

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Remember, remember the fifth of November,
the gunpowder treason and plot,
I know of no reason
why gunpowder treason
should ever be forgot...

Abstract:

The dispersion relations that naturally arise in the known emergent/analogue spacetimes typically violate analogue Lorentz invariance at high energy, but do not do so in completely arbitrary manner. This suggests that a search for arbitrary violations of Lorentz invariance is possibly overkill: There are a number of natural and physically well-motivated restrictions one can put on emergent/ analogue dispersion relations, considerably reducing the plausible parameter space.

Possible ultra-high energy violations of Lorentz invariance are one of the most popular "signals" being looked for in quantum gravity phenomenology.

Replace:

$$\omega^2 = \omega_0^2 + c^2\; k^2.$$

By:

$$F_1(\omega, k) = 0.$$

Or, (appealing to the implicit function theorem):

$$\omega = F_2(k).$$

But this might be too general to be useful...

H I:

Gravity: Not just CPT, but:

C, P, and T.

Einstein gravity: Certainly C, P, and T invariant.

Only known examples of P, T violations are in the electro-weak sector, seemingly unconnected with gravity.

Certainly no gravitational experiment has detected P or T violations.

So don't add more complications than necessary.

Working hypothesis I:

Maintain both P and T invariance.

This constrains the terms that can show up in the dispersion relation:

$$\omega^2; \qquad \omega\,(\vec{v}\cdot\vec{k}); \qquad (\vec{v}\cdot\vec{k})^2; \qquad h^{ij}\;k_i\;k_j.$$

Here "v" is a P odd, T odd, term that behaves *like* a "velocity".

Here "h" is a P even, T even, term that behaves *like* a "dielectric matrix".

Improved dispersion relation:

$$F_3\left(\ \omega^2;\ \omega\,(\vec{v}\cdot\vec{k});\ (\vec{v}\cdot\vec{k})^2;\ h^{ij}\ k_i\ k_j\ \right) = 0.$$

Improved basis:

$$\omega^2; \qquad (\omega - \vec{v}\cdot\vec{k})^2; \qquad (\vec{v}\cdot\vec{k})^2; \qquad h^{ij}\ k_i\ k_j\,.$$

Improved dispersion relation:

$$F_4\left(\omega^2;\ (\omega - \vec{v}\cdot\vec{k})^2;\ (\vec{v}\cdot\vec{k})^2;\ h^{ij}\ k_i\ k_j\ \right) = 0.$$

Both P and T invariant.

H II:

Time derivatives higher than second-order tend to be problematic.

Time derivatives higher than second-order tend to lead to ghosts and unitarity violations.

Time derivatives higher than second-order do not seem to be seen in nature.

Minor exception: Fresnel relations.

But in (almost) all known physically relevant cases, Fresnel relations factorize into second-order fragments.

Fresnel relations:

$$\omega^2\ (A\,\omega^4 + B\,\omega^2\,k^2 + C\,k^4) = 0.$$

(Two physical photon polarizations, plus the "longitudinal" mode.)

But in all known cases, (including uni-axial bi-refringent crystals), they factorize:

$$\omega^2\ (\omega^2 - c_1^2\,k^2)\ (\omega^2 - c_2^2\,k^2) = 0.$$

"Ordinary" and "extraordinary" rays... [bi-axial worse]

Net result: (Almost) all known dispersion relations are effectively second-order in time...

Working hypothesis II:

Dispersion relations are at most second-order in time.

Combine with working hypothesis I (P and T invariance):

$$F_5\left(a\,\omega^2 + b\,(\omega - \vec{v}\cdot\vec{k})^2;\ (\vec{v}\cdot\vec{k})^2;\ h^{ij}\ k_i\,k_j \right) = 0.$$

Regroup terms:

$$\vec{\bar{v}} = \frac{b}{a+b}\,\vec{v}.$$

$$a\,\omega^2 + b\,(\omega - \vec{v}\cdot\vec{k})^2 = [a+b]\left\{ \left(\omega - \vec{\bar{v}}\cdot\vec{k}\right)^2 + \frac{a}{b}\,\left(\vec{\bar{v}}\cdot\vec{k}\right)^2 \right\}.$$

Improved dispersion relation:

$$F_6\left(\left(\omega - \vec{\bar{v}}\cdot\vec{k}\right)^2;\ (\vec{\bar{v}}\cdot\vec{k})^2;\ h^{ij}\ k_i\ k_j\ \right) = 0.$$

Appeal to implicit function theorem:

$$\left(\omega - \vec{\bar{v}}\cdot\vec{k}\right)^2 = F_7\left(\ h^{ij}\ k_i\ k_j;\ (\vec{\bar{v}}\cdot\vec{k})^2\ \right).$$

Finally, drop unneeded subscripts and over-bars:

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = F\left(\ h^{ij}\ k_i\ k_j;\ (\vec{v}\cdot\vec{k})^2\ \right).$$

Under very mild conditions:

H I: P and T invariance.

H II: Second-order time derivatives.

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = F\left(\; h^{ij}\; k_i\; k_j;\; (\vec{v}\cdot\vec{k})^2 \;\right).$$

$$\left(\omega - \vec{v}\cdot\vec{k}\right) = \sqrt{F\left(\; h^{ij}\; k_i\; k_j;\; (\vec{v}\cdot\vec{k})^2 \;\right)}.$$

But this falls naturally into a minor extension of the class of dispersion relations arising in "emergent/ analogue spacetimes".

Emergent analogue spacetimes:

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = \tilde{F}\left(\, k^2 \,\right).$$

H I + H II: $$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = F\left(\, h^{ij}\, k_i\, k_j;\; (\vec{v}\cdot\vec{k})^2 \,\right).$$

Note that I have not used any "analogue model" reasoning to get to this stage --- just some very fundamental working hypotheses, of what would seem to be eminently reasonable constraints any realistic quantum gravity phenomenology should satisfy.

(I've not even used any notion of "Lorentz invariance".)

H III:

Now let's take "v" a little more seriously, and hypothesize that it is some sort of "physical velocity".

Local preferred rest frame for Lorentz breaking?

Or, (if you like to give colleagues heart attacks), "velocity of the sub-quantic aether"...

(Add your personal favourite name here...)

What you call it does not matter: If it is a physical velocity, then you can certainly go to the local rest frame where this velocity is zero.

I'm not using Lorentz transformations here, just saying that if "v" is a physical velocity then you should be able to move at speed "v", effectively putting you "at rest" with respect to whatever it is that "v" is representing...

In this "local rest frame" the dispersion relation

$$\left(\omega - \vec{\vec{v}}\cdot\vec{k}\right)^2 = F_7\left(\; h^{ij}\; k_i\, k_j;\; (\vec{\vec{v}}\cdot\vec{k})^2 \;\right),$$

specializes to [H I + H II + H III]:

$$\omega^2 = F_8\left(\; h^{ij}\; k_i\; k_j\right).$$

In the local rest frame, dropping unnecessary subscripts:

$$\omega^2 = F\left(\, h^{ij}\, k_i\, k_j \right).$$

From this I can construct the usual notion of phase velocity:

$$c^2_{\rm phase}(k^2) = \frac{F\left(\, h^{ij}\, k_i\, k_j \right)}{h^{ij}\, k_i\, k_j}.$$

$$\omega^2 = c^2_{\rm phase}(k^2)\ \left\{\, h^{ij}\, k_i\, k_j \,\right\}.$$

[H I + H II + H III] (almost done...)

H IV:

No one can stop me from making a Galilean coordinate transformation:

$$\vec{x} \longrightarrow \vec{x} + \vec{v}\, t.$$

$$t \longrightarrow t.$$

(I make no claim that this is a "symmetry", it is "merely" a convenient choice of coordinates...)

Of course this induces a change in coordinates on the cotangent space as well...

$$\omega \longrightarrow \omega - \vec{v} \cdot \vec{k}\ .$$

$$\vec{k} \longrightarrow \vec{k}\ .$$

[H I + H II + H III + H IV]

There is a convenient set of coordinates in which the dispersion relation takes the form:

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = F\left(\, h^{ij}\, k_i\, k_j \,\right).$$

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = c_{\text{phase}}(k^2)\ \left\{\, h^{ij}\, k_i\, k_j \,\right\}.$$

This should be compared to the standard "analogue spacetime" result:

$$\left(\omega - \vec{v}\cdot\vec{k}\right)^2 = \tilde{F}\left(\, k^2 \,\right).$$

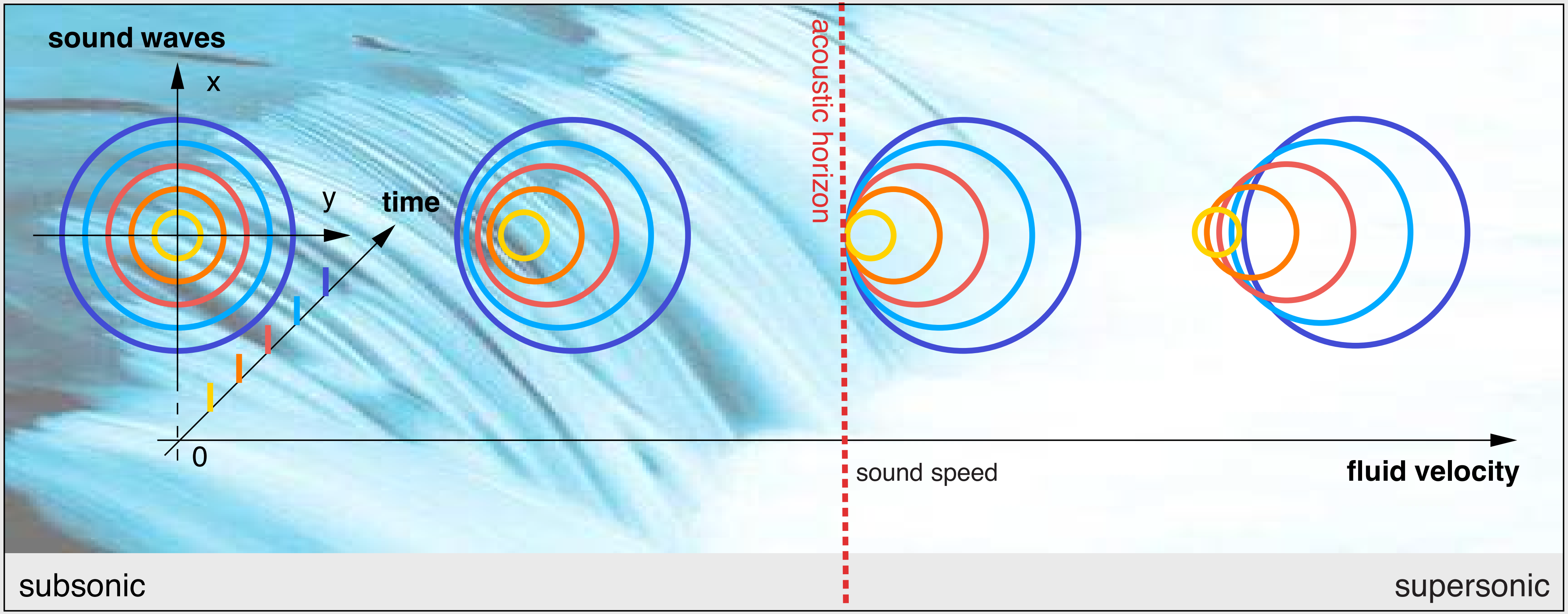
This gives me confidence that whatever insights we extract from the "emergent analogue spacetime" programme are likely to be generic to a wide class of physically reasonable quantum gravity phenomenologies.

For example, "analogue spacetimes" provide concrete models of "emergence" (the effective low-energy theory can be radically different from the high-energy microphysics).

"Analogue spacetimes" also provide controlled models of "Lorentz symmetry breaking", and extensions of the usual notions of Lorentzian geometry: "rainbow spacetimes", and more...

Acoustic spacetime:

The simplest "emergent/analogue spacetimes" are the "acoustic spacetimes"...



Consider sound waves in a moving fluid...

Acoustic spacetime:

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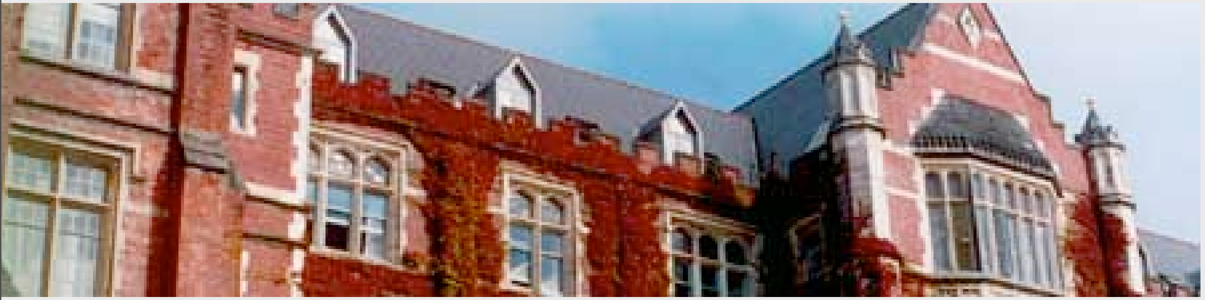
Theorem: Consider an irrotational, inviscid, barotropic perfect fluid, governed by the Euler equation, continuity equation, and an equation of state.

The dynamics of the linearized perturbations (sound, phonons) is governed by a D'Alembertian equation

$$\Delta_g \Phi = \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g} \; g^{ab} \partial_b \; \Phi \right) = 0$$

involving an "acoustic metric".

[Algebraic function of the background fields.]



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Theorem:

(3+1 dimensions)

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{\rho_0 c} \left[\begin{array}{ccc} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\,\delta^{ij} - v_0^i\,v_0^j) \end{array} \right].$$

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho_0}{c} \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{array} \right].$$

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\rho_0}{c}\left[-c^2\,\mathrm{d}t^2 + (\mathrm{d}x^i - v_0^i\,\mathrm{d}t)\,\delta_{ij}\,(\mathrm{d}x^j - v_0^j\,\mathrm{d}t)\right].$$

Back to the general case...

[H I + H II + H III + H IV]

$$c^2_{\text{phase}}(k^2) = \frac{F\left(\, h^{ij}\; k_i\; k_j \right)}{h^{ij}\; k_i\; k_j}.$$

$$\left(\omega - \vec{v} \cdot \vec{k}\right)^2 = c_{\text{phase}}(k^2)\ \left\{\, h^{ij}\; k_i\; k_j \,\right\}.$$

This lets you pick off a momentum dependent "rainbow metric".

Rainbow spacetime:

Define: $$k_a = \left(\omega;\ \vec{k}\right).$$

Rewrite the dispersion relation as:

$$g^{ab}(k^2)\ k_a\ k_b = 0.$$

Pick off the components:

$$g^{ab}(k^2) \propto \left[\begin{array}{c|c} -1 & +v^j \\ \hline +v^i & c^2_{\mathrm{phase}}(k^2)\ h^{ij} - v^i\ v^j \end{array}\right].$$

Rainbow spacetime:

$$g^{ab}(k^2) \propto \left[\begin{array}{c|c} -1 & +v^j \\ \hline +v^i & c^2_{\mathrm{phase}}(k^2)\; h^{ij} - v^i\; v^j \end{array} \right].$$

$$g_{ab}(k^2) \propto \left[\begin{array}{c|c} -\left\{ c^2_{\mathrm{phase}}(k^2) - h_{ij}\; v^i\; v^j \right\} & -v_j \\ \hline -v_i & +h_{ij} \end{array} \right].$$

Momentum-dependent "rainbow metric" depending on the phase velocity.

Rainbow spacetime:

This dispersion relation approach
is physically very transparent...

Only weakness: Conformal factor left unspecified...

This is a standard side-effect of the geometrical
quasi-particle approximation,
also shows up in geometrical acoustics
geometrical optics, and more generally
in any eikonal approximation...

PDE is better --- if you have the additional physical
information available from some other source...

Rainbow spacetime:

Similar (but distinct) steps can be taken to develop a rainbow metric based on group velocity.

Consider a wave packet centered on momentum k.

That packet will propagate with the group velocity.

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t)^2 = c^2_{\mathrm{group}}(k^2)\;\mathrm{d}t^2.$$

Rewrite as:

$$\mathrm{d}s^2 = 0 = g_{ab}(k^2)\, \mathrm{d}x^a\, \mathrm{d}x^b.$$

Pick off components:

$$g_{ab}(k^2) \propto \left[\begin{array}{c|c} -\left\{c^2_{\text{group}}(k^2) - h_{ij}\, v^i\, v^j\right\} & -v_j \\ \hline -\, v_i & +h_{ij} \end{array} \right].$$

$$g^{ab}(k^2) \propto \left[\begin{array}{c|c} -\,1 & +v^j \\ \hline +\, v^i & c^2_{\text{group}}(k^2)\; h^{ij} - v^i\, v^j \end{array} \right].$$

Momentum-dependent “rainbow metric”
depending on the group velocity.

Rainbow spacetime:

There are at least two distinct very different notions of "rainbow metric" in a [H I + H II + H III + H IV] setting.

Not now restricted to an "analogue spacetime".

They answer different questions:

* What is the dispersion relation?

* How do wave packets propagate?

If you are lucky there is a "hydrodynamic" limit:

$$\lim_{k\to 0} c^2_{\text{phase}}(k) = c^2_{\text{hydrodynamic}} = \lim_{k\to 0} c^2_{\text{group}}(k)$$

$$\neq 0!$$

Rainbow spacetime:

In general: Rainbow ==> multi-metric

$$g_{ab}(k^2) \propto \left[\begin{array}{c|c} -\{c^2(k^2) - h_{ij}\, v^i\, v^j\} & +v_j \\ \hline +\, v_i & +h^{ij} \end{array} \right].$$

$$g^{ab}(k^2) \propto \left[\begin{array}{c|c} -\,1 & +v^j \\ \hline +\, v^i & c^2(k^2)\; h^{ij} - v^i\, v^j \end{array} \right].$$

With:

$$c(k^2) \to \begin{cases} c_{\text{phase}}(k^2) \\ c_{\text{group}}(k^2) \\ c_{\text{hydrodynamic}} \\ c_{\text{signal}} \\ \infty? \end{cases}.$$

Causal structure:

Q: Is the "signal velocity" finite?

Two plausible definitions of "signal velocity".

$$c_{\text{signal}} = \lim_{k\to\infty} c_{\text{phase}}(k^2).$$

$$c_{\text{signal}} = \max_{k} \; c_{\text{group}}(k^2).$$

The first definition focusses on how discontinuities propagate, the second definition focusses on how rapidly one can transmit a packet of information.

(They are inter-related.)

Causal structure:

As long as the signal velocity is finite the global causal structure will be similar to that of general relativity, just with "signal cones" instead of light cones...

If the signal velocity is infinite then the global causal structure will be similar to that of Newtonian physics.

The distinction between "superluminal" and "subluminal" dispersion relations, while it certainly impacts on thresholds, and constrains allowable particle interactions, is of subsidiary importance when it comes to determining global causal structure.

Rainbow spacetime:

Bogoliubov dispersion relation
(eg, BECs, superconductors):

$$\omega^2 = c_0^2\, k^2 + \left(\frac{\hbar}{2m}\right)^2 k^4$$

$$c^2 = c_0^2 + \left(\frac{\hbar}{2m}\right)^2 k^2$$ (phase velocity)

Controlled breaking of Lorentz invariance...

Check group velocity to see supersonic/subsonic...

(yes, it's supersonic)

Rainbow spacetime:

Surface waves in finite depth of liquid:

[Lamb]

$$\omega^2 = g\,k\,\tanh(k\,d) = c_0^2\,k^2\,\frac{\tanh(k\,d)}{k\,d} \qquad c_0^2 = g\,d.$$

$$c^2 = c_0^2\,k^2\,\frac{\tanh(k\,d)}{k\,d}$$

(subsonic)

$$\omega^2 = c_0^2\,k^2\,\left\{1 - \frac{(k\,d)^2}{3} + \frac{2(k\,d)^2}{15} + \dots\right\}$$

So analogue models provide concrete examples for both supersonic and subsonic dispersion, and more...

Rainbow spacetime:

Surface waves in infinite depth of liquid:

$$\omega = \sqrt{g\,k}; \qquad c_{\text{phase}} = \sqrt{g/k}.$$

$$c_{\text{group}} = \frac{\partial \omega}{\partial k} = \frac{\sqrt{g/k}}{2} = \frac{c_{\text{phase}}}{2}.$$

No hydrodynamic limit...

No well-defined low-momentum spacetime...

You could argue that this is an unphysical limit...

Why does this seem to violate H I? Specifically,   P?

The real dispersion relation is this, which is P invariant:

$$\omega^2 = g\,k\;\tanh(k\,d) = c_0^2\;k^2\;\frac{\tanh(k\,d)}{k\,d}$$

The "apparent" P violation comes about in the unphysical infinite depth limit, and only for wavelengths less than the depth of the ocean...

Now let's talk quantum gravity phenomenology:

Suppose I desperately want k-cubed terms in the dispersion relation, but without violating parity invariance. How could I achieve this?

Faking a cubic term...

Consider this:

$$\omega^2 = \omega_0^2 + c^2\ k^2 + c^2 (k^4/K_1^4)\ \frac{\tanh(k/K_2)}{(k/K_2)}.$$

This dispersion relation is P invariant.

K_1 is a momentum scale characterizing Lorentz breaking.

K_2 is a momentum scale characterizing "apparent" parity breaking.

You need two different physical scales...

Need K_2 << K_1 to get anything "interesting"...

That is: sub-Planckian parity breaking...

Rainbow spacetime:

Surface waves in finite depth of liquid
+ surface tension:

$$\omega^2 = c_0^2\, k^2 \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

$$c_0^2 = g\, d.$$

$$c^2 = c_0^2 \left\{ 1 + \frac{\sigma}{\rho\, c_0^2\, d} (kd)^2 \right\} \frac{\tanh(kd)}{kd}.$$

Asymptotically supersonic, though it can be adjusted to have a subsonic dip.

Water: $\epsilon = \frac{\sigma}{\rho c_0^2 d} = \frac{\sigma}{\rho g d^2} = \frac{(0.27 \text{ cm})^2}{d^2}.$

Rainbow spacetime:

$$c^2 = c_0^2 \; \left\{1 + \epsilon\,(kd)^2\right\} \; \frac{\tanh(kd)}{kd}.$$

$$c^2 = c_0^2 \; \left\{1 + \frac{3\epsilon - 1}{3}\,(kd)^2 - \frac{5\epsilon - 2}{15}\,(kd)^4 + \mathcal{O}[(kd)^6]\right\}.$$

Can tune away the lowest order Lorentz violation...

(Water at 0.47 cm depth)

These are just some examples of the types of dispersion relation you can arrange...

Rainbow spacetime:

Can also arrange for particle masses:

$$\omega^2 = \omega_0^2 + c_0^2\; k^2 + \frac{k^4}{K^2} + \mathcal{O}[(k)^6].$$

[2 interacting BECs: Weinfurtner *et al*...]

Basic message: Lots of physically well behaved and well controlled toy models for many different types of "beyond the standard model" physics...

Conclusion:

Many interesting extensions and modifications of the general relativity notion of spacetime have concrete and well controlled models within the "analogue spacetime" framework.

The "analogue spacetime" framework is quite natural and plausible from the point of view of "quantum gravity phenomenology".

This tells us which rocks to start looking under...

"It is important to keep an open mind; just not so open that your brains fall out"

--- Albert Einstein