EUROPHYSICS LETTERS

OFFPRINT

Vol. 62 • Number 1 • pp. 1–7

Warped space-time for phonons moving in a perfect nonrelativistic fluid

* * *

U. R. Fischer and M. Visser

Published under the scientific responsability of the

EUROPEAN PHYSICAL SOCIETY

Incorporating

JOURNAL DE PHYSIQUE LETTRES • LETTERE AL NUOVO CIMENTO

EUROPHYSICS LETTERS 1 April 2003

Europhys. Lett., **62** (1), pp. 1–7 (2003)

Warped space-time for phonons moving in a perfect nonrelativistic fluid

U. R. Fischer[1,2] and M. Visser[3]

[1] *Helsinki University of Technology, Low Temperature Laboratory*
P.O. Box 2200, FIN-02015 HUT, Finland

[2] *Leopold-Franzens-Universität Innsbruck, Institut für Theoretische Physik*
Technikerstrasse 25, A-6020 Innsbruck, Austria

[3] *School of Mathematical and Computing Sciences, Victoria University of Wellington*
P.O. Box 600, Wellington, New Zealand

(received 25 November 2002; accepted 31 January 2003)

PACS. 02.40.Ky – Riemannian geometries.

Abstract. – We construct a kinematical analogue of superluminal travel in the "warped" space-times curved by gravitation, in the form of "super-phononic" travel in the curved effective space-times of perfect nonrelativistic fluids. These warp-field space-times are most easily generated by considering a solid object that is placed as an obstruction in an otherwise uniform flow. No violation of any condition on the positivity of energy is necessary, because the effective curved space-times for the phonons are ruled by the Euler and continuity equations, and not by the Einstein field equations.

Introduction. – The concept of "warp fields", or faster-than-light (FTL) propagation/travel, is usually relegated into the realm of science fiction literature. Taking warp fields more seriously, when trying to develop physical realizations within the context of Einstein gravity, one has to face the difficulty that fulfilling the Einstein equations demands "exotic matter"; matter violating the null, weak, strong, and dominant conditions on the positivity of energy [1–5]. If, on the other hand, one allows for negative energy densities (which occur, for example, in the quantum vacuum of the Casimir effect), the energy densities (and to a lesser degree the total energies [6]) actually required to construct macroscopic warp drives are astronomical [7]. One way of side-stepping these problems, and developing a concrete physical model of what a warp field might look like, is to consider effective space-time theories originating in condensed matter [8,9]: A flowing hydrodynamical background governed by the nonrelativistic Euler and continuity equations represents a curved space-time for the quasiparticle excitations moving in the fluid. The role of the speed of light is played by the speed of sound, and superluminal travel turns into super-phononic propagation due to the effective space-time curvature. The necessity of violating the energy conditions is no longer given, because the effective pseudo-Riemannian space created by the laboratory flow in absolute Newtonian space is determined by the equations of nonrelativistic hydrodynamics, and not by the Einstein field equations.

Using the curved space-time analog of quasiparticles propagating on a hydrodynamical background, we study in this paper a particularly simple and concrete physical realization of a "warp field" realized by a stationary spherical obstruction in a moving perfect nonrelativistic fluid. The effective space-time curvature near the sphere describes the fact that, when travelling between two specified points, the (absolute) Newtonian laboratory frame travel time for quasiparticles which move through the "warped" region near the sphere is either reduced or enhanced, as compared to the time the quasiparticles would need to propagate between the same two points through a flat effective space-time, that is, in a homogeneously streaming fluid.

One of the important technical issues involved in warp drives and "effective FTL" is the fact that defining FTL in a standard general relativity context is subtle and somewhat subject to coordinate artifacts [3, 10]. A particularly nice feature of the acoustic geometry presented here is that it provides a very concrete and definite physical model in which there is little room for confusion due to coordinate ambiguities, because in the present situation the background "reference metric" is unambiguous.

Metric and curvature. – For the vorticity-free flows considered here, the phonons travelling around the sphere perceive an acoustic space-time with metric [8, 9]

$$\mathrm{d}s^2 = -c_s^2\,\mathrm{d}t^2 + (\mathrm{d}\boldsymbol{x} - \boldsymbol{v}\,\mathrm{d}t)^2\,, \tag{1}$$

and with Riemann curvature tensor components given in terms of the deformation tensor [11]

$$D_{ij} = \frac{1}{2}\big(\partial_i v_j + \partial_j v_i\big). \tag{2}$$

The deformation has vanishing trace for the incompressible background flow we are assuming, $\mathrm{Tr}\,\boldsymbol{D} = 0$, and furthermore c_s is a constant. The deformation tensor corresponds in the language of general relativity to the extrinsic curvature tensor

$$K_{ij} = \frac{D_{ij}}{c_s}\,. \tag{3}$$

In the original proposal by Alcubierre [1], the extrinsic curvature tensor had nonvanishing trace, corresponding to a volume element deformation such that the space contracts in front of the spaceship and expands behind it. It was, however, recently shown by Natário [5] that $\mathrm{Tr}\,\boldsymbol{K} = \mathrm{Tr}\,\boldsymbol{D}/c_s \neq 0$ is *not* a necessary prerequisite for the warp drive, so that the assumption of an incompressible background does not impede its construction.

The nonvanishing components of the effective space-time Riemann tensor are (for an incompressible, vorticity-free background flow) [11]

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} \;=\; K_{ik}K_{jl} - K_{il}K_{jk}\,, \tag{4}$$

$$R_{\hat{t}\hat{i}\hat{t}\hat{j}} \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}K_{ij} - \big(\boldsymbol{K}^2\big)_{ij}\,. \tag{5}$$

The hats indicate that the components are in given in an orthonormal tetrad basis of the acoustic space-time, and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\cdot\nabla$ is a convective derivative. Note that if the flow is steady, the above formulae imply that the Riemann tensor scales with the flow velocity squared. The fact that the Riemann tensor goes to zero quadratically with the flow velocity tells us that in a theory linearized in the flow velocity, an irrotational, incompressible fluid flow leads to an intrinsically flat effective geometry. Phrased in more conventional language, there are no (relative acceleration) forces on the quasiparticles [12], which are linear in the velocity in such a steady flow.

The Ricci tensor is given by

$$R_{\hat{t}\hat{t}} = -\,\mathrm{Tr}\,\big(\boldsymbol{K}^2\big), \qquad R_{\hat{i}\hat{j}} = \frac{\mathrm{d}}{\mathrm{d}t}K_{ij}\,. \tag{6}$$

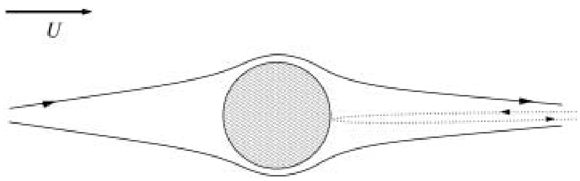

Fig. 1 – An impenetrable sphere placed in a stream with velocity $\boldsymbol{v}_\infty = v_\infty \hat{\boldsymbol{x}} = U c_s \hat{\boldsymbol{x}}$ at infinity. Quasiparticles travelling in the flow encounter a region of increased effective space-time curvature near the sphere. The dotted line is the reflected phonon trajectory along the x-axis described in the text.

while the Ricci scalar takes on a particularly simple form,

$$R = \mathrm{Tr}\left(\boldsymbol{K}^2\right). \tag{7}$$

Finally the Einstein tensor takes the form

$$G_{\hat{t}\hat{t}} = -\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{K}^2\right), \qquad G_{\hat{\imath}\hat{\jmath}} = \frac{\mathrm{d}}{\mathrm{d}t}K_{ij} - \frac{1}{2}\delta_{ij}\,\mathrm{Tr}\left(\boldsymbol{K}^2\right). \tag{8}$$

Note that $G_{tt} < 0$. This is a purely geometrical statement, which, *if* we were to impose the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$, would immediately lead to energy condition violations. Because we are not interpreting the metric in a general relativistic context, we do not use the Einstein equations. In the acoustic analogue, even though $G_{tt} < 0$ for purely geometrical reasons, energy condition violations are not implied.

Consider now the three-dimensional streaming motion of a perfect liquid past a sphere of radius a (fig. 1). Orienting the co-ordinate system such that the flow velocity at infinity $\boldsymbol{v}_\infty = v_\infty \boldsymbol{e}_x$ is in the x-direction, the velocity components for incompressible irrotational flow are [13]

$$\begin{aligned} v_x &= v_\infty - \frac{v_\infty a^3}{2r^5}\left(2x^2 - y^2 - z^2\right) = v_\infty - \frac{v_\infty a^3}{2r^3}\left(3\cos^2\theta - 1\right), \\ v_y &= -\frac{3v_\infty a^3}{2r^5}xy = -\frac{3v_\infty a^3}{4r^3}\sin 2\theta\cos\phi, \\ v_z &= -\frac{3v_\infty a^3}{2r^5}xz = -\frac{3v_\infty a^3}{4r^3}\sin 2\theta\sin\phi, \end{aligned} \tag{9}$$

where θ is the angle to the x-axis, and ϕ is an azimuth. Note that $||\boldsymbol{v}|| \to v_\infty \neq 0$ at $r \to \infty$. These formulae hold for $r \geq a$.

To get a qualitative handle on the curvature, note that the velocity field is of the form

$$\boldsymbol{v} \sim (\text{constant}) + v_\infty \frac{a^3}{r^3} \times (\text{direction-dependent factors}). \tag{10}$$

This implies that the extrinsic curvature, being given by velocity gradients, must be of the form

$$K \sim U\frac{a^3}{r^4} \times (\text{direction-dependent factors}), \tag{11}$$

where for convenience we introduce the Mach number $U = v_\infty/c_s$. Note that our incompressibility assumption forces us to work in the regime $U \ll 1$. Therefore

$$(\text{Riemann}) \sim U^2\frac{a^6}{r^8} \times (\text{direction-dependent factors}). \tag{12}$$

Note that U is dimensionless and that the curvature has the correct dimensions of $1/(\mathrm{length})^2$.

The Ricci curvature scalar (7) for the sphere flow (9) has the relatively simple anisotropic form

$$R = \frac{9U^2a^6\left(1+2\cos^2\theta\right)}{2r^8}\,. \tag{13}$$

That is, quasiparticles just in front of or behind the sphere experience the largest space-time curvature. This anisotropy of the scalar curvature should be contrasted with the effective space-time curvature around a vortex [12], or an impenetrable infinite cylinder [11], which both have isotropic R, depending only on the distance from the center of the object placed in the flow.

Super-phononic propagation. – Defining super-phononic (or in general relativity, superluminal) propagation in warp-field space-times is always tricky since, by definition, one never permits acoustic propagation outside the local sound cones of the effective geometry. What one can do, and what is done in Alcubierre's original analysis [1], is to compare two metrics placed on the same manifold. In the presence of the spherical obstruction we have the metric (1) with $\boldsymbol{v} \to \boldsymbol{v}_\infty$ at spatial infinity. As we have just seen, this metric leads to space-time curvature of the acoustic geometry. (In general, *any* inhomogeneous flow field, which is asymptotically constant, *i.e.*, yields the above Minkowski form at infinity, has nonzero curvature.) If the sphere were now to be removed, the flow would adjust itself so that the metric becomes

$$\mathrm{d}s_\infty^2 = -c_s^2\,\mathrm{d}t^2 + (\mathrm{d}\boldsymbol{x} - \boldsymbol{v}_\infty\,\mathrm{d}t)^2 \tag{14}$$

throughout the space-time. The space-time curvature of this acoustic metric is zero. It is by comparing the two metrics $\mathrm{d}s^2$ and $\mathrm{d}s_\infty^2$ that we can define the notion of super-phononic (superluminal): If the sphere is absent, the flow is simply $\boldsymbol{v}_\infty$ everywhere, the sound cones are all parallel (they are all tipped over in exactly the same way) and all have the same opening angle. Now introduce the sphere —it is an obstruction which distorts the flow. The sound cones now point in different directions at different points in the space-time.

Time advance: reflection. – For a particularly simple case, think of a phonon that propagates upstream against the flow and along the positive x-axis from x_1 to $x_0 < x_1$. In the flat reference metric this requires time $T_0 = \frac{x_1 - x_0}{c_s - v_\infty}$. In the curved space-time in the presence of the sphere, sending $x_1 \to \infty$ and $x_0 \to a$, the travel time becomes

$$T = \int_a^\infty \frac{\mathrm{d}x}{c_s - v_x(x)} = \int_a^\infty \frac{\mathrm{d}x}{c_s - v_\infty + v_\infty a^3/x^3}\,. \tag{15}$$

While the integrals for T_0 and T individually diverge, the time advance, defined as the difference $\Delta T \equiv T - T_0$, is finite:

$$\begin{aligned}\Delta T_{\mathrm{up}} &= -\frac{a}{c_s}\frac{U}{(1-U)^2}\int_1^\infty \frac{\mathrm{d}z}{z^3 + \frac{U}{1-U}} = -\frac{a}{c_s}\frac{U}{(1-U)^2}\sum_{n=0}^\infty \frac{1}{3n+2}\left(\frac{-U}{1-U}\right)^n \\ &= -\frac{a}{2c_s}\left\{U + \frac{8}{5}U^2 + \mathrm{O}\left(U^3\right)\right\}.\end{aligned} \tag{16}$$

In view of our incompressibility assumption for the background flow, keeping higher-order terms in U is meaningless.

The decrease in travel time is the acoustic analog of a "Shapiro time advance" due to the (position-dependent) tipping over of sound cones, which ultimately connects back to the presence of nontrivial space-time curvature. If we consider the surface of the sphere to be a reflector of sound rays, then upon impact and reflection the sound waves will be retarded on their downstream journey back to $r = \infty$. This retardation will not quite cancel the time advance from the upstream portion of the journey since

$$\Delta T_{\rm down} = \Delta T_{\rm up}(U \longrightarrow -U). \tag{17}$$

The net time advance is

$$\Delta T_{\rm up+down} = -\frac{8}{5}\frac{a}{c_s}U^2 + \mathrm{O}\big(U^4\big). \tag{18}$$

The fact that we are seeing a "Shapiro time advance" instead of the more usual "Shapiro time delay" characteristic of realistic sources in general relativity, is because we are *not* enforcing the Einstein equations, and more specifically not enforcing any positivity constraint on the components of the Einstein tensor.

Time advance: penetration. – A second situation where the time advance can easily be calculated, distinct from the one depicted in fig. 1, is when the sphere is taken to be a thin shell, rigid but acoustically penetrable, and filled with fluid. We now follow a photon upstream from ∞ to $-\infty$. The path divides into three zones:

– From ∞ to a, with time advance $\Delta T_{\rm up}$ as previously calculated.

– From a to $-a$: through the shell and across the "quiet zone" inside the sphere, with time advance

$$\Delta T_{\rm sphere} = \frac{2a}{c_s} - \frac{2a}{c_s - v_\infty} = -\frac{2a}{c_s}\frac{U}{1-U}\,. \tag{19}$$

– From $-a$ to $-\infty$, still an upstream battle, with time advance equal to the previously calculated $\Delta T_{\rm up}$.

The total time advance is then

$$\begin{aligned}\Delta T_{+\infty\to-\infty} &= 2\Delta T_{\rm up} + \Delta T_{\rm sphere}\\ &= -\frac{a}{c_s}\left\{3U + \frac{18}{5}U^2 + \mathrm{O}\big(U^3\big)\right\}.\end{aligned} \tag{20}$$

It should be mentioned that the sort of super-phononic propagation discussed above is in a sense "trivial", as it will be present (to some extent or another) in any acoustic metric which is both asymptotically Minkowski and has nontrivial fluid flow. In particular, the effect survives, as we have seen, for arbitrarily weak fluid flow (arbitrarily weak "warp fields"). Much more radical was Alcubierre's suggestion (in the context of general relativity) of placing an observer inside the warp bubble and letting the warp bubble travel in a superluminal manner.

Strong warp fields. – In our acoustic context "strong warp fields" correspond to $U \lesssim 1$ and more radically $U \gtrsim 1$, *i.e.* $v_\infty \gtrsim c_s$, so that (in the frame where the sphere is at rest) the asymptotic behaviour of the fluid flow is supersonic. In this situation we can no longer rely on the incompressible approximation holding for the background flow (at least, not for any Euler fluid). There are two ways of proceeding:

– For $U \lesssim 1$ we could solve for the background flow using the equation [13]

$$\nabla^2\phi = \frac{\nabla_i\phi}{c_s}\frac{\nabla_i\phi}{c_s}\nabla_i\nabla_j\phi. \tag{21}$$

For $U \ll 1$, this reduces to the usual incompressible approximation $\nabla^2\phi = \mathrm{O}(U^2)$. For $U \lesssim 1$ the background flow is distorted away from that of eq. (9), but the qualitative features of the previous discussion will survive.

For $U \gtrsim 1$, a shock wave will generically develop; precluding actual physical construction of such systems. For strong warp fields we should thus think of the analog acoustic geometry as a *gedankenexperiment* that helps us understand some of the subtleties involved with this sort of effective FTL.

– Alternatively, we could search for a physical system that has two distinct and well-separated "sound" speeds —an example of this behaviour arising for the quasiparticles present in the superfluid ^{3}He-A [14]. If the larger of these "sound" speeds (where "sound" here means any excitation having linear, *i.e.* relativistic, quasiparticle dispersion) is related to bulk compressibility via $c^2_{\mathrm{high}} = \mathrm{d}p/\mathrm{d}\rho$, while the lower "sound" speed is governed by the excitations we are interested in, then there will be a regime in which $v/c_{\mathrm{high}} \ll 1$ while $v/c_{\mathrm{low}} \gtrsim 1$ (in ^{3}He-A, $c_{\mathrm{low}}/c_{\mathrm{high}} \sim 10^{-3}$) —this would permit us to simultaneously adopt the incompressibility approximation *and* nevertheless have "supersonic" flow (with respect to "slow sound"). For the example, ^{3}He-A, this is possible for two-dimensional situations, *e.g.* the flow around a cylinder instead of around a sphere, because c_{low} can be the relevant "speed of sound" only in situations with planar symmetry [15].

Adopting either viewpoint, recall that the sound cones near spatial infinity are given by

$$\boldsymbol{v}_{\mathrm{sound}} = v_\infty \hat{\boldsymbol{x}} + c_s \hat{\boldsymbol{n}}, \tag{22}$$

and note that in this strong-field case the sound cones are tipped over so far that all sound is inexorably dragged downstream. In contrast, inside the bubble one is sheltered from this flow, and a motion that is "slower than sound" in terms of the curved space-time metric ($\mathrm{d}s^2$) may be "faster than sound" in terms of the flat space-time metric ($\mathrm{d}s^2_\infty$). Indeed, an observer at rest with respect to the sphere is in this strongly warped situation travelling "faster than sound" in terms of the flat space-time metric. The key step that allows us to make such pronouncements concerning effective "superluminal/superphononic" travel is that there are two natural metrics that can be placed on the same space-time, and that these two metrics can then easily be used for comparison purposes.

In general relativity such a "two metric" approach to space-time is considerably less natural —nevertheless, in order to make any sense of effective FTL one seems to be forced (one way or another) into a "two metric" interpretation. One could (as per Alcubierre) define the two metrics *by fiat*, effectively by agreeing to only consider a restricted class of space-time geometries. Alternatively, one can try to develop specific physical models for the "reference metric". In this article we have seen how such a reference metric naturally arises in fluid acoustics.

One of the key features of the acoustic geometry, and of analog models in general, is that they tend to inherit the notion of stable causality from the background geometry [16]. Specifically, the Newtonian time parameter t is still always timelike in the acoustic geometry, and this is very much built in at a fundamental level —because of this there is never any risk of developing closed causal curves in the acoustic geometry. We *always* have $g^{ab}\nabla_a t \nabla_b t = -1/c_s^2$, so as long as $c_s^2 > 0$ we have ∇t timelike. (And if $c_s^2 < 0$, we do not get closed timelike curves, such a situation corresponds to an elliptic equation where sound is in a sense "infinitely damped", corresponding to Euclidean signature of the metric.) In short, "chronology protection" [17, 18] is automatic for acoustic geometries and is not contingent on either the Einstein equations or quantum physics —it is built in at the foundations.

Discussion. – In this article we have presented a physical implementation of a version of Alcubierre's "warp drive space-time" in terms of a condensed-matter system for which we have absolute control over all of the fundamental physics —we have translated the notion of FTL travel in general relativistic "warp fields" into a very straightforward and simple model based on acoustic phonons in a moving fluid. Doing so has let us carry over basic insight from nonrelativistic fluid mechanics to clarify subtle issues of general relativity; and, conversely, the technical machinery of general relativity can be used as an aid to visualizing the acoustic properties of a moving fluid. This is merely one aspect of the "analogue gravity" programme, wherein a number of theorists are working on cross-cultural connections between condensed matter, general relativity, and particle physics [8, 9, 11, 12, 14, 19–21].

* * *

The research of URF was supported by the "Improving Human Potential" Programme of the European Union under grant No. HPRI 1999-CT-00050, by the Austrian Science Foundation FWF, and the ESF Programme "Cosmology in the Laboratory".

REFERENCES

[1] ALCUBIERRE M., *Class. Quantum Grav.*, **11** (1994) L73.
[2] OLUM K. D., *Phys. Rev. Lett.*, **81** (1998) 3567.
[3] VISSER M., BASSETT B. A. and LIBERATI S., *Nucl. Phys. B (Proc. Suppl.)*, **88** (2000) 267.
[4] CLARK C., HISCOCK W. A. and LARSON S. L., *Class. Quantum Grav.*, **16** (1999) 3965.
[5] NATÁRIO J., *Class. Quantum Grav.*, **19** (2002) 1157.
[6] VAN DEN BROECK C., *Class. Quantum Grav.*, **16** (1999) 3973; KRASNIKOV S., arXiv:gr-qc/0207057.
[7] PFENNING M. J. and FORD L. H., *Class. Quantum Grav.*, **14** (1997) 1743.
[8] UNRUH W. G., *Phys. Rev. Lett.*, **46** (1981) 1351.
[9] VISSER M., *Class. Quantum Grav.*, **15** (1998) 1767.
[10] GAO S. and WALD R. M., *Class. Quantum Grav.*, **17** (2000) 4999.
[11] FISCHER U. R. and VISSER M., *Ann. Phys. (N.Y.)*, **304** (2003) 22.
[12] FISCHER U. R. and VISSER M., *Phys. Rev. Lett.*, **88** (2002) 110201.
[13] LANDAU L. D. and LIFSHITZ E. M., *Fluid Mechanics*, second edition (Pergamon Press) 1987.
[14] VOLOVIK G. E., arXiv:gr-qc/0104046; *Phys. Rep.*, **351** (2001) 195.
[15] VOLOVIK G. E., *JETP Lett.*, **69** (1999) 705 (*Pis'ma Zh. Éksp. Teor. Fiz.*, **69** (1999) 662).
[16] LIBERATI S., SONEGO S. and VISSER M., *Ann. Phys. (N.Y.)*, **298** (2002) 167.
[17] HAWKING S. W., *Phys. Rev. D*, **46** (1992) 603.
[18] VISSER M., arXiv:gr-qc/0204022.
[19] SCHÜTZHOLD R. and UNRUH W. G., *Phys. Rev. D*, **66** (2002) 044019.
[20] CHAPLINE G., HOHLFELD E., LAUGHLIN R. B. and SANTIAGO D. I., *Philos. Mag. B*, **81** (2001) 235.
[21] LEONHARDT U., arXiv:gr-qc/0108085; *Phys. Rev. A*, **62** (2000) 012111.