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ESSENTIAL AND INESSENTIAL FEATURES OF HAWKING RADIATION

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There are numerous derivations of the Hawking effect available in the literature. They emphasise different features of the process, and sometimes make markedly different physical assumptions. This article presents a "minimalist" argument, and strips the derivation of as much excess baggage as possible. All that is really necessary is quantum physics plus a slowly evolving future apparent horizon (*not* an event horizon). In particular, neither the Einstein equations nor Bekenstein entropy are necessary (nor even useful) in deriving Hawking radiation.

Keywords: Hawking radiation; Bekenstein entropy; apparent horizon.

1. Introduction

Hawking radiation from black holes is a semiclassical quantum effect that has now been with us for some 28 years,[1] and whose theoretical importance is difficult to exaggerate. Over the decades, the Hawking effect has accreted a quite considerable mythology. Perhaps the two most pernicious myths attached to this effect are:

- "*Hawking radiation has something to do with gravity.*"
- "*Hawking radiation automatically implies Bekenstein entropy.*"

These myths were engendered by two historical accidents: (1) the Hawking effect was first encountered within the context of general relativity, and (2) the fact that it was discovered shortly after the notion of Bekenstein entropy (geometric entropy) had been formulated.[2,3]

Though the Hawking effect was at first partly motivated by the need for a consistent thermodynamic interpretation for the notion of Bekenstein entropy, it was rapidly appreciated that the Hawking effect is much more primitive and fundamental[4] — in particular the Hawking effect continues to make sense even in situations where geometric entropy and even gravity itself are simply not relevant.[5]

This observation underlies much of the current interest in "analog models of/for general relativity";[6,7] there is now a realistic possibility for experimental detection of the Hawking effect in condensed-matter analog systems using current or planned technology,[8–10] with "effective metrics" and "black holes" that have nothing to do with gravity itself.[11] In view of this situation, in this current article I will attempt to isolate an irreducible minimum of physical assumptions needed for the Hawking effect to arise.

Since the literature is vast, I will not be able to do justice to all known derivations of the effect — some key derivations are those due to Hawking himself,[1] the Hartle–Hawking approach using analytic continuation of the propagator across the event horizon of an eternal black hole,[12] and the Gibbons–Hawking approach using Euclideanization (Wick rotation).[13] Extremely useful general surveys are provided in Refs. 14 and 15.

A particularly relevant discussion is the early work of Damour and Ruffini,[16] who emphasize the behaviour of the "outgoing" modes as one crosses the horizon. Though their presentation is given in terms of the Kerr–Newman geometry it is easy to verify that the specifics of the geometry enter *only* in the evaluation of the surface gravity of the future Killing horizon, and that the route from surface gravity to Hawking radiation does not depend on either the Einstein equations or even the underlying physical mechanism leading to the existence of the metric.

In a slightly different vein, the discussion of Parker[17] particularly emphasizes the relationship with particle production from a dynamical vacuum state, while Gerlach,[18] Grove,[19] Hu,[20] and Brout and Parentani[14] emphasize in varying degree the near-universal role of the exponential stretching associated with many types of horizon.

More recently the contributions of Massar and Parentani,[21] Parikh and Wilczek,[22] Padmanabhan *et al*,[23] and Schutzhold[24] should be noted. They emphasize, in slightly different forms, the analyticity properties of the modes and what is effectively an imaginary contribution to the action localized at the horizon, an observation that can be traced back to the work of Damour and Ruffini.[16]

It is also important to realize that the Hawking radiation effect is independent of whatever cutoff you introduce to the high-frequency physics — this is one of the theoretical reasons for interest in analog models, because for acoustic black holes you have an explicit model for the high frequency cutoff in terms of atomic physics.[25–27] I will not have anything specific to say about this particular issue in the present paper.

Based on the results extracted from 28 years of research, as verified and made more explicit by the recent interest in analog models, it is clear that the basic physics we should be aiming for is:

- "*Hawking radiation is kinematics*,"
- "*Bekenstein entropy is geometrodynamics.*"

That is: Hawking radiation is a purely kinematic effect that depends only on the existence of a Lorentzian metric (with no particular prejudice as to how this metric arises) and some sort of horizon. Hawking radiation does not depend on the validity of the Einstein equations (as may most quickly be verified by looking at Hawking's original derivation[1] and verifying that the Einstein equations are nowhere used nor needed). In contrast, Bekenstein's geometric entropy associated with the area of the event horizon is an intrinsically geometrodynamic effect, wrapped up with the validity of the Einstein equations: Entropy equals one quarter the area (plus perturbative corrections) if and only if[a] the Einstein equations are valid (plus perturbative corrections).[5] The connection with the Einstein equations arises because when integrating the first law to evaluate the Bekenstein entropy you need to use the relationship between total mass of the black hole and its surface gravity (and hence Hawking temperature), and it is in this relationship between surface gravity and mass-energy that the Einstein equations enter.

I shall also distinguish the notion of "apparent horizon" from that of the "event horizon" (absolute horizon) and demonstrate that the existence of a locally definable apparent horizon is quite sufficient for obtaining the Hawking effect. (Remember that to define the event horizon you need to know the entire history of the spacetime out to the infinite future. You should be a little alarmed if the question of whether or not a black hole is radiating *now* depends on what it is doing in the infinite future.)

The general theme of the analysis will be to do as much as possible with the eikonal approximation (even WKB is mild overkill), and look for generic features in the modes at/near the apparent horizon. Specifically, we will look for a Boltzmann factor. Note that a good derivation should *not* depend on either grey-body factors or a past horizon. Grey-body factors are not fundamental — they are simply transmission coefficients giving the probability that modes which escape from the horizon make it out to null infinity without being backscattered. Past horizons are specific to eternal black holes and simply not relevant for astrophysical black holes. Similarly because the entropy-area law is tied to the validity of the Einstein equations we shall seek to avoid any appeal to this property. Since much of what I have to say is well-known to experts in the field (at least within the general relativity community) I will place a premium on clarity and simplicity.

2. Metric: Painlevé–Gullstrand Form

In general relativity any spherically symmetric geometry, static or not, can locally be put in the form

$$\mathrm{d}s^2 = -c(r,t)^2\, \mathrm{d}t^2 + (\mathrm{d}r - v(r,t)\, \mathrm{d}t)^2 + r^2[\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2] \tag{1}$$

[a]Modulo some technical issues if the topology of the horizon is unusual, or the surface gravity is zero.

by suitable choice of coordinates. Equivalently,

$$\mathrm{d}s^2 = -[c(r,t)^2 - v(r,t)^2]\,\mathrm{d}t^2 - 2v(r,t)\,\mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]\,. \qquad (2)$$

These are the so-called Painlevé–Gullstrand coordinates, a relatively obscure coordinate system currently enjoying a resurgence. (See Ref. 28 for a geometric discussion.) A nice feature of these coordinates is that the metric is nonsingular at the horizon. Though the basic physics is coordinate independent, we shall see that these coordinates simplify computations considerably.

In the acoustic geometries associated with sound propagation in a flowing fluid it is often convenient not to set up coordinates this precise manner but rather to use the natural coordinates inherited from the background Minkowski spacetime. Doing so results in a minor modification of the Painlevé–Gullstrand metric:

$$\mathrm{d}s^2 = \left[\frac{\rho(r,t)}{c(r,t)}\right]\left\{-c(r,t)^2\,\mathrm{d}t^2 + (\mathrm{d}r - v(r,t)\,\mathrm{d}t)^2 + r^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]\right\}\,, \qquad (3)$$

where $c(r,t)$ is now the speed of sound, $v(r,t)$ is the radial velocity of the fluid, and $\rho(r,t)$ is its density. Phonons are then massless minimally coupled scalar fields in this acoustic geometry.[6,7] Fortunately the conformal factor does not affect the surface gravity and does not affect the Hawking effect;[29] for simplicity I shall simply suppress the conformal factor, it can easily be reinstated if desired.

In matrix form the Painlevé–Gullstrand metric takes the quasi–ADM form

$$g_{\mu\nu}(t,\mathbf{x}) \equiv \left[\begin{array}{ccc} -(c^2 - v^2) & \vdots & -v\hat{r}_j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v\hat{r}_i & \vdots & \delta_{ij} \end{array}\right]. \qquad (4)$$

The inverse metric is

$$g^{\mu\nu}(t,\mathbf{x}) \equiv \frac{1}{c^2}\left[\begin{array}{ccc} -1 & \vdots & -v\hat{r}^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v\hat{r}^i & \vdots & (c^2\delta^{ij} - v^2\hat{r}^i\hat{r}^j) \end{array}\right]. \qquad (5)$$

Due to spherical symmetry, finding the apparent horizon is particularly easy: it is located at $c(r,t) = |v(r,t)|$. It is easy to see that the metric is nonsingular at the apparent horizon, with $\det(g_{\mu\nu}) = -c(r,t)^2$. To get a future apparent horizon, corresponding to an astrophysical black hole, we need $v < 0$. I make no claims as to the location or even the existence of any event horizon.

[Since for static black holes the event horizon is coincident with the apparent horizon, the general feeling is that for quasi-stationary black holes the event horizon is likely to be "near" the apparent horizon. If accretion dominates over evaporation the event horizon is likely to be just outside the apparent horizon, while if evaporation dominates over accretion it is likely to be just inside the apparent horizon. There is an unpopular minority opinion that takes the view that because of Hawking evaporation a black hole will never form a true event horizon (absolute horizon). For the derivation presented herein this entire issue is simply not relevant.]

Define a quantity:

$$g_H(t) = \frac{1}{2} \left. \frac{\mathrm{d}|c(r,t)^2 - v(r,t)^2|}{\mathrm{d}r} \right|_H = c_H \left. \frac{\mathrm{d}[c(r,t) - |v(r,t)|]}{\mathrm{d}r} \right|_H . \tag{6}$$

If the geometry is static, this reduces to the ordinary definition of surface gravity:[7]

$$\kappa = \frac{g_H}{c_H} . \tag{7}$$

If the geometry is not static this is a natural definition of the "surface gravity" of the apparent horizon.[7]

3. Eikonal and WKB Approximation (s Wave)

Consider a quantum field $\phi(r,t)$ on this Painlevé–Gullstrand background and take the eikonal (geometric optics/acoustics) approximation for the s wave

$$\phi(r,t) = \mathcal{A}(r,t)\ \exp[\mp i\varphi(r,t)] = \mathcal{A}(r,t)\ \exp\left[\mp i \left(\omega\, t - \int^r k(r')\, \mathrm{d}r'\right)\right] \tag{8}$$

whereby the field is written as a rapidly varying phase times a slowly varying envelope. The second equality above, where we have written the time dependence of the phase as ωt, is valid provided the geometry is slowly evolving on the timescale of the wave, that is, provided $\omega \gg \max\{|\dot{c}/c|, |\dot{v}/v|\}$. (I take ω to be positive.)

In the eikonal approximation the d'Alembertian equation of motion becomes

$$g^{\mu\nu}\ \partial_\mu\varphi\ \partial_\nu\varphi - i\epsilon = 0 . \tag{9}$$

Note in particular that I have used the eikonal approximation to immediately impose Feynman's "$i\epsilon$-prescription" on the field (ϵ is real, positive, and infinitesimal). The metric signature is the general relativity standard $-+++$, which is why the $i\epsilon$-prescription appears reversed relative to the particle physics standard. Also note that in invoking the prescription I have implicitly used the fact that the spacetime geometry is smooth, even at the horizon, so that it makes sense to both adopt an eikonal approximation and then within this framework use ordinary flat-space results. The use of the $i\epsilon$ prescription in this way can be traced back, at least, to the early paper of Damour and Ruffini.[16] If one prefers, the $i\epsilon$-prescription can be rephrased in terms of the analyticity of the fields on the complexified past light cone, $\mathcal{Z}(x) = \{z = x + iy\}$, where x is any point in spacetime and y lies in the past light cone. This is the Lorentz invariant generalization of analyticity in the lower half-plane, but for all practical purposes can be replaced by the $i\epsilon$ prescription.

From the preceding equation

$$\omega^2 - 2v(r,t)\omega k - [c(r,t)^2 - v(r,t)^2]k^2 + i\epsilon = 0 . \tag{10}$$

Thus,

$$(\omega - vk)^2 = c^2k^2 + i\epsilon . \tag{11}$$

Whence

$$\omega - vk = \sigma(1 + i\epsilon)ck, \qquad \sigma = \pm 1\,. \tag{12}$$

For a specific real frequency ω this gives us the wave-vector $k(r,t)$ as

$$k = \frac{\omega}{\sigma(1+i\epsilon)c + v} = \frac{\sigma\omega}{(1+i\epsilon)c + \sigma v} = \frac{\sigma(1+i\epsilon)c - v}{(1+i\epsilon)^2 c^2 - v^2}\,\omega\,. \tag{13}$$

Note that

$$\sigma = +1 \qquad \Rightarrow \qquad \text{outgoing mode}\,, \tag{14}$$

$$\sigma = -1 \qquad \Rightarrow \qquad \text{ingoing mode}\,. \tag{15}$$

Keeping track of the $i\epsilon$ is important only near the apparent horizon, where it is critical; everywhere else it can safely be set to zero.

While we do not really need to use the WKB approximation (physical acoustics/optics) it can be invoked at very little additional cost. The (approximate) conserved current is

$$J_\mu = |\mathcal{A}(r,t)|^2 \qquad (\omega, k, 0, 0)\,. \tag{16}$$

Then

$$\nabla_\mu J^\mu = 0 \qquad \Rightarrow \qquad |\mathcal{A}(r,t)| \propto \frac{1}{r}\,. \tag{17}$$

So we can write

$$\phi(r,t) \approx \frac{\mathcal{N}}{\sqrt{2\omega}\; r}\, \exp\left[\mp i\left(\omega\, t - \int^r k(r')\, \mathrm{d}r'\right)\right], \tag{18}$$

where $\mathcal{N}$ is some conveniently chosen normalization, to be discussed more fully below.

4. Near-Horizon Modes: Ingoing

The ingoing wavenumber ($\sigma = -1$) is

$$k_{\text{in}} = -\frac{\omega}{(1+i\epsilon)c - v}\,. \tag{19}$$

Thus in the vicinity of the future apparent horizon $r \approx r_H$ (with $v \approx -c$) the ingoing wavevector is approximately

$$k_{\text{in}} \to -\frac{\omega}{2c_H}\,. \tag{20}$$

Thus the ingoing modes are approximately

$$\phi(r,t)_{\text{in}} \approx \frac{\mathcal{N}_{\text{in}}}{\sqrt{2\omega} r_H}\, \exp\left[\mp i\omega\left\{t + \frac{r - r_H}{2c_H}\right\}\right]. \tag{21}$$

This means the phase velocity of the ingoing mode as it crosses the horizon (in coordinate distance per coordinate time) is $2c_H$. (Phase velocity equals group velocity because there is no dispersion.) We see that the ingoing modes contain no real surprises and are relatively uninteresting.

5. Near-Horizon Modes: Outgoing

Now consider the outgoing mode $\sigma = +1$

$$k_{\text{out}} = \frac{\omega}{(1+i\epsilon)c + v}\,. \tag{22}$$

In the vicinity of the future apparent $r \approx r_H$ (with $v \approx -c$) the outgoing wavevector is approximately

$$k_{\text{out}} \approx \frac{\omega}{[g_H/c_H](r - r_H) + i\epsilon c_H}\,. \tag{23}$$

However, since ϵ is infinitesimal, and both g_H and c_H are by hypothesis positive, we can for all practical purposes rewrite this in terms of the "principal part" and a delta function contribution. That is, near the apparent horizon

$$k_{\text{out}} \approx \frac{c_H \omega}{g_H} \left\{ \wp\left(\frac{1}{r - r_H}\right) - i\pi\delta(r - r_H) \right\}\,. \tag{24}$$

Thus we can ignore the $i\epsilon$ unless we are actually intending to cross the apparent horizon. In particular, just outside the apparent horizon

$$\int^r k = \int^r \frac{dr'\omega}{c(r') - |v(r')|} \approx \int^r \frac{dr' c_H \omega}{g_H (r' - r_H)} = \frac{c_H \omega}{g_H} \ln[r - r_H]\,. \tag{25}$$

Therefore (for $r > r_H$),

$$\begin{aligned}\phi(r,t)_{\text{out}} &\approx \mathcal{N}_{\text{out}} \frac{\exp\left(\pm i \left[\frac{\omega c_H}{g_H}\right] \ln[r - r_H]\right)}{\sqrt{2\omega} r_H} \exp\{\mp i\omega t\} \\ &\approx \mathcal{N}_{\text{out}} \frac{[r - r_H]^{\pm i\omega c_H/g_H}}{\sqrt{2\omega} r_H} \exp\{\mp i\omega t\}\,. \end{aligned} \tag{26}$$

The phase velocity of the outgoing mode as it crosses the horizon (in coordinate distance per coordinate time) is zero.

The fact that these outgoing modes have the surface gravity, $\kappa = g_H/c_H$, showing up in such a fundamental and characteristic way is already strongly suggestive; and this is really all there is to Hawking radiation. The logarithmic phase pile-up at the horizon is characteristic of many derivations of Hawking radiation (in particular [1]) and for many readers this will be sufficient to convince them that Hawking radiation is present under the current circumstances (slowly evolving apparent horizon without prejudice as to where the metric comes from). In fact, this calculation is the easiest and fastest way I know of to deduce the existence of the phase pile-up using completely elementary methods.

To properly describe the modes that escape to infinity we should normalize on the half-line $r > r_H$ using the standard Klein-Gordon norm. This results in replacing $\mathcal{N}_{\text{out}}$ with some specific normalization constant $\mathcal{N}_{\text{escape}}$ whose precise value we do not need to know.

6. Straddling the Horizon: the Boltzmann Factor

In addition to looking at outgoing modes that escape to infinity, it is also useful to consider what happens to "outgoing" modes that straddle the apparent horizon. Inside the apparent horizon the real part of $k_{\rm out}$ goes negative, indicating that while the outgoing mode is trying to escape "upstream" it is actually being overcome by the flow/shift vector and swept back "downstream". In addition the phase picks up an imaginary contribution, due ultimately to the $i\epsilon$ prescription,

$$\int_{r_-}^{r^+} k_{\rm out} \approx \int_{r_-}^{r^+} {\rm d}r' \frac{c_H\ \omega}{g_H} \left\{ \wp\left(\frac{1}{r'-r_H}\right) - i\pi\delta(r'-r_H) \right\}$$
$$= \frac{c_H\omega}{g_H}\left\{ \ln\frac{|r_+ - r_H|}{|r_- - r_H|} - i\pi \right\}. \tag{27}$$

So just inside the apparent horizon

$$\phi(r,t)_{{\rm straddle}(r<r_H)} \approx \mathcal{N}^{\pm}_{\rm straddle} \frac{|r-r_H|^{\mp i\omega c_H/g_H}}{\sqrt{2\omega}\, r_H} \exp\left\{\mp\frac{\pi\omega c_H}{g_H}\right\} \exp\left[\mp i\omega t\right]. \tag{28}$$

Compare with the situation just outside the apparent horizon

$$\phi(r,t)_{{\rm straddle}(r>r_H)} \approx \mathcal{N}^{\pm}_{\rm straddle} \frac{|r-r_H|^{\pm i\omega c_H/g_H}}{\sqrt{2\omega}\, r_H} \exp\left[\mp i\omega t\right]. \tag{29}$$

Note that this straddling mode has associated with it a normalization factor $\mathcal{N}_{\rm straddle}$, which is distinct from that of the escaping mode $\mathcal{N}_{\rm escape}$; the straddling mode is to be normalized on the entire half-line $r > 0$. In terms of the Heaviside step function

$$\phi(r,t)_{\rm straddle} \approx \mathcal{N}^{\pm}_{\rm straddle} \left[\Theta(r_H - r) \exp\left\{\mp\frac{\pi\omega c_H}{g_H}\right\} + \Theta(r-r_H) \right]$$
$$\times\ |r-r_H|^{\pm i\omega c_H/g_H} \frac{\exp\left[\mp i\omega t\right]}{\sqrt{2\omega}\, r_H}. \tag{30}$$

See the Eq. (5b) of Damour and Ruffini,[16] for an early occurrence of a very similar statement; see also the Eq. (13) of Massar and Parentani.[21] Note in particular the presence of a Boltzmann-like factor for the negative-frequency modes

$$\exp\left\{-\frac{\pi\omega c_H}{g_H}\right\} \tag{31}$$

which (we shall soon see) corresponds to the Hawking temperature

$$kT_H = \frac{\hbar g_H}{2\pi c_H}. \tag{32}$$

For many physicists the presence of this Boltzmann-like factor will be enough: The occurrence of Boltzmann factors of this type was the key to the Hartle–Hawking derivation,[12] though they were working with the full propagator and dealing with past and future horizons of a maximally extended Kruskal–Szekeres eternal black hole. Damour and Ruffini[16] demonstrated the existence of similar Boltzmann factors for mode functions evaluated at the future Killing horizon of a Kerr–Newman black hole (dispensing with the past horizon entirely). In the present situation, the same Boltzmann factor is seen to arise for any slowly evolving apparent horizon.

The imaginary contribution to the integrated wavenumber is, in slightly disguised form, equivalent to the imaginary contribution to the action that occurs in the Parikh–Wilczek[22] approach.[b] This, in a way, is also related but not identical to the imaginary contribution arising from complex paths in the approach of Padmanabhan *et al.*[23]

But it is possible to quite easily do a lot more: While I have so far carefully not specified specific values for $\mathcal{N}_{\text{escape}}$ and $\mathcal{N}_{\text{straddle}}$, the relationship between them is very simple. Since the straddling mode is to be normalized on $(0, +\infty)$, [which we actually approximate by the full line $(-\infty, +\infty)$], while the escaping mode is only normalized on the half line (r_H, ∞), for the negative-frequency mode we have[16]

$$|\mathcal{N}^{-}_{\text{straddle}}|^2 \left[\exp\left\{ + \frac{2\pi\omega c_H}{g_H} \right\} - 1 \right] = |\mathcal{N}_{\text{escape}}|^2 \,. \tag{33}$$

So the relative normalization is

$$\left| \frac{\mathcal{N}^{-}_{\text{straddle}}}{\mathcal{N}_{\text{escape}}} \right|^2 = \frac{1}{\exp\left\{ + \frac{2\pi\omega c_H}{g_H} \right\} - 1} \,. \tag{34}$$

It is this Planckian form of the normalization ratio that then leads to a Planckian distribution for the outgoing flux. (The straddling mode contains a Planckian distribution of escaping modes.)

Note the physics assumption hidden here: one is assuming that the quantum vacuum state is that corresponding to ϕ_{straddle}. That is, freely falling observers (who get to see both sides of the horizon) should not see any peculiarities as one crosses the horizon. Picking the quantum vacuum corresponding to ϕ_{straddle} implies choosing the Unruh vacuum — and when we look at this vacuum state far from the horizon we see the Planckian flux of outgoing particles.

(If for whatever reason you do not like this normalization calculation or the related thermodynamic arguments you can alternatively use the phase pile-up property directly to perform a Bogoliubov coefficient calculation in the style of Ref. 1 — all roads lead to Rome.)

7. Beyond *s* Wave

What happens if we go beyond the s wave? There is now some momentum transverse to the apparent horizon so that

$$\partial_\mu \varphi = (\omega, -k, -k_\perp) \,. \tag{35}$$

[b]Note that Parikh and Wilczek have subsequently and implicitly made use of the Einstein equations at the stage when they then relate the emission process to the entropy change. This comment also applies to the Massar–Parentani approach.

If we resolve the field in terms of partial waves

$$k_\perp^2 = \frac{\ell(\ell+1)}{r^2}\,. \tag{36}$$

Then in the eikonal approximation (I now suppress the $i\epsilon$, it has done its job and would now only serve to clutter the formulae)

$$-\omega^2 + 2v(r,t)\omega k + [c(r,t)^2 - v(r,t)^2]k^2 + c(r,t)^2 k_\perp^2 = 0\,. \tag{37}$$

That is

$$(\omega - vk)^2 = c^2k^2 + c^2k_\perp^2\,. \tag{38}$$

This is a quadratic for k as a function of ω and $k_\perp$:

$$k = \frac{\sigma\sqrt{c^2\omega^2 - (c^2-v^2)c^2k_\perp^2} - v\omega}{c^2-v^2}\,. \tag{39}$$

For ingoing modes near the apparent horizon one must evaluate using L'Hôpital's rule:

$$k_{\text{in}} \to -\frac{\omega^2 - c^2k_\perp^2}{2c_H\omega}\,. \tag{40}$$

So the ingoing modes *do* depend on $k_\perp$.

$$\phi(r,t)_{\text{in}} \approx \frac{\mathcal{N}_{\text{in}}}{\sqrt{2\omega}\,r_H}\exp\left[\mp i\omega\left\{t + \frac{(r-r_H)[\omega^2 - c^2k_\perp^2]}{2c_H\omega}\right\}\right]. \tag{41}$$

But that does not matter: The ingoing modes are not the relevant ones. For the outgoing modes, near the apparent horizon we see

$$k_{\text{out}} \to \frac{c_H\omega}{g_H(r-r_H)}\,. \tag{42}$$

That is, for the outgoing modes:

- The near-horizon asymptotic behaviour is *independent* of $k_\perp$.
- The phase pile-up is *independent* of $k_\perp$.
- Continuation of the outgoing modes across horizon is *independent* of $k_\perp$.
- The Hawking temperature *independent* of $k_\perp$.
- This behaviour is universal for all partial waves.
- Adding a mass term corresponds to:

$$c^2k_\perp^2 \to c^2k_\perp^2 + \left(\frac{mc^2}{\hbar}\right)^2\,. \tag{43}$$

In view of the above we see that the behaviour of the outgoing modes near the horizon is also completely independent of the mass and transverse momentum. Consequently the *same* universal Hawking temperature applies to all masses and all partial waves. (Again, this is the easiest way I know of to convince oneself by elementary means that restricting attention to the s-wave captures almost all the essential physics of Hawking radiation.)

Note that while the Hawking temperature is completely independent of both angular momentum and mass, the grey-body factors are another matter: they do depend on both angular momentum and mass and are responsible for effectively cutting off the higher angular momentum modes.

8. Essential Features

The only real physics input has been basic quantum physics plus the existence of a Lorentzian metric with:

- an apparent horizon,
- non-zero g_H,
- slow evolution.

The need for slow evolution of the geometry is hidden back in the approximation used to write the modes as $\exp(\pm i\omega t)$ times a position-dependent factor. This makes sense only if the geometry is quasi-static on the timescale set by ω. So for consistency we should only trust the Boltzmann factor, and the Planckian nature of the Hawking radiation, for frequencies greater than $\max\{|\dot{c}/c|, |\dot{v}/v|\}$. In particular, in order for the peak in the Planck spectrum to be meaningful we require

$$\frac{kT_H}{\hbar} \approx \omega_{\text{peak}} \gg \max\{|\dot{c}/c|, |\dot{v}/v|\}\,. \tag{44}$$

In particular we require

$$\left.\frac{\mathrm{d}[c(r,t) - |v(r,t)|]}{\mathrm{d}r}\right|_H \gg \frac{\dot{c}_H}{c_H}\,. \tag{45}$$

Near the horizon spatial gradients should dominate over temporal gradients. In particular the closer the black hole is to extremality the slower it is permitted to evolve if there is to be any hope for even a small segment of quasi-thermal spectrum.

That is it. It is truly remarkable how basic and primitive the Hawking radiation phenomenon is, and how few physical assumptions are really necessary.

9. Discussion

Can the essential conditions for the Hawking effect be further relaxed?

One obvious question is the use of spherical symmetry, which precludes direct application of the current approach to Kerr and other rotating black holes. This is a technical problem, not a fundamental problem, and working in axial-symmetric geometries will be do-able but somewhat more complicated. (For Kerr–Newman black holes the Damour–Ruffini analysis can be adapted to this end.[16]) A tricky point for general analog model geometries is that without the Einstein equations, and something like the dominant energy condition, there is no longer any reason to believe in the zeroth law: the surface gravity and Hawking temperature can then in principle vary over different parts of the apparent horizon; these complications were suppressed in the current article via the simple expedient of enforcing spherical symmetry. (For Killing horizons there are derivations of the zeroth law that do not depend on the Einstein equations,[30] but such considerations lose their force once the horizon becomes time dependent.)

Secondly, there are simple linguistic issues of definition: How far can we push the Hawking effect before we should give it another name? As argued in this article, based on the physics there is a very good case for keeping the name the same for the effect in arbitrary "effective geometries", no matter how derived. Some would even argue that the Hawking effect and Unruh effects are fundamentally identical; I prefer to view then as distinct, possibly as two sides of the same coin — the response of the quantum vacuum to externally imposed conditions.

Thirdly: What is the energy source for the Hawking radiation? The infalling particles have negative energy as seen from outside the apparent horizon. In the case of general relativity black holes the infalling particles serve to reduce the total mass-energy of the central object, and it is ultimately the total mass-energy of the black hole that provides the energy emitted in the hawking flux. In the case of an acoustic black hole the infalling negative energy phonon steals kinetic energy from the fluid flow used to generate the acoustic geometry. For effective geometries associated with "slow light" the electromagnetic control field, used to generate EIT (electromagnetically induced transparency) and so reduce the group velocity, provides energy to the system which is available to ultimately be converted to Hawking-like photons. The general message is that the Hawking effect steals energy from whatever process is used to set up the effective geometry in question.

Finally, since this point still seems to cause much confusion, I should make the explicit comment that

- *Hawking radiation is not a test of quantum gravity.*

Instead, searching for Hawking radiation is a test of the general principles of quantum field theory in curved spacetimes. As such it is an ingredient useful for testing semiclassical quantum gravity, though it does not necessarily probe quantum gravity itself. In particular, all the proposed experimental tests of Hawking radiation via "analog models" will only probe kinematic aspects of black hole physics. In order to start to address the dynamics of general relativity black holes one needs the Einstein equations (or some approximation thereto), in which case one can begin to discuss Bekenstein entropy (or some approximation thereto). This requires a whole extra layer of theoretical superstructure, and a key point of this paper is that it is important and useful to keep these notions logically distinct.

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