Spacetime geometry of static fluid spheres

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Abstract
We exhibit a simple and explicit formula for the metric of an arbitrary static spherically-symmetric perfect-fluid spacetime. This class of metrics depends on one freely specifiable monotonic non-increasing generating function. We also investigate various regularity conditions and the constraints they impose. Because we never make any assumptions as to the nature (or even the existence) of an equation of state, this technique is useful in situations where the equation of state is for whatever reason uncertain or unknown.

To illustrate the power of the method we exhibit a new form of the ‘Goldman–I’ exact solution. This is a three-parameter closed-form exact solution given in terms of algebraic combinations of quadratics. It interpolates between (and thereby unifies) at least six other reasonably well-known exact solutions.

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1. Introduction

The apparently simple problem of the general relativistic static perfect fluid sphere has by now generated hundreds of scientific articles. Good summaries of known results and commentaries regarding the extant literature can be found in the book by Kramer et al [1], and in the recent review articles by Delgaty and Lake [2] and Finch and Skea [3].

One of the more common approaches (certainly not the only approach) is to pick some barotropic equation of state \( p = p(\rho) \), pick the central pressure, apply the Tolman–Oppenheimer–Volkoff equation and integrate outwards until one reaches the surface of the ‘star’ (assumed to be characterized by the innermost zero-pressure surface \( p = 0 \)). Now, there are many physical situations in which one simply does not know the equation of state, either because of uncertainties in the basic physics (for example, there are still some uncertainties regarding the equation of state for nuclear matter in neutron stars), or more prosaically because
the chemical composition of the ‘star’ may vary throughout its bulk so that it is not meaningful
to speak of a single equation of state for the entire body\(^1\).

We therefore decided to see what explicit constraints on the spacetime geometry could be
deduced directly from the perfect fluid condition, without reference to any particular equation
of state. To start with, note that by using the coordinate freedom inherent in general relativity
any static spherically-symmetric geometry can be put into a form where there are only two
independent metric components, typical functions of the radial coordinate. The most common
such forms are given by Schwarzschild coordinates (area coordinates, curvature coordinates)
\[ ds^2 = -[\hat{g}_{rr}(\mathcal{R})]dr^2 + \hat{g}_{r\theta}(\mathcal{R})d\theta^2 + \mathcal{R}^2(d\phi^2 + \sin^2 \theta d\phi^2) \] (1.1)
and isotropic coordinates
\[ ds^2 = -[g_{rr}(r)]dr^2 + g_{r\theta}(r)[d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2)]. \] (1.2)
Now, spherical symmetry by itself automatically implies that once one calculates the Einstein
tensor and goes to an orthonormal frame\(^2\)
\[ G_{\hat{t}\hat{r}} = G_{\hat{\theta}\hat{\phi}}. \] (1.3)
If the geometry is to represent a perfect fluid, then in addition we demand pressure isotropy
\[ G_{rr} = G_{\theta\theta} = G_{\phi\phi}. \] (1.4)
This places a single differential constraint on the metric components and so we expect the
class of metrics representing a perfect fluid geometry to have only one freely-specifiable
metric component—more precisely, we expect there to be a single freely-specifiable generating
function (call it \(z(r)\)) that should characterize the entire class of metrics
\[ g[z(r)] \] (1.5)
for static perfect fluid spheres. Since the pressure isotropy condition involves derivatives of
the metric components, we expect the metric \(g[z(r)]\) to be some functional of the generating
function \(z(r)\), unavoidably involving derivatives and integrations. These comments are
of course quite standard and in some form or another implicitly underlie all extant static
spherically-symmetric perfect fluid solutions. The novelty in the current article lies in the fact
that we will make this implicit procedure explicit and thereby will be able to exhibit the most
general form of the metric for static spherically-symmetric perfect fluid spacetimes. That is,
we are seeking an explicit closed-form (algebraic integro-differential) solution to the pressure
isotropy condition.

We report that an explicit and relatively simple characterization of this type does in fact
exist. It involves a single derivative, some algebraic manipulations (of which the worst is
taking a square root) and an explicit integration. The technique can be viewed as a simple
algorithm for constructing all static spherically-symmetric perfect fluid geometries\(^3\). We also
discuss the restrictions that must be placed on the generating function in order to get ‘physically
reasonable’ geometries.

Finally we present a few specific examples where we demonstrate how various well-
known solutions fit into our scheme. We exhibit a particularly striking three-parameter
perfect-fluid solution, given in closed form in terms of algebraic functions. The solution is

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\(^1\) A somewhat different approach, explored by Baumgarte and Rendall \([4]\), consists of specifying a non-negative but
otherwise arbitrary density profile and then integrating the Tolman–Oppenheimer–Volkoff equation (without assuming
any equation of state) to determine the pressure profile. Such a procedure generates more general geometries than the
more standard approach sketched above.

\(^2\) Hatted indices are used to denote the orthonormal frame attached to a particular coordinate system.

\(^3\) In particular our technique provides a way to algorithmically generate all possible Baumgarte–Rendall
configurations \([4]\).
presented in a new manner and is, with hindsight, equivalent to the Goldman-I solution (Gold-I solution in the Delgaty–Lake classification), which we show is in turn equivalent to the Glass–Goldman solution (G–G solution). Furthermore, in various regions of parameter space the general solution reduces to at least six different previously-derived solutions. In particular, our solution includes three two-parameter sub-solutions: the interior Schwarzschild solution, the Stewart solution and (in the Delgaty–Lake classification) the Kuch5 XIII solution. It also contains, as one-parameter branches, the Einstein, de Sitter and anti-de Sitter solutions. We do not claim this list is exhaustive.

2. Perfect fluid spheres

Consider a spherically-symmetric static spacetime geometry. Without loss of generality we know we can put it into isotropic coordinates

\[ ds^2 = -[g_{tt}(r)dr^2 + g_{rr}(r)[d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2)]]. \] (2.1)

Our first key result can be phrased as a simple theorem:

**Theorem 1.** Pick an arbitrary non-increasing function \( z(r) \) (i.e., a suitably smooth function \( z(r) \) with \( z'(r) \leq 0 \), introduce a dummy integration variable \( \tau \), and formally construct the metric

\[
\begin{align*}
\begin{split}
&ds^2 = -\exp \left\{ \pm 2 \int \frac{\sqrt{-1/z^2}}{1 - 2z^2} \, d\tau \right\} \, dr^2 + \exp \left\{ -2 \int \frac{\pm \sqrt{-1/z^2} - 2z^2 \, d\tau}{1 - 2z^2} \right\} \\
&\times [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)].
\end{split}
\end{align*}
\] (2.2)

Then this metric is guaranteed to be real (by the non-increasing property of \( z(r) \)) and always describes a static spherically-symmetric distribution of perfect-fluid matter. Conversely, the spacetime metric generated by any static spherically-symmetric distribution of perfect fluid matter can be put into this form for some suitable non-increasing \( z(r) \).

**Proof.** (\( \Rightarrow \)) By explicit computation

\[ G_{\tau\tau} = G_{\theta\theta} = G_{\phi\phi} = \frac{(z^4)'}{g_{rr}(1 - z^2)^2 r^3}. \] (2.3)

The computations have been carried out and cross checked using a combination of pencil and paper, the CARTAN [5] package under Mathematica\(^4\), and the standard distribution of Maple\(^5\).

Invoking the Einstein equations, this purely geometric statement (2.3) implies

\[ p = \frac{1}{3\pi G_{\text{Newton}} g_{rr}(1 - z^2)^2 r^3}. \] (2.4)

**Proof.** (\( \Leftarrow \)) Suppose, on the other hand, we start with a static spherically-symmetric perfect fluid. Without loss of generality we can put the metric in isotropic coordinates and choose the coefficients to be

\[ ds^2 = -\exp[-2 \varphi(r)] \, dr^2 + \exp[+2 \varphi(r) + 4 \psi(r)][dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)]. \] (2.5)

\(^4\) (http://www.wolfram.com)

\(^5\) (http://www.maplesoft.com)
Then

\[ G_{\tilde{t}t} = -\frac{1}{g_{rr}} \left[ 2\psi'' + 4\psi'' + (\psi')^2 + 4(\psi')^2 + 4\psi' + \frac{4\psi'}{r} + \frac{8\psi'}{r} \right] \]  

(2.6)

\[ G_{\tilde{t}\tilde{t}} = \frac{1}{g_{rr}} \left[ 4(\psi')^2 + \frac{4\psi'}{r} - (\psi')^2 \right] \]  

(2.7)

\[ G_{\tilde{\theta}\tilde{\theta}} = G_{\tilde{\phi}\tilde{\phi}} = \frac{1}{s_{rr}} \left[ 2\psi'' + \frac{2\psi'}{r} + (\psi')^2 \right]. \]  

(2.8)

Demanding pressure isotropy yields the equation

\[(\psi')^2 + \psi'' - 2(\psi')^2 - \frac{\psi'}{r} = 0\]  

(2.9)

which is easily solved algebraically (for the derivative \(\psi'\))

\[\psi' = \pm \sqrt{2(\psi')^2 + (\psi')/r - \psi''}.\]  

(2.10)

We could satisfy this equation by picking a ‘generating function’ \(\Theta(r)\) and setting

\[\psi'(r) = \Theta(r)\]  

(2.11)

\[\psi'(r) = \pm \sqrt{2\Theta(r)^2}\]  

(2.12)

But with this particular choice of generating function it is difficult to guarantee the reality of the resulting metric. Instead we find it more useful to make the algebraic definition

\[\psi'(r) = \frac{z(r)r}{1 - z(r)r^2} \quad \text{that is } z(r) = \frac{\psi'(r)}{r[1 + r\psi'(r)]}.\]  

(2.13)

With this definition for the generating function \(z(r)\), it is now a simple matter to verify that the isotropy condition (2.10) is equivalent to

\[\psi'(r) = \pm \frac{\sqrt{-rr'}}{1 - zr^2}.\]  

(2.14)

Integrating and substituting, we get the form of the metric given in the statement of the theorem, with now a very simple condition on \(z(r)\) (the non-increasing condition) being sufficient to guarantee reality of the metric.

Aside. We also mention, because it is relatively simple, that for this entire class of metrics

\[\rho' = \mp \frac{\sqrt{-rr'}}{1 - zr^2} (\rho + p)\]  

(2.15)

The choice of sign for the square root will be fixed once we demand positivity of density at the origin. In contrast, we note that the corresponding formula for \(G_{tt}\) is quite messy. It is better, but still less than ideal, to consider \(G_{tt} + 3G_{\tilde{t}t} = 2R_{tt}\) which can be cast into any of the equivalent forms

\[G_{tt} + 3G_{\tilde{t}t} = \mp \frac{5z' + 3rz' + 2r^3(z')^2 + r(1 - rz^2)z''}{g_{rr}\sqrt{-rz'}(1 - rz^2)^2} \]  

(2.16)

\[\mp \frac{4z' + (1 - rz^2)^2}{g_{rr}\sqrt{-rz'}(1 - rz^2)^2} \]  

(2.17)

\[\pm \frac{2}{g_{rr}(1 - rz^2)^2} \left[ (r^2\sqrt{-rz'})' + \sqrt{-rz'} \left( \frac{zr}{1 - rz^2} \right)' \right] \]  

(2.18)

\[\pm \frac{2}{g_{rr}} \left[ \frac{(\sqrt{-rz'})'}{1 - rz^2} + \frac{2\sqrt{-rz'}}{r(1 - rz^2)^2} + \sqrt{-rz'} \left( \frac{1}{1 - rz^2} \right)' \right] \]  

(2.19)
\[
= \pm \frac{2}{8\pi G_{\text{Newton}} g_{rr}} \left[ \left( \frac{\sqrt{-r z'}}{1 - z r^2} \right)' + \frac{2\sqrt{-r z'}}{r (1 - z r^2)^2} \right] \\
= \pm \frac{2}{8\pi G_{\text{Newton}} g_{rr}} \left[ \frac{1}{r^2} \left( \frac{r^2 \sqrt{-r z'}}{1 - z r^2} \right)' + \frac{2 z r \sqrt{-r z'}}{(1 - z r^2)^2} \right].
\] (2.20)

We agree that none of these formulae are stunningly pleasant, but despite considerable effort this is (in general) the best we have been able to do.

Once we apply the Einstein equations

\[
\rho + 3p = \pm \frac{2}{8\pi G_{\text{Newton}} g_{rr}} \left[ \frac{1}{r^2} \left( \frac{r^2 \sqrt{-r z'}}{1 - z r^2} \right)' + \frac{2 z r \sqrt{-r z'}}{(1 - z r^2)^2} \right].
\] (2.22)

**Comment.** We have not used any equation of state anywhere in the derivation. Furthermore, we have not yet applied any regularity conditions to the metric—so far it could represent a perfect-fluid sphere such as a star, or a completely liquid planet (e.g., Jupiter). It could also represent the fluid portion of a mostly liquid planet surrounding a solid core (e.g., Saturn), or a black hole surrounded by a spherically-symmetric perfect-fluid halo (an example of a so-called ‘dirty black hole’ [6] or ‘hairy black hole’), or even a traversable wormhole [7, 8] supported by an ‘exotic’ perfect fluid.

We also wish to contrast this explicit formula for the metric (equation (2.2)) with the more traditional implicit formulation of the problem. (Typically along the lines of ‘solve a certain differential equation for one of the metric components and implicitly substitute the result back into the metric ansatz’.)

3. Regularity conditions

We now investigate the effect of placing various regularity conditions on the geometry and the fluid.

3.1. Regularity of the geometry at the origin

If we focus on what is perhaps the astrophysically most interesting case, that of a perfect-fluid star (or a completely liquid planet), then we want to impose some regularity conditions at the centre. As a minimum we want the geometry to be regular, which at the most elementary level requires [2]

\[
g_{tt}(r = 0) = \text{finite} \quad g'_{tt}(r = 0) = 0
\] (3.1)

and

\[
g_{rr}(r = 0) = \text{finite} \quad g'_{rr}(r = 0) = 0.
\] (3.2)

We can then, without loss of generality, rescale \( r \) to set \( g_{rr}(r = 0) = 1 \) (this only works because we are using isotropic coordinates); it is convenient not to rescale \( g_{tt}(r = 0) \). Then these geometric regularity conditions can be satisfied by: (1) specifying the lower limit of integration to be the origin; then (2) setting the integration constants by defining

\[
ds^2 = -\exp(-2\phi(0)) \exp \left\{ \pm 2 \int_0^r \frac{\sqrt{-r z'}}{1 - z r'^2} \, dr \right\} \, dt^2 + \exp \left\{ -2 \int_0^r \frac{\pm \sqrt{-r z'} - 2 r z'}{1 - z r'^2} \, dr \right\} \\
\times [d\tau^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)]
\] (3.3)
and finally (3) demanding that both
\[
\frac{\sqrt{-rz''}}{1 - rz} \to 0 \quad \text{and} \quad \frac{rz}{1 - rz^2} \to 0 \quad \text{as} \quad r \to 0.
\] (3.4)
This requires both \( z(r) \) and \( z'(r) \) to be finite at the origin.

### 3.2. Finiteness of central pressure and density

The central pressure, derived from equation (2.4) using the condition of geometric regularity at the origin, is
\[
p_c = \frac{4z(0)}{8\pi G_{\text{Newton}}}
\] (3.5)
which gives no additional constraint beyond regularity of the geometry itself. Indeed
\[
z(0) = 2\pi G_{\text{Newton}} p_c.
\] (3.6)
On the other hand, by considering \( \rho + 3p \) as one naers the origin we can derive additional constraints. Suppose we expand \( z(r) \) in a power series
\[
z(r) = z(0) + z'(0)r + \frac{1}{2}z''(0)r^2 + O(r^3).
\] (3.7)
Then, evaluating the numerator and denominator of equation (2.16) separately
\[
[\rho + 3p](r) = \pm 1 \frac{5z'(0) + 6z''(0)r + O(r^2)}{8\pi G_{\text{Newton}} \sqrt{-rz''(0)} - rz'(0)r^2(1 + O(r))}.
\] (3.8)
So if the central value of \( \rho + 3p \) is to be finite we must have
\[
z'(0) = 0
\] (3.9)
in which case
\[
[\rho + 3p](r) = \pm 1 \frac{6z''(0) + O(r)}{8\pi G_{\text{Newton}} \sqrt{-z''(0)}(1 + O(r))} = \pm 1 \frac{6}{8\pi G_{\text{Newton}}} \sqrt{-z''(0)} + O(r).
\] (3.10)
Thus, if both central pressure and central density are to be finite we must have
\[
z(r) \approx z(0) + \frac{1}{2}z''(0)r^2 + O(r^3)
\] (3.11)
with
\[
\rho_c + 3p_c = \frac{\pm 6}{8\pi G_{\text{Newton}}} \sqrt{-z''(0)}.
\] (3.12)
So in summary, finiteness of central pressure and central density implies
\[
z(0) = \text{finite}
\] (3.13)
while
\[
z'(0) = 0
\] (3.14)
and
\[
z''(0) = -\frac{(8\pi G_{\text{Newton}})^2}{36}[\rho_c + 3p_c]^2.
\] (3.15)
3.3. Positivity of central pressure and density

For the central pressure to be positive we additionally require
\[ z(0) > 0. \]  \quad (3.16)

For the central density to be positive we need, first, to take the positive root in the expression for \( \rho + 3p \). This implies a specific choice for the \( \pm \) throughout the entire body of the ‘star’. That is, we must take
\[
\begin{align*}
    ds^2 &= -\exp\{-2\phi(0)\} \exp\left\{ 2 \int_0^r \frac{\sqrt{-\rho}}{1 - z\rho^2} \, d\mathcal{F} \right\} \, dt^2 + \exp\left\{ -2 \int_0^r \frac{\sqrt{-\rho} z^2 - 2\rho z}{1 - z\rho^2} \, d\mathcal{F} \right\} \\
    &\quad \times [dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)].
\end{align*}
\]  \quad (3.17)

Second, we must demand
\[ \sqrt{-z''(0)} > 2z(0). \]  \quad (3.18)

That is
\[ z''(0) < -4z(0)^2. \]  \quad (3.19)

3.4. Positivity of pressure (and density)

Enforcing positivity of pressure is easy: the pressure is proportional to \((zr^4)'\) multiplied by quantities that are guaranteed positive. So the pressure will be positive as long as
\[ (zr^4)' > 0. \]  \quad (3.20)

The innermost surface at which \((zr^4)' = 0\) is defined to be the surface of the star, denoted by \( r_{\text{surface}} \).

The positivity of pressure throughout the star then implies
\[ z(r) + 4 \frac{z'(r)}{r} > 0 \quad \Rightarrow \quad z(r) > -4 \frac{z'(r)}{r} > 0. \]  \quad (3.21)

Thus the function \( z(r) \) must be positive at least as far out as the surface of the star.

Guaranteeing positivity of density throughout the star is much more difficult to achieve. Mathematically, this is because the density (in contrast to the pressure) depends on second derivatives \( z''(r) \) of the generating function \( z(r) \). Physically, this arises because we have not specified any equation of state. Because of this it should not be too surprising that we cannot say everything about the ‘star’—the surprise perhaps is just how much we can do without an equation of state.

3.5. Monotonic decrease of pressure and density

Similarly, guaranteeing a monotonic decrease of pressure and density throughout the star is generically difficult to achieve. On the other hand, what can be done very easily is to derive conditions for the gravitational potential to be monotonically increasing as we move from the centre (that is, to keep the local gravitational force pointing downwards). Given the sign choice made to keep the central density positive, we need now only add the condition
\[ z(r)r^2 < 1 \]  \quad (3.22)

throughout the interior of the star. (This condition also prevents singularities in the integration used to define the metric.)
But given our sign choice for the root, we already know

\[ p' = -\frac{\sqrt{-p^\prime}}{1 - zr^2}(\rho + p). \]  

(3.23)

Thus, assuming monotonicity for the gravitational potential implies

\[ \text{sign}(p') = -\text{sign}(\rho + p). \]  

(3.24)

So under these assumptions the pressure will be monotonically decreasing if and only if the null energy condition (NEC) is satisfied [9].

3.6. Positivity of total mass

To calculate the total mass out to radius \( r \) we use the Hernandez–Misner mass formula [11]

\[ m(r) = \frac{1}{2} R_{\phi\phi\phi\phi} g^{\phi\phi} = \frac{1}{2} g_{\phi\phi} g^{\phi\phi}. \]  

(3.25)

This formula is valid for any spherically-symmetric spacetime in any spherically-symmetric coordinate system [11]. Applied to the metric (3.17) in isotropic coordinates one obtains

\[ m(r) = \sqrt{g_{\phi\phi}(r)} r \left\{ \frac{2\sqrt{-r^\phi}(1 + z(r)r^2) - 4r z(r) + z'(r)r^2}{2(1 - z(r)r^2)^2} \right\}. \]  

(3.26)

The surface of the star is located at \( r_{\text{surface}} \), where \( \langle r^4 \rangle = 0 \). In particular, \( z'(r_{\text{surface}}) = -4z_{\text{surface}}/r_{\text{surface}} \). Then a brief computation shows that the total mass of the ‘star’ is

\[ m(r_{\text{surface}}) = \sqrt{g_{\phi\phi}(r_{\text{surface}})} r_{\text{surface}} \left\{ \frac{2\sqrt{-z_{\text{surface}}}(1 + z_{\text{surface}}r_{\text{surface}}^2) - 4z_{\text{surface}} r_{\text{surface}}}{(1 - z_{\text{surface}} r_{\text{surface}})^2} \right\}. \]  

(3.27)

Both the numerator and denominator can be factorized to yield

\[ m(r_{\text{surface}}) = \sqrt{g_{\phi\phi}(r_{\text{surface}})} \frac{2r_{\text{surface}} \sqrt{-z_{\text{surface}}}}{(1 + \sqrt{-z_{\text{surface}}} r_{\text{surface}})^2} = R_{\text{surface}} \frac{2r_{\text{surface}} \sqrt{-z_{\text{surface}}}}{(1 + \sqrt{-z_{\text{surface}}} r_{\text{surface}})^2}. \]  

(3.28)

Here \( R_{\text{surface}} = \sqrt{g_{\phi\phi}(r_{\text{surface}})} \) is the radius of the ‘star’ in curvature coordinates (Schwarzschild coordinates). Since each of these factors is manifestly positive, so is the total mass.

The ‘compactness’ can be defined by

\[ \chi \equiv \frac{2m(r_{\text{surface}})}{R_{\text{surface}}} = \frac{4r_{\text{surface}} \sqrt{-z_{\text{surface}}}}{1 + \sqrt{-z_{\text{surface}}} r_{\text{surface}}^2}. \]  

(3.29)

Since \( 4x/(1 + x)^2 \leq 1 \), we have \( \chi \in [0, 1] \) and the very sensible result

\[ m(r_{\text{surface}}) \leq R_{\text{surface}}/2. \]  

(3.30)

That is:

\[ R_{\text{Schwarzschild}} = 2m(r_{\text{surface}}) = 2G_{\text{Newton}} M_{\text{physical}} \leq R_{\text{surface}}. \]  

(3.31)

These observations are not enough to guarantee that the density is positive everywhere, but they do place powerful constraints on its behaviour.
3.7. Volume-averaged strong-energy condition

Another simple constraint on the stress–energy distribution for a static perfect-fluid sphere that can be extracted without specifying an equation of state is a certain type of 'weighted volume average' of the strong-energy condition. Specifically, consider

\[
\int g_{rr} \left( \rho + \frac{3}{2} p \right) r^2 dr = \frac{1}{4\pi G_{\text{Newton}}} \int r^2 dr \left[ \frac{1}{r^2} \left( \frac{r^2 - r_c^2}{1 - z^2} \right) + \frac{2 z r \sqrt{1 - z^2}}{(1 - z^2)^2} \right].
\]

We have already chosen the positive sign for the root, since we want the central density to be positive. Note that this is not the 'proper volume average' (which would correspond to \( \int g_{rr}^{-3/2} r^2 dr \)) but is instead weighted by a compensating factor \( g_{rr}^{-1/2} \), chosen to simplify the mathematical analysis. (The occurrence and usefulness of this and related weighted volume averages is quite common in spherically-symmetric systems.)

Then, using the regularity conditions we have already deduced for the origin, for any value of \( r_* \), we deduce

\[
\int_0^{r_*} g_{rr} \left( \rho + \frac{3}{2} p \right) r^2 dr = \frac{1}{4\pi G_{\text{Newton}}} \left[ \left( \frac{r_*^2 - r_c^2}{1 - z^2} \right) \right] + 2 \int r^3 dr \frac{z \sqrt{1 - z^2}}{(1 - z^2)^2},
\]

The first term is non-negative by the non-increasing property of \( z(r) \), which was required just to enforce reality of the metric (plus the constraint \( z(r)r^2 < 1 \), which was imposed to keep the local gravitational force pointing downwards). The second term is positive definite for the same reason, so we have, for all \( r_* \),

\[
\int_0^{r_*} g_{rr} \left( \rho + \frac{3}{2} p \right) r^2 dr > 0.
\]

This is not the strong-energy condition (SEC, \( \rho + 3p > 0 \)) itself, but is, at least, a weighted volume average thereof [9]. This implies that the energy density is not permitted to become too violently negative.

3.8. Subluminal speed of sound

It is traditional to compute the quantity

\[
\left( \frac{dp}{d\rho} \right)_{\text{fluid}} \equiv \frac{dp}{dr} \left[ \frac{dp}{dr} \right]^{-1},
\]

and to demand that this be less than or equal to \( c^2 \), on the grounds that this quantity is alleged to represent the physical speed of sound (which certainly should be subluminal). This assertion is dangerously misleading and cannot be justified without significant additional technical assumptions above and beyond those that have so far been made.

Specifically, let us assume that the fluid is described by some equation of state

\[
p = p(\rho, X).
\]

Here \( X \) stands for some collection of variables characterizing the fluid, possibly chemical concentrations, entropy density, temperature or the like. Then

\[
\frac{dp}{dr} = \frac{\partial p}{\partial \rho} \frac{d\rho}{dX} \frac{dX}{dr} + \frac{\partial p}{\partial X} \frac{dX}{dr}.
\]

Thus

\[
\left( \frac{dp}{d\rho} \right)_{\text{fluid}} = \frac{\partial p}{\partial \rho} \frac{d\rho}{dX} + \frac{\partial p}{\partial X} \frac{dX}{dr} \left[ \frac{d\rho}{dr} \right]^{-1}.
\]
That is
\[ \left. \frac{dp}{d\rho} \right|_{\text{fluid}} = c_s^2(X) + \frac{\partial p}{\partial X} \left|_{\rho} \right. \left( \frac{dX}{d\rho} \right) \left|_{\text{fluid}} . \right. \]  
(3.39)

In other words \((dp/d\rho)_{\text{fluid}}\) can be related to the (constant \(X\)) speed of sound \(c_s(X)\) if and only if you add extremely powerful additional assumptions. (Such as \(\partial p/\partial X = 0\), implying an \textit{a priori} exactly barotropic equation of state. Or \(dX/dr = 0\), implying for instance either thorough mixing of the entire fluid mass, an adiabatic star or an isothermal star.) Without such additional assumptions no particular conclusion regarding the relationship between \((dp/d\rho)_{\text{fluid}}\) and the physical speed of sound can be drawn [10]. Given our philosophy in this article (we wish to see what can be deduced without making assumptions about the equation of state), such assumptions would be completely opposite to our purpose, and so we do not seek to impose the condition \(dp/d\rho \leq c_s^2\).

3.9. Summary

We can summarize the essential core of these regularity conditions in the following theorem:

**Theorem 2.** Let \(z(r)\) be a positive non-increasing function \((z'(r) \leq 0)\) such that:
1. \(z(0)\) is finite
2. \(z'(0) = 0\)
3. \(z''(0) < -4z(0)^2\)
4. \(z(r) < 1/r^2\)

and consider the metric (guaranteed to be real)
\[
ds^2 = -\exp[-2\phi(0)] \exp \left( 2 \int_0^r \frac{\sqrt{-rz}}{1-zr^2} \, dr \right) \, dt^2 + \exp \left( -2 \int_0^r \frac{\sqrt{-r^2z - 2r} \, d\rho}{1-zr^2} \right) \times [dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2)]. \]
(3.40)

Then this metric represents a static perfect-fluid sphere with:
1. regular geometry at the origin
2. finite and positive pressure and density at the origin
3. a local gravitational field that always points downward.

Conversely, any static perfect fluid sphere satisfying these last three conditions can be cast into the preceding form with a generating function \(z(r)\) satisfying the first four conditions.

Furthermore, under the conditions enunciated above, if the system is additionally isolated (so that the pressure drops to zero at some finite radius), then the total mass is guaranteed to be both positive and bounded.

**Proof.** (\(\Leftrightarrow\)) This theorem is just a codification of the most salient of the preceding results.

\hfill \Box

4. Examples

While the metric given in equation (3.40) is guaranteed to be a perfect fluid for a very wide class of generating functions \(z(r)\), it is only for a much more restricted class of generating functions that the relevant integrals can be performed in terms of elementary functions. We now present several examples where this can be done.
4.1. Schwarzschild exterior geometry

The Schwarzschild exterior solution corresponds to

\[ z(r) = \frac{(m/2)^2}{r^4} \quad m = G_{\text{Newton}} M_{\text{physical}} \]  

(4.1)

together with choosing the positive sign for the root. A brief computation leads to \( \rho = 0 \), \( \rho = 0 \) and the isotropic form of the Schwarzschild exterior metric.

\[ ds^2 = -\left(1 - \frac{m}{2r}\right)^2 \, dt^2 + \left(1 + \frac{m}{2r}\right)^4 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]. \]  

(4.2)

Note that the exterior Schwarzschild does not satisfy the regularity requirements for a ‘normal’ fluid sphere—in particular, \( r = 0 \) is not a ‘point’ but instead corresponds (in these isotropic coordinates) to a second asymptotically flat region. Because the geometry is not regular at the origin, we cannot use equation (3.40), (3.17) or even (3.3). Instead, we must back-track all the way to (2.2).

4.2. Einstein universe

The Einstein universe corresponds to

\[ z = \frac{1}{R^2} \]  

(3.3)

with either sign for the root. A brief computation leads to

\[ \rho = \frac{12}{8\pi G_{\text{Newton}} R^2} \quad p = -\frac{4}{8\pi G_{\text{Newton}} R^2} \]  

(4.4)

and the isotropic form of the Einstein metric:

\[ ds^2 = -dt^2 + \frac{dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)}{(1 + r^2/R^2)^2}. \]  

(4.5)

Note that if the density is positive the pressure is negative, and vice versa—the Einstein universe does not satisfy the regularity requirements for a ‘normal’ fluid sphere.

4.3. de Sitter

The de Sitter geometry corresponds to

\[ z(r) = \frac{\frac{3R^2}{R^2} + r^2}{R^2 + 3r^2} \]  

(3.6)

Choose the positive sign for the root. A brief computation yields

\[ \rho = -\frac{12}{8\pi G_{\text{Newton}} R^2} \quad p = \frac{12}{8\pi G_{\text{Newton}} R^2} \]  

(4.7)

and the isotropic form of the de Sitter metric:

\[ ds^2 = -\left(1 + r^2/R^2\right)^2 \, dt^2 + \frac{dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)}{(1 - r^2/R^2)^2}. \]  

(4.8)
4.4. Anti-de Sitter

The anti-de Sitter geometry corresponds to

\[ z(r) = \frac{1}{2} \left( \frac{3R^2 - r^2}{R^2} \right) \]  

(4.9)

Choose the negative sign for the root in (3.3). A brief computation yields

\[ \rho = \frac{12}{8\pi G_{\text{Newton}} R^2} \quad p = -\frac{12}{8\pi G_{\text{Newton}} R^2} \]  

(4.10)

and the isotropic form of the anti-de Sitter metric:

\[ ds^2 = - \left( \frac{1 - r^2/R^2}{1 + r^2/R^2} \right)^2 dt^2 + \frac{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)}{(1 + r^2/R^2)^2} \]  

(4.11)

4.5. The general quadratic ansatz

Suppose we consider the general quadratic ansatz

\[ z(r) = \frac{A_i + B_i r^2}{C_i + D_i r^2} \]  

(4.12)

We were led to this ansatz by considering the form of \( z(r) \) for the de Sitter and anti-de Sitter cases. By construction, for any choice of \( A_i, B_i, C_i \) and \( D_i \) we get a real metric satisfying the perfect-fluid constraint. (Due to rescaling invariance—multiply both numerator and denominator by any fixed constant—only three of these four coefficients are actually physically meaningful.) If the central pressure is to be positive then we must have \( z(0) = A_i / B_i > 0 \). Then we can without loss of generality take

\[ z(r) = \frac{A_i + B_i r^2}{1 + C_i r^2} \]  

(4.13)

Now in order to justify calling the geometry an ‘exact solution’ we need an explicit formula for the metric. Inserting this quadratic ansatz into (3.35) the integrals can be done in closed form. Expressed in terms of \( A_2, B_2 \) and \( C_2 \) the resulting metric is a rather messy combination of quadratics in \( r \) raised to various real exponents.

It is much more convenient to introduce new parameters \( S, R \) and \( \eta \) and write

\[ ds^2 = - \left( \frac{1 \pm r^2/S^2}{1 \pm r^2/R^2} \right)^{2\eta/(2^n - 1)} dr^2 + \frac{1 \pm r^2/S^2}{1 \pm r^2/R^2} \left( \frac{2^n - 2 \eta}{2^n - 1} \right)^{\eta/(2^n - 1)} \]  

(4.14)

This is a perfect-fluid solution for arbitrary \( S, R \) and \( \eta \); there are additionally two independent sign choices that can be made, one associated with each of the parameters \( S \) and \( R \). For definiteness of presentation we discuss the case ++ but it is trivial to flip the signs as required.

This solution can (after yet another redefinition of parameters) be seen to be equivalent to the Goldman-I solution [12], called Gold-I in the Delgaty–Lake [2] classification. The pressure and density are rational functions of position.

A brief computation yields

\[ p = \frac{G_{ij}}{8\pi G_{\text{Newton}}} = \frac{G_{\phi\phi}}{8\pi G_{\text{Newton}}} = \frac{G_{\phi\phi}}{8\pi G_{\text{Newton}}} \]  

(4.15)

\[ = \frac{4 q_1(r)}{8\pi G_{\text{Newton}} (24 - 1)^2 R^4 S^4 g_{ij}(r)(1 + r^2/S^2)^2(1 + r^2/R^2)^2} \]  

(4.15)
This verifies that it is a perfect-fluid solution. Here \(q_1(r)\) is the quartic

\[
q_1(r) = [(2n^2 - 1)S^2R^2(R^2 - 2n^2S^2) + n^2(R^2 - S^2)^2 - 2S^2R^2(2n^2 - 1)^2]r^2 - [(2n^2 - 1)(2n^2 - 2S^2)]r^4.
\]

(4.16)

Similarly,

\[
\rho = \frac{G_{ii}}{8\pi G_{\text{Newton}}} = \frac{4q_2(r)}{8\pi G_{\text{Newton}}(2n^2 - 1)^2R^4S^4\delta_{rr}(r)(1 + r^2/S^2)(1 + r^2/R^2)^2}
\]

(4.17)

where \(q_2(r)\) is the quartic

\[
q_2(r) = [3(2n^2 - 1)S^2R^2[(n - 1)R^2 + n(2n - 1)S^2]] + [n(n - 1)(2n - 1)(R^2 - S^2)^2
+ 6(2n - 1)^2S^2R^2]r^2 + [3(2n^2 - 1)(n[2n - 1]R^2 + [n - 1]S^2)]r^4.
\]

(4.18)

A somewhat simpler quartic is obtained if we consider

\[
\rho + 3p = \frac{G_{ii} + 3G_{ij}}{8\pi G_{\text{Newton}}} = \frac{4n(R^2 - S^2)q_3(r)}{8\pi G_{\text{Newton}}(2n^2 - 1)^2R^4S^4\delta_{rr}(r)(1 + r^2/S^2)(1 + r^2/R^2)^2}
\]

(4.19)

since then

\[
q_3(r) = [3(2n^2 - 1)S^2R^2] + [(2n^2 + 1)(R^2 - S^2)]r^2 - [3(2n^2 - 1)]r^4.
\]

(4.20)

In particular, at the centre of the star

\[
\rho_c = \frac{1}{8\pi G_{\text{Newton}}} \frac{R^2 - 2n^2S^2}{(2n^2 - 1)S^2R^2}
\]

(4.21)

while

\[
\rho_c = \frac{1}{8\pi G_{\text{Newton}}} \frac{12[(n - 1)R^2 + n(2n - 1)S^2]}{(2n^2 - 1)S^2R^2}
\]

(4.22)

and

\[
\rho_c + 3p_c = \frac{1}{8\pi G_{\text{Newton}}} \frac{12n(R^2 - S^2)}{(2n^2 - 1)S^2R^2}.
\]

(4.23)

From this we can use the positivity of central pressure and density to constrain the parameters.

We can find the surface of the star by locating the first zero \(p(r)\) [or \(q_1(r)\)], and then use the Hernández–Misner formula to deduce the total mass. While a closed-form exact algebraic mass formula exists, it is too unwieldy to be reproduced here.

To summarize the situation so far: the generating function technique developed in this paper has helped us in several ways. It led us to consider the quadratic ansatz, realize it was explicitly integrable and find a simple form for the metric. We shall now show that this quadratic ansatz (the Gold-I solution) is also equivalent to the G–G solution, and that it furthermore contains many interesting special cases: interior Schwarzschild, Stewart, Kuch5 XIII, de Sitter, anti-de Sitter and Einstein among them.

4.6. Glass–Goldman: G–G

The G–G (Glass–Goldman) geometry [12], in the form reported by Delgaty–Lake [2], is

\[
ds^2 = \left(\frac{2r^2 - B^2D - 1 - C}{2r^2 - B^2D - 1 + C}\right)^{2B/C} dr^2 + \frac{1}{B^2(2 + D) - (B^2D + 1)r^2 + r^4}
\]

\[
\times \left(\frac{2r^2 - B^2D - 1 - C}{2r^2 - B^2D - 1 + C}\right)^{-(B^2D + 2B - 1)/C} \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]
\]

(4.24)
where
\[ C = \sqrt{(B^2 D - 1)^2 - 8B^2}. \] (4.25)

Thus at first glance it appears to be a two-parameter solution to the perfect-fluid field equations. There is a subtlety here in the fact that G–G have implicitly chosen their \( r \) coordinate to be dimensionless, effectively hiding a dimensional parameter in their conventions, that is
\[ \kappa_{\text{this paper}} = \kappa_0 r_{\text{CGG}} \] (4.26)
with \( \kappa_0 \) some arbitrary but fixed distance scale. Then translating the \( B, D, \kappa_0 \) variables to our notation
\[ S^2 = -\kappa_0^2 (1 + B^2 D + C)/2 \] (4.27)
\[ R^2 = -\kappa_0^2 (1 + B^2 D - C)/2 \] (4.28)
\[ n(R, S) = \sqrt{\frac{1}{2} \frac{R^2 + \kappa_0^2}{S^2 + \kappa_0^2}} \quad \text{or} \quad \frac{1}{2} \frac{S^2 + \kappa_0^2}{R^2 + \kappa_0^2} \] (4.29)
depending on one’s choice for the sign of the square root in the definition of \( C \). That is, despite appearances the G–G solution is equivalent to the Gold-I solution and is equivalent to our general quadratic ansatz.

From the point of view of (4.14), this is a reflection of the fact that this solution is scale-covariant under \( r \to \lambda r, S \to \lambda S, R \to \lambda R \).

4.7. Schwarzschild interior geometry

The Schwarzschild interior geometry is a special case of the quadratic ansatz. It corresponds to making both sign choices positive ++ and setting \( n = 1 \). It is now easy to check that the metric is
\[ ds^2 = \frac{(1 + r^2 / S^2)^2}{(1 + r^2 / R^2)^2} dr^2 + \frac{r^2}{(1 + r^2 / R^2)^2} (d\theta^2 + \sin^2 \theta \, d\phi^2) \text{.} \] (4.30)

A brief computation yields
\[ \rho = \frac{1}{8\pi G_{\text{Newton}}} \frac{12}{R^2} p = \frac{1}{8\pi G_{\text{Newton}}} \frac{4}{S^2 R^4} \frac{R^2 (R^2 - 2S^2) - (2R^2 - S^2) r^2}{1 + r^2 / S^2}. \] (4.31)
The central pressure is
\[ p_c = \frac{4}{8\pi G_{\text{Newton}}} \frac{R^2 - 2S^2}{R^2 S^2}. \] (4.32)
The stellar surface is located at
\[ r_{\text{surface}} = R \sqrt{\frac{R^2 - 2S^2}{2R^2 - S^2}}. \] (4.33)
The total mass is
\[ m = \frac{2}{27} R \left[ \frac{(2R^2 - S^2)^{3/2} (2R^2 - 2S^2)^{3/2}}{(R^2 - S^2)^3} \right]. \] (4.34)
We mention that the generating function is
\[ z(r) = \frac{R^2 - 2S^2 - r^2}{R^2 S^2 + (2R^2 - S^2) r^2}. \] (4.35)
Note that \( S \to 0 \) is a singular limit of the interior Schwarzschild geometry where the central pressure goes to infinity; the central core of the stellar model is on the verge of becoming
a black hole. On the other hand, as $S^2 \to R^2/2$ from below the stellar surface moves inward and the star vanishes.

If we drive $S$ out of this ‘regular’ range and in particular force $S^2 \to \infty$ then one obtains the Einstein universe (see above). Finally, $S^2 \to -R_0^2$ and $R^2 \to +R_0^2$ (that is, the $-+$ sign choice) corresponds to the anti-de Sitter universe with scale factor $R_0$ (see above); while $S^2 \to +R_0^2$, $R^2 \to -R_0^2$ (that is $++$) corresponds to the de Sitter universe with scale factor $R_0$.

4.8. Stewart

To obtain Stewart’s geometry [14] we choose the signs to be $-+$ and pick $n = -1$; it is also convenient (but not mandatory) to interchange the roles of $R$ and $S$. When written in this form we can see that it is very closely related to the interior Schwarzschild solution. It is now easy to check that the metric is

$$ds^2 = -\frac{(1 - r^2/S^2)^2}{(1 - r^2/R^2)^2} dt^2 + \frac{(1 - r^2/R^2)^4}{(1 - r^2/S^2)^6} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (4.36)$$

A brief computation yields

$$\rho = \frac{1}{8\pi G_{\text{Newton}}} \frac{12 R^6 (S^2 - r^2)^4 [S^2 (2 S^2 - 3 R^2) - (2 R^2 - 3 S^2) r^2]}{S^{12} (R^2 - r^2)^5}$$

and

$$p = \frac{1}{8\pi G_{\text{Newton}}} \frac{4 R^6 (S^2 - r^2)^4 [S^2 (2 R^2 - S^2) + (R^2 - 2 S^2) r^2]}{S^{12} (R^2 - r^2)^5}, \quad (4.38)$$

The central pressure is

$$\rho_c = \frac{4}{8\pi G_{\text{Newton}}} \frac{2 R^2 - S^2}{R^2 S^2}$$

which implies $R^2 > S^2/2$. In contrast, the central density is

$$\rho_c = \frac{12}{8\pi G_{\text{Newton}}} \frac{2 S^2 - 3 R^2}{R^2 S^2}$$

which implies $S^2 > 3 R^2/2$. Combined this provides a rather tight constraint

$$\frac{1}{2} S^2 < R^2 < \frac{7}{4} S^2. \quad (4.41)$$

The stellar surface is located at

$$r_{\text{surface}} = S \sqrt{\frac{2 R^2 - S^2}{2 S^2 - R^2}}. \quad (4.42)$$

The total mass is

$$m = \frac{2}{27} S \left[ \frac{(2 S^2 - R^2)^{3/2} (2 R - S^2)^{3/2}}{R^3 (R^2 - S^2)} \right]. \quad (4.43)$$

We mention that the generating function is

$$z(r) = \frac{2 R^2 - S^2 - r^2}{R^2 S^2 + (R^2 - 2 S^2) r^2} \quad (4.44)$$
4.9. Kuchowicz: Kuch5 XIII

To obtain the Kuch5 XIII geometry [15], we simply let $R \to \infty$; it is then convenient (but not mandatory) to relabel $S$ as $R$. It is now easy to check that the metric is

$$ds^2 = -(1 + r^2/R^2)^2n/(2n^2 - 1) dr^2 + (1 + r^2/R^2)^n(2 - 2n)/(2n^2 - 1) [d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2)].$$

(4.45)

A brief computation yields

$$\rho = \frac{1}{8\pi G_{\text{Newton}}} \frac{4(n - 1)[3(2n^2 - 1) + (2n^2 - n)r^2/R^2](1 + r^2/R^2)^{-2n(2n - 1)/(2n^2 - 1)}}{(2n^2 - 1)^2 R^2}.\]$$

(4.46)

and

$$p = \frac{1}{8\pi G_{\text{Newton}}} \frac{4((2n^2 - 1) + n^2r^2/R^2)(1 + r^2/R^2)^{-2n(2n - 1)/(2n^2 - 1)}}{(2n^2 - 1)^2 R^2},\]$$

(4.47)

The central pressure is

$$p_c = \frac{4}{8\pi G_{\text{Newton}}} \frac{1}{R^2(2n^2 - 1)}$$

(4.48)

which implies $n^2 > 1/2$. In contrast, the central density is

$$\rho_c = \frac{12}{8\pi G_{\text{Newton}}} \frac{n - 1}{R^2(2n^2 - 1)}$$

(4.49)

which further implies $n > 1$. Because of these constraints, there is no stellar surface; pressure remains positive for all values of $r$ and the solution is actually cosmological. (This ultimately can be traced back to the fact that $b_2 = 0$, which means we are dealing with a singular solution of our general three-parameter result.)

We mention

$$z(r) = \frac{1}{R^2(2n^2 - 1) + 2n^2r^2}.$$  

(4.50)

Also, it is formally possible to replace $R^2 \to -R^2$ at the cost of reversing the positivity conditions ($2n^2 < 1; n < 1$).

5. Discussion and conclusions

We have explicitly characterized the spacetime metrics corresponding to the class of all static spherically-symmetric perfect-fluid geometries in a relatively straightforward manner. This observation is useful whenever there is some uncertainty regarding the actual equation of state one wishes to use. The first theorem we presented is applicable to all static spherically-symmetric perfect-fluid geometries without further restriction, while the second theorem encodes the most important of the regularity conditions that are relevant to an isolated static fluid sphere (such as a star).

Though the formulae we present do involve an integration, it is particularly noteworthy that in our representation the metric is explicit. Furthermore, it is easy to keep the metric real and particularly easy to find the surface of the ‘star’. Some (but not all) of the standard regularity conditions are easy to enforce and can be interpreted as extra restrictions on the class of ‘generating functions’ $z(r)$.

Throughout this article we chose to work in isotropic coordinates, because we found them to be the most useful. (See Glass and Goldman for an earlier, and rather different, use of the
ideas of isotropic coordinates and generating functions [13].) The use of isotropic coordinates is not a matter of deep principle and we do not rule out the possibility that there may still be other (possibly even simpler) representations in other coordinate systems. For instance, the recent work of Fodor [16] in Schwarzschild coordinates is particularly intriguing.

In closing we reiterate that while a tremendous amount is already known concerning static spherically-symmetric spacetimes (see in particular [1–3]), the particular approach adopted in the present article falls well outside any of the standard schemes.

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References


[5] See also http://grtensor.org/solutions for updates


Among the approaches we have encountered in the literature, the closest in spirit to the current approach is the Baumgarte–Rendall analysis of [4].