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Acoustic black holes: horizons, ergospheres and Hawking radiation

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Abstract. It is a deceptively simple question to ask how acoustic disturbances propagate in a non-homogeneous flowing fluid. Subject to suitable restrictions, this question can be answered by invoking the language of Lorentzian differential geometry. This paper begins with a pedagogical derivation of the following result: if the fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent), then the equation of motion for the velocity potential describing a sound wave is identical to that for a minimally coupled massless scalar field propagating in a $(3+1)$-dimensional Lorentzian geometry

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\,\partial_\nu\psi\right) = 0.$$

The *acoustic metric* $g_{\mu\nu}(t, \boldsymbol{x})$ governing the propagation of sound depends algebraically on the density, flow velocity, and local speed of sound. Even though the underlying fluid dynamics is Newtonian, non-relativistic, and takes place in flat space plus time, the fluctuations (sound waves) are governed by an effective $(3+1)$-dimensional Lorentzian spacetime geometry. This rather simple physical system exhibits a remarkable connection between classical Newtonian physics and the differential geometry of curved $(3+1)$-dimensional Lorentzian spacetimes, and is the basis underlying a deep and fruitful analogy between the black holes of Einstein gravity and supersonic fluid flows. Many results and definitions can be carried over directly from one system to another. For example, it will be shown how to define the ergosphere, trapped regions, acoustic apparent horizon, and acoustic event horizon for a supersonic fluid flow, and the close relationship between the acoustic metric for the fluid flow surrounding a point sink and the Painlevé–Gullstrand form of the Schwarzschild metric for a black hole will be exhibited. This analysis can be used either to provide a concrete non-relativistic analogy for black-hole physics, or to provide a framework for attacking acoustics problems with the full power of Lorentzian differential geometry.

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1. Introduction

In 1981 Unruh developed a way of mapping certain aspects of black-hole physics into problems in the theory of supersonic acoustic flows [1]. The connection between these two seemingly disparate systems is both surprising and powerful, and has been independently rediscovered several times over the ensuing decade and a half [2]. Over the last six years, a respectable body of work has been developed using this analogy to investigate micro-physical models that might underly the Hawking radiation process from black holes (or acoustic holes—'dumb holes'), and to investigate the extent to which the Hawking

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radiation process may be independent of the physics of extremely high-energy trans-Planckian modes [3–14].

In this paper, I wish to take another look at the derivation of the relationship between curved spacetimes and acoustics in flowing fluids, to provide a pedagogically clear and precise derivation using a minimum of technical assumptions, and to develop the analogy somewhat further in directions not previously envisaged. In particular, I will show how to define the notions of ergo-region, trapped regions, acoustic apparent horizons and acoustic event horizons (both past and future) for supersonic fluid flows, and show that in general it is necessary to keep these notions distinct.

As a particular example of a simple model exhibiting such behaviour, I write down the acoustic metric appropriate to a draining bathtub ($(2+1)$ dimensions), and the equivalent vortex filament sink ($(3+1)$ dimensions).

I shall further show that the relationship between the Schwarzschild geometry and the acoustic metric is clearest when the Schwarzschild metric is written in the Painlevé–Gullstrand† form [15–19], and that while the relationship is very close it is not exact. (It is in fact impossible to obtain an acoustic metric that is *identical* to the Schwarzschild metric, the best that one can achieve is to obtain an acoustic metric that is conformally related to the Schwarzschild metric.) If all one is interested in is either the Hawking temperature or the behaviour in the immediate region of the event horizon, then the analogy is much closer in that the conformal factor can be neglected.

For an arbitrary steady flow the 'surface gravity' (*mutatis mutandis*, the Hawking temperature) of an acoustic horizon will be shown to be proportional to a combination of the normal derivative of the local speed of sound and the normal derivative of the normal component of the fluid velocity at the horizon. In general, the 'surface gravity' is

$$g_H = \frac{1}{2}\,\frac{\partial(c^2 - v_\perp^2)}{\partial n} = c\,\frac{\partial(c - v_\perp)}{\partial n}. \tag{1}$$

(This generalizes the result of Unruh [1, 5] to the case where the speed of sound is position dependent and/or the acoustic horizon is not the null surface of the time translation Killing vector. This result is also compatible with that deduced for the solid-state black holes of Reznik [14], and with the 'dirty black holes' of [20].)

Finally, I shall show how to formulate the notion of a static (as opposed to merely stationary) acoustic metric and exhibit the constraint that must be satisfied in order to put the acoustic metric into Schwarzschild coordinates. I point out that while this is a perfectly acceptable and correct mathematical step, and a perfectly reasonable thing to do in general relativity, it is (I claim) a good way to get confused when doing acoustics—from the Newtonian view underlying the equations of fluid motion, the Schwarzschild coordinate system corresponds to a very peculiar way of synchronizing (or rather, de-synchronizing) your clocks.

To begin the discussion, I address the deceptively simple question of how acoustic disturbances propagate in a non-homogeneous flowing fluid. It is well known that for a static homogeneous inviscid fluid the propagation of sound waves is governed by the simple equation [21–24]

$$\partial_t^2 \psi = c^2 \nabla^2 \psi. \tag{2}$$

(Here c denotes the speed of sound.) Generalizing this result to a fluid that is non-homogeneous, or to a fluid that is in motion, possibly even in non-steady motion, is more subtle than it at first would appear.

† This is also often called the Lemaître form of the Schwarzschild metric.

An important aspect of this paper is to provide a pedagogical proof of the following theorem:

Theorem. If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent) then the equation of motion for the velocity potential describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a $(3+1)$-dimensional Lorentzian geometry

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\,\partial_\nu\psi\right) = 0. \tag{3}$$

Under these conditions, the propagation of sound is governed by an *acoustic metric* $g_{\mu\nu}(t,\boldsymbol{x})$. This acoustic metric describes a $(3+1)$-dimensional Lorentzian (pseudo-Riemannian) geometry. The metric depends algebraically on the density, velocity of flow and local speed of sound in the fluid. Specifically

$$g_{\mu\nu}(t,\boldsymbol{x}) \equiv \frac{\rho}{c}\left[\begin{array}{c|c} -(c^2-v^2) & -\boldsymbol{v} \\ \hline -\boldsymbol{v} & I \end{array}\right]. \tag{4}$$

(Here I is the 3×3 identity matrix.) In general, when the fluid is non-homogeneous and flowing, the *acoustic Riemann tensor* associated with this Lorentzian metric will be non-zero.

It is quite remarkable that even though the underlying fluid dynamics is Newtonian, non-relativistic, and takes place in flat space plus time, the fluctuations (sound waves) are governed by a curved $(3+1)$-dimensional Lorentzian (pseudo-Riemannian) spacetime geometry.

For practitioners of general relativity, this paper describes a very simple and concrete physical model for certain classes of Lorentzian spacetimes, including black holes. On the other hand, the discussion of this paper is also potentially of interest to practitioners of continuum mechanics and fluid dynamics in that it provides a simple concrete introduction to Lorentzian differential geometric techniques.

2. Fluid dynamics

2.1. Fundamental equations

The fundamental equations of fluid dynamics [21–24] are the equation of continuity

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v}) = 0, \tag{5}$$

and Euler's equation

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \equiv \rho\left[\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right] = \boldsymbol{F}. \tag{6}$$

I start the analysis by assuming the fluid to be inviscid (zero viscosity), with the only forces present being those due to pressure, plus Newtonian gravity, and with the inclusion of any arbitrary gradient-derived and possibly even time-dependent externally-imposed body force, then

$$\boldsymbol{F} = -\boldsymbol{\nabla}p - \rho\boldsymbol{\nabla}\phi - \rho\boldsymbol{\nabla}\Phi. \tag{7}$$

Here ϕ denotes the Newtonian gravitational potential, while Φ denotes the potential of the external driving force (which may in fact be zero)†.

† These two terms are lumped together without comment in [1], and are neglected in [5].

Using standard manipulations, the Euler equation can be rewritten as

$$\partial_t \boldsymbol{v} = \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) - \frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \phi + \Phi\right). \tag{8}$$

Now take the flow to be *vorticity free*, that is, *locally irrotational*†. Introduce the velocity potential ψ such that $\boldsymbol{v} = -\boldsymbol{\nabla}\psi$, at least locally‡. If one further takes the fluid to be *barotropic*§ (this means that ρ is a function of p only), it becomes possible to define

$$h(p) = \int_0^p \frac{\mathrm{d}p'}{\rho(p')}; \qquad \text{so that} \qquad \nabla h = \frac{1}{\rho}\,\nabla p. \tag{9}$$

Thus the specific enthalpy, $h(p)$, is a function of p only. Euler's equation now reduces to

$$-\partial_t\psi + h + \tfrac{1}{2}(\nabla\psi)^2 + \phi + \Phi = 0. \tag{10}$$

This is a version of Bernoulli's equation in the presence of external driving forces.

2.2. Fluctuations

Now linearize these equations of motion around some assumed background (ρ_0, p_0, ψ_0). Set $\rho = \rho_0 + \epsilon\rho_1 + \mathrm{O}(\epsilon^2)$, $p = p_0 + \epsilon p_1 + \mathrm{O}(\epsilon^2)$ and $\psi = \psi_0 + \epsilon\psi_1 + \mathrm{O}(\epsilon^2)$. The gravitational potential ϕ, and driving potential Φ, are taken to be fixed and external‖. Sound is *defined* to be these linearized fluctuations in the dynamical quantities. Please note that this is the *standard definition* of sound and more generally of acoustical disturbances. In principle, of course, one is really interested in solving the complete equations of motion for the fluid variables (ρ, p, ψ). In practice, it is both traditional and extremely useful to separate the exact motion, described by the exact variables, (ρ, p, ψ), into some average bulk motion, (ρ_0, p_0, ψ_0), plus low amplitude acoustic disturbances, $(\epsilon\rho_1, \epsilon p_1, \epsilon\psi_1)$. See, for example, [21–24].

Since this is a subtle issue that I have seen cause considerable confusion in the past, let me be even more explicit by asking the rhetorical question: *'How can we tell the difference between a wind gust and a sound wave?'* The answer is that the difference is to some extent a matter of convention—sufficiently low-frequency long-wavelength disturbances (wind gusts) are conventionally lumped in with the average bulk motion. Higher-frequency, shorter-wavelength disturbances are conventionally described as acoustic disturbances. If you wish to be hyper-technical, we can introduce a high-pass filter function to define the bulk motion by suitably averaging the exact fluid motion. There are no deep physical principles at stake here—merely an issue of convention.

† The irrotational condition is automatically satisfied for the superfluid component of physical superfluids. This point has been emphasized by Comer [25], who has also pointed out that in superfluids there will be multiple acoustic metrics (and multiple acoustic horizons) corresponding to first and second sound. Even for normal fluids, vorticity free flows are common, especially in situations of high symmetry.

‡ It is sufficient that the flow be vorticity free, $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, so that velocity potentials exist on an atlas of open patches—this enables us to handle vortex filaments, where the vorticity is concentrated into a thin vortex core, provided we do not attempt to probe the vortex core itself. ψ does not need to be globally defined.

§ An unstated assumption of this type is implicit, though not explicit, in the analysis of [1]. On the other hand, [5] explicitly makes the stronger assumption that the fluid is *isentropic*. (That is, the specific entropy density is taken to be constant throughout the fluid.) This is a stronger assumption than is actually required, and the weaker barotropic assumption used here is sufficient. In particular, the present derivation also applies to isothermal perturbations of an isothermal fluid.

‖ Fixed means that I do not allow the back-reaction to modify the gravitational or driving potentials. Fixed does not necessarily mean time independent, as I explicitly wish to allow the possibility of time-dependent external driving forces.

The place where we are making a specific physical assumption that restricts the validity of our analysis is in the requirement that the amplitude of the high-frequency short-wavelength disturbances be small. This is the assumption underlying the linearization programme, and this is why sufficiently high-amplitude sound waves must be treated by direct solution of the full equations of fluid dynamics.

Linearizing the continuity equation results in the pair of equations

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \boldsymbol{v}_0) = 0, \tag{11}$$

$$\partial_t \rho_1 + \nabla \cdot (\rho_1 \boldsymbol{v}_0 + \rho_0 \boldsymbol{v}_1) = 0. \tag{12}$$

Now, the barotropic condition implies

$$h(p) = h(p_0 + \epsilon p_1 + \mathrm{O}(\epsilon^2)) = h_0 + \epsilon \frac{p_1}{\rho_0} + \mathrm{O}(\epsilon^2). \tag{13}$$

Using this result in linearizing the Euler equation, we obtain

$$-\partial_t \psi_0 + h_0 + \tfrac{1}{2}(\nabla\psi_0)^2 + \phi + \Phi = 0. \tag{14}$$

$$-\partial_t \psi_1 + \frac{p_1}{\rho_0} - \boldsymbol{v}_0 \cdot \nabla \psi_1 = 0. \tag{15}$$

This last equation may be rearranged to yield

$$p_1 = \rho_0(\partial_t \psi_1 + \boldsymbol{v}_0 \cdot \nabla \psi_1). \tag{16}$$

Use the barotropic assumption to give the relation

$$\rho_1 = \frac{\partial \rho}{\partial p}\, p_1 = \frac{\partial \rho}{\partial p}\, \rho_0\, (\partial_t \psi_1 + \boldsymbol{v}_0 \cdot \nabla \psi_1). \tag{17}$$

Now substitute this consequence of the linearized Euler equation into the linearized equation of continuity. We finally obtain, up to an overall sign, the wave equation:

$$-\partial_t \left(\frac{\partial \rho}{\partial p}\, \rho_0\, (\partial_t \psi_1 + \boldsymbol{v}_0 \cdot \nabla \psi_1) \right) + \nabla \cdot \left(\rho_0\, \nabla \psi_1 - \frac{\partial \rho}{\partial p}\, \rho_0\, \boldsymbol{v}_0\, (\partial_t \psi_1 + \boldsymbol{v}_0 \cdot \nabla \psi_1) \right) = 0. \tag{18}$$

This wave equation describes the propagation of the linearized scalar potential ψ_1. Once ψ_1 is determined, (16) determines p_1, and (17) then determines ρ_1. Thus this wave equation completely determines the propagation of acoustic disturbances. The background fields p_0, ρ_0 and $\boldsymbol{v}_0 = -\nabla \psi_0$, which appear as time-dependent and position-dependent coefficients in this wave equation, are constrained to solve the equations of fluid motion for an externally-driven, barotropic, inviscid and irrotational flow. Apart from these constraints, they are otherwise permitted to have *arbitrary* temporal and spatial dependences.

Now, written in this form, the physical import of this wave equation is somewhat less than pellucid. To simplify things algebraically, observe that the local speed of sound is defined by

$$c^{-2} \equiv \frac{\partial \rho}{\partial p}. \tag{19}$$

Now construct the symmetric 4×4 matrix

$$f^{\mu\nu}(t, \boldsymbol{x}) \equiv \frac{\rho_0}{c^2} \left[\begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^i & (c^2 \delta^{ij} - v_0^i v_0^j) \end{array} \right]. \tag{20}$$

(Greek indices run from 0 to 3, while Latin indices run from 1 to 3.) Then, introducing $(3+1)$-dimensional spacetime coordinates $x^\mu \equiv (t; x^i)$, the above wave equation (18) is easily rewritten as

$$\partial_\mu (f^{\mu\nu}\, \partial_\nu \psi_1) = 0. \tag{21}$$

This remarkably compact formulation is completely equivalent to (18) and is a much more promising stepping-stone for further manipulations. The remaining steps are a straightforward application of the techniques of curved space $(3+1)$-dimensional Lorentzian geometry.

3. Lorentzian geometry

In any Lorentzian (that is, pseudo-Riemannian) manifold, the curved space scalar d'Alembertian is given in terms of the metric $g_{\mu\nu}(t, \boldsymbol{x})$ by (see, for example, [26–30])

$$\Delta\psi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \psi\right). \tag{22}$$

The inverse metric $g^{\mu\nu}(t, \boldsymbol{x})$ is pointwise the matrix inverse of $g_{\mu\nu}(t, \boldsymbol{x})$, while $g \equiv \det(g_{\mu\nu})$. Thus one can rewrite the physically-derived wave equation (18) in terms of the d'Alembertian, provided one identifies

$$\sqrt{-g}\, g^{\mu\nu} = f^{\mu\nu}. \tag{23}$$

This implies, on the one hand

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4\, g^{-1} = g \tag{24}$$

and, on the other hand, from the explicit expression (20), expanding the determinant in minors

$$\det(f^{\mu\nu}) = \left(\frac{\rho_0}{c^2}\right)^4 \left[(-1)(c^2 - v_0^2) - (-v_0)^2\right] \left[c^2\right] \left[c^2\right] = -\frac{\rho_0^4}{c^2}, \tag{25}$$

thus

$$g = -\frac{\rho_0^4}{c^2}; \qquad \sqrt{-g} = \frac{\rho_0^2}{c}. \tag{26}$$

We can therefore pick off the coefficients of the inverse acoustic metric†

$$g^{\mu\nu}(t, \boldsymbol{x}) \equiv \frac{1}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\delta^{ij} - v_0^i v_0^j) \end{bmatrix}. \tag{27}$$

We could now determine the metric itself simply by inverting this 4×4 matrix. On the other hand, it is even easier to recognize that one has in front of one an example of the Arnowitt–Deser–Misner split of a $(3+1)$-dimensional Lorentzian spacetime metric into space + time, more commonly used in discussing initial value data in Einstein's theory of gravity—general relativity (see, for example, [28] pp 505–8). The acoustic metric is

$$g_{\mu\nu} \equiv \frac{\rho_0}{c} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta_{ij} \end{bmatrix}. \tag{28}$$

† There is a minor typographic error, a missing factor of c, in [5].

Equivalently, the acoustic interval can be expressed as

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\rho_0}{c}\big[-c^2\,\mathrm{d}t^2 + (\mathrm{d}x^i - v_0^i\,\mathrm{d}t)\,\delta_{ij}\,(\mathrm{d}x^j - v_0^j\,\mathrm{d}t)\big]. \qquad (29)$$

A few brief comments should be made before proceeding:

- Observe that the signature of this metric is indeed $(-+++)$, as it should be to be regarded as Lorentzian.
- It should be emphasized that there are two distinct metrics relevant to the current discussion:
  - The *physical spacetime metric* is just the usual flat metric of Minkowski space
  $$\eta_{\mu\nu} \equiv (\mathrm{diag}[-c_{light}^2, 1, 1, 1])_{\mu\nu}. \qquad (30)$$
  (Here c_{light} denotes the speed of light.) The fluid particles couple only to the physical metric $\eta_{\mu\nu}$. In fact, the fluid motion is completely non-relativistic $\|v_0\| \ll c_{light}$.
  - Sound waves, on the other hand, do not 'see' the physical metric at all. Acoustic perturbations couple only to the *acoustic metric* $g_{\mu\nu}$.

  The geometry determined by the acoustic metric does, however, inherit some key properties from the existence of the underlying flat physical metric.
- For instance, the topology of the manifold does not depend on the particular metric considered. The acoustic geometry inherits the underlying topology of the physical metric $\mathbb{R}^4$ with possibly a few regions excised (due to imposed boundary conditions).
- The acoustic geometry automatically inherits the property of 'stable causality' [29, 30]. Note that
$$g^{\mu\nu}\,(\nabla_\mu t)\,(\nabla_\nu t) = -\frac{1}{\rho_0 c} < 0. \qquad (31)$$
This precludes some of the more entertaining causality-related pathologies that sometimes arise in general relativity. (For a discussion of causal pathologies see, for example, [31]).
- Other concepts that translate immediately are those of 'ergo-region', 'trapped surface', 'apparent horizon' and 'event horizon'. These notions will be developed fully in the following section.
- The properly normalized 4-velocity of the fluid is
$$V^\mu = \frac{(1; v_0^i)}{\sqrt{\rho_0\,c}}. \qquad (32)$$
This is related to the gradient of the natural time parameter by
$$\nabla_\mu t = (1, 0, 0, 0); \qquad \nabla^\mu t = -\frac{(1; v_0^i)}{\rho_0\,c} = -\frac{V^\mu}{\sqrt{\rho_0 c}}. \qquad (33)$$
Thus the integral curves of the fluid velocity field are orthogonal (in the Lorentzian metric) to the constant time surfaces. The acoustic proper time along the fluid flow lines (streamlines) is
$$\tau = \int \sqrt{\rho_0 c}\,\mathrm{d}t, \qquad (34)$$
and the integral curves are geodesics of the acoustic metric if and only if $\rho_0 c$ is position independent.

- Observe that in a completely general (3+1)-dimensional Lorentzian geometry the metric has six degrees of freedom per point in spacetime. (A 4×4 symmetric matrix has 10 independent components; then subtract four coordinate conditions). In contrast, the acoustic metric is more constrained. Being specified completely by the three scalars $\psi_0(t, \boldsymbol{x})$, $\rho_0(t, \boldsymbol{x})$ and $c(t, \boldsymbol{x})$, the acoustic metric has at most three degrees of freedom per point in spacetime. The equation of continuity actually reduces this to two degrees of freedom which can be taken to be $\psi_0(t, \boldsymbol{x})$ and $c(t, \boldsymbol{x})$.
- A point of notation: where the general relativist uses the word 'stationary' the fluid dynamicist uses the phrase 'steady flow'. The general-relativistic word 'static' translates to a rather messy constraint on the fluid flow (to be discussed more fully below).
- Finally, I should add that in Einstein gravity the spacetime metric is related to the distribution of matter by the nonlinear Einstein–Hilbert differential equations. In contrast, in the present context, the acoustic metric is related to the distribution of matter in a simple algebraic fashion.

4. Ergo-regions, trapped surfaces and acoustic horizons

Let us start with the notion of an ergo-region: consider integral curves of the vector $K^\mu \equiv (\partial/\partial t)^\mu = (1, 0, 0, 0)^\mu$. (If the flow is steady then this is the time translation Killing vector. Even if the flow is not steady the background Minkowski metric provides us with a natural definition of 'at rest'.) Then†

$$g_{\mu\nu}\,(\partial/\partial t)^\mu\,(\partial/\partial t)^\nu = g_{tt} = -[c^2 - v^2]. \tag{35}$$

This changes sign when $\|\boldsymbol{v}\| > c$. Thus any region of supersonic flow is an ergo-region. (And the boundary of the ergo-region may be deemed to be the ergosphere.) The analogue of this behaviour in general relativity is the ergosphere surrounding any spinning black hole—it is a region where space 'moves' with superluminal velocity relative to the fixed stars [28–30].

A trapped surface in acoustics is defined as follows: take any closed 2-surface. If the fluid velocity is everywhere inward-pointing and the normal component of the fluid velocity is everywhere greater than the local speed of sound, then no matter what direction a sound wave propagates, it will be swept inward by the fluid flow and be trapped inside the surface. The surface is then said to be outer-trapped. (For comparison with the usual situation in general relativity see [29, pp 319–23] or [30, pp 310–1].) Inner-trapped surfaces (anti-trapped surfaces) can be defined by demanding that the fluid flow is everywhere outward-pointing with supersonic normal component. It is only because of the fact that the background Minkowski metric provides a natural definition of 'at rest' that we can adopt such a simple definition. In ordinary general relativity we need to develop additional machinery, such as the notion of the 'expansion' of bundles of ingoing and outgoing null geodesics, before defining trapped surfaces—that the above definition is equivalent to the usual one follows from the discussion on pp 262–3 of Hawking and Ellis [29]. The acoustic trapped region is now defined as the region containing outer-trapped surfaces, and the acoustic (future) apparent horizon as the boundary of the trapped region. (We can also define anti-trapped regions and past apparent horizons, but these notions are of limited utility in general relativity.)

The event horizon (absolute horizon) is defined, as in general relativity, by demanding that it be the boundary of the region from which null geodesics (phonons) cannot escape.

† Henceforth, in the interests of notational simplicity, I shall drop the explicit subscript 0 on background field quantities unless there is risk of confusion.

This is actually the future event horizon. A past event horizon can be defined in terms of the boundary of the region that cannot be reached by incoming phonons—strictly speaking this requires us to define notions of past and future null infinities, but I will simply take all relevant incantations as understood. In particular, the event horizon is a null surface, the generators of which are null geodesics.

In all stationary geometries the apparent and event horizons coincide, and the distinction is immaterial. In time-dependent geometries the distinction is often important. When computing the surface gravity I shall restrict attention to stationary geometries (steady flow). In fluid flows of high symmetry, (spherical symmetry, plane symmetry) the ergosphere may coincide with the acoustic apparent horizon, or even the acoustic event horizon. This is the analogue of the result in general relativity that for static (as opposed to stationary) black holes the ergosphere and event horizon coincide.

5. Vortex geometries

As an example of a fluid flow where the distinction between ergosphere and acoustic event horizon is critical, consider the 'draining bathtub' fluid flow. I model a draining bathtub by a $(2+1)$-dimensional flow with a sink at the origin. The equation of continuity implies that for the radial component of the fluid velocity we must have

$$\rho\, v^{\hat{r}} \propto \frac{1}{r}. \tag{36}$$

In the tangential direction, the requirement that the flow be vorticity free (apart from a possible delta-function contribution at the vortex core) implies, via Stokes' theorem, that

$$v^{\hat{\theta}} \propto \frac{1}{r}. \tag{37}$$

On the other hand, assuming conservation of angular momentum (this places a constraint on the external body forces by assuming the absence of external torques) implies the slightly different constraint

$$\rho\, v^{\hat{\theta}} \propto \frac{1}{r}. \tag{38}$$

Combining these constraints, the background density ρ must be constant (position-independent) throughout the flow (which automatically implies that the background pressure p and speed of sound c are also constant throughout the fluid flow). Furthermore, for the background velocity potential we must then have

$$\psi(r,\theta) = A\ln(r/a) + B\theta. \tag{39}$$

Note that, as we have previously hinted, the velocity potential is not a true function (because it has a discontinuity when going through 2π radians). The velocity potential must be interpreted as being defined patch-wise on overlapping regions surrounding the vortex core at $r = 0$. The velocity of the fluid flow is

$$\boldsymbol{v} = \frac{(A\hat{r} + B\hat{\theta})}{r}. \tag{40}$$

Dropping a position-independent prefactor, the acoustic metric for a draining bathtub is explicitly given by

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\,\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\,\mathrm{d}t\right)^2, \tag{41}$$

or, equivalently,

$$\mathrm{d}s^2 = -\left(c^2 - \frac{A^2+B^2}{r^2}\right)\mathrm{d}t^2 - 2\frac{A}{r}\,\mathrm{d}r\,\mathrm{d}t - 2B\,\mathrm{d}\theta\,\mathrm{d}t + \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2. \tag{42}$$

A similar metric, restricted to $A = 0$ (no radial flow), and generalized to an anisotropic speed of sound, has been exhibited by Volovik [32], that metric being a model for the acoustic geometry surrounding physical vortices in superfluid ^{3}He. (For a survey of the many analogies and similarities between the physics of superfluid ^{3}He and the standard electroweak model see [33], this reference is also useful as background to understanding the Lorentzian geometric aspects of ^{3}He fluid flow.) Note that the metric given above is *not* identical to the metric of a spinning cosmic string, which would instead take the form [31]

$$\mathrm{d}s^2 = -c^2(\mathrm{d}t - \tilde{A}\,\mathrm{d}\theta)^2 + \mathrm{d}r^2 + (1-\tilde{B})r^2\,\mathrm{d}\theta^2. \tag{43}$$

In conformity with previous comments, the vortex fluid flow is seen to possess an acoustic metric that is stably causal and which does not involve closed timelike curves. (At large distances it is possible to *approximate* the vortex geometry by a spinning cosmic string [32], but this approximation becomes progressively worse as the core is approached.)

The ergosphere forms at

$$r_{ergosphere} = \frac{\sqrt{A^2+B^2}}{c}. \tag{44}$$

Note that the sign of A is irrelevant in defining the ergosphere and ergo-region: it does not matter if the vortex core is a source or a sink.

The acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, that is at

$$r_{horizon} = \frac{|A|}{c}. \tag{45}$$

The sign of A now makes a difference. For $A < 0$ we are dealing with a future acoustic horizon (acoustic black hole), while for $A > 0$ we are dealing with a past event horizon (acoustic white hole).

Though this construction has been phrased in $(2+1)$ dimensions we are of course free to add an extra dimension by going to $(3+1)$ dimensions and interpreting the result as a superposition of an ordinary vortex filament and a line source (or line sink).

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\,\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\,\mathrm{d}t\right)^2 + \mathrm{d}z^2. \tag{46}$$

6. Slab geometries

A popular model for the investigation of event horizons in the acoustic analogy is the one-dimensional slab geometry where the velocity is always along the z direction and the velocity profile depends only on z. The continuity equation then implies that $\rho(z)v(z)$ is a constant, and the acoustic metric becomes

$$\mathrm{d}s^2 \propto \frac{1}{v(z)c(z)}\left[-c(z)^2\,\mathrm{d}t^2 + \{\mathrm{d}z - v(z)\,\mathrm{d}t\}^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right], \tag{47}$$

that is

$$\mathrm{d}s^2 \propto \frac{1}{v(z)c(z)}\left[-\left\{c(z)^2 - v(z)^2\right\}\mathrm{d}t^2 - 2v(z)\,\mathrm{d}z\,\mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{48}$$

If we set $c = 1$ and ignore the conformal factor, we have the toy model acoustic geometry discussed by Unruh [5, p 2828, equation (8)] Jacobson [8, p 7085, equation (4)], Corley and Jacobson [9] and Corley [11]. (Since the conformal factor is regular at the event horizon, we know that the surface gravity and Hawking temperature are independent of this conformal factor [34].) In the general case it is important to realize that the flow can go supersonic for either of two reasons: the fluid could speed up, or the speed of sound could decrease. When it comes to calculating the 'surface gravity' both of these effects will have to be taken into account.

7. The Painlevé–Gullstrand line element

To see how close the acoustic metric can get to reproducing the Schwarzschild geometry, it is first useful to introduce one of the more exotic representations of the Schwarzschild geometry: the Painlevé–Gullstrand line element, which is simply an unusual choice of coordinates on the Schwarzschild spacetime†. In modern notation the Schwarzschild geometry in ingoing (+) and outgoing (−) Painlevé–Gullstrand coordinates may be written as:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\mathrm{d}r \pm \sqrt{\frac{2GM}{r}}\,\mathrm{d}t\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{49}$$

or, equivalently,

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 \pm \sqrt{\frac{2GM}{r}}\,\mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{50}$$

This representation of the Schwarzschild geometry is not particularly well known and has been rediscovered several times this century. See, for instance, Painlevé [15], Gullstrand [16], Lemaître [17], the related discussion by Israel [18], and more recently, the paper by Kraus and Wilczek [19]. The Painlevé–Gullstrand coordinates are related to the more usual Schwarzschild coordinates by

$$t_{PG} = t_S \pm \left[4M\tanh^{-1}\left(\sqrt{\frac{2GM}{r}}\right) - 2\sqrt{2GMr}\right], \tag{51}$$

or, equivalently,

$$\mathrm{d}t_{PG} = \mathrm{d}t_S \pm \frac{\sqrt{2GM/r}}{1 - 2GM/r}\,\mathrm{d}r. \tag{52}$$

With these explicit forms at hand, it becomes an easy exercise to check the equivalence between the Painlevé–Gullstrand line element and the more usual Schwarzschild form of the line element. It should be noted that the + sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the future singularity of the maximally extended Schwarzschild spacetime. It thus covers the future horizon and the black-hole singularity. On the other hand, the − sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the past singularity. It thus covers the past horizon and the white-hole singularity.

As emphasized by Kraus and Wilczek, the Painlevé–Gullstrand line element exhibits a number of features of pedagogical interest. In particular, the constant-time spatial slices are completely flat—the curvature of space is zero, and all the spacetime curvature of the

† The Painlevé–Gullstrand line element is often called the Lemaître line element.

Schwarzschild geometry has been pushed into the time–time and time–space components of the metric.

Given the Painlevé–Gullstrand line element, it might seem trivial to force the acoustic metric into this form: simply take ρ and c to be constants, and set $v = \sqrt{2GM/r}$. While this certainly forces the acoustic metric into the Painlevé–Gullstrand form, the problem with this is that this assignment is incompatible with the continuity equation $\nabla \cdot (\rho \boldsymbol{v}) \neq 0$ that was used in deriving the acoustic equations.

The best we can actually do is this: pick the speed of sound c to be a position-independent constant, which we normalize to unity ($c = 1$). Now set $v = \sqrt{2GM/r}$, and use the continuity equation $\nabla \cdot (\rho \boldsymbol{v}) = 0$ to deduce $\rho |\boldsymbol{v}| \propto 1/r^2$ so that $\rho \propto r^{-3/2}$. Since the speed of sound is taken to be constant, we can integrate the relation $c^2 = \mathrm{d}p/\mathrm{d}\rho$ to deduce that the equation of state must be $p = p_\infty + c^2 \rho$ and that the background pressure satisfies $p - p_\infty \propto c^2 r^{-3/2}$. Overall the acoustic metric is now

$$\mathrm{d}s^2 \propto r^{-3/2} \left[-\mathrm{d}t^2 + \left(\mathrm{d}r \pm \sqrt{\frac{2GM}{r}}\, \mathrm{d}t \right)^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 \right) \right]. \tag{53}$$

The net result is conformal to the Painlevé–Gullstrand form of the Schwarzschild geometry but not identical to it. For many purposes this is quite good enough: we have an event horizon, we can define surface gravity, we can analyse Hawking radiation. Since surface gravity and Hawking temperature are conformal invariants [34] this is sufficient for analysing basic features of the Hawking radiation process. The only way in which the conformal factor can influence the Hawking radiation is through back-scattering off the acoustic metric. (The phonons are minimally coupled scalars, not conformally coupled scalars so there will in general be effects on the frequency-dependent greybody factors.)

If we focus attention on the region near the event horizon, the conformal factor can simply be taken to be a constant, and we can ignore all these complications.

8. The canonical acoustic black hole

We can turn this argument around and ask: given a spherically symmetric flow of incompressible fluid, what is the acoustic metric? What is the corresponding line element in Schwarzschild coordinates? If we start by assuming incompressibility and spherical symmetry then, since ρ is position independent, the continuity equation implies $v \propto 1/r^2$. But if ρ is position independent then (because of the barotropic assumption) so is the pressure, and hence the speed of sound as well. So we can define a normalization constant r_0 and set

$$v = c\, \frac{r_0^2}{r^2}. \tag{54}$$

The acoustic metric is therefore, up to an irrelevant position-independent factor,

$$\mathrm{d}s^2 = -c^2\, \mathrm{d}t^2 + \left(\mathrm{d}r \pm c\, \frac{r_0^2}{r^2}\, \mathrm{d}t \right)^2 + r^2 (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2). \tag{55}$$

If we make the coordinate change

$$\mathrm{d}\tau = \mathrm{d}t \pm \frac{r_0^2/r^2}{c[1 - (r_0^4/r^4)]}\, \mathrm{d}r, \tag{56}$$

then

$$\mathrm{d}s^2 = -c^2 [1 - (r_0^4/r^4)]\, \mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1 - (r_0^4/r^4)} + r^2 (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2). \tag{57}$$

This is not any of the standard geometries typically considered in general relativity but is, in the sense described above, the canonical acoustic black hole.

It is very important to realize that a time-dependent version of this canonical acoustic metric is very easy to set up experimentally [35], since the time-dependent version of this canonical black hole metric is exactly the acoustic metric that is set up around a spherically-symmetric bubble with oscillating radius. For a bubble of radius R we have

$$r_0 = R\sqrt{\frac{\dot{R}}{c}}. \tag{58}$$

We should only use this canonical metric for the fluid region outside the bubble, and only in the approximation that the ambient fluid is incompressible (e.g. water). For the typically gaseous and necessarily compressible medium inside the bubble (e.g. air) we should use a separate acoustic metric. The two acoustic metrics need not be continuous across the bubble wall.

It is experimentally easy to generate (non-stationary) acoustic apparent horizons in this manner: in cavitating bubbles (typically air bubbles in water) it is experimentally easy to get the bubble wall moving at supersonic speeds (up to Mach 10 in extreme cases). Once the bubble wall is moving supersonically an acoustic apparent horizon forms. It first forms at the bubble wall itself but then will typically detach itself from the bubble wall (since the apparent horizon will continue to be the surface at which the fluid achieves Mach 1) as the bubble wall goes supersonic. Since the bubble must eventually stop its collapse and re-expand, there is strictly speaking no acoustic event horizon (no absolute horizon) in this experimental situation, merely a temporary apparent horizon. (The apparent horizon must, by construction, last less than one sound-crossing time for the collapsing bubble.)

To set up a geometry of this particular type with a true event horizon (or at the very least, an apparent horizon that lasts for many sound-crossing times) requires a rather different physical setup: a big tank of fluid with a long thin pipe leading to the centre. Then apply pressure to the tank till the outflow of fluid escaping through the pipe goes supersonic, being careful to maintain laminar flow and avoid turbulence. This would appear to be a technologically challenging project.

9. Hawking radiation and 'surface gravity'

Establishing the existence of acoustic Hawking radiation follows directly from the original Hawking argument [36, 37] once one realizes that the acoustic fluctuations effectively couple to the Lorentzian acoustic metric introduced above. The only subtlety arises in correctly identifying the 'surface gravity' of an acoustic black hole. Because of the definition of an event horizon in terms of phonons (null geodesics) that cannot escape the acoustic black hole, the event horizon is automatically a null surface, and the generators of the event horizon are automatically null geodesics.

In the case of acoustics there is one particular parametrization of these null geodesics that is 'most natural', which is the parametrization in terms of the Newtonian time coordinate of the underlying physical metric. This allows us to unambiguously define a 'surface gravity' even for non-stationary (time-dependent) acoustic event horizons, by calculating the extent to which this natural time parameter fails to be an affine parameter for the null generators of the horizon. (This part of the construction fails in general relativity where there is no universal natural time coordinate unless there is a timelike Killing vector—this is why extending the notion of surface gravity to non-stationary geometries in general relativity is so difficult.)

When it comes to explicitly calculating the surface gravity in terms of suitable gradients of the fluid flow, it is nevertheless very useful to limit attention to situations of steady flow (so that the acoustic metric is stationary). This has the added bonus that for stationary geometries the notion of 'acoustic surface gravity' in acoustics is unambiguously equivalent to the general relativity definition.

It is also useful to take cognizance of the fact that the situation simplifies considerably for static (as opposed to merely stationary) acoustic metrics.

9.1. Static acoustic geometries

To set up the appropriate framework, write the general stationary acoustic metric in the form

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-c^2\,\mathrm{d}t^2 + (\mathrm{d}\boldsymbol{x} - \boldsymbol{v}\,\mathrm{d}t)^2\right]. \tag{59}$$

The time translation Killing vector is simply $K^\mu = (1; \mathbf{0})$, with

$$K^2 \equiv g_{\mu\nu} K^\mu K^\nu \equiv -\|K\|^2 = -\frac{\rho}{c}[c^2 - v^2]. \tag{60}$$

The metric can also be written as

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-(c^2 - v^2)\,\mathrm{d}t^2 - 2\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + (\mathrm{d}\boldsymbol{x})^2\right]. \tag{61}$$

Now suppose that the vector $\boldsymbol{v}/(c^2 - v^2)$ is integrable, then we can define a new time coordinate by

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}}{c^2 - v^2}. \tag{62}$$

Substituting this back into the acoustic line element gives†

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-(c^2 - v^2)\,\mathrm{d}\tau^2 + \left\{\delta_{ij} + \frac{v^i v^j}{c^2 - v^2}\right\}\mathrm{d}x^i\,\mathrm{d}x^j\right]. \tag{63}$$

In this coordinate system the absence of the time–space cross-terms makes it manifest that the acoustic geometry is in fact static (the Killing vector is hypersurface orthogonal). The condition that an acoustic geometry be static, rather than merely stationary, is thus seen to be

$$\boldsymbol{\nabla}\times\left\{\frac{\boldsymbol{v}}{(c^2 - v^2)}\right\} = 0, \tag{64}$$

that is

$$\boldsymbol{v}\times\boldsymbol{\nabla}(c^2 - v^2) = 0. \tag{65}$$

This requires the fluid flow to be parallel to another vector that is not quite the acceleration but is closely related to it. (Note that, because of the vorticity-free assumption, $\frac{1}{2}\nabla v^2$ is just the 3-acceleration of the fluid; it is the occurrence of a possibly position-dependent speed of sound that complicates the above.)

Once we have a static geometry, we can of course directly apply all of the standard tricks [38] for calculating the surface gravity developed in general relativity. We set up a system of fiducial observers (FIDOS) by properly normalizing the time-translation Killing vector

$$V_{FIDO} \equiv \frac{K}{\|K\|} = \frac{K}{\sqrt{(\rho/c)[c^2 - v^2]}}. \tag{66}$$

† The corresponding formula in [5] is missing a factor of c and a bracket.

The 4-acceleration of the FIDOS is defined as $A_{FIDO} \equiv (V_{FIDO} \cdot \nabla) V_{FIDO}$, and using the fact that K is a Killing vector, it may be computed in the standard manner

$$A_{FIDO} = +\frac{1}{2} \frac{\nabla \|K\|^2}{\|K\|^2}, \tag{67}$$

that is

$$A_{FIDO} = \frac{1}{2} \left[\frac{\nabla (c^2 - v^2)}{(c^2 - v^2)} + \frac{\nabla (\rho/c)}{(\rho/c)} \right]. \tag{68}$$

The surface gravity is now defined by taking the norm $\|A_{FIDO}\|$, multiplying by the lapse function, $\|K\| = \sqrt{(\rho/c)[c^2 - v^2]}$, and taking the limit as one approaches the horizon: $|v| \to c$, remember this is the static case. The net result is

$$\|A_{FIDO}\| \, \|K\| = \tfrac{1}{2} \boldsymbol{v} \cdot \nabla (c^2 - v^2) + \mathrm{O}(c^2 - v^2), \tag{69}$$

so that the surface gravity is given in terms of a normal derivative by†

$$g_H = \frac{1}{2} \frac{\partial (c^2 - v^2)}{\partial n} = c \, \frac{\partial (c - v)}{\partial n}. \tag{70}$$

This is not quite Unruh's result [1, 5], since he implicitly took the speed of sound to be a position-independent constant. The fact that ρ drops out of the final result for the surface gravity can be justified by appeal to the known conformal invariance of the surface gravity [34]. Though derived in a totally different manner, this result is also compatible with the expression for 'surface-gravity' obtained in the solid-state black holes of Reznik [14], wherein a position dependent (and singular) refractive index plays a role analogous to the acoustic metric. As a further consistency check, one can go to the spherically symmetric case and check that this reproduces the results for 'dirty black holes' enunciated in [20].

Since this is a static geometry, the relationship between the Hawking temperature and surface gravity may be verified in the usual fast-track manner—using the Wick rotation trick to analytically continue to Euclidean space [39]. If you do not like Euclidean signature techniques (which are in any case only applicable to equilibrium situations) you should go back to the original Hawking derivations [36, 37].

One final comment to wrap up this section: the coordinate transform we used to put the acoustic metric into the explicitly static form is perfectly good mathematics, and from the general relativity point of view is even a simplification. However, from the point of view of the underlying Newtonian physics of the fluid, this is a rather bizarre way of deliberately de-synchronizing your clocks to take a perfectly reasonable region—the boundary of the region of supersonic flow—and push it out to the infinite future. From the fluid dynamics point of view, this coordinate transformation is correct but perverse, and it is easier to keep a good grasp on the physics by staying with the original Newtonian time coordinate.

9.2. Stationary but non-static acoustic geometries

If the fluid flow does not satisfy the integrability condition which allows us to introduce an explicitly static coordinate system, then defining the surface gravity is a little trickier. The situation is somewhat worse than for general relativity since in the acoustic case we have no reason to believe that anything like the zeroth law of black-hole mechanics holds [40], nor do we have any reason to believe that stationary event horizons have to be Killing horizons.

† Because of the background Minkowski metric there can be no possible confusion as to the definition of this normal derivative.

Recall that the zeroth law of black-hole mechanics (constancy of the surface gravity over the event horizon) is proved in general relativity by appealing to the Einstein equations and imposing suitable energy conditions. In the acoustic paradigm we have no analogue for the Einstein equations and no particular reason to suspect the existence of anything like a zeroth law. Sufficiently convoluted supersonic flows would seem to be able to set up almost any pattern of surface gravity one wants.

Similarly, in general relativity the fact that stationary but non-static black holes possess Killing horizons is related to the axisymmetry that is deduced from the fact that non-axisymmetric black holes are expected to lose energy via gravitational radiation, and so dynamically relax to an axisymmetric configuration—in the fluid dynamic models discussed here I have explicitly allowed for external driving forces and explicitly excluded back-reaction effects, therefore there is no particular reason to expect acoustic black holes to dynamically relax to axisymmetry. In particular, this means that even for stationary acoustic geometries there is no particular reason to expect the acoustic event horizon in general to be a Killing horizon.

So what does survive of our usual general relativistic notions for acoustic event horizons in stationary but non-static geometries? Recall that, by construction, the acoustic apparent horizon is in general defined to be a 2-surface for which the normal component of the fluid velocity is everywhere equal to the local speed of sound, whereas the acoustic event horizon is characterized by the boundary of those null geodesics (phonons) that do not escape to infinity. In the stationary case these notions coincide, and it is still true that the horizon is a null surface, and that the horizon can be ruled by an appropriate set of null curves. Suppose we have somehow isolated the location of the acoustic horizon, then in the vicinity of the horizon we can split up the fluid flow into normal and tangential components

$$\boldsymbol{v} = \boldsymbol{v}_\perp + \boldsymbol{v}_\parallel; \qquad \text{where} \qquad \boldsymbol{v}_\perp = v_\perp \hat{n}. \tag{71}$$

Here (and for the rest of this section) it is essential that we use the natural Newtonian time coordinate inherited from the background Newtonian physics of the fluid. In addition $\hat{n}$ is a unit vector field that at the horizon is perpendicular to it, and away from the horizon is some suitable smooth extension. (For example, take the geodesic distance to the horizon and consider its gradient.) We only need this decomposition to hold in some open set encompassing the horizon and do not need to have a global decomposition of this type available. Furthermore, by definition we know that $v_\perp = c$ at the horizon. Now consider the vector field

$$L^\mu = (1; v_\parallel^i). \tag{72}$$

Since the spatial components of this vector field are by definition tangent to a constant time slice through the horizon, the integral curves of this vector field will be generators for the horizon. Furthermore, the norm of this vector (in the acoustic metric) is

$$\|L\|^2 = -\frac{\rho}{c}\left[-(c^2 - v^2) - 2\boldsymbol{v}_\parallel \cdot \boldsymbol{v} + \boldsymbol{v}_\parallel \cdot \boldsymbol{v}_\parallel\right] \propto (c^2 - v_\perp^2). \tag{73}$$

In particular, on the acoustic horizon L^μ defines a null vector field, the integral curves of which are generators for the acoustic horizon. I shall now verify that these generators are geodesics, though the vector field L is not normalized with an affine parameter, and in this way shall calculate the surface gravity. (For clarity, I will drop the conformal factor because it is known that it will not affect the surface gravity [34].)

Consider the quantity $(L \cdot \nabla)L$ and calculate

$$L^\alpha \nabla_\alpha L^\mu = L^\alpha(\nabla_\alpha L_\beta - \nabla_\beta L_\alpha) g^{\beta\mu} + \tfrac{1}{2}\nabla_\beta(L^2) g^{\beta\mu}. \tag{74}$$

To calculate the first term note that

$$L_\mu = \frac{\rho}{c}\,(-[c^2 - v_\perp^2];\, \boldsymbol{v}_\perp). \tag{75}$$

Thus

$$L_{[\alpha,\beta]} = -\begin{bmatrix} 0 & \vdots & -\nabla_i\left[(\rho/c)(c^2-v_\perp^2)\right] \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ +\nabla_j\left[(\rho/c)(c^2-v_\perp^2)\right] & \vdots & \left((\rho/c)\,v^\perp\right)_{[i,j]} \end{bmatrix} \tag{76}$$

and so

$$L^\alpha L_{[\beta,\alpha]} = \left(v_\parallel\cdot\nabla\left[\frac{\rho}{c}(c^2-v_\perp^2)\right];\, \nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] + v_\parallel^i\left(\frac{\rho}{c}\,v^\perp\right)_{[j,i]}\right). \tag{77}$$

On the horizon, where $c = v_\perp$, this simplifies tremendously to

$$(L^\alpha L_{[\beta,\alpha]})|_{horizon} = -\frac{\rho}{c}\left(0;\, \nabla_j(c^2-v_\perp^2)\right). \tag{78}$$

Similarly, for the second term we have

$$\nabla_\beta(L^2) = \left(0;\, \nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right]\right). \tag{79}$$

On the horizon this again simplifies to

$$\nabla_\beta(L^2)|_{horizon} = +\frac{\rho}{c}\left(0;\, \nabla_j(c^2-v_\perp^2)\right). \tag{80}$$

There is partial cancellation between the two terms, and so

$$L^\alpha\nabla_\alpha L^\mu = +\frac{1}{2c^2}\left(v^j\nabla_j[(c^2-v_\perp^2)];\, (c^2\delta^{ij}-v^iv^j)\nabla_j[(c^2-v_\perp^2)]\right). \tag{81}$$

But, as we have already seen, at the horizon the gradient term is purely normal, thus

$$L^\alpha\nabla_\alpha L^\mu = +\frac{1}{2c}\frac{\partial(c^2-v_\perp^2)}{\partial n}\,(1;\, v_\parallel^i). \tag{82}$$

Comparing this with the standard definition of surface gravity [30]†

$$L^\alpha\nabla_\alpha L^\mu = +\frac{g_H}{c}L^\mu, \tag{83}$$

we finally have

$$g_H = \frac{1}{2}\frac{\partial(c^2-v_\perp^2)}{\partial n} = c\,\frac{\partial(c-v_\perp)}{\partial n}. \tag{84}$$

This is in agreement with the previous calculation for static acoustic black holes and, insofar as there is overlap, is also consistent with the results of Unruh [1, 5], Reznik [14], and the results for 'dirty black holes' [20]. From the construction it is clear that the surface gravity is a measure of the extent to which the Newtonian time parameter inherited from the underlying fluid dynamics fails to be an affine parameter for the null geodesics on the horizon.

† There is an issue of normalization here. On the one hand, we want to be as close as possible to general relativistic conventions. On the other hand, we would like the surface gravity to really have the dimensions of an acceleration. The convention adopted here is the best compromise I have come up with.

10. Geometric acoustics

Up to now, we have been developing general machinery to force acoustics into Lorentzian form. This can be justified either with a view to using fluid mechanics to teach us more about general relativity, or to using the techniques of Lorentzian geometry to teach us more about fluid mechanics.

For example, given the machinery developed so far, taking the short wavelength/high-frequency limit to obtain geometrical acoustics is now easy. Sound rays (phonons) follow the *null geodesics* of the acoustic metric. Compare this to general relativity where in the geometrical optics approximation light rays (photons) follow *null geodesics* of the physical spacetime metric. Since null geodesics are insensitive to any overall conformal factor in the metric [28–30], one might as well simplify life by considering a modified conformally related metric

$$h_{\mu\nu} \equiv \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^i & \vdots & \delta^{ij} \end{array} \right]. \tag{85}$$

This immediately implies that, in the geometric acoustics limit, sound propagation is insensitive to the density of the fluid. In this limit, acoustic propagation depends only on the local speed of sound and the velocity of the fluid. It is only for specifically wave related properties that the density of the medium becomes important.

We can rephrase this in a language more familiar to the acoustics community by invoking the eikonal approximation. Express the linearized velocity potential, ψ_1, in terms of an amplitude a and phase φ by $\psi_1 \sim a\mathrm{e}^{\mathrm{i}\varphi}$. Then, neglecting variations in the amplitude a, the wave equation reduces to the *eikonal equation*

$$h^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi = 0. \tag{86}$$

This eikonal equation is blatantly insensitive to any overall multiplicative prefactor (conformal factor).

As a sanity check on the formalism, it is useful to re-derive some standard results. For example, let the null geodesic be parametrized by $x^\mu(t) \equiv (t; \boldsymbol{x}(t))$. Then the null condition implies

$$h_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0 \iff -(c^2 - v_0^2) - 2v_0^i \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^i}{\mathrm{d}t} = 0$$

$$\iff \left\| \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \boldsymbol{v}_0 \right\| = c. \tag{87}$$

Here the norm is taken in the flat physical metric. This has the obvious interpretation that the ray travels at the speed of sound c relative to the moving medium.

Furthermore, if the geometry is stationary one can do slightly better. Let $x^\mu(s) \equiv (t(s); \boldsymbol{x}(s))$ be some null path from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$, parametrized in terms of physical arc length (i.e. $\|\mathrm{d}\boldsymbol{x}/\mathrm{d}s\| \equiv 1$). Then the tangent vector to the path is

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \left(\frac{\mathrm{d}t}{\mathrm{d}s}; \frac{\mathrm{d}x^i}{\mathrm{d}s} \right). \tag{88}$$

The condition for the path to be null (though not yet necessarily a null geodesic) is

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0. \tag{89}$$

Using the explicit algebraic form for the metric, this can be expanded to show

$$-(c^2 - v_0^2)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - 2v_0^i\left(\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) + 1 = 0. \tag{90}$$

Solving this quadratic gives

$$\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) = \frac{-v_0^i\left(\mathrm{d}x^i/\mathrm{d}s\right) + \sqrt{c^2 - v_0^2 + \left(v_0^i\,\mathrm{d}x^i/\mathrm{d}s\right)^2}}{c^2 - v_0^2}. \tag{91}$$

Therefore, the total time taken to traverse the path is

$$\begin{aligned} T[\gamma] &= \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} (\mathrm{d}t/\mathrm{d}s)\,\mathrm{d}s \\ &= \int_\gamma \frac{1}{c^2 - v_0^2}\left\{\sqrt{(c^2 - v_0^2)\,\mathrm{d}s^2 + (v_0^i\,\mathrm{d}x^i)^2} - v_0^i\,\mathrm{d}x^i\right\}. \end{aligned} \tag{92}$$

If we now recall that extremizing the total time taken is Fermat's principle for sound rays, we see that we have checked the formalism for stationary geometries (steady flow) by reproducing the discussion on p 262 of Landau and Lifshitz [22].

As a second example of the insights arising from the Lorentzian point of view, consider the 'reciprocity theorem'. Suppose a pulse of sound is emitted at time t_1 at position $\boldsymbol{x}_1$. The disturbance propagates according to the inhomogeneous differential equation

$$\Delta\psi = \frac{1}{\sqrt{-g}}\delta^4(x^\mu - x_1^\mu) = \frac{c}{\rho^2}\,\delta(t - t_1)\,\delta^3(\boldsymbol{x} - \boldsymbol{x}_1). \tag{93}$$

The solution to this is the retarded scalar Green function

$$\psi(x)|_{source\ at\ x_1} = G_R(x, x_1). \tag{94}$$

The Green function has well known symmetry properties that are completely unaffected by any time dependence in the underlying acoustic metric. We may, in the usual manner, construct advanced and retarded Green functions that vanish outside the past and future sound cones, respectively. Then

$$G_R(x_2, x_1) = G_A(x_1, x_2). \tag{95}$$

So that the reciprocity theorem *for the velocity potential* is valid in absolute generality

$$\psi(x_2)|_{source\ at\ x_1} = \psi(x_1)|^{time\ reversed}_{source\ at\ x_2}. \tag{96}$$

To get a reciprocity theorem for *pressure* one has to recall that

$$p_1 = \rho_0(\partial_t\psi_1 + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\psi_1). \tag{97}$$

Then, by restricting to the case of a fluid at rest ($\boldsymbol{v}_0 = 0$, $\partial_t\rho_0 = 0$, $\partial_t p_0 = 0$), using the time translation invariance of the Green functions and the time reversal property $G_A \rightleftharpoons G_R$, one has

$$\left[\frac{p_1}{\rho_0}\right](\boldsymbol{x}_2, t_2 - t_1)|_{source\ at\ x_1} = \left[\frac{p_1}{\rho_0}\right](\boldsymbol{x}_1, t_2 - t_1)|_{source\ at\ x_2}. \tag{98}$$

This result is still much more general than the usual reciprocity theorem.

11. Limitations

The derivation of the wave equation involved two key assumptions—that the flow is irrotational flow and the fluid is barotropic.

The d'Alembertian equation of motion for acoustic disturbances, though derived only under the assumption of irrotational flow†, continues to make perfectly good sense in its own right if the background velocity field $\boldsymbol{v}_0$ is given some vorticity. This leads one to hope that it *might* be possible to find a suitable generalization of the present derivation that will work for flows with non-zero vorticity. In this regard, note that if the vorticity is everywhere confined to thin vortex filaments, the present derivation already works everywhere outside the vortex filaments themselves.

The technical problem with flows with non-zero vorticity is that the vorticity in the background flow couples to the perturbations and generates vorticity in the fluctuations. Then sound waves can no longer be represented simply by a scalar potential and a much more complicated mathematical structure results. (Phonons are no longer simply minimally coupled scalar fields and the appropriate generalization is sufficiently unpleasant as to be intractable.)

The restriction to a barotropic fluid (ρ a function of p only) is in fact also related to issues of vorticity. Examples of barotropic fluids are:

- Isothermal fluids subject to isothermal perturbations.
- Fluids in convective equilibrium subject to adiabatic perturbations.

See for example [21], section 311, pp 547–8, and section 313 pp 554–6. Failure of the barotropic condition implies that the perturbations cannot be vorticity free and thus requires more sophisticated analysis.

If the fluid is in addition inviscid, then the analysis of this paper implies a hidden Lorentz invariance in the acoustic equations. This hidden Lorentz invariance is more than just a formal quirk: if one has 'atoms' held together by phonons (Cooper pairs?), then these atoms, and complex systems built up out of such atoms, will see (hear) an acoustic special relativity that is as real to them as Einstein's special relativity is to us. Furthermore, these systems would with additional observation detect (hear) an acoustic general relativity—but instead of the Einstein–Hilbert equations of our general relativity they would experience an acoustic general relativity governed by the hydrodynamic equations.

If the fluid has non-zero viscosity then there will be violations of this acoustic Lorentz symmetry. These violations are momentum dependent and, as I shall discuss in the next section, they are small at low momentum.

12. Viscosity: breaking the Lorentz symmetry

After this long build-up, emphasizing the hidden Lorentzian geometry hiding in (inviscid vorticity-free barotropic) fluid dynamical equations, I will now show how to explicitly break the Lorentz symmetry. From the atomic perspective underlying continuum fluid mechanics the eventual breakdown of the Lorentz symmetry governing the notion of the phonons is no great surprise: eventually, once the wavelength of the phonons is less than the mean interatomic spacing in the fluid, we should certainly expect modifications to the phonon dispersion relation [1, 3]. Specific *ad hoc* mutilations of the dispersion relation have been considered by Jacobson [3], Unruh [5], Corley and Jacobson [9] and Corley [11]. I shall

† But remember that irrotational flow is automatic for superfluids [25], and is natural in situations of high symmetry.

now show that a similar, but not identical, breakdown of acoustic Lorentz invariance can be deduced directly from the continuum equations merely by adding the effects of viscosity.

Of course, the fundamental equations of fluid dynamics, the equation of continuity (5) and Euler's equation (6) are unaltered. What changes is the expression for the driving force in Euler's equation so that (7) becomes [21, section 328, pp 576–7]

$$\boldsymbol{F} = -\boldsymbol{\nabla} p - \rho \boldsymbol{\nabla}\phi - \rho \boldsymbol{\nabla}\Phi + \rho\,\nu \left(\nabla^2 \boldsymbol{v} + \tfrac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\right). \tag{99}$$

Here ν denotes kinematic viscosity. I again take the flow to be *vorticity free*, and again choose the fluid to be *barotropic*. Repeating the steps that led to (10) now show that Euler's equation reduces to

$$-\partial_t \psi + h + \tfrac{1}{2}(\nabla\psi)^2 + \phi + \Phi + \tfrac{4}{3}\nu\,\nabla^2\psi = 0. \tag{100}$$

This again is a well known equation, simply being Burgers' equation subject to external driving forces [41]. (In obtaining this equation it is necessary to assume that the kinematic viscosity ν is position independent. In addition it is common practice, though not universal, to absorb the $\frac{4}{3}$ into a modified definition of kinematic viscosity.)

Linearization proceeds as previously. For the continuity equation there are no changes, while linearizing the Euler equation (Burgers' equation) yields

$$-\partial_t \psi_0 + h_0 + \tfrac{1}{2}(\nabla\psi_0)^2 + \phi + \Phi + \tfrac{4}{3}\nu\,\nabla^2\psi_0 = 0, \tag{101}$$

$$-\partial_t \psi_1 + \frac{p_1}{\rho_0} - \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\psi_1 + \tfrac{4}{3}\nu\,\nabla^2\psi_1 = 0. \tag{102}$$

Rearranging

$$p_1 = \rho_0\left(\partial_t\psi_1 + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\psi_1 - \tfrac{4}{3}\nu\nabla^2\psi_1\right). \tag{103}$$

As before, we substitute this linearized Euler equation into the linearized continuity equation, to obtain the physical wave equation:

$$-\partial_t\left(\frac{\partial\rho}{\partial p}\,\rho_0\,\left(\partial_t\psi_1 + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\psi_1 - \tfrac{4}{3}\nu\,\nabla^2\psi_1\right)\right) + \boldsymbol{\nabla}\cdot\left(\rho_0\,\boldsymbol{\nabla}\psi_1 - \frac{\partial\rho}{\partial p}\,\rho_0\,\boldsymbol{v}_0\,\left(\partial_t\psi_1 + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\psi_1 - \tfrac{4}{3}\nu\,\nabla^2\psi_1\right)\right) = 0. \tag{104}$$

Using the same matrix $f^{\mu\nu}$ defined previously, the above wave equation is easily rewritten as†

$$\partial_\mu(f^{\mu\nu}\,\partial_\nu\psi_1) = -\tfrac{4}{3}\rho_0\,\nu\,\left(\frac{\partial}{\partial t} + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\right)\left[c^{-2}\nabla^2\psi_1\right]. \tag{105}$$

In terms of the d'Alembertian associated with the acoustic metric this reads

$$\Delta\psi_1 = -\frac{4}{3}\frac{\nu c}{\rho_0}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\right)\left[c^{-2}\nabla^2\psi_1\right]. \tag{106}$$

The convective derivative appearing here may easily be converted into four-dimensional form by utilizing the acoustic 4-velocity for the fluid. Recall that

$$V^\mu = \frac{(1;\,\boldsymbol{v})}{\sqrt{\rho_0 c}}. \tag{107}$$

† I have used the continuity equation for the background fluid flow to pull the factor ρ_0 outside the convective derivative.

It is easy to see that this is a timelike unit vector in the acoustic metric, so that

$$\Delta\psi_1 = -\frac{4}{3}\frac{\nu c^2}{\sqrt{\rho_0 c}}\left(V^\mu \nabla_\mu\right)\left[c^{-2}\nabla^2\psi_1\right]. \tag{108}$$

The $\nabla^2\psi_1$ term explicitly couples only to the flat spatial metric and can be written in terms of the acoustic metric by noting that

$$g^{\mu\nu} = -V^\mu V^\nu + \frac{c}{\rho}\ {}^{(3)}g^{\mu\nu}_{space}. \tag{109}$$

It is the explicit appearance of the fluid 4-velocity in the above expressions that justifies my claim that viscosity breaks the acoustic Lorentz invariance.

Sanity check 1. If the background fluid flow is at rest and homogeneous ($\boldsymbol{v}_0 = 0$, and with ρ_0 and c independent of position) then this viscous wave equation reduces to

$$\partial_t^2\psi_1 = c^2\nabla^2\psi_1 + \tfrac{4}{3}\nu\,\partial_t\nabla^2\psi_1. \tag{110}$$

This equation may be found, for instance, in section 359 pp 646–8 of Lamb [21].

Sanity check 2. Take the eikonal approximation in the form

$$\psi_1 = a(x)\exp(-\mathrm{i}[\omega t - \boldsymbol{k}\cdot\boldsymbol{x}]), \tag{111}$$

with $a(x)$ a slowly varying function of position. Furthermore, agree to ignore derivatives of the metric. Then the viscous wave equation in the eikonal approximation reduces to

$$-(\omega - \boldsymbol{v}\cdot\boldsymbol{k})^2 + c^2k^2 - \mathrm{i}\nu\,\tfrac{4}{3}(\omega - \boldsymbol{v}\cdot\boldsymbol{k})k^2 = 0. \tag{112}$$

This lets us write down a dispersion relation for sound waves

$$\omega = \boldsymbol{v}\cdot\boldsymbol{k} \pm \sqrt{c^2k^2 - \left(\frac{2\nu k^2}{3}\right)^2} - \mathrm{i}\frac{2\nu k^2}{3}. \tag{113}$$

The first term simply arises from the bulk motion of the fluid. The second term specifically introduces dispersion due to viscosity, while the third term is specifically dissipative. The *ad hoc* models introduced in Jacobson [3], Unruh [5], Corley and Jacobson [9] and Corley [11] are exactly recovered by ignoring the dissipation due to viscosity but retaining the dispersion due to viscosity.

Note that the violation of Lorentz invariance is suppressed at low momentum. This is in agreement with general arguments of Nielsen *et al* [42–44], though it should be borne in mind that Nielsen *et al* were dealing with interacting quantum field theories and the context here is, if not purely classical, at worst one of free phonons propagating on a fixed classical background. (An alternative model for the breakdown of Lorentz invariance has been discussed by Everett [45, 46].) The violations of Lorentz symmetry become significant once

$$k \approx k_0 \equiv \frac{c}{\nu}. \tag{114}$$

But from the atomic theory of (normal) fluids

$$\nu \approx \frac{\text{mean free path}^2}{\text{mean free time}} \approx c \times \text{mean free path}. \tag{115}$$

This gives the very reasonable result

$$k_0 \approx \frac{1}{\text{mean free path}}, \tag{116}$$

verifying that the breakdown of acoustic Lorentz invariance is explicitly linked to the atomic nature of matter.

13. Precursors

It is perhaps surprising that anything new can be said about so venerable a subject as fluid dynamics. Certainly there are precursors to the discussion of this paper in the fluid dynamics literature. For instance, take the background to be static, so that $\boldsymbol{v}_0 = 0$, while $\partial_t \rho_0 = 0 = \partial_t p_0$, though p_0 and hence c are permitted to retain arbitrary spatial dependences. Then the wave equation derived in this paper reduces to

$$\partial_t^2 \psi = c^2 \frac{1}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} \psi). \tag{117}$$

This equation is in fact well known. It is equivalent, for instance, to equation (13) of section 313 of Lamb's classic *Hydrodynamics* [21]. See also equation (1.4.5) of the recent book by DeSanto [47]. The superficially similar wave equations discussed by Landau and Lifshitz [22] (see section 74, equation (74.1)), and by Skudrzyk [24] (see p 282), utilize somewhat different physical assumptions concerning the behaviour of the fluid.

In a somewhat different vein, the modern study of classical continuum mechanics has greatly benefited from the use of three-dimensional Riemannian geometry to describe the physics of the spatial configurations of elastic media and other continua [48–50]. Analyses of this type have traditionally treated space and time on quite separate footings.

The most direct precursor of the results derived in this paper are due to Unruh [1] and Jacobson [3], and in the body of work prompted by those papers [4–14].

14. Summary and discussion

Acoustic waves in an inviscid fluid can, under the assumptions of irrotational barotropic flow, be described by an equation of motion involving the scalar d'Alembertian of a suitable Lorentzian geometry. For inhomogeneous flows this Lorentzian geometry will exhibit non-zero Riemann curvature.

Traditionally, Lorentzian geometries have been of interest to physics only within the confines of Einstein's theory of gravitation. The results of this paper provide the general relativity community with a very down to earth physical model for certain classes of Lorentzian geometry. This is of interest both pedagogically and because it extends the usefulness of Lorentzian differential geometry beyond the confines of Einstein gravity.

Particularly intriguing is the fact that while the underlying physics of fluid dynamics is completely non-relativistic, Newtonian, and sharply separates the notions of space and time, one nevertheless sees that the acoustic fluctuations couple to a full-fledged Lorentzian *spacetime*.

As discussed by Unruh [1] (and in the subsequent papers [3–14]), an acoustic event horizon will emit Hawking radiation in the form of a thermal bath of phonons at a temperature

$$kT_H = \frac{\hbar\, g_H}{2\pi c}. \tag{118}$$

(Yes, this really is the speed of sound in the above equation, and g_H is really normalized to have the dimensions of a physical acceleration.) Using the numerical expression

$$T_H = (1.2 \times 10^{-9}\,\mathrm{K\,m}) \left[\frac{c}{1000\,\mathrm{m\,s^{-1}}} \right] \left[\frac{1}{c} \frac{\partial (c - v_\perp)}{\partial n} \right], \tag{119}$$

it is clear that experimental verification of this acoustic Hawking effect will be rather difficult. (Though, as Unruh has pointed out [1], this is certainly technologically easier than building (general relativistic) micro-black holes in the laboratory.)

A particularly important side effect of this entire analysis is that it forces us to re-examine all of black-hole physics to cleanly separate what is intrinsic to general relativity from what is generic to Lorentzian geometries. The acoustic analogue for black-hole physics accurately reflects half of general relativity—the kinematics due to the fact that general relativity takes place in a Lorentzian spacetime. The aspect of general relativity that does not carry over to the acoustic model is the dynamics—the Einstein equations. Thus the acoustic model provides a very concrete and specific model for separating the kinematic aspects of general relativity from the dynamic aspects.

In particular, perhaps the most important lesson to be learned is this: Hawking radiation from event horizons is a purely kinematic effect that occurs in any Lorentzian geometry with an event horizon and is independent of any dynamical equations imposed on the Lorentzian geometry. On the other hand, the classical laws of black-hole mechanics [40] are intrinsically results of the dynamical equations (Einstein equations) that have no analogue in the acoustic model. Thus Hawking radiation persists even in the absence of the laws of black-hole mechanics and, in particular, the existence or otherwise of Hawking radiation is now seen to be divorced from the issue of the existence or otherwise of the *laws of black-hole thermodynamics*. Hawking radiation is a purely kinematical effect that will be there regardless of whether or not it makes any sense to assign an entropy to the event horizon—and attempts at deriving black-hole entropy from the Hawking radiation phenomenon are thereby seen to require specific dynamical assumptions about the (at least approximate) applicability of the Einstein equations.

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