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Questions of Canonicity

Abstract. We review progress on the long standing question of whether every *canonical* modal logic must be characterized by an *elementary* class of Kripke frames, or equivalently, whether every canonical variety of Boolean algebras with operators must be generating by the complex algebras of an elementary class of relational structures. We verify that this does hold for certain families $\mathbf{RA}_n$, $\mathsf{S}\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ and $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ of varieties related to relation algebras and cylindric algebras. Canonical extensions of structures are shown to be free objects in certain categories of structures with topology, and to be associated with a monad on the category of sets that generalizes Manes' Theorem to relational structures.

Keywords: Boolean algebra with operators, canonical extension, elementary class, variety, relation algebra, free object.

Overview

This paper is about the concepts of the *canonical structure* of a Boolean algebra with operators, the *canonical extension* of such an algebra and of a relational structure, and the closely related notion of the *canonical frame* of a propositional modal logic. These ideas originated in work of Alfred Tarski and Bjarni Jónsson from the late 1940's.* They have played a significant role in modal and algebraic logic in the subsequent decades, and continue to be of importance. Our aim is to review their place in the study both of modal logics and of varieties of algebras, to present some new results, and to suggest questions for further research.

In the first Section we discuss the concepts of "canonicity" and "elementarity" for logics and varieties, and survey the author's generalizations and strengthenings of the 1973 result of Kit Fine that elementary modal logics are canonical. We also review progress on the fundamental converse question, still unanswered, of whether canonicity implies elementarity. This concerns the extent to which notions of canonical object account for the correspondence manifest in numerous cases between *equationally* definable classes (varieties) of algebras that are closed under canonical extensions, and logics characterized by classes of frames that are *elementary*, i.e. first-order definable.

Section 2 studies three infinite families of canonically-closed varieties of Boolean algebras with operators—called $\mathbf{RA}_n$, $\mathsf{S}\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ and $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$—

*See [29] for the historical background.

Studia Logica **0**: 1–36, 2003.

which are related to relation algebras and cylindric algebras, and which have been the focus of recent attention [24]. We show that each of these varieties is generated by the class of complex algebras of some elementary class of structures.

Section 3 uses topology and category theory to obtain a characterization of the operation of forming the canonical extension of a relational structure. These extensions are free objects in certain categories of topological structures, and are associated with a monad on the category **Set** of sets and functions. This analysis lifts to the relational structural setting the famous theorem of Manes that the category of compact Hausdorff spaces is isomorphic to the category of algebras for the ultrafilter monad on **Set**.

Section 4 poses questions for further research.

1. Canonicity

By an *operator* on a Boolean algebra we will mean any finitary function that, in each argument, is additive and preserves the least element 0. "Additive" means that it preserves the Boolean sum (join) operation. A *Boolean algebra with operators* (BAO) is, as the name suggests, an algebra comprising a Boolean algebra with a specified collection of operators.

In their seminal work [32] that founded the study of BAO's, Jónsson and Tarski showed that a relational structure $\mathfrak{S}$ of any kind has a *complex algebra* $\mathsf{Cm}\,\mathfrak{S}$, which is a BAO based on the Boolean algebra of all subsets of $\mathfrak{S}$. Each $n+1$-ary relation R of $\mathfrak{S}$ determines an n-ary operator f_R of $\mathsf{Cm}\,\mathfrak{S}$. $f_R(X_1,\dots,X_n)$ is the subset

$$\{x : R(x_1,\dots,x_n,x) \text{ for some } x_1 \in X_1,\dots,x_n \in X_n\}.^1$$

In the converse direction, for each BAO $\mathfrak{A}$ there is a certain structure, which we will denote $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ and call the *canonical structure of* $\mathfrak{A}$, such that there is an isomorphic embedding $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$ representing $\mathfrak{A}$ as an algebra of subsets of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. In this way Jónsson and Tarski lifted the Stone representation of Boolean algebras to BAO's, showing that each n-ary operator of $\mathfrak{A}$ is represented by the operator defined from an $n+1$-ary relation of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$.

We will write $\mathsf{Em}\,\mathfrak{A}$ for the complex algebra $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$, which is commonly known as the *canonical embedding algebra*, or *canonical extension* of $\mathfrak{A}$. Jónsson and Tarski called it the *perfect extension* of $\mathfrak{A}$, although to

[1] We are following the *algebraist's convention* of assigning a special role to the last coordinate of R. The *logician's convention* is to use the first coordinate and write $R(x, x_1,\dots,x_n)$ in this definition.

be precise they gave an axiomatic algebraic definition of *a* perfect extension, proving that any two such extensions are isomorphic over $\mathfrak{A}$. Here we will adopt what has become the standard approach of defining $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ as the structure whose members are the ultrafilters of $\mathfrak{A}$, with each operator f of $\mathfrak{A}$ determining the relation R_f of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ such that

$$R_f(p_1,\ldots,p_{n+1}) \quad \text{iff} \quad \{f_R(x_1,\ldots,x_n) : x_1 \in p_1,\ldots,x_n \in p_n\} \subseteq p_{n+1}.$$

We include here the case $n = 0$ of f being a nullary operation, identifiable with an element d of $\mathfrak{A}$. Then $R_f(p)$ iff $d \in p$, so that R_f becomes the subset $\{p : d \in p\}$ of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$.

It was shown in [32, Theorem 2.18] that any equation satisfied by $\mathfrak{A}$ that is *positive*, in the sense of not involving Boolean complementation, must also be satisfied by $\mathsf{Em}\,\mathfrak{A}$. This was the beginning of many investigations of properties preserved under the passage from $\mathfrak{A}$ to $\mathsf{Em}\,\mathfrak{A}$, investigations that are still ongoing after half a century. We can formulate such questions by asking whether a class $\mathcal{W}$ of algebras is closed under Em, i.e. whether $\mathfrak{A} \in \mathcal{W}$ implies $\mathsf{Em}\,\mathfrak{A} \in \mathcal{W}$. $\mathcal{W}$ is called *canonical* if it has this closure property. Canonicity has been of particular interest for varieties (equational classes) of modal algebras, relation algebras, and cylindric algebras [2, 12, 13, 15, 18, 21, 24, 29, 30, 31].

In the mid-1960's John Lemmon and Dana Scott defined, for each normal propositional modal logic Λ, a certain structure $\mathfrak{S}^\Lambda$ whose points are the maximally Λ-consistent sets of formulas, and showed that $\mathfrak{S}^\Lambda$ can be used as the basis for completeness theorems for many logics [34]. $\mathfrak{S}^\Lambda$ is known as the *canonical frame* for Λ and is commonly invoked for logics whose language has denumerably many variables. But more generally, for each infinite cardinal κ we can define a frame $\mathfrak{S}^\Lambda_\kappa$ whose points are the maximally Λ-consistent sets of formulas from a language with κ-many variables. These frames are intimately connected with the notion of a canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$, as follows from the fact that the underlying set of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is the set of ultrafilters of $\mathfrak{A}$. Maximally consistent sets of formulas correspond to ultrafilters of the Lindenbaum algebra of a logic, and Lindenbaum algebras are the ones that are free. Thus the canonical frame $\mathfrak{S}^\Lambda_\kappa$ is isomorphic to the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$, where $\mathfrak{A}^\Lambda_\kappa$ is the free algebra on κ many generators in the variety of all algebras that validate the logic Λ.[2]

$\mathfrak{S}^\Lambda_\kappa$ supports a model that falsifies all non-theorems of Λ, so if it validates Λ, then it characterizes the logic. Thus Λ is called a *canonical logic* if it is valid in $\mathfrak{S}^\Lambda_\kappa$ for all κ. Most completeness theorems using canonical

[2] See [9] for more information on the origin of canonical frames.

frames proceed by showing that $\mathfrak{S}^\Lambda_\kappa$ satisfies some first-order definable condition that guarantees the validity of Λ (thereby showing Λ is canonical). In 1973 Kit Fine provided a theoretical basis for this procedure by proving the following fundamental theorem for a monomodal logic Λ:

> *if Λ is characterized by some elementary class of frames, then it is canonical, i.e. is valid in $\mathfrak{S}^\Lambda_\kappa$ for all κ.*

[3, Theorem 3]. We will say that Λ is *elementary* if it is characterized by some elementary class of frames. An elementary class is one that is the class of all models of some set of first-order sentences, and can be characterized as a class that is closed under ultraproducts while its *complement* is closed under ultrapowers. Only the closure of the class under ultraproducts is needed for Fine's proof that an elementary logic is canonical.[3]

Fine's theorem has been generalized by the present author in two main directions, firstly broadening the domain of application, and secondly strengthening the conclusion. The broader domain comes from reformulating the result as one about varieties of algebras, and proving it for BOA's of any type. To describe this we will say that a variety $\mathcal{V}$ is *generated by* a class $\mathcal{K}$ of relational structures (of appropriate type) if $\mathcal{V}$ is the smallest variety that includes the class

$$\mathsf{Cm}\,\mathcal{K} = \{\mathfrak{A} : \mathfrak{A} \cong \mathsf{Cm}\,\mathfrak{S} \text{ for some } \mathfrak{S} \in \mathcal{K}\}$$

of all (isomorphic copies of) complex algebras of structures in $\mathcal{K}$. This means that $\mathcal{V} = \mathsf{HSP}\,\mathsf{Cm}\,\mathcal{K}$, where H, S, and P denote the operations of closure of a class of algebras under (isomorphic copies of) homomorphic images, subalgebras, and direct products, respectively. We write $\mathsf{Var}\,\mathcal{K}$ for the variety $\mathsf{HSP}\,\mathsf{Cm}\,\mathcal{K}$ generated by $\mathcal{K}$. In general a variety may be generated by many different classes of structures: if $\mathcal{K}$ generates $\mathcal{V}$, then so does any class $\mathcal{K}'$ with $\mathcal{K} \subseteq \mathcal{K}' \subseteq \mathbb{S}\mathsf{tr}\mathcal{V}$, where

$$\mathbb{S}\mathsf{tr}\mathcal{V} = \{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in \mathcal{V}\}$$

is the class of all *structures for* $\mathcal{V}$, or $\mathcal{V}$*-structures*. A variety is *elementarily generated* if it is equal to $\mathsf{Var}\,\mathcal{K}$ for some elementary class $\mathcal{K}$.

THEOREM 1.1. *If $\mathcal{K}$ is closed under ultraproducts, then the variety $\mathsf{Var}\,\mathcal{K}$ generated by $\mathcal{K}$ is canonical, i.e. $\mathfrak{A} \in \mathsf{Var}\,\mathcal{K}$ implies $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathcal{K}$.* ■

[3]Closure under ω-saturated elementary extensions was also used in [3], but that follows from closure under ultraproducts (indeed under ultrapowers) [1, Section 6.1].

The replacement of an elementary class in the hypothesis of this theorem by the weaker notion of one closed under ultraproducts is not a true strengthening, because when $\mathcal{K}$ is closed under ultraproducts, the closure of $\mathcal{K}$ under elementary equivalence gives an elementary class that also generates $\mathsf{Var}\,\mathcal{K}$ [15, p. 581]. The proof of this uses the Keisler-Shelah ultrapower theorem, in a similar manner to the proof of Theorem 1.4 below.

Underlying the proof of Theorem 1.1 is a categorical *duality* between BAO's and relational structures [11, 12, 15] that lifts the correspondences $\mathfrak{A} \mapsto \mathbb{C}\mathsf{st}\,\mathfrak{A}$ and $\mathfrak{S} \mapsto \mathsf{Cm}\,\mathfrak{S}$ to functors that take any BAO-homomorphism $\mathfrak{A}_1 \to \mathfrak{A}_2$ to a *bounded morphism*[4] $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \to \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ of structures, and any bounded morphism $\mathfrak{S}_1 \to \mathfrak{S}_2$ to a BAO-homomorphism $\mathsf{Cm}\,\mathfrak{S}_2 \to \mathsf{Cm}\,\mathfrak{S}_1$. A homomorphism $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ induces the bounded morphism $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \to \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ that maps each ultrafilter p of $\mathfrak{A}_2$ to the ultrafilter $f^{-1}p$ of $\mathfrak{A}_1$. A bounded morphism $\theta : \mathfrak{S}_1 \to \mathfrak{S}_2$ induces the homomorphism $\mathsf{Cm}\,\mathfrak{S}_2 \to \mathsf{Cm}\,\mathfrak{S}_1$ that takes each set X in $\mathsf{Cm}\,\mathfrak{S}_2$ to its inverse image $\theta^{-1}X$ in $\mathsf{Cm}\,\mathfrak{S}_1$.

These functors map injections to surjections and vice versa. Thus an injective $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ induces a surjective $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \twoheadrightarrow \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$, and hence an injective $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_1 \rightarrowtail \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_2$, i.e. $\mathsf{Em}\,\mathfrak{A}_1 \rightarrowtail \mathsf{Em}\,\mathfrak{A}_2$. Similarly a surjective homomorphism $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ lifts to a surjective homomorphism $\mathsf{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathsf{Em}\,\mathfrak{A}_2$.

Dual to the algebraic class operations H, S, and P are operations $\mathbb{S}$, $\mathbb{H}$, and $\mathbb{U}\mathsf{d}$ of closure of a class of relational structures under "inner" substructures,[5] bounded epimorphic images, and disjoint unions of structures, respectively. We also write $\mathbb{P}\mathsf{u}$ and $\mathbb{P}\mathsf{w}$ for closure of classes of structures under ultraproducts and ultrapowers, and Pu and Pw for closure of classes of *algebras* under these operations.[6] Closure under *ultraroots* is denoted by $\mathbb{R}\mathsf{u}$. Thus $\mathbb{R}\mathsf{u}\,\mathcal{K}$ is the class

$$\{\mathfrak{S} : \Pi_U \mathfrak{S} \in \mathcal{K} \text{ for some ultrafilter } U\}$$

of all structures $\mathfrak{S}$ that have some ultrapower $\Pi_U \mathfrak{S}$ in $\mathcal{K}$. A class is elementary iff it is closed under ultraproducts and ultraroots. When $\mathcal{K}$ is closed under ultraproducts, $\mathbb{R}\mathsf{u}\,\mathcal{K}$ is the closure of K under elementary equivalence, and hence is the smallest elementary class including $\mathcal{K}$ [15, 4.11].

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[4]Also known as a "p-morphism" in the case of modal frames. The technical definition of bounded morphism is given in Section 3.2 below.

[5]A substructure $\mathfrak{S}_1$ of $\mathfrak{S}_2$ is *inner* if the inclusion $\mathfrak{S}_1 \hookrightarrow \mathfrak{S}_2$ is a bounded morphism. The notion is sometimes called "generated subframe" in modal logic.

[6]In general, "sans serif" capitals E, H, P, S ... will be used as the first letter in symbolic names for operations on *algebras*, while "blackboard bold" letters $\mathbb{C}$, $\mathbb{E}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{U}$ ... occur likewise in names of operations on *structures*.

Another important operation is $\mathbb{U}\mathsf{b}$, the closure under *bounded unions.* A structure $\mathfrak{S}$ is the bounded union of structures $\mathfrak{S}_i$ if $\mathfrak{S}$ is the union of these $\mathfrak{S}_i$'s and each $\mathfrak{S}_i$ is an inner substructure of $\mathfrak{S}$. $\mathbb{U}\mathsf{b}$ is dual to the operation of forming subdirect products of algebras [17, p. 415]. Any structure is the bounded union of the inner substructures generated by each of its points. This dualizes the representation of an algebra as a subdirect product of subdirectly irreducible algebras, since the complex algebra of a structure generated by a point is always subdirectly irreducible ([12, Theorem 3.3.1], [39]).

Dual to the notion of the canonical extension $\mathsf{Em}\,\mathfrak{A} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$ of a BAO is the canonical extension $\mathbb{E}\mathsf{x}\,\mathfrak{S} = \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S}$ of a structure $\mathfrak{S}$. The points of $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ are the ultrafilters on the underlying set of $\mathfrak{S}$, and by identifying each point of $\mathfrak{S}$ with the principal ultrafilter it generates, we may view $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ as an extension of $\mathfrak{S}$ itself. Note that $\mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S} = \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S}$. Any surjective or injective bounded morphism $\mathfrak{S}_1 \to \mathfrak{S}_2$ lifts to a bounded morphism $\mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \to \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$ of the same kind.

Johan van Benthem [43, 3.6] reworked the proof of Fine's theorem to show, for any monomodal logic Λ, that

> if Λ is characterized by some elementary class of frames, then whenever a frame $\mathfrak{S}$ validates Λ, so does $\mathbb{E}\mathsf{x}\,\mathfrak{S}$.

To describe properties of class operations we use the notation $\mathbb{X} \leq \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are combinations of operations, to mean that $\mathbb{X}\mathcal{C} \subseteq \mathbb{Y}\mathcal{C}$ for all appropriate classes $\mathcal{C}$. Then we put $\mathbb{X} = \mathbb{Y}$ when $\mathbb{X} \leq \mathbb{Y}$ and $\mathbb{Y} \leq \mathbb{X}$.

There are numerous relationships holding between class operations that can be expressed in this notation, such as

$$\mathbb{S}\,\mathbb{H} \leq \mathbb{H}\mathbb{S}\,, \quad \mathsf{P}\,\mathsf{Cm} = \mathsf{Cm}\,\mathbb{U}\mathsf{d}\,, \quad \mathbb{U}\mathsf{d} \leq \mathbb{U}\mathsf{b} = \mathbb{H}\mathbb{U}\mathsf{d}\,, \quad \mathbb{P}\mathsf{u}\,\mathbb{S} \leq \mathbb{S}\,\mathbb{P}\mathsf{u}\,,$$

and the facts that for any variety of BOA's $\mathcal{V}$, $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under $\mathbb{S}$, $\mathbb{H}$, $\mathbb{U}\mathsf{d}$, $\mathbb{U}\mathsf{b}$ and $\mathbb{R}\mathsf{u}$, while the complement of $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under $\mathbb{E}\mathsf{x}$ and $\mathbb{P}\mathsf{w}$. One straightforward application of the duality is that $\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S} \leq \mathbb{S}\,\mathbb{H}\mathbb{C}\mathsf{st}$. An extensive catalogue of such results is given in [15, 17].

A crucial property connecting canonical extensions with ultrapowers is

$$\mathbb{E}\mathsf{x} \leq \mathbb{H}\mathbb{P}\mathsf{w}\,, \tag{1.i}$$

which follows from the fact that any structure $\mathfrak{S}$ has an ultrapower $\Pi_U\mathfrak{S}$ for which there is a surjective bounded morphism $\Pi_U\mathfrak{S} \twoheadrightarrow \mathbb{E}\mathsf{x}\,\mathfrak{S}$ (see [12, Theorem 3.6.1] or [13, Lemma 3.1]). For this to hold it suffices that the ultrafilter U be chosen so that the ultrapower is ω-saturated. This is essentially a structural version of the key construction in the proof of Fine's theorem.

Our first proof of Theorem 1.1, given in [12, Theorem 3.6.7], used a number of ingredients, including a diagonal construction of ultraproducts of subdirectly irreducible quotients of complex algebras and an important result of Jónsson [28] about the representation of subdirectly irreducible algebras in congruence-distributive varieties. It also required (1.i), the relationship $\mathsf{Pu}\,\mathsf{Cm} \leq \mathsf{S}\,\mathsf{Cm}\,\mathbb{P}\mathsf{u}$, and the fact that $\mathsf{Var}\,\mathcal{K}$ is canonical if, and only if, its class $\mathbb{S}\mathsf{tr}\mathsf{Var}\,\mathcal{K}$ of structures is closed under $\mathbb{E}\mathsf{x}$.

A different and more structurally revealing proof of Theorem 1.1 was developed in [13] and analysed further in [15] and [17]. It uses the relationship

$$\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d} \leq \mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}, \tag{1.ii}$$

expressing that an ultrapower of a disjoint union of structures from $\mathcal{K}$ can be obtained as a bounded epimorphic image of a disjoint union of ultraproducts of families of members of $\mathcal{K}$. This in turn is a consequence of the more general result $\mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{b} \leq \mathbb{U}\mathsf{b}\,\mathbb{P}\mathsf{u}$ [15, Theorem 2.4], which will be used below in Theorem 1.6. By combining (1.i), (1.ii) and the result that $\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S} \leq \mathbb{S}\,\mathbb{H}\mathbb{C}\mathsf{st}$, it can be proven that any class $\mathcal{K}$ of structures has

$$\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\,\mathcal{K},$$

which shows how the canonical structures of BAO's from the variety generated by $\mathcal{K}$ can be built from members of $\mathcal{K}$ (see [15, p. 580] or [17, p. 419]). Then if $\mathcal{K}$ is closed under ultraproducts, so that $\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathcal{K}$, it follows that

$$\mathfrak{A} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{K} \quad \text{implies} \quad \mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\,\mathcal{K},$$

showing how canonical structures mediate between the dual operations on algebras and structures. But then

$$\mathsf{Em}\,\mathsf{Var}\,\mathcal{K} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathsf{Var}\,\mathcal{K} \subseteq \mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathsf{Var}\,\mathcal{K} \subseteq \mathsf{Var}\,\mathcal{K},$$

which proves Theorem 1.1. A refinement of this analysis then establishes [15, 4.12] that:

> if $\mathcal{K}$ is closed under ultraproducts, and L is any class satisfying
>
> $$\mathbb{C}\mathsf{st}\,\mathsf{Var}\,K \subseteq L = \mathbb{P}\mathsf{u}\,L \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\,K,$$
>
> then $M = \mathbb{R}\mathsf{u}\,L$ is an elementary class satisfying $\mathsf{Var}\,K = \mathsf{S}\,\mathsf{Cm}\,M$. In particular, this holds when $L = \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,K$ or $L = \mathbb{H}\mathbb{S}\,\mathbb{U}\mathsf{d}\,K$.

In fact something much stronger than results like this one and Theorem 1.1 can be proven by these methods. Instead of just showing that some given variety $\mathsf{Var}\,\mathcal{K}$ is canonical, we can use the assumption that $\mathbb{P}\mathsf{u}\,\mathcal{K} = \mathcal{K}$ to prove that a certain class of algebras built from $\mathcal{K}$ *is a variety* as well as being canonical. Moreover we can weaken the hypothesis on $\mathcal{K}$ for this purpose, and obtain the following result [15, Theorem 4.5]:

THEOREM 1.2. *If* $\mathbb{P}\mathsf{u}\,\mathcal{K} \subseteq \mathbb{H}\mathbb{S}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$*, then* $\mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$ *is a canonical variety (and hence is the variety generated by* $\mathcal{K}$*).* ∎

(Since $\mathbb{S}\,\mathbb{U}\mathsf{d} = \mathbb{U}\mathsf{d}\,\mathbb{S}$ and $\mathsf{Cm}\,\mathbb{U}\mathsf{d} = \mathsf{P}\,\mathsf{Cm}$ in general, $\mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\,\mathbb{U}\mathsf{d}\,\mathcal{K}$ is equal to $\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathbb{S}\,\mathcal{K}$.) Theorem 1.2 has been be applied to give structural proofs that the classes $\mathbf{RCA}_\alpha$ of representable cylindric algebras and $\mathbf{ICrs}_\alpha$ of cylindric-relativised set algebras of any dimension α form canonical varieties [15, 4.6].

As a consequence of 1.2, if $\mathcal{K}$ is any class of structures closed under $\mathbb{P}\mathsf{u}$, $\mathbb{S}$, and $\mathbb{U}\mathsf{d}$ (e.g. any elementary class closed under $\mathbb{S}$ and $\mathbb{U}\mathsf{d}$), then $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ is a canonical variety [15, 4.3]. This has an elegant application to the class **RRA** of representable relation algebras. A *proper relation algebra* is a collection of binary relations on some set X (i.e. a subset of $\mathcal{P}(X \times X)$) that forms a field of sets (hence a Boolean algebra) that is closed under the binary operator $R_1|R_2$ of composition of relations and the unary converse operator R^{-1}, and contains the identity relation on X as a distinguished element. Any algebra isomorphic to a proper relation algebra is called a *representable* relation algebra. Thus members of **RRA** are of the type $\mathfrak{A} = (\mathfrak{A}_0, ;, \smile, 1')$ with $\mathfrak{A}_0$ a Boolean algebra having a binary operator ; called *composition*, a unary operator $\smile$ called *conversion*, and a member 1' of $\mathfrak{A}_0$ called the *identity element*. **RRA** is a subclass of the variety **RA** of *relation algebras*, the latter being defined by a (finite) set of equations [33, 4.1]. **RA** is a canonical variety [33] and $\mathbb{S}\mathsf{tr}\mathbf{RA}$ is an elementary class [36]. Tarski used metamathematical arguments in [40, 2.4] to show that **RRA** is an equational class. Some years later J. D. Monk proved that **RRA** is canonical by exploiting its connections with cylindric algebra theory (see [24, p. 123] for more information about this).

The structures whose complex algebras are of **RA**-type have the form $\mathfrak{S} = (S, C, R, I)$ with $C \subseteq X^3$, $R \subseteq X^2$ and $I \subseteq X$. In the algebra $\mathsf{Cm}\,\mathfrak{S}$, I is the identity element, composition is the operator defined by C, i.e.

$$X;Y = \{z \in S : \exists x \in X\, \exists y \in Y\, C(x,y,z)\},$$

while conversion is defined from R by

$$X^{\smile} = \{z \in S : \exists x \in X\, (xRz)\}.$$

Jónsson and Tarski showed in [33, 5.8] that **RRA** $= \mathsf{S}\,\mathsf{Cm}\,\mathcal{GB}$, where $\mathcal{GB}$ is a class of such structures, called *generalized Brandt groupoids*, that are defined by a set of conditions that are expressible in first-order logic. Thus $\mathcal{GB}$ is an elementary class. They also showed that $\mathfrak{S}$ is a generalized Brandt groupoid iff $\mathsf{Cm}\,\mathfrak{S}$ is a relation algebra and the ternary relation C of $\mathfrak{S}$ is functional, in the sense that if $C(x,y,z_1)$ and $C(x,y,z_2)$ then $z_1 = z_2$ (see Theorems 5.5 and 5.7 and comment on page 159 of [33]).

THEOREM 1.3. **RRA** *is a variety and is canonical.*

PROOF. Since $\mathcal{GB}$ is an elementary class it is $\mathbb{P}\mathsf{u}$-closed, so to prove $\mathsf{S}\,\mathsf{Cm}\,\mathcal{GB}$ is a canonical variety it is enough by Theorem 1.2 to prove that $\mathcal{GB}$ is closed under $\mathbb{U}\mathsf{d}$ and $\mathbb{S}$.

We use the fact just mentioned that $\mathfrak{S} \in \mathcal{GB}$ iff $\mathsf{Cm}\,\mathfrak{S} \in \mathbf{RA}$ and C is functional in $\mathfrak{S}$. But $\mathsf{Cm}\,\mathfrak{S} \in \mathbf{RA}$ iff $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathbf{RA}$, and $\mathbb{S}\mathsf{tr}\mathbf{RA}$ is closed under $\mathbb{U}\mathsf{d}$ and $\mathbb{S}$ (indeed this holds of $\mathbb{S}\mathsf{tr}\mathcal{V}$ for any variety $\mathcal{V}$). The condition of functionality of C is readily seen to be preserved by disjoint unions, and is preserved by arbitrary substructures, not just inner ones. These observations suffice to show that $\mathbb{U}\mathsf{d}\,\mathcal{GB} = \mathbb{S}\,\mathcal{GB} = \mathcal{GB}$. ∎

Thus **RRA** can be axiomatized by a set $\mathcal{E}$ of equations that are *collectively* canonical in the sense that if an algebra $\mathfrak{A}$ is a model of all members of $\mathcal{E}$, then so is $\mathsf{Em}\,\mathfrak{A}$. Recently Ian Hodkinson and Yde Venema [26] have shown that no set of equational axioms for **RRA** can consist of equations that are *individually* canonical (an equation ϵ being individually canonical if $\mathfrak{A} \models \epsilon$ implies $\mathsf{Em}\,\mathfrak{A} \models \epsilon$).

The closure properties of a class of the form $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ can be given a finer analysis. Firstly, the condition $\mathbb{P}\mathsf{w}\,\mathcal{K} = \mathcal{K}$ is sufficent on its own to ensure that $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ is a canonical class of algebras [12, 3.6.3]. Then if $\mathbb{S}\,\mathcal{K} = \mathcal{K}$ as well, $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ is closed under homomorphic images. Adding $\mathbb{U}\mathsf{d}\,\mathcal{K} = \mathcal{K}$ to these makes $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ a canonical variety [15, 4.3].

This analysis has application to the class **GRA** of *group relation algebras.* A group $(S, \circ, {}^{-1}, e)$ may be viewed as an **RA**-type structure $\mathfrak{S}$ with ternary relation C being the graph $\{(x, y, x \circ y) : x, y \in S\}$ of the group operation; $R = \{(x, x^{-1}) : x \in S\}$ being the graph of the group inverse; and $I = \{e\}$ where e is the group identity. If $\mathcal{G}\mathsf{p}$ is the class of groups construed as such structures, then $\mathbf{GRA} = \mathsf{S}\,\mathsf{Cm}\,\mathcal{G}\mathsf{p}$ by definition. Each member of $\mathsf{Cm}\,\mathcal{G}\mathsf{p}$ is a representable relation algebra, so $\mathbf{GRA} \subseteq \mathbf{RRA}$. But $\mathcal{G}\mathsf{p}$ is an elementary class, hence in particular closed under ultrapowers, so from what was said in the previous paragraph, it follows directly that **GRA** is a canonical class by itself.

$\mathcal{G}\mathfrak{p}$ is also closed under $\mathbb{S}$. To see this we need to know some detail of what "inner substructure" means in this context. Given structures $\mathfrak{S}_i = (S_i, C_i, R_i, I_i)$ for $i = 1, 2$, if $\mathfrak{S}_1$ is an inner substructure of $\mathfrak{S}_2$ then in particular it is a substructure of $\mathfrak{S}_2$ and the following holds for all $x, y, z \in S_2$:

$$\text{if } C_2(x, y, z) \text{ and } z \in S_1, \text{ then } x, y \in S_1.$$

But if $\mathfrak{S}_2 \in \mathcal{G}\mathfrak{p}$, with $I_2 = \{e\}$, then for any $z \in S_1$ we have $C_2(z, e, z)$ from the group law $z \circ e = z$, so $e \in S_1$ by the inner substructure condition. Then for any $x \in S_2$, $C_2(x, x^{-1}, e)$ and so $x \in S_1$. Hence $\mathfrak{S}_1 = \mathfrak{S}_2$. This shows that a group structure has no proper inner substructures, and therefore trivially $\mathbb{S}\,\mathcal{G}\mathfrak{p} = \mathcal{G}\mathfrak{p}$. Since $\mathcal{G}\mathfrak{p}$ is closed under ultraproducts, Theorem 1.2 then yields that the class $\mathsf{S}\,\mathsf{Cm}\,\mathbb{U}\mathsf{d}\,\mathcal{G}\mathfrak{p}$, i.e. $\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{G}\mathfrak{p}$, is a canonical variety. Hence it is the variety generated by **GRA**.

Tarski [40] proved that **GRA** is a universal class and consists of the simple algebras in $\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{G}\mathfrak{p}$. Steven Givant [8] has shown that this is a manifestation of the following general phenomenon:

> if $\mathcal{W}$ is a class of simple algebras in a discriminator variety $\mathcal{V}$, and there exists a class $\mathcal{K}$ of structures that is closed under ultraproducts and has $\mathsf{Cm}\,\mathcal{K} \subseteq \mathcal{W} \subseteq \mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$, then $\mathsf{S}\,\mathsf{P}\,\mathcal{W}$ is a canonical variety and $\mathsf{S}\,\mathcal{W}$ is the universal class of simple algebras in $\mathsf{S}\,\mathsf{P}\,\mathcal{W}$.

■

The second way in which we have generalized the theorem of Fine is to strengthen its conclusion, showing that if a logic Λ is elementary, then it is validated not only by its canonical frames but also by other structures that have some first-order connection with these frames. This relates to the vital question of whether the converse of Fine's theorem is true:

> *must every canonical logic Λ be elementary, i.e. be characterized by some elementary class of frames?*

A natural candidate for an elementary characterizing class here is the class $\mathcal{K}^\Lambda_\omega$ of all frames that are first-order equivalent to the canonical Λ-frame $\mathfrak{S}^\Lambda_\omega$ based on a language with denumerably many variables. $\mathcal{K}^\Lambda_\omega$ is the class of all models of the *first-order theory* of $\mathfrak{S}^\Lambda_\omega$, i.e. the set of all first-order sentences true in $\mathfrak{S}^\Lambda_\omega$. The relevance of this class is shown by the following result:

> if Λ is characterized by some class of frames that is closed under ultraproducts, then it is valid in all members of the elementary class $\mathcal{K}_\omega^\Lambda$ [14, Theorem 11.3.1].

Of course strengthening the conclusion of a statement makes the converse weaker, and the converse of this last theorem is immediate: if Λ is valid in all members of $\mathcal{K}_\omega^\Lambda$ then it is characterized by an elementary class, namely $\mathcal{K}_\omega^\Lambda$, since $\mathfrak{S}_\omega^\Lambda \in \mathcal{K}_\omega^\Lambda$ and all non-theorems of Λ are falsified in $\mathfrak{S}_\omega^\Lambda$.

The algebraic version of this result is formulated for a variety $\mathcal{V}$ of BAO's, and invokes the free $\mathcal{V}$-algebra $\mathfrak{A}_\omega^\mathcal{V}$ on denumerably many generators:

> $\mathcal{V}$ is generated by some class of structures that is closed under ultraproducts if, and only if, it is generated by the elementary class of all models of $Th\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$, the first-order theory of the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ [15, Theorem 4.15].

Now the models of $Th\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ are just the structures elementarily equivalent to $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$, and elementary equivalence has a structural characterization, due to Keisler and Shelah: elementarily equivalent structures are those that have isomorphic U-ultrapowers for some ultrafilter U. From this we get another characterization of elementary generation of varieties:

THEOREM 1.4. *A variety $\mathcal{V}$ is elementarily generated if, and only if, every ultrapower of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ is in $\mathbb{S}\mathsf{tr}\mathcal{V}$.*

PROOF. Let $\mathcal{K}$ be the elementary class of all models of $Th\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$.

If $\mathcal{V}$ is elementarily generated, then as explained above it is generated by $\mathcal{K}$, so $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V} \in \mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathcal{V}$. Since $\mathbb{P}\mathsf{w}\mathcal{K} \subseteq \mathcal{K}$, this puts all ultrapowers of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ into $\mathbb{S}\mathsf{tr}\mathcal{V}$.

Conversely, suppose $\mathbb{S}\mathsf{tr}\mathcal{V}$ contains the ultrapower $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ for any ultrafilter U. If $\mathfrak{S} \in \mathcal{K}$ then by the Keisler-Shelah theorem there is an ultrafilter U such that $\Pi_U\mathfrak{S}$ and $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ are isomorphic. By hypothesis $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V} \in \mathbb{S}\mathsf{tr}\mathcal{V}$, and so $\Pi_U\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$. But $\mathbb{S}\mathsf{tr}\mathcal{V}$ is always closed under ultraroots, so this forces $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$. Thus we have $\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathcal{V}$, and so

$$\mathsf{Var}\mathcal{K} \subseteq \mathsf{Var}\mathbb{S}\mathsf{tr}\mathcal{V} \subseteq \mathcal{V}.$$

Hence $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V} \in \mathsf{Var}\mathcal{K} \subseteq \mathcal{V}$. Since $\mathfrak{A}_\omega^\mathcal{V}$ generates $\mathcal{V}$ (as the free algebra), and is a subalgebra of $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ ($= \mathsf{Em}\,\mathfrak{A}_\omega^\mathcal{V}$), it follows that $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^\mathcal{V}$ generates $\mathcal{V}$. Hence $\mathcal{V} = \mathsf{Var}\mathcal{K}$, and $\mathcal{V}$ is elementarily generated by $\mathcal{K}$. ■

An even stronger generalization of Fine's theorem can be obtained by restricting attention to *quasi-modal* sentences. These are first-order sentences of the syntactic form $\forall v\varphi$, with φ being constructed from amongst atomic formulas and the constants $\bot$ (False) and $\top$ (True) using at most $\wedge$ (conjunction), $\vee$ (disjunction), and *bounded* universal and existential quantifiers

$$\forall v_0 \cdots \forall v_{n-1}(R(v_0, \dots, v_{n-1}, v) \to \psi)$$
$$\exists v_0 \cdots \exists v_{n-1}(R(v_0, \dots, v_{n-1}, v) \wedge \psi)$$

with v distinct from $v_0, \dots, v_{n-1}$. The relevance of quasi-modal sentences, and the reason for the name, is that they are precisely those first-order sentences whose satisfaction is preserved by the basic modal-validity preserving operations of $\mathbb{S}$, $\mathbb{H}$, and $\mathbb{U}\mathsf{d}$ [44, 12]. As a consequence of Theorem 1.2 we have that

> if $\mathcal{K}$ is the class of all models of a set of quasi-modal first-order sentences, then $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ is a canonical variety.

By the *quasi-modal theory* of a structure $\mathfrak{S}$ we mean the set of all quasi-modal first-order sentences that are true in $\mathfrak{S}$.

It transpires that for any logic Λ, there is no quasi-modally-expressible property that can differentiate canonical Λ-frames: the structures $\mathfrak{S}^{\Lambda}_{\kappa}$ have exactly the same quasi-modal first-order theory for all κ. We will denote this unique quasi-modal theory of canonical Λ-frames by Ψ^{Λ}. Moreover, if Λ is not canonical, then it always has a largest canonical proper sublogic Λ^c and a largest elementary sublogic Λ^e (with $\Lambda^e \subseteq \Lambda^c$), and the quasi-modal theories Ψ^{Λ^c} and Ψ^{Λ^c} of these other logics are identical to Ψ^{Λ}. These results are all proven in [19].

Our main strengthening of Fine's theorem is

THEOREM 1.5. *[14, 11.4.2] If a modal logic Λ is characterized by some class of frames that is closed under ultraproducts, then it is valid in all models of the first-order theory Ψ^{Λ} (including all the canonical frames $\mathfrak{S}^{\Lambda}_{\kappa}$).* ■

The algebraic version of this theorem states that if a variety $\mathcal{V}$ of BAO's is generated by some $\mathbb{P}\mathsf{u}$-closed class of structures, then $\mathbb{S}\mathsf{tr}\mathcal{V}$ contains all models of the quasi-modal first-order theory $\Psi^{\mathcal{V}}$ of the canonical structures $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\kappa}$. The proof of these results depends on a careful analysis of the preservation theorem for quasi-modal sentences given in [15, Section 7].[7] This shows that if $\Psi^{\mathcal{K}}$ is the set of all quasi-modal sentences true of a class $\mathcal{K}$ of structures that is closed under ultraproducts, and $Mod\,\Psi^{\mathcal{K}}$ is the class of all models of $\Psi^{\mathcal{K}}$, then

$$\mathbb{C}\mathsf{st}\,\mathsf{Var}\,\mathcal{K} \subseteq Mod\,\Psi^{\mathcal{K}} = \mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{H}\mathbb{S}\,\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\,\mathcal{K}.$$

[7] Quasi-modal sentences were called "pseudo-equational" in [14, 15].

It seems reasonable to ask whether an elementary generating class for $\mathcal{V}$ might be $\mathbb{S}\mathsf{tr}\mathcal{V}$, the class of all structures $\mathfrak{S}$ with $\mathsf{Cm}\,\mathfrak{S} \in \mathcal{V}$. The version of this for a modal logic Λ is the class $\mathbb{S}\mathsf{tr}\Lambda$ of all structures (frames) that validate Λ. But is it known that there are *canonical* logics Λ for which $\mathbb{S}\mathsf{tr}\Lambda$ is not an elementary class. The first such example, due to Fine [3], was an elementary Λ, characterized by a subclass of $\mathbb{S}\mathsf{tr}\Lambda$ defined by a single first-order sentence, for which $\mathbb{S}\mathsf{tr}\Lambda$ itself is not even closed under elementary equivalence. Another example with a long history is the canonical variety **RRA** of representable relation algebras discussed in Theorem 1.3. Robin Hirsch and Ian Hodkinson have shown [24, 14.2.3] that $\mathbb{S}\mathsf{tr}\mathbf{RRA}$ is not closed under ultraproducts.

The full story about elementarity of $\mathbb{S}\mathsf{tr}\Lambda$ was revealed by the discovery of van Benthem [42] that for $\mathbb{S}\mathsf{tr}\Lambda$ to be an elementary class it suffices that it be closed under elementary equivalence. It follows that another sufficient condition is closure under ultrapowers. His proof involved a model-theoretic compactness argument (discussed in [16]), but a simple structural explanation was then found in the present author's observation that there is an injective bounded morphism

$$\prod_U \mathfrak{S}_i \rightarrowtail \prod_U (\coprod_I \mathfrak{S}_i)$$

of any U-ultraproduct of a family $\{\mathfrak{S}_i : i \in I\}$ of structures into the associated U-ultrapower of their disjoint union $\coprod_I \mathfrak{S}_i$. This makes the ultraproduct isomorphic to an inner substructure of the ultrapower, and establishes the relationship

$$\mathbb{P}\mathsf{u} \leq \mathbb{S}\,\mathbb{P}\mathsf{w}\mathbb{U}\mathsf{d} \tag{1.iii}$$

(see [12, 3.8.3] for details). Now if a class of the form $\mathbb{S}\mathsf{tr}\Lambda$ or $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultrapowers, then since it is always closed under $\mathbb{S}$ and $\mathbb{U}\mathsf{d}$, it follows from (1.iii) that it is also closed under ultraproducts. But $\mathbb{S}\mathsf{tr}\Lambda$ (or $\mathbb{S}\mathsf{tr}\mathcal{V}$) is always closed under ultraroots $\mathbb{R}\mathsf{u}$, so this now shows that it is an elementary class.

Note that if $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultrapowers, and hence is an elementary class, it does not necessarily follow that $\mathcal{V}$ is canonical. The point is that $\mathbb{S}\mathsf{tr}\mathcal{V}$ might not generate $\mathcal{V}$, so Theorem 1.1 does not apply. If this is the case, then no class of structures can generate $\mathcal{V}$, and $\mathcal{V}$ is what is known as an *incomplete* variety. This notion corresponds to that of a logic that is not characterized by its class $\mathbb{S}\mathsf{tr}\Lambda$ of validating frames, and so has no completeness theorem with respect to the relational semantics. An extreme example, and the first one discovered, is the tense logic of S. K. Thomason

[41] that is consistent but not validated by any frame. This defines a non-trivial variety for which $\mathbb{S}\mathsf{tr}\mathcal{V}$ is the empty class.

In view of (1.iii) it is natural to consider whether $\mathbb{P}\mathsf{u}$-closure can be replaced by $\mathbb{P}\mathsf{w}$-closure in variations on Fine's theorem:

THEOREM 1.6. *A variety of BAO's is generated by some $\mathbb{P}\mathsf{u}$-closed class of structures if, and only if, it is generated by a class that is closed under $\mathbb{P}\mathsf{w}$ and $\mathbb{U}\mathsf{d}$.*

PROOF. Let $\mathcal{V}$ be generated by $\mathcal{K}$, and suppose that $\mathcal{K}$ is closed under $\mathbb{P}\mathsf{w}$ and $\mathbb{U}\mathsf{d}$. Then as $\mathcal{K} \subseteq \mathbb{S}\,\mathcal{K} \subseteq \mathsf{Var}\,\mathcal{K}$, $\mathcal{V}$ is also generated by $\mathbb{S}\,\mathcal{K}$, so it suffices to show that $\mathbb{S}\,\mathcal{K}$ is $\mathbb{P}\mathsf{u}$-closed. But in general $\mathbb{P}\mathsf{u}\mathbb{S} \leq \mathbb{S}\,\mathbb{P}\mathsf{u}$, so using (1.iii) we get that

$$\mathbb{P}\mathsf{u}\mathbb{S}\,\mathcal{K} \subseteq \mathbb{S}\,\mathbb{P}\mathsf{u}\mathcal{K} \subseteq \mathbb{S}\,\mathbb{S}\mathbb{P}\mathsf{w}\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{S}\,\mathcal{K}$$

as desired, because $\mathbb{S}\,\mathbb{S} = \mathbb{S}$ and $\mathcal{K}$ is closed under $\mathbb{P}\mathsf{w}$ and $\mathbb{U}\mathsf{d}$.

Conversely, let $\mathcal{V}$ be generated by $\mathcal{K}$ with $\mathcal{K}$ closed under $\mathbb{P}\mathsf{u}$. Then $\mathcal{V}$ is generated by $\mathbb{H}\mathbb{U}\mathsf{d}\,\mathcal{K}$, which is the same as $\mathbb{U}\mathsf{b}\,\mathcal{K}$, so it suffices to prove that the latter is closed under $\mathbb{P}\mathsf{w}$ and $\mathbb{U}\mathsf{d}$. Now as an instance of $\mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{b} \leq \mathbb{U}\mathsf{b}\,\mathbb{P}\mathsf{u}$ (see after (1.ii)) we have that $\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{b} \leq \mathbb{U}\mathsf{b}\,\mathbb{P}\mathsf{u}$, so as $\mathbb{P}\mathsf{u}\,\mathcal{K} \subseteq \mathcal{K}$ in this case we immediately get $\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{b}\,\mathcal{K} \subseteq \mathbb{U}\mathsf{b}\,\mathcal{K}$.

But in general $\mathbb{U}\mathsf{d}\,\mathbb{H} \leq \mathbb{H}\mathbb{U}\mathsf{d}$ and $\mathbb{U}\mathsf{d}\,\mathbb{U}\mathsf{d} = \mathbb{U}\mathsf{d}$, so $\mathbb{U}\mathsf{d}\,\mathbb{U}\mathsf{b} = \mathbb{U}\mathsf{d}\,\mathbb{H}\mathbb{U}\mathsf{d} \leq \mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{U}\mathsf{d} = \mathbb{H}\mathbb{U}\mathsf{d} = \mathbb{U}\mathsf{b}$, showing that $\mathbb{U}\mathsf{b}\,\mathcal{K}$ is closed under $\mathbb{U}\mathsf{d}$ for any class $\mathcal{K}$. ■

The converse of Fine's theorem remains an open question. Is every canonical logic elementary, and more generally is every canonical variety of BAO's generated by some $\mathbb{P}\mathsf{u}$-closed class of structures? In view of the analysis given here, we see that to answer this converse question negatively it would be enough to exhibit a canonical logic Λ that was not valid in some frame elementarily equivalent to $\mathfrak{S}^{\Lambda}_{\omega}$, or even more weakly was not valid in some model of the theory Ψ^{Λ} (see 1.5). Another perspective is provided by the following parallel to Theorem 1.6:

THEOREM 1.7. *A variety $\mathcal{V}$ of BAO's is canonical if, and only if, it is generated by a class that is closed under $\mathbb{E}\mathsf{x}$ and $\mathbb{U}\mathsf{d}$.*

PROOF. Let $\mathcal{V}$ be canonical. Then $\mathcal{V}$ is generated by its class $\mathbb{S}\mathsf{tr}\mathcal{V}$ of structures. For if $\mathfrak{A} \in \mathcal{V}$ then $\mathfrak{A}$ is a subalgebra of $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A} = \mathsf{Em}\,\mathfrak{A} \in \mathcal{V}$, which shows that $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\mathcal{V}$. Hence $\mathcal{V} = \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathcal{V}$. But $\mathbb{S}\mathsf{tr}\mathcal{V}$ is always closed under $\mathbb{U}\mathsf{d}$, and in this case is closed under $\mathbb{E}\mathsf{x}$, for if $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$, then $\mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S} = \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} \in \mathcal{V}$, showing that $\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$.

Conversely, let $\mathcal{V}$ be generated by $\mathcal{K}$ with $\mathcal{K}$ closed under $\mathbb{E}\mathsf{x}$ and $\mathbb{U}\mathsf{d}$. Then we show that $\mathcal{V}$ is closed under Em. If $\mathfrak{A} \in \mathcal{V}$, then

$$\mathfrak{A} \in \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{K} = \mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathcal{K},$$

as $\mathcal{K}$ is $\mathbb{U}\mathsf{d}$-closed. Thus $\mathfrak{A}$ is a homomorphic image of some BAO $\mathfrak{B}$ that is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ for some $\mathfrak{S} \in \mathcal{K}$. This gives a diagram of homomorphisms

$$\mathfrak{A} \twoheadleftarrow \mathfrak{B} \rightarrowtail \mathsf{Cm}\,\mathfrak{S},$$

which lifts by duality to

$$\mathsf{Em}\,\mathfrak{A} \twoheadleftarrow \mathsf{Em}\,\mathfrak{B} \rightarrowtail \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S}.$$

But $\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathcal{K}$ by $\mathbb{E}\mathsf{x}$-closure of $\mathcal{K}$, so $\mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathsf{Cm}\,\mathcal{K} \subseteq \mathcal{V}$. Closure of $\mathcal{V}$ under S and H then guarantees that $\mathsf{Em}\,\mathfrak{A} \in \mathcal{V}$ as desired. ∎

Comparing Theorems 1.6 and 1.7 shows that a canonical variety that is not elementarily generated, if there is such a thing, would be one that has a $\mathbb{E}\mathsf{x}$-$\mathbb{U}\mathsf{d}$-closed generating class but no $\mathbb{P}\mathsf{w}$-$\mathbb{U}\mathsf{d}$-closed generating class.

Yet another perspective on the converse question is provided by considering *finite* validating frames for a logic Λ. We proved in [19, 9.2] that if Λ is elementary then it is valid in any ultraproduct of finite Λ-frames. The algebraic version is that

> if a variety $\mathcal{V}$ is elementarily generated, then $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultraproducts of its finite members.

The reason for this is that if $\mathfrak{S}$ is a finite member of $\mathbb{S}\mathsf{tr}\mathcal{V}$, then the finite algebra $\mathsf{Cm}\,\mathfrak{A}$ is a homomorphic image of the free $\mathcal{V}$-algebra $\mathfrak{A}^{\mathcal{V}}_{\omega}$, so $\mathbb{E}\mathsf{x}\,\mathfrak{S} = \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S}$ is isomorphic to an inner substructure of $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\omega}$ and hence is a model of the quasi-modal first-order theory $\Psi^{\mathcal{V}}$ of $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\omega}$. But in the finite case, $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ is isomorphic to $\mathfrak{S}$. Thus all finite $\mathcal{V}$-structures are models of $\Psi^{\mathcal{V}}$, so any ultraproduct of finite $\mathcal{V}$-structures is a model of $\Psi^{\mathcal{V}}$, and therefore belongs to $\mathbb{S}\mathsf{tr}\mathcal{V}$ (see remark after Theorem 1.5).

The situation is well illustrated by the variety **RRA**, which is canonical and elementarily generated. Hence $\mathbb{S}\mathsf{tr}$**RRA** is closed under ultraproducts of *finite* structures, but not closed under arbitrary ultraproducts because it is not elementary.

Thus to show that a canonical variety $\mathcal{V}$ is not elementarily generated it would be enough to show that it is not closed under ultraproducts of its finite members. Equivalently, to show that a canonical logic Λ is not elementary it would be enough to show that it is falsifiable in some ultraproduct of finite Λ-frames.

2. Three Cases of Elementary Generation

All canonical varieties $\mathcal{V}$ of BAO's that have been studied to date are known to be elementarily generated. Indeed most have been defined in the first place by equations that are preserved by the $\mathsf{Em}\,\mathfrak{A}$ construction, with the proof of this consisting of a demonstration that the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ satisfies some elementary conditions that force $\mathsf{Em}\,\mathfrak{A}$ to validate these defining equations for $\mathcal{V}$.

However there are some classes of algebras that have turned out to be canonical varieties, but which were defined in some structural way that is not obviously equational or expressible by first-order conditions on generating structures. Such examples arise particularly in the area of relation algebras and cylindric algebras, involving varieties that are defined by some representability conditions, as in the case of **RRA** we discussed above.

In this section we consider three such types of canonical variety whose elementary generation has not previously been shown in the literature. To show this we call on the lessons of Section 1, from which we learned that to prove that variety $\mathcal{V}$ is elementarily generated it is enough to prove that it is generated by a $\mathbb{P}\mathsf{u}$-closed class, or that $\mathbb{S}\mathsf{tr}\mathcal{V}$ contains all ultrapowers of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_\omega^{\mathcal{V}}$.

2.1. $\mathbf{RA}_n$

The classes $\mathbf{RA}_n$ for $3 \leq n \leq \omega$ were introduced by Roger Maddux in [37].[8] $\mathbf{RA}_\omega$ is identical to **RRA**, so we will focus on the case of finite n. Algebras for these classes are of the same similarity type $\mathfrak{A} = (\mathfrak{A}_0, ;, \breve{}, 1')$ as for **RRA**, with $\mathfrak{A}_0$ a Boolean algebra having operators ; (binary) and $\breve{}$ (unary) and a distinguished element 1'. The structures dual to these algebras have the form $\mathfrak{S} = (S, C, R, I)$ with $C \subseteq S^3$, $R \subseteq S^2$ and $I \subseteq S$, as discussed prior to Theorem 1.3.

There is a canonical variety **NA** of *nonassociative relation algebras* defined by equations that are weaker than those for **RA** (so $\mathbf{RA} \subseteq \mathbf{NA}$) but strong enough to imply that if x is an atom of $\mathfrak{A}$ then so is $x\breve{}$, and $x\breve{}\breve{} = x$ [36, 3.4]. This implies that if $\mathsf{Cm}\,\mathfrak{S} \in \mathbf{NA}$, then the relation R of $\mathfrak{S}$ is total and functional and defines a function of period two. We can replace R by this function $f : S \to S$ that has $ffx = x$ [35, p. 56]. Then the conversion operation in $\mathsf{Cm}\,\mathfrak{S}$ is just given by f-images:

$$X\breve{} = \{fx : x \in X\}.$$

[8] They were originally called MA_n.

$\mathbb{S}\mathsf{tr}\mathbf{NA}$ is an elementary class [35, 36].

Maddux introduced the notion of an *n-dimensional basis* for an atomic **NA**-algebra as a collection of $n \times n$ matrices of atoms of $\mathfrak{A}$ with certain properties (see below). $\mathbf{RA}_n$ is defined as the class of subalgebras of those complete and atomic **NA**-algebras that have an n-dimensional basis.

Now if $\mathfrak{A}$ is complete and atomic, then $\mathfrak{A}$ is isomorphic to $\mathsf{Cm}\,\mathfrak{S}$ where $\mathfrak{S}$ is a structure based on the set of atoms of $\mathfrak{A}$.[9] Then matrices of atoms of $\mathfrak{A}$ become matrices of points of $\mathfrak{S}$, and conditions on atoms of $\mathfrak{A}$ become conditions expressible in the language of $\mathfrak{S}$. We will define bases in these terms here. An $n \times n$ matrix will be described as $a = (a_{ij})$, where a_{ij} is the ij-th entry of a.

An *n-dimensional basis* for a structure of type $\mathfrak{S} = (S, C, f, I)$ is a set M of $n \times n$ matrices of elements of S satisfying the following.

(a) If $a \in M$, then $a_{ii} \in I$, $fa_{ij} = a_{ji}$ and $C(a_{ik}, a_{kj}, a_{ij})$ for all $i, j, k < n$.

(b) If $a \in M$, $x, y \in S$, $i, j < n$ and $C(x, y, a_{ij})$, then for all $k < n$ with $k \notin \{i, j\}$ there exists some $b \in M$ with $b_{ik} = x$, $b_{kj} = y$, and $b_{lm} = a_{lm}$ whenever $k \notin l, m$ (so b agrees with a outside of row k and column k).

(c) For all $x \in S$ there is some $a \in M$ with $a_{01} = x$.

Let $\mathcal{B}_n$ be the class of all structures $\mathfrak{S}$ such that $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathbf{NA}$ (i.e. $\mathsf{Cm}\,\mathfrak{S} \in \mathbf{NA}$) and $\mathfrak{S}$ has an n-dimensional basis. Then $\mathbf{RA}_n = \mathsf{S}\,\mathsf{Cm}\,\mathcal{B}_n$ by definition.

We are going to show that $\mathcal{B}_n$ is closed under $\mathbb{P}\mathsf{u}$, $\mathbb{S}$ and $\mathbb{U}\mathsf{d}$. Hence by Theorem 1.2, $\mathsf{S}\,\mathsf{Cm}\,\mathcal{B}_n$ is a canonical variety, and so is the variety generated by $\mathcal{B}_n$. This achieves our goal of showing that $\mathbf{RA}_n$ has a $\mathbb{P}\mathsf{u}$-closed generating class, and therefore an elementary one. But it also provides a new proof of the result [37] that $\mathbf{RA}_n$ is a variety and is canonical.

$\mathbb{S}\mathsf{tr}\mathbf{NA}$ is closed under $\mathbb{S}$ and $\mathbb{U}\mathsf{d}$, since **NA** is a variety, and is $\mathbb{P}\mathsf{u}$-closed because it is elementary. Thus the burden of the proof is to show that the property of a structure having an n-dimensional basis is preserved by these three constructions.

For the case of $\mathbb{P}\mathsf{u}$, an $n \times n$ matrix will be viewed as an n^2-tuple

$$(a_{00}, \ldots, a_{ij}, \ldots, a_{n-1\,n-1})$$

indexed by the pairs ij with $i, j < n$ (ordered lexicographically, say). Then a set M of $n \times n$ matrices over S becomes an n^2-ary relation $M \subseteq S^{n^2}$. For

[9] The definition of this structure appears in Section 4 below.

a fixed n, the three conditions (a)–(c) can be expressed in the first-order language of the structure $(\mathfrak{S}, M)$. For instance (c) is expressible by

$$\forall x \exists a_{00} \ldots \exists a_{n-1\,n-1}\ (M(a_{00}, \ldots, a_{n-1\,n-1}) \wedge x = a_{01})$$

(or equivalently $\forall x \exists a_{00} \ldots \exists a_{n-1\,n-1} M(a_{00}, x, a_{03}, \ldots, a_{n-1\,n-1})$). Conjoining the three sentences expressing (a)–(c) gives a single first-order sentence φ_n that is true in $(\mathfrak{S}, M)$ iff M is an n-dimensional basis for $\mathfrak{S}$. Then regarding M as a relation variable we form the *second-order* sentence $\exists M \varphi_n$ which it true in $\mathfrak{S}$ iff $\mathfrak{S}$ has an n-dimensional basis.

But syntactically $\exists M \varphi_n$ is a Σ_1^1 second-order sentence, and it is a standard fact that the truth of such sentences is preserved by ultraproducts [1, 4.1.14]. Therefore $\mathcal{B}_n$ is closed under ultraproducts.

For the case of closure under $\mathbb{S}$ we use the fact that, in this context, if $\mathfrak{S}_1$ is an inner substructure of $\mathfrak{S}_2$ then it is a substructure of $\mathfrak{S}_2$ such that $f_2^{-1}(S_1) \subseteq S_1$, i.e. $f_2 x \in S_1$ implies $x \in S_1$ for all $x \in S_2$, and the following holds for all $x, y, z \in S_2$:

$$\text{if } C_2(x, y, z) \text{ and } z \in S_1, \text{ then } x, y \in S_1$$

(as noted above in the discussion of **GRA**). The condition on f_2 ensures that S_1 is closed under f_2, for if $x \in S_1$, then $f_2 f_2 x \in S_1$ as f_2 is of period two, hence $f_2 x \in S_1$ by the inner condition.

If $\mathfrak{S}_1$ is an inner substructure of $\mathfrak{S}_2$ and $\mathfrak{S}_2$ has an n-dimensional basis M_2, let M_1 be the set of matrices from M_2 whose entries all belong to S_1. Then M_1 is an n-dimensional basis for $\mathfrak{S}_1$. The key to proving this is the observation that as soon as a matrix $a \in M_2$ has *at least one* entry in S_1 then all its entries are in S_1, so $a \in M_1$. For, if $a_{ij} \in S_1$, then for any $k < n$ we have $C_2(a_{ik}, a_{kj}, a_{ij})$ by (a), and hence $a_{ik}, a_{kj} \in S_1$ by the inner substructure condition. Thus all members of the row and column of a_{ij} are in S_1. Now we can repeat that argument for each member of the row (or column) of a_{ij} to deduce that all entries of a are in S_1. Using this observation it is now straightforward to show that the conditions (a)–(c) are preserved in passing from M_2 to M_1. Thus $\mathcal{B}_n$ is closed under $\mathbb{S}$.

For the case of $\mathbb{U}\mathsf{d}$, let $\mathfrak{S}$ be the disjoint union of a family $\{\mathfrak{S}_\lambda : \lambda \in I\}$ with each $\mathfrak{S}_\lambda$ having an n-dimensional basis M_λ. $\mathfrak{S}$ is the union of a family of pairwise disjoint copies of the $\mathfrak{S}_\lambda$'s, so we can assume that these are already pairwise disjoint. Each is then an inner substructure of $\mathfrak{S}$. The union of all the M_λ's is easily checked to be an n-dimensional basis for $\mathfrak{S}$.

This completes the proof that $\mathcal{B}_n$ is closed under $\mathbb{P}\mathsf{u}$, $\mathbb{S}$ and $\mathbb{U}\mathsf{d}$, and hence that $\mathbf{RA}_n$ is an elementarily generated canonical variety. Note that we have not shown that $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$ is an elementary class, and indeed this is not in general true. The intersection of all the $\mathbf{RA}_n$'s is the variety $\mathbf{RRA}$ [37], so the intersection of all the $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$'s is $\mathbb{S}\mathsf{tr}\mathbf{RRA}$ which is not an elementary class. Since the intersection of a family of elementary classes is elementary, some $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$ must fail to be elementary. In fact there must be some m such that $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$ fails to be elementary for all $n > m$. This is because the $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$'s form a nested sequence with $\mathbb{S}\mathsf{tr}\mathbf{RA}_n \supseteq \mathbb{S}\mathsf{tr}\mathbf{RA}_{n+1}$, so if there were arbitrarily large n with $\mathbb{S}\mathsf{tr}\mathbf{RA}_n$ elementary, then $\mathbb{S}\mathsf{tr}\mathbf{RRA}$ would be an intersection of elementary classes.

2.2. $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$

Let α and β be ordinals with $\beta < \alpha$. $\mathbf{CA}_\alpha$ is the variety of *cylindric algebras of dimension* α, and consists of BAO's of the type $\mathfrak{A} = (\mathfrak{A}_0, c_{ij}, d_{ij})_{i,j<\alpha}$, with each c_{ij} a unary operator called a *cylindrification* and each d_{ij} a member of of $\mathfrak{A}_0$ called a *diagonal element*. $\mathbf{CA}_\alpha$ can be defined by positive equations, and is a canonical variety for which $\mathbb{S}\mathsf{tr}\mathbf{CA}_\alpha$ is an elementary class [21, 2.7].

The members of $\mathbb{S}\mathsf{tr}\mathbf{CA}_\alpha$ are structures of the form $\mathfrak{S} = (S, R_i, E_{ij})_{i,j<\alpha}$ with R_i an equivalence relation on S, and E_{ij} a subset of S that is the diagonal element d_{ij} in $\mathsf{Cm}\,\mathfrak{S}$. Cylindrifications in $\mathsf{Cm}\,\mathfrak{S}$ are defined by

$$c_i X = \{y \in S : \exists x \in X(xR_i y)\}.$$

If $\mathfrak{A}$ is as above, then the algebra

$$\mathsf{Rd}_\beta\mathfrak{A} = (\mathfrak{A}_0, c_{ij}, d_{ij})_{i,j<\beta}$$

will be called the *β-reduct* of $\mathfrak{A}$. It is a member of $\mathbf{CA}_\beta$ and any of its subalgebras is a *subreduct* of $\mathfrak{A}$. The set

$$Cl_{\alpha-\beta}\mathfrak{A} = \{x \text{ in } \mathfrak{A}_0 : c_i x = x \text{ for all } i \in \alpha - \beta\}$$

is the underlying set of a subalgebra of $\mathsf{Rd}_\beta\mathfrak{A}$, comprising those elements that are left fixed by all cylindrifications c_i with $i \geq \beta$. This subalgebra is called the *neat β-reduct* of $\mathfrak{A}$, and denoted $\mathsf{Nr}_\beta\mathfrak{A}$. $\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ is the class of all neat β-reducts of members of $\mathbf{CA}_\alpha$, and its closure $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ under subalgebras is the class of all *neat β-subreducts* of $\mathbf{CA}_\alpha$-alebras.

$\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ was shown by Monk to be a canonical variety by direct algebraic constructions proving that it is closed under H, S, P and Em (see [21, 2.6.32 and 2.7.24]). These varieties play a significant role in cylindric algebra theory. In particular, $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_{\beta+\omega}$ coincides with the class $\mathbf{RCA}_\beta$ of all representable cylindric algebras of dimension β.

Now fix a member $\mathfrak{B} = (\mathfrak{B}_0, c_{ij}, d_{ij})_{i,j<\beta}$ of $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$. We will show that any ultrapower of $\mathbb{C}\mathsf{st}\,\mathfrak{B}$ belongs to $\mathbb{S}\mathsf{tr}\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$. Symbolically, this says that

$$\mathbb{P}\mathsf{w}\,\mathbb{C}\mathsf{st}\,\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha \subseteq \mathbb{S}\mathsf{tr}\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha.$$

By our Theorem 1.4 that is (more than) enough to show that $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ is elementarily generated.

Take $\mathfrak{A} \in \mathbf{CA}_\alpha$ such that $\mathfrak{B}$ is a subalgebra of the neat β-reduct $\mathsf{Nr}_\beta\mathfrak{A}$. Then $\mathfrak{B}$ is a subalgebra of the β-reduct $\mathsf{Rd}_\beta\mathfrak{A}$ of $\mathfrak{A}$, and $\mathfrak{B}_0$ is a Boolean subalgebra of $\mathfrak{A}_0$ included in $Cl_{\alpha-\beta}\mathfrak{A}$. To establish some notation, put

$$\mathbb{C}\mathsf{st}\,\mathfrak{B} = (S^{\mathfrak{B}}, R_i^{\mathfrak{B}}, E_{ij}^{\mathfrak{B}})_{i,j<\beta}$$
$$\mathbb{C}\mathsf{st}\,\mathfrak{A} = (S^{\mathfrak{A}}, R_i^{\mathfrak{A}}, E_{ij}^{\mathfrak{A}})_{i,j<\alpha}$$

and then

$$\mathbb{R}\mathsf{d}_\beta\mathbb{C}\mathsf{st}\,\mathfrak{A} = (S^{\mathfrak{A}}, R_i^{\mathfrak{A}}, E_{ij}^{\mathfrak{A}})_{i,j<\beta}.$$

Here $S^{\mathfrak{B}}$ is the set of all ultrafilters of $\mathfrak{B}_0$, and $S^{\mathfrak{A}}$ the set of ultrafilters of $\mathfrak{A}_0$. The structure $\mathbb{R}\mathsf{d}_\beta\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is by definition the *β-reduct* of the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. The relation $R_i^{\mathfrak{A}}$ is defined by

$$pR_i^{\mathfrak{A}}q \text{ iff } \{c_i x : x \in p\} \subseteq q.$$

By duality, the inclusion of $\mathfrak{B}$ into $\mathsf{Rd}_\beta\mathfrak{A}$ induces a surjection $\theta : S^{\mathfrak{A}} \twoheadrightarrow S^{\mathfrak{B}}$, given by $\theta(x) = x \cap B_0$, where B_0 is the underlying set of $\mathfrak{B}_0$. θ is a bounded morphism from $\mathbb{C}\mathsf{st}\,\mathsf{Rd}_\beta\mathfrak{A}$ onto $\mathbb{C}\mathsf{st}\,\mathfrak{B}$.

Now let U be an ultrafilter on a set I. For a structure $\mathfrak{S}$, the elements of the U-ultrapower $\Pi_U\mathfrak{S}$ will be denoted f^U, where $f \in S^I$. Here $f^U = \{g \in S^I : \{k \in I : f(k) = g(k)\} \in U\}$. The ultrapower construction lifts the bounded epimorphism $\theta : \mathbb{C}\mathsf{st}\,\mathsf{Rd}_\beta\mathfrak{A} \to \mathbb{C}\mathsf{st}\,\mathfrak{B}$ just defined to the map

$$\theta_U : \Pi_U\mathbb{C}\mathsf{st}\,\mathsf{Rd}_\beta\mathfrak{A} \to \Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$$

having $\theta_U(f^U) = (\theta \circ f)^U$. θ_U is a bounded epimorphism, and so by duality it induces an injective $\mathbf{CA}_\beta$-homomorphism

$$\theta_U^+ : \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B} \rightarrowtail \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathsf{Rd}_\beta\mathfrak{A},$$

which acts by $X \mapsto \theta_U^{-1}(X)$.

Now because reduct formations just involve deleting operations and relations without changing the underlying set of the algebra or structure, it is evident that they commute with many operations. In particular we can show such facts as $\mathbb{C}\mathsf{st}\,\mathsf{Rd}_\beta\mathfrak{A} = \mathbb{R}\mathsf{d}_\beta\mathbb{C}\mathsf{st}\,\mathfrak{A}$, $\Pi_U\mathbb{R}\mathsf{d}_\beta\mathfrak{S} = \mathbb{R}\mathsf{d}_\beta\Pi_U\mathfrak{S}$, and $\mathsf{Cm}\,\mathbb{R}\mathsf{d}_\beta\mathfrak{S} = \mathsf{Rd}_\beta\mathsf{Cm}\,\mathfrak{S}$. Using these to re-write the codomain of θ_U^+ we get

$$\theta_U^+ : \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B} \rightarrowtail \mathsf{Rd}_\beta\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}.$$

Since $\mathbf{CA}_\alpha$ is canonical, $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ belongs to the elementary class $\mathbb{S}\mathsf{tr}\mathbf{CA}_\alpha$, and therefore so does the ultrapower $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$. Hence $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$ belongs to $\mathbf{CA}_\alpha$. So, if we can show that

(†) θ_U^+ maps $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ into $Cl_{\alpha-\beta}\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$,

this will make $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ isomorphic to a subalgebra of the neat β-reduct $\mathsf{Nr}_\beta\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$, and hence a member of $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$. That puts $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ into $\mathbb{S}\mathsf{tr}\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$, showing that the latter contains all ultrapowers of $\mathbb{C}\mathsf{st}\,\mathfrak{B}$, as desired.

For each element X of $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$, put $X^+ = \theta_U^+(X) = \theta_U^{-1}(X)$. To prove (†) we have to show that $c_iX^+ = X^+$ in $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$ for any $i \geq \beta$. Let R_i^U be the relation in the ultrapower $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$ that determines c_i. By definition,

$$f^U R_i^U g^U \quad \text{iff} \quad \{k \in I : f(k)R_i^{\mathfrak{A}}g(k) \text{ in } \mathbb{C}\mathsf{st}\,\mathfrak{A}\} \in U.$$

Since $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is a cylindric algebra it is immediate that $X^+ \subseteq c_iX^+$. For the converse inclusion, let $g^U \in c_iX^+$, so that $f^U R_i^U g^U$ for some $f^U \in X^+$. Hence $\theta_U(f^U) \in X$ and the set

$$J = \{k \in I : f(k)R_i^{\mathfrak{A}}g(k)\}$$

belongs to U. Now for any $k \in J$, if $x \in \theta(f(k)) = f(k) \cap B_0$, then $c_ix \in g(k)$ as $x \in f(k)$ and $f(k)R_i^{\mathfrak{A}}g(k)$; while $x \in B_0 \subseteq Cl_{\alpha-\beta}\mathfrak{A}$ so $c_ix = x$; hence $x \in g(k) \cap B_0 = \theta(g(k))$. This proves that $\theta(f(k)) \subseteq \theta(g(k))$ and therefore that $\theta(f(k)) = \theta(g(k))$ as both are ultrafilters of $\mathfrak{B}_0$. Thus the functions $\theta \circ f$ and $\theta \circ g$ agree on the set $J \in U$, which is enough to force $(\theta \circ f)^U = (\theta \circ g)^U$, i.e. $\theta_U(f^U) = \theta_U(g^U)$. But $\theta_U(f^U) \in X$, and so $g^U \in \theta_U^{-1}(X) = X^+$.

This establishes that $c_iX^+ = X^+$, completing the proof of (†), hence the proof that $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ is a $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$-structure, and thereby completing the proof that $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ is elementarily generated.

2.3. $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$

From any cylindric algebra $\mathfrak{A} = (\mathfrak{A}_0, c_{ij}, d_{ij})_{i,j<\alpha}$ with $\alpha \geq 3$ an $\mathbf{RA}$-type algebra $\mathfrak{Ra}\mathfrak{A}$ can be defined on the underlying set of the neat 2-reduct $\mathsf{Nr}_2\mathfrak{A}$

of $\mathfrak{A}$. This uses the *substitution operators* of $\mathfrak{A}$: for any distinct $i, j < \alpha$, let $s^i_j b = c_i(d_{ij} \cdot b)$. Then s^i_j is an operator on $\mathfrak{A}_0$ that is *completely additive* in the sense that if B is any subset that has a join ΣB in $\mathfrak{A}_0$, then $s^i_j \Sigma B = \Sigma\{s^i_j b : b \in B\}$. This follows from the complete additivity of the cylindrifications c_i of $\mathfrak{A}$ and the Boolean principle that $a \cdot \Sigma B = \Sigma\{a \cdot b : b \in B\}$ (i.e. complete additivity of the Boolean product (meet) operation).

The set $Cl_{\alpha-2}\mathfrak{A}$ underlying $\mathsf{Nr}_2\mathfrak{A}$ is closed under the operations ; , $\breve{}$, 1' defined by

$$a;b = c_2(s^1_2 a \cdot s^0_2 b), \qquad a^{\smile} = s^2_0 s^0_1 s^1_2 a, \qquad 1\text{'} = d_{01}. \tag{2.iv}$$

From the complete additivity of c_i and s^i_j it follows readily that $\breve{}$ is completely additive and ; is completely additive in each of its arguments. Define

$$\mathfrak{Ra}\mathfrak{A} = (Cl_{\alpha-2}\mathfrak{A}, \; ; \; , \; \breve{} \; , \; 1\text{'}).$$

Henkin and Tarski, who developed the $\mathfrak{Ra}$ construction, proved that $\mathfrak{Ra}\mathfrak{A}$ is a relation algebra whenever $\alpha \geq 4$ [22, 5.3.8]. For $\alpha = 3$ this can fail (see [24, p. 191] for discussion). For $\alpha \geq 3$, the class $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ is a canonical variety, as may be shown by a similar proof to that for $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_a$ [24, 5.48].

In order to give a structural proof that $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ is elementarily generated, we introduce a new construction $\mathbb{M}\mathfrak{S}$ of $\mathbf{CA}_\alpha$-structures that dualizes the operations (2.iv) on cylindric algebras, and which plays an analogous role for $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ to that played by $\mathbb{R}\mathrm{d}_\beta$ for $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$. First note that these operations (2.iv) are defined on the whole of the Boolean algebra $\mathfrak{A}_0$, so we can form the algebra

$$\mathsf{M}\,\mathfrak{A} = (\mathfrak{A}_0, \; ; \; , \; \breve{} \; , \; 1\text{'}).$$

Then $\mathfrak{Ra}\mathfrak{A}$ is the subalgebra of $\mathsf{M}\,\mathfrak{A}$ based on $Cl_{\alpha-2}\mathfrak{A}$. (The letter M could stand for "monoid", since 1' is an identity element for the operation ; , which is associative when $\alpha \geq 4$.)

Now for a $\mathbf{CA}_\alpha$-structure $\mathfrak{S} = (S, R_i, E_{ij})_{i,j<\alpha}$, put

$$\mathbb{M}\mathfrak{S} = (S, C, R, E_{01}),$$

where $C \subseteq S^3$ and $R \subseteq S^2$ are defined by

$$C(x, y, z) \quad \text{iff} \quad z \in \{x\};\{y\} \text{ in } \mathsf{M}\,\mathsf{Cm}\,\mathfrak{S},$$
$$xRy \quad \text{iff} \quad y \in \{x\}^{\smile} \text{ in } \mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}.$$

LEMMA 2.1. $\mathsf{Cm}\,\mathbb{M}\mathfrak{S} = \mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}$.

PROOF. $\mathsf{Cm}\,\mathbb{M}\mathfrak{S}$ and $\mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}$ are both based on the Boolean powerset algebra of S, and both have E_{01} as their distinguished element 1'. Let f be the binary operator on $\mathsf{Cm}\,\mathbb{M}\mathfrak{S}$ determined by C, i.e.

$$f(X,Y) = \{z \in S : \exists x \in X\, \exists y \in Y\, C(x,y,z)\}.$$

But since the operation $X;Y$ given by the definition of $\mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}$ is completely additive in X and in Y we get

$$X;Y = \bigcup_{x\in X, y\in Y} \{x\};\{y\} = \{z \in S : \exists x \in X\, \exists y \in Y\, (z \in \{x\};\{y\})\,\}$$

which is equal to $f(X,Y)$ by definition of C.

Similarly, if $gX = \{y \in S : \exists x \in X\, (xRy)\}$ is the unary operator on $\mathsf{Cm}\,\mathbb{M}\mathfrak{S}$ determined by R, then by complete additivity of $\smile$ in $\mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}$ we get $X^{\smile} = \bigcup_{x\in X}\{x\}^{\smile} = gX$. Thus the operators of $\mathsf{Cm}\,\mathbb{M}\mathsf{S}$ and $\mathsf{M}\,\mathsf{Cm}\,\mathfrak{S}$ are identical. ■

LEMMA 2.2. *$\Pi_U \mathbb{M}\mathfrak{S} = \mathbb{M}\Pi_U \mathfrak{S}$, for any ultrafilter U.*

PROOF. Let U be an ultrafilter on set I. The ultrapowers $\Pi_U\mathfrak{S}$ and $\Pi_U\mathbb{M}\mathfrak{S}$, and the structure $\mathbb{M}\Pi_U\mathfrak{S}$, are all based on the ultrapower $\Pi_U S$ of set S. We have $\Pi_U\mathbb{M}\mathfrak{S} = (\Pi_U S, C^U, R^U, E_{01}^U)$, where

$$\begin{array}{ll} C^U(f^U, g^U, h^U) & \text{iff } \{k \in I : C(f(k), g(k), h(k))\} \in U, \\ f^U R g^U & \text{iff } \{k \in I : f(k) R g(k)\} \in U, \\ f^U \in E_{01}^U & \text{iff } \{k \in I : f(k) \in E_{01}\} \in U. \end{array}$$

$\Pi_U\mathfrak{S} = (\Pi_U S, R_i^U, E_{ij}^U)_{i,j<\alpha}$ is defined likewise. Then

$$\mathbb{M}\Pi_U\mathfrak{S} = (\Pi_U S, C', R', E_{01}^U),$$

where C' and R' are defined from $\mathsf{M}\,\mathsf{Cm}\,\Pi_U\mathfrak{S}$ by the $\mathbb{M}$ construction. So all we have to show is that $C' = C^U$ and $R' = R^U$.

Now the relation $C(x,y,z)$ in $\mathbb{M}\mathfrak{S}$ is definable by a formula $\varphi_C(x,y,z)$ in the first-order language of any $\mathbf{CA}_\alpha$-structure $\mathfrak{S}$. From the way that the cylindrification c_i is defined from the relation R_i, we have that $u \in s^i_j\{v\}$ in $\mathsf{Cm}\,\mathfrak{S}$ iff $\exists w(wR_iu$ and $w \in E_{ij} \cap \{v\})$, which is equivalent to $(vR_iu \wedge v \in E_{ij})$. So for $\varphi_C(x,y,z)$ we can use the (quasi-modal) formula

$$\exists u(uR_2z \wedge xR_1u \wedge x \in E_{12} \wedge yR_0u \wedge y \in E_{02}).$$

By the fundamental theorem of satisfaction in ultraproducts (Łoś's theorem),

$$\Pi_U \mathfrak{S} \models \varphi_C(f^U, g^U, h^U) \quad \text{iff} \quad \{k : \mathfrak{S} \models \varphi_C(f(k), g(k), h(k))\} \in U,$$

which just says that

$$C'(f^U, g^U, h^U) \quad \text{iff} \quad \{k : C(f(k), g(k), h(k))\} \in U,$$

implying that $C' = C^U$. Similarly there is a formula $\varphi_R(x, y)$ in the language of $\mathbf{CA}_\alpha$-structures that defines the binary relation R of $\mathbb{M}\mathfrak{S}$, and this can be used in the same way to show that $R' = R^U$. ∎

LEMMA 2.3. $\mathbb{M}\mathbb{C}\mathsf{st}\,\mathfrak{A} = \mathbb{C}\mathsf{st}\,\mathsf{M}\,\mathfrak{A}$.

PROOF. Let $\mathbb{C}\mathsf{st}\,\mathsf{M}\,\mathfrak{A} = (S_0, C_0, R_0, E_0)$, where S_0 is the set of ultrafilters of $\mathfrak{A}_0$, C_0 is the ternary relation on S_0 defined by the operator ; of $\mathsf{M}\,\mathfrak{A}$, R_0 the binary relation defined by $\breve{}$, and $E_0 = \{p \in S_0 : 1\text{'} \in p\}$ is the set (unary relation) determined by the identity element 1' of $\mathsf{M}\,\mathfrak{A}$..

C_0 and R_0 in turn define operators ; , $\breve{}$ on the power set of S_0. All that we will need to know about those operators is how they act on singleton subsets $\{p\}$ of S_0, which is, by definition,

$$\{p\};\{q\} = \{r \in S_0 : C_0(p, q, r)\}, \qquad (2.\text{v})$$

$$\{p\}^{\smile} = \{q \in S_0 : pR_0q\}. \qquad (2.\text{vi})$$

Next let $\mathbb{M}\mathbb{C}\mathsf{st}\,\mathfrak{A} = (S_0, C, R, E)$. Here C, R, E are defined from the operations ; , $\breve{}$, 1' of $\mathsf{M}\,\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A} = \mathsf{M}\,\mathsf{Em}\,\mathfrak{A}$, and hence from the cylindric algebra operations of $\mathsf{Em}\,\mathfrak{A}$. In particular, E is the 01-diagonal element of $\mathsf{Em}\,\mathfrak{A}$, which is the set $\{p \in S_0 : d_{01} \in p\}$. Since $d_{01} = 1\text{'}$ (2.iv), this shows that $E = E_0$.

The relation C is defined by

$$C(p, q, r) \quad \text{iff} \quad r \in c_2(s_2^1\{p\} \cdot s_2^0\{q\}) \text{ in } \mathsf{Em}\,\mathfrak{A}.$$

To show that $C = C_0$ we introduce the algebra $\mathfrak{A}^+ = (\mathfrak{A}, ; , \breve{}, 1\text{'})$, the expansion of $\mathfrak{A}$ itself by the operations of (2.iv). The $\mathbf{CA}_\alpha$-reduct of $\mathsf{Em}\,\mathfrak{A}^+$ is just $\mathsf{Em}\,\mathfrak{A}$, and the $\{; , \breve{}, 1\text{'}\}$-reduct is $\mathsf{Em}\,\mathsf{M}\,\mathfrak{A}$. Since the equations in (2.iv) are all positive (do not involve Boolean complementation), they are preserved by canonical extensions, and so they continue to hold in $\mathsf{Em}\,\mathfrak{A}^+$. Therefore

$$\{p\};\{q\} = c_2(s_2^1\{p\} \cdot s_2^0\{q\}) \text{ in } \mathsf{Em}\,\mathfrak{A}^+.$$

Thus $r \in c_2(s_2^1\{p\} \cdot s_2^0\{q\})$ iff $r \in \{p\};\{q\}$ iff (by 2.v) $C_0(p,q,r)$. This proves $C = C_0$.

Similarly, R is defined by

$$pRq \quad \text{iff} \quad q \in s_0^2 s_1^0 s_2^1\{p\} \text{ in } \mathsf{Em}\,\mathfrak{A}.$$

But $\{p\}^{\smile} = s_0^2 s_1^0 s_2^1\{p\}$ in $\mathsf{Em}\,\mathfrak{A}^+$, by preservation of the equations (2.iv) again, so from (2.vi) we get pRq iff pR_0q, and so $R = R_0$. ■

We can now go ahead and apply the three Lemmas 2.1–2.3 to show that if $\mathfrak{B} \in \mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$ then any ultrapower $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ of the canonical structure of $\mathfrak{A}$ is a $\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$-structure, i.e.

$$\mathbb{P}\mathsf{w}\,\mathbb{C}\mathsf{st}\,\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha \subseteq \mathbb{S}\mathsf{tr}\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha,$$

implying the desired result that $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_\alpha$ is elementarily generated. The reasoning is exactly parallel to the $\mathsf{S}\,\mathsf{Nr}_\beta\mathbf{CA}_\alpha$ case, and makes use of some of the work already done for that case.

Since $\mathfrak{B} \in \mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$, there is some $\mathfrak{A} \in \mathbf{CA}_\alpha$ such that $\mathfrak{B}$ is a subalgebra of $\mathfrak{Ra}\mathfrak{A}$, and hence of $\mathsf{M}\,\mathfrak{A}$. By duality this induces the bounded epimorphism $\theta : \mathbb{C}\mathsf{st}\,\mathsf{M}\,\mathfrak{A} \twoheadrightarrow \mathbb{C}\mathsf{st}\,\mathfrak{B}$, given by $\theta(x) = x \cap B_0$, where B_0 is the underlying set of $\mathfrak{B}$. Then for any ultrafilter U, θ lifts to the bounded epimorphism

$$\theta_U : \Pi_U\mathbb{C}\mathsf{st}\,\mathsf{M}\,\mathfrak{A} \to \Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$$

having $\theta_U(f^U) = (\theta \circ f)^U$, which in turn induces the injective homomorphism

$$\theta_U^+ : \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B} \rightarrowtail \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathsf{M}\,\mathfrak{A}$$

acting by $X \mapsto \theta_U^{-1}(X)$. Using 2.3, 2.2 and then 2.1, we rewrite the codomain of θ_U^+, to get

$$\theta_U^+ : \mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B} \rightarrowtail \mathsf{M}\,\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}.$$

But, as explained in Section 2.2, $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbf{CA}_\alpha$ because $\mathbf{CA}_\alpha$ is canonical and $\mathbb{S}\mathsf{tr}\mathbf{CA}_\alpha$ is elementary. So, if we can show that

(‡) θ_U^+ maps $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ into $Cl_{\alpha-2}\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$,

this will make $\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ isomorphic to a subalgebra of $\mathfrak{Ra}\mathsf{Cm}\,\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{A}$, and hence a member of $\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$, putting $\Pi_U\mathbb{C}\mathsf{st}\,\mathfrak{B}$ into $\mathbb{S}\mathsf{tr}\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$ as desired. But since $B_0 \subseteq Cl_{\alpha-2}\mathfrak{A}$, (‡) is just the case $\beta = 2$ of the result (†) proven in Section 2.2.

Finally, we note that $\mathbb{S}\mathsf{tr}\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha$ is not itself elementary in general. For infinite α this follows because in that case $\mathsf{S}\,\mathfrak{Ra}\mathbf{CA}_\alpha = \mathbf{RRA}$ [24, p. 192].

For $3 \leq n < \omega$, the $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_n$'s form a nested sequence with $\mathsf{S}\mathfrak{Ra}\mathbf{CA}_n \supseteq \mathsf{S}\mathfrak{Ra}\mathbf{CA}_{n+1}$, and with the intersection of the sequence being $\mathbf{RRA}$. From the fact that $\mathbb{S}\mathsf{tr}\mathbf{RRA}$ is not elementary it then follows, analogously to the $\mathbf{RA}_n$ case, that for cofinitely many n, $\mathbb{S}\mathsf{tr}\mathsf{S}\mathfrak{Ra}\mathbf{CA}_n$ must fail to be elementary.

3. Canonical Extensions as Free Objects

In this final Section we disuss the sense in which the operation $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ can be characterized in category-theoretic terms by a universal property of freeness. A natural setting for this involves structures that carry a topology. Before describing that setting we first review the analogous categorical characterization of algebras constructed from formulas.

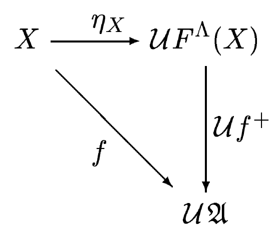
3.1. Lindenbaum Algebras are Free

Let Λ be a normal propositional modal logic. For each set X, the Lindenbaum algebra construction produces a Λ-algebra $F^\Lambda(X)$, built from the language for Λ that takes X as its set of propositional variables. Two formulas φ, ψ of this language are said to be Λ-*equivalent* if the biconditional $\varphi \leftrightarrow \psi$ is a theorem of Λ. This defines an equivalence relation on the set of formulas that is a congruence for the logical connectives. $F^\Lambda(X)$ is the resulting quotient algebra, based on the set of equivalence classes $|\varphi|$ for all formulas φ. In $F^\Lambda(X)$, $|\varphi| = 1$ precisely when φ is a Λ-theorem.

We will now write $\mathcal{U}\mathfrak{A}$ for the underlying set of an algebra $\mathfrak{A}$. A function of the form $f : X \to \mathcal{U}\mathfrak{A}$ is a *valuation*, assigning a value in $\mathfrak{A}$ to each variable from X. This extends to give a value $f(\varphi) \in \mathcal{U}\mathfrak{A}$ to each formula by interpreting the connectives of φ by the corresponding operations in $\mathfrak{A}$. φ is *valid in* $\mathfrak{A}$ when $f(\varphi) = 1$ for all such valuations. $\mathfrak{A}$ is a Λ-*algebra* when all Λ-theorems are valid in $\mathfrak{A}$.

As is well known, $F^\Lambda(X)$ is a Λ-algebra. It has a special valuation $\eta_X : X \to \mathcal{U}F^\Lambda(X)$ defined by $\eta_X(p) = |p|$ for all $p \in X$. Then $\eta_X(\varphi) = |\varphi|$ for all formulas φ, showing that $F^\Lambda(X)$ invalidates all non-theorems of Λ under η_X. If $\mathfrak{A}$ is any Λ-algebra, then a valuation $f : X \to \mathcal{U}\mathfrak{A}$ will assign the same value to any two Λ-equivalent formulas, so a function $f^+ : \mathcal{U}F^\Lambda(X) \to \mathcal{U}\mathfrak{A}$ is well-defined by putting $f^+(|\varphi|) = f(\varphi)$. Then f^+ is a homomorphism from $F^\Lambda(X)$ to $\mathfrak{A}$ that has $f^+(|p|) = f(p)$ for all variables $p \in X$, and is the only homomorphism with this property. This is because f^+ is determined by its values on the $|p|$'s, since they generate the algebra $F^\Lambda(X)$. To sum up:

(3.i) For any Λ-algebra $\mathfrak{A}$ and any function $f : X \to \mathcal{U}\mathfrak{A}$, there is exactly one homomorphism $f^+ : F^\Lambda(X) \to \mathfrak{A}$ such that $\mathcal{U}f^+ \circ \eta_X = f$:



Here we use the notation $\mathcal{U}f^+$ when f^+ is being considered as a set function, forgetting that it is also a homomorphism. In universal algebraic terms, (3.i) states that the algebra $F^\Lambda(X)$ is *free over* X *in the variety* $\mathcal{V}^\Lambda$ *of all* Λ *algebras*. In categorical terms, the maps $\mathfrak{A} \mapsto \mathcal{U}\mathfrak{A}$ and $g \mapsto \mathcal{U}g$ constitute the *forgetful functor* $\mathcal{V}^\Lambda \to \mathbf{Set}$, where **Set** is the category of sets and functions, while $\mathcal{V}^\Lambda$ is viewed as a category whose arrows are the homomorphisms between Λ-algebras. In categorical terms, (3.i) states that the pair $(F^\Lambda(X), \eta_X)$ is *free over* X *with respect to the functor* $\mathcal{U}$. This property characterizes the pair uniquely up to a unique isomorphism, by a standard argument (see Theorem 3.2 below).

3.2. Topological Structures

By a *type* we will mean a function of the form $\tau : I \to \{1, 2, 3, \dots\}$. A *structure of type* τ has the form $\mathfrak{S} = (S, \{R_i : i \in I\})$ with R_i being a $\tau_i + 1$-ary relation on S. Then the complex algebra $\mathsf{Cm}\,\mathfrak{S}$ is a BAO *of type* τ, having the τ_i-ary operator f_{R_i} corresponding to each R_i, as defined in Section 2. The *dual* operation $f^\delta_{R_i}$ to f_{R_i} is defined by

$$f^\delta_{R_i}(X_1, \dots, X_{\tau_i}) = -f_{R_i}(-X_1, \dots, -X_{\tau_i}).$$

Given τ-structures $\mathfrak{S}_k = (S_k, \{R^k_i : i \in I\})$ for $k = 1, 2$, a *weak morphism* $\theta : \mathfrak{S}_1 \to \mathfrak{S}_2$ is a function $\theta : S_1 \to S_2$ satisfying, for each $i \in I$,

$$R^1_i(x_1, \dots, x_{\tau_i+1}) \quad \text{implies} \quad R^2_i(\theta(x_1), \dots, \theta(x_{\tau_i+1})).$$

A *bounded morphism* is a weak morphism that also satisfies

$R^2_i(y_1, \dots, y_{\tau_i}, \theta(x))$ implies $R^1_i(x_1, \dots, x_{\tau_i}, x)$ for some $x_1, \dots, x_{\tau_i} \in S_1$ such that $\theta(x_j) = y_j$ for all $j \le \tau_i$.

By a *topological τ-structure* we will mean a τ-structure $\mathfrak{S}$ that carries a topology $\mathcal{T}^{\mathfrak{S}}$ such that the following hold for each $i \in I$:

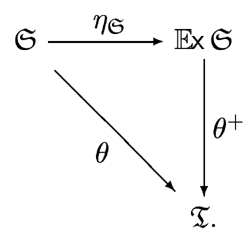
- if $X_1, \ldots, X_{\tau_i}$ are $\mathcal{T}^{\mathfrak{S}}$-open subsets of S, then so are $f_{R_i}(X_1, \ldots, X_{\tau_i})$ and $f^{\delta}_{R_i}(X_1, \ldots, X_{\tau_i})$;
- for each $x \in S$, the set $R_i^{-1}(x) = \{(x_1, \ldots, x_{\tau_i}) : R_i(x_1, \ldots, x_{\tau_i}, x)\}$ is closed in the product topology on S^{τ_i}.

The canonical extension $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ of any τ-structure becomes a topological structure under the *Stone topology*, which has as base the sets $\|X\|$ for all $X \subseteq S$, where $\|X\|$ is the set of ultrafilters on S to which X belongs. The Stone topology makes $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ into a compact, Hausdorff and totally connected space (i.e. a *Stone space* [27]).

There is a natural injection $\eta_{\mathfrak{S}} : \mathfrak{S} \to \mathbb{E}\mathrm{x}\,\mathfrak{S}$ for which $\eta_{\mathfrak{S}}(x) = \{X : x \in X\}$ is the principal ultrafilter on S generated by x. $\eta_{\mathfrak{S}}$ is continuous with respect to the given topology on $\mathfrak{S}$ and the Stone topology on $\mathbb{E}\mathrm{x}\,\mathfrak{S}$, and is always a weak morphism. But $\eta_{\mathfrak{S}}$ is not in general bounded. In fact it is bounded precisely when the inverse-image sets $R_i^{-1}(x)$ are all finite.

We can now state an analogue of (3.i) for topological τ-structures $\mathfrak{S}$:

THEOREM 3.1. *For any compact Hausdorff τ-structure $\mathfrak{T}$ and any continuous function θ from $\mathfrak{S}$ to $\mathfrak{T}$, there is a unique continuous function $\theta^+ : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{T}$ such that $\theta^+ \circ \eta_{\mathfrak{S}} = \theta$:*



Moreover, if θ is a weak or bounded morphism, then so is θ^+, respectively.

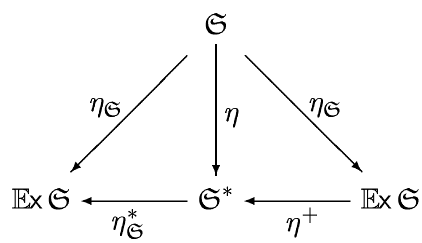
■

The function θ^+ is provided by the standard topological theory of convergence of ultrafilters. An ultrafilter on $\mathfrak{T}$ *converges* to a point y of $\mathfrak{T}$ if it contains all of the open neighbourhoods of y in $\mathfrak{T}$. Since $\mathfrak{T}$ is compact and Hausdorff, every ultrafilter on $\mathfrak{T}$ converges to exactly one point, defining a map $\varepsilon_{\mathfrak{T}} : \mathbb{E}\mathrm{x}\,\mathfrak{T} \to \mathfrak{T}$. But by duality, θ lifts to the map $\mathbb{E}\mathrm{x}\,\theta : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathbb{E}\mathrm{x}\,\mathfrak{T}$ such that for each ultrafilter p on $\mathfrak{S}$, $\mathbb{E}\mathrm{x}\,\theta(p)$ is the ultrafilter $\{Y : \theta^{-1}Y \in p\}$ on $\mathfrak{T}$. Then $\theta^+(p)$ is defined to be the unique point of $\mathfrak{T}$ to which $\mathbb{E}\mathrm{x}\,\theta(p)$ converges. Thus $\theta^+ = \varepsilon_{\mathfrak{T}} \circ \mathbb{E}\mathrm{x}\,\theta$.

Now $\mathbb{E}\mathrm{x}\,\theta$ is continuous, weak, or bounded whenever θ has the corresponding property. But $\varepsilon_{\mathfrak{T}}$ is always a continuous bounded morphism for compact Hausdorff $\mathfrak{T}$, and so Theorem 3.1 follows from these observations. It leads to the following characterization of the canonical structure $\mathbb{E}\mathrm{x}\,\mathfrak{S}$.

THEOREM 3.2. *Let $\mathfrak{S}^*$ be a compact Hausdorff τ-structure with a continuous weak morphism $\eta : \mathfrak{S} \to \mathfrak{S}^*$ such that for any compact Hausdorff τ-structure $\mathfrak{T}$ and any continuous weak morphism $\theta : \mathfrak{S} \to \mathfrak{T}$ there is a unique continuous weak morphism $\theta^* : \mathfrak{S}^* \to \mathfrak{T}$ with $\theta^* \circ \eta = \theta$. Then there exists a unique homeomorphic isomorphism $\rho : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{S}^*$ such that $\rho \circ \eta_{\mathfrak{S}} = \eta$.*

PROOF. Consider the diagram



η^+ is the continuous weak morphism resulting from taking η as θ in Theorem 3.1, while $\eta^*_{\mathfrak{S}}$ is the one resulting from taking $\eta_{\mathfrak{S}}$ as θ in the assumed property of $(\mathfrak{S}^*, \eta)$. The composition $\eta^*_{\mathfrak{S}} \circ \eta^+ : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathbb{E}\mathrm{x}\,\mathfrak{S}$ is a continuous weak morphism, and hence by the uniqueness expressed in Theorem 3.1 must be the identity function on $\mathbb{E}\mathrm{x}\,\mathfrak{S}$. Similarly, $\eta^+ \circ \eta^*_{\mathfrak{S}}$ is the identity function on $\mathfrak{S}^*$. Thus we put $\rho = \eta^+$ to obtain a bijective continuous weak morphism whose inverse $\eta^*_{\mathfrak{S}}$ is also a continuous weak morphism. This ensures that ρ is a homeomorphism of topological spaces and an isomorphism of relational structures. ■

Given the importance of bounded morphisms in our theory, it may seem surprising that only weak morphisms are involved in this characterization of canonical extensions of structures. But the explanation lies in the observation that in order for a bijection $\theta : \mathfrak{S}_1 \to \mathfrak{S}_2$ to be an isomorphism it suffices that both θ and θ^{-1} be *weak* morphisms, for that is enough to ensure that in general

$$R^1_i(x_1, \ldots, x_{\tau_i+1}) \quad \text{iff} \quad R^2_i(\theta(x_1), \ldots, \theta(x_{\tau_i+1})).$$

Theorem 3.1 also admits a more categorical description. Let $\mathbf{TopSt}_\tau$ be the category of topological τ-structures and continuous weak morphisms, and $\mathbf{CHSt}_\tau$ its full subcategory of compact Hausdorff structures. Then 3.1 states that the pair $(\mathbb{E}\mathrm{x}\,\mathfrak{S}, \eta_{\mathfrak{S}})$ is *free over $\mathfrak{S}$ with respect to the inclusion functor* $\mathbf{CHSt}_\tau \to \mathbf{TopSt}_\tau$. The assignments $\mathfrak{S} \mapsto \mathbb{E}\mathrm{x}\,\mathfrak{S}$ and $\theta \mapsto \mathbb{E}\mathrm{x}\,\theta$

give a functor $\mathbb{E}\mathrm{x} : \mathbf{TopSt}_\tau \to \mathbf{CHSt}_\tau$ that is left adjoint to this inclusion, making $\mathbf{CHSt}_\tau$ into what is known as a *reflective* subcategory of $\mathbf{TopSt}_\tau$ [23, 36.1].

There appears to be no corresponding analysis of the subcategories of $\mathbf{TopSt}_\tau$ and $\mathbf{CHSt}_\tau$ whose arrows are bounded morphisms. The obstacle is that the functions $\eta_{\mathfrak{S}}$ are in general only weak, and not bounded, morphisms, so do not belong to those subcategories.

A different, non-topological, approach to characterizing $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ can be developed from categorical structure associated with $\mathbb{E}\mathrm{x}$ as a functor on the category of τ-structures and weak morphisms (technically, this categorical structure is known as a *monad*). By a *retraction* onto a structure $\mathfrak{S}$ we will mean any function $\rho : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{S}$ such that $\rho \circ \eta_{\mathfrak{S}}$ is the identity function on $\mathfrak{S}$, i.e. $x = \rho(\eta_{\mathfrak{S}}(x))$ in general. Given retractions ρ_1, ρ_2 onto structures $\mathfrak{S}_1, \mathfrak{S}_2$ respectively, a function $\theta : S_1 \to S_2$ is called *retraction preserving* if $\theta \circ \rho_1 = \rho_2 \circ \mathbb{E}\mathrm{x}\,\theta$:

$$\begin{array}{ccc} \mathbb{E}\mathrm{x}\,\mathfrak{S}_1 & \xrightarrow{\mathbb{E}\mathrm{x}\,\theta} & \mathbb{E}\mathrm{x}\,\mathfrak{S}_2 \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ \mathfrak{S}_1 & \xrightarrow{\theta} & \mathfrak{S}_2 \end{array}$$

There is a natural retraction onto any canonical extension $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ given by the function $\mu_{\mathfrak{S}} : \mathbb{E}\mathrm{x}\,\mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathbb{E}\mathrm{x}\,\mathfrak{S}$ having, for any ultrafilter q on $\mathbb{E}\mathrm{x}\,\mathfrak{S}$,

$$\mu_{\mathfrak{S}}(q) = \{X \subseteq S : \|X\| \in q\},$$

where $\|X\| = \{p \in \mathbb{E}\mathrm{x}\,\mathfrak{S} : X \in p\}$ as above. $\mu_{\mathfrak{S}}$ is a bounded morphism of τ-structures.

By a *retraction structure* we will mean a τ-structure $\mathfrak{S}$ with a specified retraction $\rho_{\mathfrak{S}} : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{S}$ which is itself a bounded morphism that is retraction preserving from $(\mathbb{E}\mathrm{x}\,\mathfrak{S}, \mu_{\mathfrak{S}})$ to $(\mathfrak{S}, \rho_{\mathfrak{S}})$:

$$\begin{array}{ccc} \mathbb{E}\mathrm{x}\,\mathbb{E}\mathrm{x}\,\mathfrak{S} & \xrightarrow{\mathbb{E}\mathrm{x}\,\rho_{\mathfrak{S}}} & \mathbb{E}\mathrm{x}\,\mathfrak{S} \\ {\scriptstyle \mu_{\mathfrak{S}}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathfrak{S}}} \\ \mathbb{E}\mathrm{x}\,\mathfrak{S} & \xrightarrow{\rho_{\mathfrak{S}}} & \mathfrak{S} \end{array}$$

The category $\mathbf{Ret}_\tau$ has retraction structures as objects and retraction preserving weak morphisms as arrows, while $\mathbf{St}_\tau$ is the category of τ-structures and weak morphisms. It turns out that for any τ-structure $\mathfrak{S}$, the canonical extension $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ is a retraction structure under the retraction $\mu_{\mathfrak{S}}$. Thus $\mathbb{E}\mathrm{x}$ becomes a functor from $\mathbf{St}_\tau$ to $\mathbf{Ret}_\tau$ that is left adjoint to the forgetful functor $\mathcal{U} : \mathbf{Ret}_\tau \to \mathbf{St}_\tau$ which forgets about retractions and retraction preservation. The result which guarantees this is the following analogue of Theorem 3.1.

THEOREM 3.3. *For any retraction τ-structure $\mathfrak{T}$ and any weak morphism θ from $\mathfrak{S}$ to $\mathcal{U}\mathfrak{T}$, there is a unique retraction reserving weak morphism $\theta^+ :$ $\mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{T}$ such that $\mathcal{U}\theta^+ \circ \eta_{\mathfrak{S}} = \theta$:*

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\eta_{\mathfrak{S}}} & \mathbb{E}\mathrm{x}\,\mathfrak{S} \\ & {\scriptstyle\theta}\searrow & \downarrow {\scriptstyle \mathcal{U}\theta^+} \\ & & \mathcal{U}\mathfrak{T}. \end{array}$$

■

Now if $\mathfrak{S}$ is a compact Hausdorff τ-structure, then there is a function $\varepsilon_{\mathfrak{S}} : \mathbb{E}\mathrm{x}\,\mathfrak{S} \to \mathfrak{S}$, discussed earlier in relation to Theorem 3.1, with $\varepsilon_{\mathfrak{S}}(p)$ being the unique point of $\mathfrak{S}$ to which the ultrafilter p converges. $\varepsilon_{\mathfrak{S}}$ is a retraction and a bounded morphism for which the diagram

$$\begin{array}{ccc} \mathbb{E}\mathrm{x}\,\mathbb{E}\mathrm{x}\,\mathfrak{S} & \xrightarrow{\mathbb{E}\mathrm{x}\,\varepsilon_{\mathfrak{S}}} & \mathbb{E}\mathrm{x}\,\mathfrak{S} \\ {\scriptstyle \mu_{\mathfrak{S}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\mathfrak{S}}} \\ \mathbb{E}\mathrm{x}\,\mathfrak{S} & \xrightarrow{\varepsilon_{\mathfrak{S}}} & \mathfrak{S} \end{array}$$

commutes. Thus $\mathfrak{S}$ is a retraction structure under $\varepsilon_{\mathfrak{S}}$. The assignment $\mathfrak{S} \mapsto (\mathfrak{S}, \varepsilon_{\mathfrak{S}})$ gives rise to an isomorphism between the categories $\mathbf{CHSt}_\tau$ and $\mathbf{Ret}_\tau$. This generalizes the celebrated result of Manes (see [38] or [27, Section III.2]) that the category of compact Hausdorff topological spaces and continuous functions is isomorphic to the category of algebras for the ultrafilter monad on $\mathbf{Set}$.

Details of the theory surveyed in Section 3.2, with proofs of Theorems 3.1 and 3.3, will be given in [10].

4. Further Questions

The material reviewed in this paper suggests further problems for research. In the absence of a resolution of the converse of Fine's theorem, we may seek partial solutions, in the form of theorems showing that canonicity implies elementarity for various classes of logics or varieties. There are a number of such theorems in the literature. For example, Fine [4] showed this for the class of monomodal logics that are characterised by some class of transitive frames that is closed under subframes (not just inner ones). The transitivity restriction here was removed by Frank Wolter in his dissertation (see [46, Theorem 3.11]). Wolter also proved [45] that canonicity implies elementarity for all normal extensions of the standard bimodal linear tense logic.

In a different direction, a general result can be obtained by considering *atom structures*. If a BAO $\mathfrak{A}$ is atomic, it determines a structure $\mathbb{A}\mathrm{t}\,\mathfrak{A}$ based on the set of atoms of $\mathfrak{A}$, with each operator f of $\mathfrak{A}$ determining the relation R_f of $\mathbb{A}\mathrm{t}\,\mathfrak{A}$ having

$$R_f(x_1,\dots,x_{n+1}) \quad \text{iff} \quad f(x_1,\dots,x_n) \geq x_{n+1}.$$

For any variety $\mathcal{V}$, the class $\mathbb{A}\mathrm{t}\,\mathcal{V}$ of such atom structures of atomic members of $\mathcal{V}$ is closed under ultraproducts and includes $\mathbb{S}\mathsf{tr}\mathcal{V}$. We showed in [15] that if a canonical variety has $\mathbb{A}\mathrm{t}\,\mathcal{V} = \mathbb{S}\mathsf{tr}\mathcal{V}$, then $\mathbb{A}\mathrm{t}\,\mathcal{V}$ is an elementary class generating $\mathcal{V}$. However there are canonical varieties, and even elementarily generated ones, for which $\mathbb{A}\mathrm{t}\,\mathcal{V} \supsetneq \mathbb{S}\mathsf{tr}\mathcal{V}$. Examples include the variety of algebras for the modal logic K4+$(\Box\Diamond\varphi \to \Diamond\Box\varphi)$ [15, p. 592], and the variety **RRA** [25].

Another condition that is sufficient for canonicity of $\mathcal{V}$ to imply elementary generation is *singleton-persistence*. This means that if $\mathcal{V}$ contains the subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ generated by the atoms (i.e. the singleton subsets of $\mathfrak{S}$), then it contains $\mathsf{Cm}\,\mathfrak{S}$ itself. Singleton persistence is strong enough to force $\mathbb{S}\mathsf{tr}\mathcal{V}$ to be an elementary class [18].

The question remains: are there other results on of this kind, providing conditions on some class of varieties, or some family of classes of structures that generate varieties, that ensure that canonicity implies elementary generation? For instance, can we show this for varieties that are generated by their finite members? Another important question concerns *how much* canonicity is enough. Is validity of a logic Λ in the canonical structure $\mathfrak{S}^{\Lambda}_{\omega}$ sufficient to make Λ be elementary? If so, then by Fine's theorem validity of Λ in $\mathfrak{S}^{\Lambda}_{\omega}$ would imply its validity in $\mathfrak{S}^{\Lambda}_{\kappa}$ for all infinite κ. But not even that is known.

We may also raise these questions in contexts other than that of BAO's, such for distributive lattices with operators, where canonical extensions have been studied in [6, 7] and for which a duality can be developed using partially ordered relational structures [12, 20]. Then there are other kinds of intensional logic that have relational semantics and notions of canonical model, such as intuitionistic logic (with or without modalities), and various species of substructural logic (linear logic, relevant logic etc).

Then there is the question of whether the categorical/topological description of $\mathbb{E}\mathrm{x}\,\mathfrak{S}$ in Section 3 can be adapted to these other contexts. Finally, it is notable that there is now available a notion of canonical extension for any kind of lattice with additional operations [5]. A duality theory for such algebras has yet to be developed, and it remains to be seen whether an analogue of $\mathbb{E}\mathrm{x}$ can be successful formulated at that level of generality.

References

[1] C. C. Chang and H. J. Keisler. *Model Theory.* North-Holland, Amsterdam, 1973.

[2] Maarten de Rijke and Yde Venema. Sahlqvist's theorem for Boolean algebras with operators with an application to cylindric algebras. *Studia Logica*, 54:61–78, 1995.

[3] Kit Fine. Some connections between elementary and modal logic. In Stig Kanger, editor, *Proceedings of the Third Scandinavian Logic Symposium*, pages 15–31. North-Holland, 1975.

[4] Kit Fine. Logics containing $K4$. Part II. *The Journal of Symbolic Logic*, 50(3):619–651, 1985.

[5] Mai Gehrke and John Harding. Bounded lattice expansions. *Journal of Algebra*, 239:345–371, 2001.

[6] Mai Gehrke and Bjarni Jónsson. Bounded distributive lattices with operators. *Mathematica Japonica*, 40(2):207–215, 1994.

[7] Mai Gehrke and Bjarni Jónsson. Monotone bounded distributive lattice expansions. *Mathematica Japonica*, 52(2):197–213, 2000.

[8] Steven Givant. Universal classes of simple relation algebras. *The Journal of Symbolic Logic*, 64(2):575–589, 1999.

[9] Robert Goldblatt. Mathematical modal logic: A view of its evolution. To appear. Manuscript available at `http://www.mcs.vuw.ac.nz/~rob`.

[10] Robert Goldblatt. Topological frames and retraction. In preparation.

[11] Robert Goldblatt. *Metamathematics of Modal Logic.* PhD thesis, Victoria University, Wellington, February 1974. Included in [14].

[12] Robert Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44:173–242, 1989.

[13] Robert Goldblatt. On closure under canonical embedding algebras. In H. Andréka, J.D. Monk, and I. Németi, editors, *Algebraic Logic*, volume 54 of *Colloquia Mathematica Societatis János Bolyai*, pages 217–229. North-Holland, 1991.

[14] Robert Goldblatt. *Mathematics of Modality.* CSLI Lecture Notes No. 43. CSLI Publications, Stanford, California, 1993. Distributed by Cambridge University Press.

[15] Robert Goldblatt. Elementary generation and canonicity for varieties of Boolean algebras with operators. *Algebra Universalis*, 34:551–607, 1995.

[16] Robert Goldblatt. Reflections on a proof of elementarity. In Jelle Gerbrandy, Maarten Marx, Maarten de Rijke, and Yde Venema, editors, *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday.* Amsterdam University Press, 1999. Electronically available at: `http://turing.wins.uva.nl/~j50/cdrom/`.

[17] Robert Goldblatt. Algebraic polymodal logic: A survey. *Logic Journal of the IGPL,* Special Issue on Algebraic Logic edited by István Németi and Ildikó Sain, 8(4):393–450, July 2000. Electronically available at: `http://www3.oup.co.uk/igpl/Volume_08/Issue_04/`.

[18] Robert Goldblatt. Persistence and atomic generation for varieties of Boolean algebras with operators. *Studia Logica*, 68(2):155–171, 2001.

[19] Robert Goldblatt. Quasi-modal equivalence of canonical structures. *The Journal of Symbolic Logic*, 66:497–508, 2001.

[20] G. Hansoul. A duality for Boolean algebras with operators. *Algebra Universalis*, 17:34–49, 1983.

[21] Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric Algebras I.* North-Holland, 1971.

[22] Leon Henkin, J. Donald Monk, and Alfred Tarski. *Cylindric Algebras II.* North-Holland, 1985.

[23] H. Herrlich and G. Strecker. *Category Theory.* Allyn and Bacon, Newton, MA, 1973.

[24] Robin Hirsch and Ian Hodkinson. *Relational Algebras by Games*, volume 147 of *Studies in Logic.* Elsevier, 2002.

[25] Ian Hodkinson. Atom structures of cylindric algebras and relation algebras. *Annals of Pure and Applied Logic*, 89:117–148, 1997.

[26] Ian Hodkinson and Yde Venema. Canonical varieties with no canonical axiomatisation. Electronically available at `http://www.doc.ic.ac.uk/~imh`, 2003.

[27] P. T. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[28] Bjarni Jónsson. Algebras whose congruence lattices are distributive. *Math. Scand.*, 21:110–121, 1967.

[29] Bjarni Jónsson. A survey of Boolean algebras with operators. In *Algebras and Orders*, volume 389 of *NATO ASI Series*, pages 239–286. Kluwer Academic Publishers, 1993.

[30] Bjarni Jónsson. On the canonicity of Sahlqvist identities. *Studia Logica*, 53:473–491, 1994.

[31] Bjarni Jónsson. The preservation theorem for canonical extensions of Boolean algebras with operators. In K. A. Baker and R. Wille, editors, *Lattice Theory and its Applications, in Celebration of Garrett Birkhoff's 80th Birthday*, volume 23 of *Research and Expositions in Mathematics*, pages 121–130. Heldermann Verlag, 1995.

[32] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators, part I. *American Journal of Mathematics*, 73:891–939, 1951.

[33] Bjarni Jónsson and Alfred Tarski. Boolean algebras with operators, part II. *American Journal of Mathematics*, 74:127–162, 1952.

[34] E. J. Lemmon and D. Scott. Intensional logic. Preliminary draft of initial chapters by E. J. Lemmon, Stanford University (later published as *An Introduction to Modal Logic*, American Philosophical Quarterly Monograph Series, no. 11 (ed. by Krister Segerberg), Basil Blackwell, Oxford, 1977), July 1966.

[35] Roger Maddux. *Topic in Relation Algebras.* PhD thesis, University of California, Berkeley, 1978.

[36] Roger Maddux. Some varieties containing relation algebras. *Transaction of the American Mathematical Society*, 272(2):501–526, 1982.

[37] Roger Maddux. A sequent calculus for relation algebras. *Annals of Pure and Applied Logic*, 25:73–101, 1983.

[38] Ernest G. Manes. *Algebraic Theories.* Springer-Verlag, 1976.

[39] Giovanni Sambin. Subdirectly irreducible modal algebras and initial frames. *Studia Logica*, 62:269–282, 1999.

[40] Alfred Tarski. Contributions to the theory of models III. *Koninklijkle Nederlandse Akademie van Wetenschappen, Proceedings, Series A*, 58:56–64, 1955. (Indagationes Mathematicae, vol. 17).

[41] S. K. Thomason. Semantic analysis of tense logic. *The Journal of Symbolic Logic*, 37:150–158, 1972.

[42] J. F. A. K. van Benthem. Modal formulas are either elementary or not $\Sigma\Delta$-elementary. *The Journal of Symbolic Logic*, 41:436–438, 1976.

[43] J. F. A. K. van Benthem. Some kinds of modal completeness. *Studia Logica*, 39:125–141, 1980.

[44] J. F. A. K. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Naples, 1983.

[45] Frank Wolter. Properties of tense logics. *Mathematical Logic Quarterly*, 42:481–500, 1996.

[46] Frank Wolter. The structure of lattices of subframe logics. *Annals of Pure and Applied Logic*, 86:47–100, 1997.

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