Monadic Bounded Algebras

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9 June 2010

Abstract

We introduce the equational notion of a monadic bounded algebra (**MBA**), intended to capture algebraic properties of bounded quantification. The variety of all **MBA**'s is shown to be generated by certain algebras of two-valued propositional functions that correspond to models of monadic free logic with an existence predicate. Every **MBA** is a subdirect product of such functional algebras, a fact that can be seen as an algebraic counterpart to semantic completeness for monadic free logic. The analysis involves the representation of **MBA**'s as powerset algebras of certain directed graphs with a set of "marked" points.

It is shown that there are only countably many varieties of **MBA**'s, all are generated by their finite members, and all have finite equational axiomatisations classifying them into fourteen kinds of variety. The universal theory of each variety is decidable.

Finitely generated **MBA**'s are shown to be finite, with the free algebra on r generators having exactly $2^{3\cdot 2^r \cdot 2^{2^r-1}}$ elements. An explicit procedure is given for constructing this freely generated algebra as the powerset algebra of a certain marked graph determined by the number r.

1 Introduction and Overview

A *monadic* algebra in the sense of Halmos [16] is a Boolean algebra **B** equipped with a closure operator $\exists$ whose range is a subalgebra of **B**. This operator abstracts algebraic properties of the standard existential quantifier "for some". The name "monadic" comes from the connection with predicate logics for languages having one-placed predicates and a single quantifier.

This paper introduces a more general notion of *monadic bounded* algebra (**MBA**), intended to capture algebraic properties of *bounded* quantification, as in "for some member of E", where E is an explicity referenced set that bounds the range of the standard quantifier. This kind of quantification occurs in *free* logic: logic free from existence assumptions. There the existential generalisation principle

$$\varphi(t) \to \exists v\varphi(v)$$

is regarded as invalid, as the term t may be undefined, or refer to an entity that does not exist, such as the present King of France. Typically, free logics have a monadic existence predicate $\mathcal{E}$ and admit existential generalisation in the modified form

$$\varphi(t) \wedge \mathcal{E}t \to \exists v \varphi(v).$$

The formula $\mathcal{E}t$ may be read "t exists". In systems with an identity predicate, it may be introduced definitionally as $\exists v(v \approx t)$. One such approach occurs in [23], where models of free logic are obtained by adding a "null entity" to a standard relational structure, allowing terms to take values in the expanded model, but constraining the quantifier $\exists$ to range over the original structure.

There are also systems in which $\mathcal{E}t$ is defined to be $t \approx t$. This happens with the intuitionistic logic of sheaves, where t may be identified with a partially defined function, like a section of a bundle, and $\mathcal{E}t$ is interpreted as the domain of t, viewed a measure of the "extent" of t, or the degree to which it exists (see [7], [14, §§11.8, 11.9]). It also happens in a version of first-order dynamic logic in [10], where there are identities between certain Boolean expressions that are interpreted in a system of three truth values, representing *True*, *False* and *Undefined*.

$\mathcal{E}$ may also be taken as a primitive notion, as in quantified modal logics without identity, where different possible worlds are assigned different domains of existing individuals (e.g. [18, Chapter 16]). Another case is the logic of partial terms of [3, §VI.1], having formulas $t\downarrow$, read "t is defined" or "t denotes", where this is viewed as expressing a property of the term t, rather than a property, like existence, of the object denoted.

Returning to the notion of monadic algebra, we note that the basic examples of these, in terms of which all others can be represented, are algebras of Boolean-valued functions. They are motivated by the relationship between monadic predicates and propositional functions. For example, the predicate "is human" determines the function assigning to each individual x the proposition asserting that x is human. A propositional function can be viewed as having the form $p : X \to \mathbf{B}$, where X is a set of individuals and $\mathbf{B}$ a Boolean algebra of propositions. Existential quantification then produces the (constant) function $\exists p : X \to \mathbf{B}$ defined by

$$\exists p(y) = \bigvee \{p(x) \mid x \in X\},$$

where $\bigvee$ is the join operation in $\mathbf{B}$. Of course this requires the existence of the join. A *functional monadic algebra* is a subalgebra $\mathbf{A}$ of the Boolean algebra of all functions $X \to \mathbf{B}$, such that for each $p \in \mathbf{A}$, this join exists and $\exists p \in \mathbf{A}$. If $\mathbf{B}$ is the two-element Boolean algebra $\mathbf{2}$, then $\mathbf{A}$ is called a *model*, being an algebraic counterpart to the notion of model for monadic predicate logic.

To define a notion of "functional **MBA**", based on X and $\mathbf{B}$, we specify a subset X_E of X, thought of as the set of "existing" members of X, and define

$$\exists p(y) = \bigvee \{p(x) \mid x \in X_E\}.$$

The existence predicate is represented by a function $E : X \to \mathbf{B}$, with $E(x)$ thought of as the proposition "x exists". This suggests we should have

$$\exists p(y) = \bigvee \{E(x) \wedge p(x) \mid x \in X\},$$

and indeed we will require both of these last two equations for $\exists p(y)$ to hold. We also require that if $x \in X_E$, then $E(x) = \mathbf{1}$, the greatest element of $\mathbf{B}$, representing the truth value *True*. But we do not require that E be the characteristic function of X_E, which would mean additionally that if $x \notin X_E$, then $E(x) = \mathbf{0}$, the least element of $\mathbf{B}$, representing *False*. We allow that x may exist "partially" or "to some extent".

The precise definition of a functional **MBA** is provided in the next section, giving rise to the class **FMBA** of algebras isomorphic to a functional **MBA**. A *model* is defined as a functional **MBA** for which $\mathbf{B}$ is $\mathbf{2}$ and E is the characteristic function of X_E. **Mod** is the subclass of **FMBA** consisting of algebras isomorphic to a model.

The situation with these new functional algebras is different to that of the monadic ones. The class of all monadic algebras is a *variety*, i.e. is definable by equations, and every algebra satisfying these equations is isomorphic to a functional one [17, p. 70]. So the class of algebras isomorphic to functional monadic algebras is a variety. The point of difference for us is that **FMBA** is not a variety: every functional **MBA** has $\exists E \in \{\mathbf{0}, \mathbf{1}\}$, a property that is not definable by equations. Consequently, we focus on the variety generated by **FMBA**. This is the same as the variety generated by **Mod**. We call this variety **MBA**, and refer to its members as *monadic bounded algebras*, or "**MBA***'s*". The aim of this paper is to study **MBA** and its subvarieties.

In Section 3, we define **MBA** axiomatically as the class of algebras of type $(\mathbf{B}, \exists, E)$, with E a distinguished element, that satisfy six given equations. To verify that this is the variety generated by **Mod** and **FMBA**, we show in Section 5 that every monadic bounded algebra is isomorphic to a subdirect product of models. The proof uses aspects of the representation theory for Boolean algebras with operators, including their construction as *complex algebras*, i.e. algebras of subsets of relational structures. It also uses a characterisation of the members of **Mod** as being those **MBA**'s that are *basic*, meaning that their quantifier $\exists$ has $\exists p = \mathbf{1}$ whenever $p \wedge E \neq \mathbf{0}$. This is a natural generatisation of the notion of *simple* monadic algebra [17, p. 47], as the monadic algebras are just the **MBA**'s for which $E = \mathbf{1}$.

To construct **MBA**'s as complex algebras, we introduce in Section 4 the notion of a *bounded graph* as a certain kind of directed graph $\mathcal{F}$ with a set E of "marked" vertices. The complex algebra $\mathbf{P}\mathcal{F}$ of $\mathcal{F}$ is its Boolean algebra of subsets, with the distinguished member E and the operation $\exists$ having $x \in \exists X$ iff there is a edge from x to an element of X. Of particular importance are bounded graphs that are generated from a single point. There are three types of these, which we call *monadic*, *spiked*, and *vacuous*. These are depicted in Figure 1, where a circle denotes the set E of marked points, while $\bullet$ denotes an unmarked one. A monadic graph is just the complete graph on E, while a vacuous one consists of a single unmarked point with $E = \emptyset$. If $\mathcal{F}$ is of one of these three types we call $\mathbf{P}\mathcal{F}$ a *special* **MBA**. Every subalgebra of a special one is basic, hence isomorphic to a model, and is also subdirectly irreducible. When $\mathcal{F}$ is monadic, $\mathbf{P}\mathcal{F}$ is a simple monadic algebra. We prove that every **MBA** is isomorphic to a subdirect product of sub-special ones.

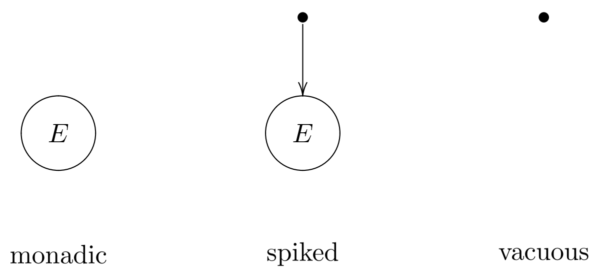


Figure 1: The three types of point-generated bounded graph

The filtration technique is used in Section 6 to collapse complex algebras to finite ones and thereby prove that every variety of **MBA**'s is generated by its finite special members. Then in Section 7 we show that every such variety is generated by at most three special algebras taken from a countable list that has two infinite algebras and isomorphic representatives of all finite ones. It follows that there are only countably many varieties of **MBA**'s, and for each of them we provide a characterisation by an explicit finite set of equations. This analysis classifies the varieties of **MBA**'s into fourteen kinds, described in Table 1. From the information provided by filtration, together with the nature of the special generating algebras, it is possible to show that the equational theory, and indeed the universal theory, of any variety of **MBA**'s is algorithmically decidable.

In Section 8 we prove that every finitely-generated **MBA** is finite, having at most $2^{3\cdot 2^r\cdot 2^{2^r-1}}$ elements, where r is the number of generators. Moreover, this upper bound is shown to be attained by the free **MBA** on r generators. We give an explicit recipe for constructing the free algebra as the complex algebra of a finite graph $\mathcal{G}^r$. This graph is defined as the disjoint union of a collection of bounded graphs $\mathcal{G}^r_J$, each one associated with a different subset J of $\{0, \ldots, 2^r - 1\}$ and having $E = J$ together with 2^r unmarked points connected to the marked ones (see (8.7)).

We turn now to filling in the details of this outline.

2 Functional Algebras

We assume familiarity with the theory of Boolean algebras, and with universal algebra in general. Notationally, we use $\wedge$, $\vee$, and $'$ for the meet, join, and complement operations of a Boolean algebra; $\mathbf{0}$ and $\mathbf{1}$ for its least (zero) and greatest (unit) elements (sometimes with a superscript to indicate the algebra as in $\mathbf{0}^{\mathbf{B}}$, $\mathbf{1}^{\mathbf{B_1}}$ etc.); and $\bigwedge$ and $\bigvee$ for the meet and join operation on sets of elements (when these operations are defined). $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$ denotes the standard two-element Boolean algebra. We write $p - q$ for $p \wedge q'$ and $p + q$ for the symmetric difference $(p - q) \vee (q - p)$, and denote by $I(p)$ the principal ideal $\{q \mid q \leq p\}$ generated by an element p.

Let $\mathbf{B}$ be a Boolean algebra, X a set, and $X_E \subseteq X$. We think of the members of X_E as the "actual" members of X, and the members of the complement $X - X_E$ as the "virtual", or "possible", members of X. The set $\mathbf{B}^X$ of all functions from X to $\mathbf{B}$ is a Boolean algebra with respect to the pointwise operations. A Boolean subalgebra $\mathbf{A}$ of $\mathbf{B}^X$ with a distinguished member E of $\mathbf{A}$ is called a *functional monadic bounded algebra*, or a $\mathbf{B}$-valued functional monadic bounded algebra with *domain* (X, X_E) and *distinguished function* E iff

(1) $E(x) = \mathbf{1}^{\mathbf{B}}$ for every $x \in X_E$;

(2) for every $p \in \mathbf{A}$, both $\bigvee \{p(x) \mid x \in X_E\}$ and $\bigvee \{p(x) \wedge E(x) \mid x \in X\}$ exist in $\mathbf{B}$ and are equal; and

(3) for every $p \in \mathbf{A}$, $\mathbf{A}$ contains the constant function $\exists p$ on X, defined by

$$\exists p(y) = \bigvee \{p(x) \mid x \in X_E\}.$$

If $\mathbf{B} = \mathbf{2}$, and $E(x) = \mathbf{1}^{\mathbf{B}}$ iff $x \in X_E$, then we call $\mathbf{A}$ a *model*. Thus a model is a $\mathbf{2}$-valued functional monadic bounded algebra whose distinguished function E is the *characteristic function* of the set X_E. We write **Mod** for the class of all algebras that are isomorphic to some model. A functional monadic bounded algebra will usually be called a "functional **MBA**", and we write **FMBA** for the class of all algebras that are isomorphic to some functional **MBA**. Thus **Mod** $\subseteq$ **FMBA**.

Example 2.1. To illustrate the notion of model, let $\mathfrak{A}$ be a relational structure interpreting a language for free monadic logic, with an existence predicate $\mathcal{E}$, and a single individual variable[1], which we write v, with corresponding existential quantifier $\exists v$. Let $|\mathfrak{A}|$ be the underlying set of $\mathfrak{A}$, and $\mathcal{E}^{\mathfrak{A}}$ be the subset of $|\mathfrak{A}|$ interpreting the existence predicate $\mathcal{E}$ in $\mathfrak{A}$. Write $\mathfrak{A} \models \varphi[a]$ to mean that formula φ is satisfied in $\mathfrak{A}$ when v is given value a, and $\mathfrak{A} \models \varphi$ to mean that $\mathfrak{A} \models \varphi[a]$ holds for all elements a of $\mathfrak{A}$ (i.e. φ is valid in $\mathfrak{A}$). Satisfaction for quantified formulas is defined by:

$$\mathfrak{A} \models \exists v\varphi \text{ iff } \mathfrak{A} \models \varphi[a] \text{ for some } a \in \mathcal{E}^{\mathfrak{A}}.$$

Define an equivalence relation $\equiv_{\mathfrak{A}}$ on the set of all formulas by putting $\varphi \equiv_{\mathfrak{A}} \psi$ iff $\mathfrak{A} \models \varphi \leftrightarrow \psi$, and let $[\varphi]$ be the $\equiv_{\mathfrak{A}}$-equivalence class of φ, for each formula φ. The set of all such equivalence classes forms a Boolean algebra $\mathbf{B}_{\mathfrak{A}}$ whose operations are naturally induced by the logical connectives. For each formula φ, define $\widehat{\varphi} : |\mathfrak{A}| \to \mathbf{2}$ to be the characteristic function of the subset $\{a \mid \mathfrak{A} \models \varphi[a]\}$ of $\mathfrak{A}$ defined by φ. Then the collection $\mathbf{A}_{\mathfrak{A}}$ of all such functions $\widehat{\varphi}$ is a *model* as defined above, a $\mathbf{2}$-valued functional **MBA** with domain $(|\mathfrak{A}|, \mathcal{E}^{\mathfrak{A}})$, i.e.

[1] It is possible to dispense with the variable, taking monadic predicate symbols themselves as atomic formulas, and allowing formation of formulas $\exists\varphi$ where φ is a formula. This is done, for example, in the system LPC1 of [19, Chapter 24]. But the use of a variable helps in dealing with individual constants under substitution.

$X = |\mathfrak{A}|$ and $X_E = \mathcal{E}^{\mathfrak{A}}$. The distinguished element E of $\mathbf{A}_{\mathfrak{A}}$ is $\widehat{\mathcal{E}v}$, which is the characteristic function of the set $\{a \mid \mathfrak{A} \models \mathcal{E}v[a]\} = \mathcal{E}^{\mathfrak{A}}$. We have

$$\bigvee_{a\in\mathcal{E}^{\mathfrak{A}}} \widehat{\varphi}(a) = \bigvee_{a\in|\mathfrak{A}|} \left(\widehat{\varphi}(a) \wedge \widehat{\mathcal{E}v}(a)\right),$$

and $\widehat{\exists v\varphi}(b) = \bigvee_{a\in\mathcal{E}^{\mathfrak{A}}} \widehat{\varphi}(a)$, for every $b \in |\mathfrak{A}|$, so the operation $\exists : \mathbf{A}_{\mathfrak{A}} \to \mathbf{A}_{\mathfrak{A}}$ has $\exists\widehat{\varphi} = \widehat{\exists v\varphi}$. The map $\widehat{\varphi} \mapsto [\varphi]$ is a well-defined Boolean algebra isomorphism between $\mathbf{A}_{\mathfrak{A}}$ and $\mathbf{B}_{\mathfrak{A}}$ under which E (i.e. $\widehat{\mathcal{E}v}$) corresponds to $[\mathcal{E}v]$. Similarly, $\widehat{\varphi} \mapsto \{a : \mathfrak{A} \models \varphi[a]\}$ is an isomorphism between the model $\mathbf{A}_{\mathfrak{A}}$ and the Boolean set algebra of all definable subsets of $\mathfrak{A}$, under which E corresponds to $\mathcal{E}^{\mathfrak{A}}$.

Note that if $\widehat{\varphi} \wedge E \neq \mathbf{0}$ in $\mathbf{A}_{\mathfrak{A}}$, then there must be an $a \in X_E$ with $\widehat{\varphi}(a) = \mathbf{1}$, which is enough to ensure that $\exists\widehat{\varphi} = \mathbf{1}$. □

Example 2.2. Functional **MBA**'s in which E is not the characteristic function of X_E can be constructed from classes of structures $\mathfrak{A}$ of the kind just considered. Suppose our language has individual constants; let D be the set of all such constants, and fix some set $C \subseteq D$. Let S_C be the class of all structures in which the actual elements are precisely the elements defined by members of C, i.e. $\mathfrak{A} \in S_C$ iff $\mathcal{E}^{\mathfrak{A}} = \left\{c^{\mathfrak{A}} \mid c \in C\right\}$. This time, define an equivalence relation $\equiv_C$ on the set of all formulas by putting $\varphi \equiv_C \psi$ iff $\mathfrak{A} \models \varphi \leftrightarrow \psi$ for all $\mathfrak{A} \in S_C$. Let $\mathbf{B}_C$ be the resulting Boolean algebra of equivalence classes $[\varphi]$. This has $\mathbf{1}^{\mathbf{B}_C} = \{[\varphi] \mid S_C \models \varphi\}$ and $\mathbf{0}^{\mathbf{B}_C} = \{[\varphi] \mid S_C \models \neg\varphi\}$.

For each φ, define $\widetilde{\varphi} : D \to \mathbf{B}_C$ by putting $\widetilde{\varphi}(c) = [\varphi(c)]$, where $\varphi(c)$ is the formula obtained from φ by replacing all free occurrences of the variable v in φ by the constant symbol c. Then

$$\bigvee_{c\in C} \widetilde{\varphi}(c) = \bigvee_{c\in D} (\widetilde{\varphi}(c) \wedge \widetilde{\mathcal{E}v}(c)),$$

or, in other words, $\bigvee_{c\in C}[\varphi(c)] = \bigvee_{c\in D}[\varphi(c) \wedge \mathcal{E}c]$. Also $[\exists v\varphi] = \bigvee_{c\in C}[\varphi(c)]$, so $\widetilde{\exists v\varphi}(d) = \bigvee_{c\in C} \widetilde{\varphi}(c)$ for all $d \in D$. Moreover, if $c \in C$, then $\mathcal{E}c$ is valid in every member of S_C, so $\widetilde{\mathcal{E}v}(c) = [\mathcal{E}c] = \mathbf{1}^{\mathbf{B}_C}$.

Thus the collection $\mathbf{A}_C$ of all functions $\widetilde{\varphi}$ is a $\mathbf{B}_C$-valued functional **MBA** whose domain (X, X_E) has $X = D$ and $X_E = C$, and whose distinguished function E is $\widetilde{\mathcal{E}v}$. This function need not be $\mathbf{2}$-valued, so need not be a characteristic function at all. Provided that $C \subsetneq D$, we can take a constant $d \notin C$ and make structures in S_C in which $\mathcal{E}d$ is true and others in which it is false, so neither $\mathcal{E}d$ nor $\neg\mathcal{E}d$ is valid in S_C, and hence $\mathbf{1}^{\mathbf{B}_C} \neq \widetilde{\mathcal{E}v}(d) \neq \mathbf{0}^{\mathbf{B}_C}$. In fact this construction of $\mathbf{B}_C$ only requires S_C to be *some* class of structures in which $\mathcal{E}^{\mathfrak{A}} = \left\{c^{\mathfrak{A}} \mid c \in C\right\}$, not necessarily all of them. In that more general case there may even be a $d \notin C$ with $\widetilde{\mathcal{E}v}(d) = \mathbf{1}^{\mathbf{B}_C}$.

In the model $\mathbf{A}_{\mathfrak{A}}$ of the previous example we saw that $\widehat{\varphi} \wedge E \neq \mathbf{0}$ implies $\exists\widehat{\varphi} = \mathbf{1}$. The corresponding property can fail for $\mathbf{A}_C$: it may have $\widetilde{\varphi} \wedge E \neq \mathbf{0}$ but $\exists\widetilde{\varphi} \neq \mathbf{1}$. To have $\widetilde{\varphi} \wedge E \neq \mathbf{0}$ only requires that some $\mathfrak{A} \in S_C$ have $\mathfrak{A} \models \varphi[c]$ for some $c \in C$; whereas to have $\exists\widetilde{\varphi} = \mathbf{1}$ it requires that *every* $\mathfrak{A} \in S_C$ have $\mathfrak{A} \models \varphi[c]$ for some $c \in C$. □

Theorem 2.3. *In a functional monadic bounded algebra, $\exists E$ is either* $\mathbf{0}$ *or* $\mathbf{1}$*.*

Proof. Let $\mathbf{A}$ be any $\mathbf{B}$-valued functional **MBA**. Putting $p = E$ in the definition of a functional **MBA** shows that the join $\bigvee\{E(x) \mid x \in X_E\}$ exists in $\mathbf{B}$ and is equal to $\exists E(y)$ for any $y \in X$.

Now if $X_E = \emptyset$, then $\exists E(y) = \bigvee\{E(x) \mid x \in X_E\} = \bigvee\emptyset = \mathbf{0}^{\mathbf{B}}$ for all $y \in X$, so $\exists E = \mathbf{0}^{\mathbf{A}}$.

But if $X_E \neq \emptyset$, then there is some $x_0 \in X_E$, and for any y,

$$\exists E(y) = \bigvee\{E(x) \mid x \in X_E\} \geq E(x_0) = \mathbf{1}^{\mathbf{B}}.$$

Hence $\exists E = \mathbf{1}^{\mathbf{A}}$. □

Now any functional **MBA** has $\exists\mathbf{0} = \mathbf{0}$ and $E \leq \exists E$. These imply that $\exists E = \mathbf{0}$ iff $E = \mathbf{0}$, and so the condition "$\exists E$ is $\mathbf{0}$ or $\mathbf{1}$" is equivalent to "$E = \mathbf{0}$ or $\exists E = \mathbf{1}$". These relationships also hold in the abstract algebras we define next.

3 Abstract MBA's

A *monadic bounded algebra*, is a triple $\mathbf{A} = (\mathbf{B}, E, \exists)$, where $\mathbf{B}$ is a Boolean algebra, $E \in \mathbf{B}$, and $\exists$ is a unary operation on $\mathbf{B}$, called the *quantifier*, such that for all $p, q \in \mathbf{B}$,

(ax1) $\exists\mathbf{0} = \mathbf{0}$,

(ax2) $\exists(p \wedge \exists q) = \exists p \wedge \exists q$,

(ax3) $p \wedge E \leq \exists p$,

(ax4) $\exists p \leq \exists(p \wedge E)$,

(ax5) $\exists(p \vee q) \leq \exists p \vee \exists q$,

(ax6) $\exists\exists p \leq \exists p$.

All of these axioms can be putting in equational form, as $p \leq q$ iff $p \wedge q = p$. We write **MBA** for the class of all monadic bounded algebras, and also refer to any of its members as "an **MBA**". Every functional monadic bounded algebra, as defined in the last section, is an **MBA** as just defined. The verification of this is left as an instructive exercise for the reader.

Example 3.1. *Relativised monadic algebras*

A *monadic algebra* in the sense of Halmos [17] is a pair $\mathbf{A} = (\mathbf{B}, \exists)$ where the operation $\exists$ on Boolean algebra $\mathbf{B}$ satisfies (ax1) and (ax2) above as well as $p \leq \exists p$ in place of (ax3). Taking an arbitrary element E of $\mathbf{A}$ here, define $\exists^E : \mathbf{B} \to \mathbf{B}$ by $\exists^E p = \exists(p \wedge E)$. Then $\mathbf{A}^E = (\mathbf{B}, E, \exists^E)$ is an **MBA**, representing the notion of bounded quantification mentioned in the Introduction. We may call $\mathbf{A}^E$ a *relativised monadic algebra.*[2]

In particular, when $E = \mathbf{1}$, then $\exists^E = \exists$, i.e. when $(\mathbf{B}, \exists)$ is a monadic algebra then $(\mathbf{B}, \mathbf{1}, \exists)$ is an **MBA**. From our perspective, a monadic algebra is just an **MBA** in which $E = \mathbf{1}$, so **MBA**'s may be considered as a generalization of monadic algebras. □

In any **MBA**, the three inequalities (ax4)–(ax6) can be strengthened to equations. We now demonstrate these and other elementary consequences of the **MBA** axioms:

Theorem 3.2. *Suppose* $\mathbf{A}$ *is an* **MBA** *and* $p, q \in \mathbf{A}$.

(1) *If* $p \leq \exists q$, *then* $\exists p \leq \exists q$.

(2) *If* $p \leq q$, *then* $\exists p \leq \exists q$ (*i.e.* $\exists$ *is monotone*).

(3) $\exists p = \exists(p \wedge E)$.

(4) $\exists(p \vee q) = \exists p \vee \exists q$.

(5) $\exists\exists p = \exists p$.

(6) $p \in \exists(\mathbf{A})$ *iff* $\exists p = p$, *where* $\exists(\mathbf{A})$ *is the range of the quantifier* $\exists$.

(7) $\exists(p \wedge E') = \mathbf{0}$.

(8) $\exists(E') = \mathbf{0}$.

(9) $\exists E = \exists\mathbf{1}$.

(10) $E \leq \exists E$.

(11) $\exists p \leq \exists E$.

(12) $\exists(\exists E)' = \mathbf{0}$.

(13) $\exists(\exists p)' \leq (\exists p)'$.

(14) $\exists p - \exists q \leq \exists(p - q)$.

(15) $\exists p + \exists q \leq \exists(p + q)$.

(16) $\exists(p \wedge (\exists q)') = \exists p \wedge (\exists q)'$.

(17) $\exists((\exists p)') = \exists E \wedge (\exists p)'$.

Proof. (1) If $p \leq \exists q$, then $\exists p = \exists(p \wedge \exists q) = \exists p \wedge \exists q$ by (ax2), so $\exists p \leq \exists q$.

(2) If $p \leq q$, then $p \wedge E \leq q \wedge E \leq \exists q$ by (ax3), so by (1), $\exists(p \wedge E) \leq \exists q$. But by (ax4), $\exists p \leq \exists(p \wedge E)$, so then $\exists p \leq \exists q$.

[2] "Bounded monadic algebra" might be more appropriate, but that could be confused with "monadic bounded algebra"!

(3) $\exists(p \wedge E) \leq \exists p$ by monotonicity of $\exists$ (2), so with (ax4) this gives (3).

(4) By (2), $\exists p \leq \exists(p \vee q)$ and $\exists q \leq \exists(p \vee q)$, hence $\exists p \vee \exists q \leq \exists(p \vee q)$. Together with (ax5) this gives (4).

(5) $\exists p = \exists(p \wedge E)$ by (3), and $\exists(p \wedge E) \leq \exists\exists p$ by (ax3) and (2), so $\exists p \leq \exists\exists p$. Together with (ax6) this gives (5).

(6) From (5).

(7) By (3), $\exists(p \wedge E') = \exists(p \wedge E' \wedge E) = \exists(p \wedge \mathbf{0}) = \mathbf{0}$ by (ax1).

(8) Put $p = \mathbf{1}$ in (7).

(9) $\exists E = \exists(\mathbf{1} \wedge E) = \exists\mathbf{1}$ by (3).

(10) Put $p = E$ in (ax3).

(11) By (2), $\exists p \leq \exists\mathbf{1} = \exists E$ by (9).

(12) By (11) and then (ax2), $\exists(\exists E)' = \exists(\exists E)' \wedge \exists E = \exists((\exists E)' \wedge \exists E) = \exists\mathbf{0} = \mathbf{0}$.

(13) By (ax2), $\exists(\exists p)' \wedge \exists p = \exists((\exists p)' \wedge \exists p) = \mathbf{0}$ by (ax1). Hence $\exists(\exists p)' \leq (\exists p)'$.

(14) Since $p \vee q = (p - q) \vee q$, it follows by (4) that $\exists p \vee \exists q = \exists(p - q) \vee \exists q$. Forming the meet of both sides of this equation with $(\exists q)'$, we obtain $\exists p - \exists q = \exists(p - q) - \exists q \leq \exists(p - q)$.

(15) $\exists p + \exists q = (\exists p - \exists q) \vee (\exists q - \exists p) \leq \exists(p - q) \vee \exists(q - p)$ by (14). But by (4), $\exists(p - q) \vee \exists(q - p) = \exists((p - q) \vee (q - p)) = \exists(p + q)$.

(16) By $\exists$-monotonicity, $\exists(p \wedge (\exists q)') \leq \exists p \wedge \exists(\exists q)' \leq \exists p \wedge (\exists q)'$ by item (13). For the converse inequality, $\exists p \wedge (\exists q)' = \exists p - \exists q = \exists p - \exists\exists q$ by (5). But by (14), $\exists p - \exists\exists q \leq \exists(p - \exists q) = \exists(p \wedge (\exists q)')$.

(17) By (3) and then (16), $\exists((\exists p)') = \exists(E \wedge (\exists p)') = \exists E \wedge (\exists p)'$.

□

A monadic algebra has $\exists\mathbf{1} = \mathbf{1}$, but this can fail in a general **MBA**. Indeed there are significant **MBA**'s in which $\exists\mathbf{1} = \mathbf{0} \neq \mathbf{1}$, implying that $\exists p = \mathbf{0}$ for all p. The simplest example has $\mathbf{B} = \mathbf{2}$ with $E = \mathbf{0}$ and $\exists\mathbf{1} = \exists\mathbf{0} = \mathbf{0}$. (We call this the *vacuous* **MBA** for reasons that will emerge in Section 4.) Also, monadic algebras satisfy the equation $\exists(\exists p)' = (\exists p)'$, whereas the general **MBA** only has $\exists(\exists p)' \leq (\exists p)'$, as in (13) of the last Theorem. The example just given has $\exists(\exists\mathbf{1})' = \mathbf{0}$ while $(\exists\mathbf{1})' = \mathbf{1}$.

Since $\exists(E') = \mathbf{0}$ in general, $\exists$-monotonicity implies that the quantifier of an **MBA** takes the constant value $\mathbf{0}$ on the ideal $I(E') = \{p \mid p \leq E'\}$ generated by E'. In a monadic algebra, this is just the trivial ideal $\{\mathbf{0}\}$. The quantifier of a monadic algebra is called *simple* if it takes the constant value $\mathbf{1}$ outside of $\{\mathbf{0}\}$, i.e. if $\exists p = \mathbf{1}$ for all $p \neq \mathbf{0}$. A monadic algebra is itself called *simple* if $\{\mathbf{0}\}$ is its largest (hence only) proper *monadic* ideal, where a monadic ideal is a Boolean ideal that is closed under $\exists$. These two usages of "simple" are equivalent: a monadic algebra is simple iff its quantifier is simple [17, p. 47]. Moreover, a monadic algebra is simple iff it is isomorphic to a model, i.e. to a $\mathbf{2}$-valued functional monadic algebra having $X_E = \emptyset$ [17, p. 48]. This gives an abstract characterisation, up to isomorphism, of those monadic algebras that are models.

We now give a similar characterisation of **MBA**'s that are models. In an **MBA**, we continue to say that a monadic ideal is any Boolean ideal closed under $\exists$. Such ideals are in bijective correspondence with the congruences of the algebra, with a congruence $\sim$ corresponding to the monadic ideal $\{p \mid p \sim \mathbf{0}\}$, and a monadic ideal I corresponding to the congruence defined by $p \sim q$ iff $p + q \in I$.

Every ideal of the form $I(\exists q)$ is monadic, since $p \leq \exists q$ implies $\exists p \leq \exists q$. Also, every ideal I that is included in $I(E')$ is monadic, since $p \leq E'$ implies $\exists p = \mathbf{0} \in I$. We say that an ideal I is *virtual* if it is a subset of $I(E')$, which is equivalent to having $p \wedge E = \mathbf{0}$ for all $p \in I$. An **MBA** is called *basic* if all of its proper monadic ideals are virtual.

The quantifier $\exists$ of an **MBA** is called *basic* if it takes the constant value $\mathbf{1}$ outside of $I(E')$, i.e. if $\exists p = \mathbf{1}$ whenever $p \wedge E \neq \mathbf{0}$. Both of these uses of "basic" generalise the use of "simple" for monadic algebras. At the end of Example 2.2 we saw that the quantifier of a functional **MBA** of the form $\mathfrak{A}_C$ may not be basic, whereas the quantifier of a model of the form $\mathbf{A}_{\mathfrak{A}}$ as in Example 2.1 is always basic.

Theorem 3.3. *An* **MBA** *is basic iff its quantifier is basic.*

Proof. Let $\mathbf{A}$ be a basic **MBA**, and suppose $p \wedge E \neq \mathbf{0}$. We have to show $\exists p = \mathbf{1}$ to prove that $\exists$ is basic. But we have $p \wedge E \not\leq E'$, and $p \wedge E \in I(\exists p)$ by (ax2), so the monadic ideal $I(\exists p)$ is not included in $I(E')$, hence is not virtual. Since $\mathbf{A}$ is basic, $I(\exists p)$ is not proper, so contains $\mathbf{1}$, giving $\exists p = \mathbf{1}$ as required.

Conversely, assume that $\exists$ is basic. Let I be a monadic ideal of $\mathbf{A}$ that is not virtual. Then there is a $p \in I$ with $p \wedge E \neq \mathbf{0}$. Hence $\exists p = \mathbf{1}$ as $\exists$ is basic. But $\exists p \in I$ as I is monadic, so this shows that I is not proper. That proves that $\mathbf{A}$ is basic. □

We can now characterise **Mod** as consisting precisely of the basic **MBA**'s:

Theorem 3.4. *An* **MBA** *is basic iff it is isomorphic to a model.*

Proof. First we show that any model is basic, hence so is any algebra isomorphic to a model. So let $\mathbf{A}$ be a $\mathbf{2}$-valued functional **MBA** with domain (X, X_E) and distinguished function E. Suppose $p \wedge E \neq \mathbf{0}$ in $\mathbf{A}$, Then there is some $x_0 \in X$ with $p(x_0) \wedge E(x_0) \neq \mathbf{0}$ in $\mathbf{2}$. Hence $\bigvee\{E(x) \wedge p(x) \mid x \in X\} = \mathbf{1}$ in $\mathbf{2}$. But this join is the constant value of the function $\exists p$, so we get $\exists p = \mathbf{1}$ in $\mathbf{A}$, proving that the quantifier of $\mathbf{A}$ is basic, as required.

For the converse, let $\mathbf{A} = (\mathbf{B}, E, \exists)$ be any basic **MBA**. By the Stone representation of $\mathbf{B}$ there is a set X and a Boolean monomorphism $f : \mathbf{B} \to \mathbf{2}^X$ making $\mathbf{B}$ isomorphic to a subalgebra $\mathbf{A}^f$ of the functional Boolean algebra $\mathbf{2}^X$.

Let E^f be the $\mathbf{2}$-valued function $f(E)$, and $X_{E^f} = \{x \in X \mid E^f(x) = \mathbf{1}\}$, so E^f is the characteristic function of X_{E^f}.

Now if $q \in \mathbf{2}^X$, then for any $x \in X$ we have $q(x) \wedge E^f(x) = q(x)$ if $x \in X_{E^f}$, and $q(x) \wedge E^f(x) = \mathbf{0}$ otherwise. Hence

$$\bigvee \{q(x) \mid x \in X_{E^f}\} = \bigvee \left\{q(x) \wedge E^f(x) \mid x \in X\right\} \tag{3.1}$$

(these joins always exist in $\mathbf{2}$). In particular, this equation holds for $q \in \mathbf{A}^f$.

Let $\exists^f$ be the functional quantifier that is induced on $\mathbf{2}^X$ by X_{E^f}, i.e. $\exists^f q$ is the constant function on X whose value is given by either of the joins in (3.1). We will show that $\exists^f f(p) = f(\exists p)$ for all p in $\mathbf{A}$. This implies that $\mathbf{A}^f$ is closed under $\exists^f$, and so is a model, being a $\mathbf{2}$-valued functional **MBA** with domain (X, X_{E^f}) and distinguished function E^f. Moreover, f is an **MBA**-isomorphism between $\mathbf{A}$ and the model $\mathbf{A}^f$.

There are two cases for the proof that $\exists^f f(p) = f(\exists p)$. The first is when $p \wedge E = \mathbf{0}$ in $\mathbf{A}$. Then $\exists p = \mathbf{0}$ and hence $f(\exists p) = \mathbf{0}$ in $\mathbf{A}^f$. But also by the Boolean homomorphism f we get $f(p) \wedge E^f = \mathbf{0}$ in $\mathbf{A}^f$. This means that for all $x \in X$, $f(p)(x) \wedge E^f(x) = \mathbf{0}$ in $\mathbf{2}$. Since $\exists^f f(p)$ is the constant function whose value is the join of all the elements $f(p)(x) \wedge E^f(x)$ (see (3.1)), this gives $\exists^f f(p) = \mathbf{0} = f(\exists p)$ in $\mathbf{A}^f$ as required.

The other case is when $p \wedge E \neq \mathbf{0}$. As the quantifier of $\mathbf{A}$ is basic, this implies $\exists p = \mathbf{1}$, hence $f(\exists p) = \mathbf{1}$ in $\mathbf{A}^f$. Also, under the monomorphism f we get $f(p) \wedge E^f \neq \mathbf{0}$ in $\mathbf{A}^f$, so there is some $x \in X$ with $f(p)(x) \wedge E^f(x) \neq \mathbf{0}$ in $\mathbf{2}$, hence $f(p)(x) \wedge E^f(x) = \mathbf{1}$. That is enough to ensure that $\exists^f f(p) = \mathbf{1}$ in $\mathbf{A}^f$, hence $\exists^f f(p) = f(\exists p)$ in this case as well. □

Corollary 3.5. *Every subalgebra and every homomorphic image of a basic* **MBA** *is basic, hence is isomorphic to a model.*

Proof. Using Theorem 3.3, if an **MBA** is basic, then its quantifier is basic, so the restriction of this quantifier to any subalgebra is also basic, making the subalgebra basic. Hence the subalgebras are isomorphic to models by the Theorem just proved.

Also, if $\mathbf{A}_2$ is the image of some basic **MBA** $\mathbf{A}_1$ under a homomorphism f, then if $f(p) \wedge E \neq \mathbf{0}$ in $\mathbf{A}_2$, then $p \wedge E \neq \mathbf{0}$ in $\mathbf{A}_1$, hence $\exists p = \mathbf{1}$ in $\mathbf{A}_1$ as $\mathbf{A}_1$ is basic, so $\exists f(p) = \mathbf{1}$ in $\mathbf{A}_2$. This shows that $\mathbf{A}_2$ is basic, hence isomorphic to a model.

In more general terms: the property of being basic is definable by a positive universal condition, namely "for all p, either $p \wedge E = \mathbf{0}$ or $\exists p = \mathbf{1}$", so is preserved by subalgebras and homomorphic images. □

4 Complex Algebras of Bounded Graphs

Jónsson and Tarski [20] showed that a Boolean algebra with join-preserving operators can be represented as a *complex* algebra, i.e. an algebra of subsets of a relational structure, and that some equational properties of complex algebras correspond to first-order properties of their underlying relational structures. Subsequently there was developed a categorical duality between Boolean algebra with operators and their homomorphisms on the one hand, and relational structures and certain *bounded morphisms* on the other [9, 11, 13]. This theory has been applied to cylindric algebras, relation algebras, varieties of modal algebras, and eventually to non-Boolean lattices with operators. Here we will use the methodology to analyse and characterise varieties of **MBA**'s.

By a *marked graph* we mean a structure $\mathcal{F} = (W, R, E)$, where W is a set, R a binary relation on W, and E a subset of W. We view (W, R) as a directed graph in which E is a set of *marked* vertices or points. For each point x, let $R[x] = \{y \mid xRy\}$, the *R-image set* of x. A binary relation is completely specified by specifying the image sets of all points. If $x \in R[x]$, we say that x is *reflexive*. The relation R induces an operation $\langle R \rangle$ on the powerset $\mathcal{P}(W)$ of W, given by

$$\langle R \rangle X = \{x \in W \mid R[x] \cap X \neq \emptyset\},$$

for all $X \subseteq W$. Let $\mathbf{P}W$ be the Boolean set algebra on $\mathcal{P}W$, in which $X \wedge Y$ is the intersection $X \cap Y$, $X \vee Y$ is the union $X \cup Y$, X' is the set complement $W - X$, $\mathbf{0} = \emptyset$ and $\mathbf{1} = W$. Then the *complex algebra* of $\mathcal{F}$ is $\mathbf{P}\mathcal{F} = (\mathbf{P}W, E, \langle R \rangle)$.

Now we define a *bounded graph* to be a marked graph having the following properties:

- R is transitive.
- R is *Euclidean*: xRy and xRz implies yRz.
- R-image points are marked: xRy implies $y \in E$.
- Marked points are reflexive: $x \in E$ implies xRx.

These are strong constraints on the structure of a graph. The last two properties imply that R-image points are reflexive: xRy implies yRy. Moreover, if xRx, then x is an R-image point and so is marked. Thus E is precisely the set of all reflexive points, hence is completely determined by the relation R.

In a bounded graph, $R[x] \subseteq E$ and R is *universal* on $R[x]$, for if $y, z \in R[x]$, then yRz by the Euclidean property. Also, $R[x]$ is *R-closed*, in the sense that if $y \in R[x]$ and yRz, then $z \in R[x]$. This is immediate by transitivity of R.

R-transitivity ensures that $\langle R \rangle \langle R \rangle X \subseteq \langle R \rangle X$ for arbitrary $X \subseteq W$ [20, Theorem 3.5]. So $\langle R \rangle$ satisfies axiom (ax6) for the quantifier of an **MBA**. In fact we have:

Theorem 4.1. *A marked graph $\mathcal{F}$ is bounded iff its complex algebra $\mathbf{P}\mathcal{F}$ is an* **MBA**.

Proof. Let $\mathcal{F}$ be bounded. The equations $\langle R \rangle \emptyset = \emptyset$ and $\langle R \rangle (X \cup Y) = \langle R \rangle X \cup \langle R \rangle Y$ hold without any constraint on R. The first is (ax1), and the second implies (ax5). Also $\langle R \rangle$ is monotonic, i.e. $X \subseteq Y$ implies $\langle R \rangle X \subseteq \langle R \rangle Y$.

(ax2): By $\langle R \rangle$-monotonicity, $\langle R \rangle (X \cap \langle R \rangle Y) \subseteq \langle R \rangle X \cap \langle R \rangle \langle R \rangle Y \subseteq \langle R \rangle X \cap \langle R \rangle Y$. For the converse inclusion, if $x \in \langle R \rangle X \cap \langle R \rangle Y$, then $xRy \in X$ and

$xRz \in Y$ for some y, z. By the Euclidean property yRz, showing $y \in \langle R\rangle Y$. Hence $xRy \in X \cap \langle R\rangle Y$, so $x \in \langle R\rangle(X \cap \langle R\rangle Y)$.

(ax3): If $x \in X \cap E$, then x is marked and hence reflexive, so $xRx \in X$, giving $x \in \langle R\rangle X$. Thus $X \cap E \subseteq \langle R\rangle X$.

(ax4): Let $x \in \langle R\rangle X$. Then $xRy \in X$ for some y. Then y is marked, so $xRy \in X \cap E$, giving $x \in \langle R\rangle(X \cap E)$. Thus $\langle R\rangle X \subseteq \langle R\rangle(X \cap E)$.

Since (ax5) and (ax6) have already been observed to hold, this completes the proof that $\mathbf{P}\mathcal{F}$ is an **MBA**.

Conversely, let $\mathbf{P}\mathcal{F}$ be an **MBA**. Then R-transitivity follows from (ax6): if $xRyRz$, then $x \in \langle R\rangle\langle R\rangle\{z\} \subseteq \langle R\rangle\{z\}$, so xRz.

For the Euclidean property, if xRy and xRz, then $x \in \langle R\rangle\{y\} \cap \langle R\rangle\{z\} \subseteq \langle R\rangle(\{y\} \cap \langle R\rangle\{z\})$ by (ax2). Hence xRy' for some $y' \in \{y\} \cap \langle R\rangle\{z\}$. Then $y = y' \in \langle R\rangle\{z\}$, implying yRz.

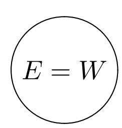
To see that R-image points are marked: if xRy, then $x \in \langle R\rangle\{y\} \subseteq \langle R\rangle(\{y\} \cap E)$ by (ax4). Hence xRy' for some $y' \in \{y\} \cap E$. Then $y = y' \in E$ as required.

To see that marked points are reflexive: if $x \in E$, then $x \in \{x\} \cap E \subseteq \langle R\rangle(\{x\})$ by (ax3), implying xRx. That completes the proof that $\mathcal{F}$ is bounded. □

We make extensive use of *point-generated* subgraphs of a bounded graph $\mathcal{F}$. If x is a point of $\mathcal{F}$, let $\mathcal{F}^x = (W^x, R^x, E^x)$, where $W^x = \{x\} \cup R[x]$, R^x is the restriction of R to W^x, and $E^x = W^x \cap E$. It is readily checked that $\mathcal{F}^x$ is a bounded graph, which we call the subgraph of $\mathcal{F}$ *generated by* x. The function $f^x : \mathcal{P}W \to \mathcal{P}W^x$ defined by $f^x(X) = W^x \cap X$ is a Boolean algebra homomorphism from $\mathbf{P}W$ onto $\mathbf{P}W^x$. By R-transitivity, W^x is R-closed, i.e. if $y \in W^x$ and yRz, then $z \in W^x$. (In fact $R[x]$ is the least R-closed subset of W containing x.) This ensures that $W^x \cap \langle R\rangle X = \langle R^x\rangle(W^x \cap X)$, i.e. $f^x(\langle R\rangle X) = \langle R^x\rangle(f^x(X))$, so f^x is an **MBA**-homomorphism from $\mathbf{P}\mathcal{F}$ onto $\mathbf{P}\mathcal{F}^x$.

We now describe three types of point-generated bounded graph, and then show that these are all the types there are.

Type I: For any set W, let $\mathcal{F}_W = (W, W \times W, W)$. This is the complete graph on W, with every point being marked/reflexive and R-related to every other point. Thus $R[x] = W$ for all x, so this graph is generated by each of its points. We visualise it as



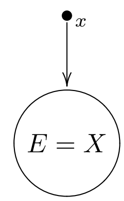
Since $\mathbf{P}\mathcal{F}_W$ has $E = \mathbf{1}$, it is a monadic algebra. We may call $\mathcal{F}_W$ the *monadic* bounded graph on W.

Type II: Take any set X and object $x \notin X$, and define the marked graph

$$\mathcal{F}_X^x = (\{x\} \cup X, (\{x\} \cup X) \times X, X).$$

This is a bounded graph generated by the point x, which is unmarked. $R[x]$ is X, the set of marked points. Indeed $R[y] = X = E$ for every y.

A Type II graph is given by this construction with $X \neq \emptyset$. Then we can visualise $\mathcal{F}^x_X$ as



We call this a *spiked* bounded graph.

Type III: Put $X = \emptyset$ in the above construction to form $\mathcal{F}^x_\emptyset = (\{x\}, \emptyset, \emptyset)$. This is a one-element graph consisting of the unmarked point x with $R[x] = \emptyset$. We call this a *vacuous* bounded graph. Its complex algebra is a copy of $\mathbf{2}$, with $E = \mathbf{0}$ and $\exists\mathbf{1} = \exists\mathbf{0} = \mathbf{0}$.

We also describe the algebra $\mathbf{P}\mathcal{F}^x_X$ as having the type of its underlying graph. Thus if $X \neq \emptyset$, $\mathbf{P}\mathcal{F}^x_X$ is a *spiked* **MBA**, while $\mathbf{P}\mathcal{F}^x_\emptyset$ is a *vacuous* **MBA**. All vacuous **MBA**'s are isomorphic.

Theorem 4.2. *Let x be any point of a bounded graph $\mathcal{F}$.*

(1) *If x is reflexive, then $\mathcal{F}^x$ is the monadic graph on W^x (Type I).*

(2) *If x is not reflexive, and $R[x]$ is non-empty, then $\mathcal{F}^x$ is a spiked bounded graph (Type II).*

(3) *If x is not reflexive, and $R[x] = \emptyset$, then $\mathcal{F}^x = \mathcal{F}^x_\emptyset$, the vacuous graph on $\{x\}$ (Type III).*

Proof. In general, $W^x = \{x\} \cup R[x] = \{x\} \cup R^x[x]$ and $R^x[x] \subseteq E^x$. If xRx, then $x \in R^x[x]$, so $W^x = R^x[x]$, on which R^x is universal, and $E^x = W^x$, so $\mathcal{F}^x$ is of Type I as described.

If x is not reflexive, then $x \notin R^x[x]$ and $E^x = R^x[x]$. So if $R^x[x] \neq \emptyset$ we have a spiked graph (Type II) for $\mathcal{F}^x$,and otherwise we have the vacuous graph on $\{x\}$. □

Corollary 4.3. *If $\mathcal{F}$ is a point-generated bounded graph, then the* **MBA** $\mathbf{P}\mathcal{F}$ *is basic.*

Proof. If $\mathcal{F}$ is the monadic graph $\mathcal{F}_W$, then in fact $\mathbf{P}\mathcal{F}$ is *simple*. For if $X \neq \emptyset$ in $\mathbf{P}\mathcal{F}$, then $\langle R\rangle(X) = W = \mathbf{1}$ because R is universal on W. So the quantifier of $\mathbf{P}\mathcal{F}$ is simple.

If $\mathcal{F}$ is of the form $\mathcal{F}^x_X$, and $Y \cap E \neq \mathbf{0}$ in $\mathbf{P}\mathcal{F}$, then taking any $z \in Y \cap E$ we have $yRz \in Y$ for all $y \in W$ (since $R[y] = E$), and so $\langle R\rangle(Y) = W = \mathbf{1}$. Thus the quantifier of $\mathbf{P}\mathcal{F}$ is basic. □

Theorem 4.4. *If $\mathcal{F}$ is a point-generated bounded graph, then any subalgebra of* $\mathbf{P}\mathcal{F}$ *is subdirectly irreducible.*

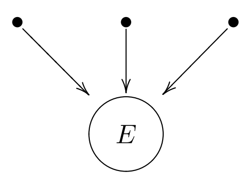
Proof. It is known that the complex algebra of any point-generated relational structure is subdirectly irreducible [11, Theorem 3.3.1]. Here we can give a particularly direct proof, using special features of $\mathcal{F}$, that applies to any subalgebra of $\mathbf{P}\mathcal{F}$.

Let $\mathcal{F}$ be of the form $\mathcal{F}^x$ with generating point x, and let $\mathbf{A}$ be subalgebra of $\mathbf{P}\mathcal{F}$. Then E belongs to $\mathbf{A}$. To show $\mathbf{A}$ is subdirectly irreducible, it is enough to show that there is a smallest non-identity congruence on $\mathbf{A}$ [5, Theorem 8.4], or equivalently for **MBA**'s, that there is a smallest monadic ideal of $\mathbf{A}$ that is *non-zero*, i.e. is not equal to $\{\mathbf{0}\}$.

Now if $\mathcal{F}^x$ is monadic, then $\mathbf{P}\mathcal{F}$ is a simple monadic algebra, hence so is $\mathbf{A}$, so the only monadic ideals of $\mathbf{A}$ are $\{\emptyset\}$ and $\mathbf{A}$ itself. But in this case x is marked, so $\mathbf{A}$ is non-zero, as it contains $E \neq \emptyset$. Thus $\mathbf{A}$ is the *unique*, hence smallest, non-zero monadic ideal of $\mathbf{A}$.

Alternatively, $\mathcal{F}^x$ is spiked or vacuous, with x as its only unmarked point. Thus $\{x\} = E'$, so $\{x\}$ belongs to $\mathbf{A}$. Let $I_x = \{\emptyset, \{x\}\}$ be the principal ideal of $\mathbf{A}$ generated by $\{x\}$. Then I_x is a monadic ideal, because $\langle R\rangle\{x\} = \emptyset \in I_x$. Now let I be any non-zero monadic ideal of $\mathbf{A}$. Then I has some member $Y \neq \emptyset$. Take any $y \in Y$. If $y = x$, then $\{x\} \subseteq Y$, so $I_x \subseteq I$. If $y \neq x$, then $xRy \in Y$, so $\{x\} \subseteq \langle R\rangle Y \in I$, and again $I_x \subseteq I$. This shows that $\mathbf{A}$ has a smallest non-zero monadic ideal, namely I_x. □

The disjoint union of any family of bounded graphs is a bounded graph, but not one that is point-generated. There are also *connected* bounded graphs that are not point-generated, such as one of the form



having three irreflexive points x such that $R[x] = E$. We could have any number of such irreflexive points like this and still have a bounded graph. Such graphs will be used in Section 8 to construct freely generated **MBA**'s.

A *bounded morphism* from $\mathcal{F}_1 = (W_1, R_1, E_1)$ to $\mathcal{F}_2 = (W_2, R_2, E_2)$ is a function $f : W_1 \to W_2$ such that

- xR_1y implies $f(x)R_2f(y)$;
- $f(x)R_2z$ implies $\exists y(xR_1y$ and $f(y) = z)$; and
- $x \in E_1$ iff $f(x) \in E_2$.

Now *any function* $f : W_1 \to W_2$ induces the pulling-back map $Y \mapsto f^{-1}(Y) = \{x \in W_1 \mid f(x) \in Y\}$ from $\mathcal{P}(W_2)$ to $\mathcal{P}(W_1)$. This preserves the Boolean set operations. If f is a bounded morphism, then $f^{-1}(\langle R_2\rangle(Y)) = \langle R_1\rangle(f^{-1}(Y))$, and $f^{-1}(E_2) = E_1$, so we have a homomorphism $f^{-1} : \mathbf{P}\mathcal{F}_2 \to \mathbf{P}\mathcal{F}_1$ of complex algebras.

If f is surjective, then f^{-1} is injective and makes $\mathbf{P}\mathcal{F}_2$ is isomorphic to its image. This yields the important fact that if there is a bounded morphism from $\mathcal{F}_1$ onto $\mathcal{F}_2$, then $\mathbf{P}\mathcal{F}_2$ is isomorphic to a subalgebra of $\mathbf{P}\mathcal{F}_1$. We use this fact below.

Surjective bounded morphisms may be referred to as bounded *epi*morphisms. These are particularly simple to describe between point-generated bounded graphs. *Any* surjective function from W_1 onto W_2 is a bounded epimorphism $\mathcal{F}_{W_1} \twoheadrightarrow \mathcal{F}_{W_2}$ from the monadic graph on W_1 to the monadic graph on W_2. In the non-monadic case, a surjection from $\mathcal{F}^{x_1}_{X_1}$ to $\mathcal{F}^{x_2}_{X_2}$ is a bounded epimorphism iff it preserves the generators and the marked points, i.e. $f(x_1) = x_2$ and $f(X_1) \subseteq X_2$. There can be no bounded epimorphism between two point-generated bounded graphs of different types (monadic, spiked or vacuous).

If f is injective, then f^{-1} is surjective and makes $\mathbf{P}\mathcal{F}_1$ a homomorphic image of $\mathbf{P}\mathcal{F}_2$. For instance the function that includes X into $\{x\} \cup X$ is an injective bounded morphism $\mathcal{F}_X \rightarrowtail \mathcal{F}^x_X$, showing that $\mathbf{P}\mathcal{F}_X$ is a homomorphic image of $\mathbf{P}\mathcal{F}^x_X$. This even holds when $X = \emptyset$, in which case $\mathbf{P}\mathcal{F}_X$ has one element and is a homomorphic image of every **MBA**.

5 Representation of MBA's

The complex algebra $\mathbf{P}\mathcal{F}^x$ of any point-generated bounded graph will be called a *special* **MBA**, while a *subspecial* **MBA** is any subalgebra of a special one. These can be used to represent all **MBA**'s in the following sense.

Theorem 5.1. *Every* **MBA** *is isomorphic to a subdirect product of subspecial* **MBA***'s.*

Proof. Let $\mathbf{A}$ be an **MBA** with distinguished element E and quantifier $\exists$. We apply the Jónsson-Tarski embedding of $\mathbf{A}$ into the complex algebra of a marked graph, in a form due to Dana Scott (see [21, pp. 191, 204–206]), and then use the point-generated subgraphs of this graph to subdirectly represent the complex algebra by special algebras.

Let $\mathcal{F}_{\mathbf{A}} = (W_{\mathbf{A}}, R_{\mathbf{A}}, E_{\mathbf{A}})$, where $W_{\mathbf{A}}$ is the set of ultrafilters of $\mathbf{A}$; $xR_{\mathbf{A}}y$ iff $\{\exists p \mid p \in y\} \subseteq x$; and $E_{\mathbf{A}} = \{x \in W_{\mathbf{A}} \mid E \in x\}$. A function $\phi : \mathbf{A} \to \mathcal{P}(W_{\mathbf{A}})$ is defined by $\phi(p) = \{x \in W_{\mathbf{A}} \mid p \in x\}$. Hence $E_{\mathbf{A}} = \phi(E)$. This ϕ is an injective homomorphism from $\mathbf{A}$ into the complex algebra $\mathbf{P}\mathcal{F}_{\mathbf{A}}$ [21, Theorem 32].

Now $\mathcal{F}_{\mathbf{A}}$ is a bounded graph. The fact that $\mathbf{A}$ satisfies $\exists\exists p \leq \exists p$ (ax6) ensures that $R_{\mathbf{A}}$ is transitive. For the Euclidean property, if $xR_{\mathbf{A}}y$ and $xR_{\mathbf{A}}z$, we get $yR_{\mathbf{A}}z$ from result $\exists(\exists p)' \leq (\exists p)'$ of Theorem 3.2(13), For if $p \in z$ then $\exists p \in x$ by $xR_{\mathbf{A}}z$, hence $(\exists p)' \notin x$, so $\exists(\exists p)' \notin x$, therefore $(\exists p)' \notin y$ by $xR_{\mathbf{A}}y$, giving $\exists p \in y$ as required.

That $R_{\mathbf{A}}$-image points are marked follows from $\exists E' = \mathbf{0}$ (Theorem 3.2(8)). For if $xR_{\mathbf{A}}y$, then $\exists E' \notin x$, hence $E' \notin y$, giving $E \in y$ and therefore $y \in E_{\mathbf{A}}$.

That marked points are reflexive follows by (ax3). For if $x \in E_{\mathbf{A}}$ and $p \in x$, then $p \wedge E \in x$, hence by (ax3) $\exists p \in x$, as required to show $x R_{\mathbf{A}} x$.

Since $\mathcal{F}_{\mathbf{A}}$ is a bounded graph, its complex algebra $\mathbf{P}\mathcal{F}_{\mathbf{A}}$ is an **MBA** (Theorem 4.1), and has the ϕ-image $\phi(\mathbf{A})$ as a subalgebra isomorphic to $\mathbf{A}$. Each point x of $W_{\mathbf{A}}$ generates the bounded subgraph $\mathcal{F}^x_{\mathbf{A}}$ with its complex algebra $\mathbf{P}\mathcal{F}^x_{\mathbf{A}}$ being a special **MBA**, and being the image of $\mathbf{P}\mathcal{F}_{\mathbf{A}}$ under the **MBA**-homomorphism $f^x : X \mapsto W^x_{\mathbf{A}} \cap X$. Let $\mathbf{A}_x = f^x(\phi(\mathbf{A}))$, the f^x-image of $\phi(\mathbf{A})$. Then $\mathbf{A}_x$ is a subalgebra of $\mathbf{P}\mathcal{F}^x_{\mathbf{A}}$, hence is *subspecial*. Each f^x acts as a homomorphism from $\phi(\mathbf{A})$ onto $\mathbf{A}_x$ and the collection of f^x's for all $x \in W_{\mathbf{A}}$ induces the homomorphism

$$f : \phi(\mathbf{A}) \longrightarrow \prod_{x \in W_{\mathbf{A}}} \mathbf{A}_x$$

into the direct product of the $\mathbf{A}_x$'s that is defined by $f(X)(x) = f^x(X)$. This map is injective, since if $X \neq \emptyset$ in $\phi(\mathbf{A})$, taking any $x \in X$ gives $x \in W^x \cap X = f^x(X)$, so $f(X)(x) \neq \mathbf{0}$ in $\mathbf{A}_x$, hence $f(X) \neq \mathbf{0}$ in the direct product.

Let $\mathbf{A}^*$ be the f-image of $\phi(\mathbf{A})$. Since each f^x maps $\phi(\mathbf{A})$ *onto* $\mathbf{A}_x$, the projection of $\prod_{x \in W_{\mathbf{A}}} \mathbf{A}_x$ to each $\mathbf{A}_x$ maps $\mathbf{A}^*$ onto $\mathbf{A}_x$. So $\mathbf{A}^*$ is a subdirect product of the subspecial **MBA**'s $\mathbf{A}_x$, and being isomorphic to $\phi(\mathbf{A})$, is isomorphic to $\mathbf{A}$. □

Corollary 5.2. *Every* **MBA** *is isomorphic to a subdirect product of models.*

Proof. In the construction of the Theorem, each special algebra $\mathbf{P}\mathcal{F}^x_{\mathbf{A}}$ is basic by Corollary 4.3, and hence each subspecial $\mathbf{A}_x$ is isomorphic to a model by Corollary 3.5. □

Now Theorem 4.4 showed that every subspecial **MBA** is subdirectly irreducible. Hence Theorem 5.1 gives a concrete realisation for **MBA**'s of the universal fact that any algebra is a subdirect product of subdirectly irreducible algebras.

Corollary 5.2 can be viewed as providing an algebraic version of a semantic completeness theorem for monadic free predicate calculus, from the perspective described in [17, p. 47]. According to this approach, we take a *logic* to be a pair $(\mathbf{A}, \mathbf{I})$ where $\mathbf{A}$ is an **MBA** and $\mathbf{I}$ is a monadic ideal of $\mathbf{A}$. The members p of $\mathbf{I}$ are called the *refutable* members of the logic. If $p' \in \mathbf{I}$ then p is called *provable.*

For example, in the functional **MBA** $\mathbf{A}_C$ described in Example 2.2, let $\mathbf{I}_C$ be the set of all elements $\widetilde{\varphi}$ such that the formula $\neg\varphi$ is a theorem of monadic free predicate calculus. Then $(\mathbf{A}_C, \mathbf{I}_C)$ is a logic in this sense, and its provable elements are those $\widetilde{\varphi}$ such that φ is a theorem of monadic free predicate calculus.

An *interpretation* of a logic $(\mathbf{A}, \mathbf{I})$ is a homomorphism f from $\mathbf{A}$ into some model such that $f(p) = \mathbf{0}$ for all $p \in \mathbf{I}$; and hence $f(p) = \mathbf{1}$ for all provable p. This is expressed by saying that every refutable element is *false* in the interpretation, and every provable element is *true.* A member of $\mathbf{A}$ is called *valid* if it is true in every interpretation of the logic, and *universally invalid* if it is false in every interpretation. The definition of interpretation implies that every refutable element of the logic is universally invalid, hence every provable element is valid. The logic is called *semantically complete* if the converse holds, i.e. if every universally invalid element is refutable, or equivalently if every valid element is provable.

Now form the quotient algebra $\mathbf{A}/\mathbf{I}$ of $\mathbf{A}$ by the congruence corresponding to $\mathbf{I}$, and let $\eta : \mathbf{A} \to \mathbf{A}/\mathbf{I}$ be the quotient homomorphism. Take any $p \in \mathbf{A}$ that is not refutable, i.e. $p \notin \mathbf{I}$. Then $\eta(p) \neq \mathbf{0}$ in $\mathbf{A}/\mathbf{I}$. From Corollary 5.2 it follows that there is a homomorphism g from $\mathbf{A}/\mathbf{I}$ into some model such that $g(\eta(p)) \neq \mathbf{0}$. Let f be the composition of η and g. Then $f(p) \neq \mathbf{0}$. Now every member of $\mathbf{I}$ goes to $\mathbf{0}$ under η, and g preserves $\mathbf{0}$. So f is an interpretation of the logic $(\mathbf{A}, \mathbf{I})$ in which p is not false. Hence p is not universally invalid.

So this application of Corollary 5.2 shows that every logic is semantically complete. But if $\mathbf{A}$ is any MBA, then $(A, \{\mathbf{0}\})$ is a logic. Its semantical completeness implies that for any $p \neq \mathbf{0}$ in $\mathbf{A}$ there is a homomorphism $f_p : \mathbf{A} \to \mathbf{A}_p$ into some model $\mathbf{A}_p$ such that $f_p(p) \neq \mathbf{0}$. Now the image of f_p is a subalgebra of $\mathbf{A}_p$, hence is itself isomorphic to a model (Corollary 3.5). So we may assume that f_p is surjective. But then the direct product of all the f_p's gives an injective homomorphism $\mathbf{A} \longrightarrow \prod_{p \neq \mathbf{0}} \mathbf{A}_p$ whose image is a subdirect product of the models $\mathbf{A}_p$.

Thus the statement of Corollary 5.2 is equivalent to the statement that every logic $(\mathbf{A}, \mathbf{I})$ is semantically complete.

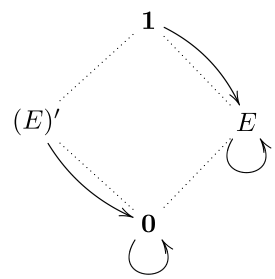
6 Varieties of MBA's

A class of algebras is a *variety* if it consists of all algebras satisfying some particular set of equations. Equivalently, by a celebrated theorem of Birkhoff, a variety is any non-empty class of algebras that is closed under homomorphic images, subalgebras and direct products. We write $V(K)$ for the variety generated by a class K, i.e. $V(K)$ is the smallest variety that includes K. It consists of those algebras that satisfy the equations that hold of all members of K. Any equationally defined class contains all one-element algebras of its kind, since a one-element algebra satisfies all equations whatsoever.

MBA has an equational definition, hence is a variety, with $\mathsf{Mod} \subseteq \mathsf{FMBA} \subseteq \mathsf{MBA}$. But by Corollary 5.2, each MBA is isomorphic to a subalgebra of a direct product of models, so belongs to $V(\mathsf{Mod})$. Thus

$$V(\mathsf{Mod}) = V(\mathsf{FMBA}) = \mathsf{MBA}.$$

Now Mod is precisely the class of basic MBA's (Theorem 3.4), so by Corollary 3.5, it is closed under homomorphic images and subalgebras. But Mod is not a variety, since it is not closed under direct products. To see why, take the one-element bounded graphs $\mathcal{F}_{\{0\}}$ and $\mathcal{F}^0$. The first is the monadic graph, and the second the vacuous graph, on $\{0\}$. Let $\mathbf{A}$ be the direct product $\mathbf{P}\mathcal{F}_{\{0\}} \times \mathbf{P}\mathcal{F}^0$, the four-element **MBA** depicted as



where $\mathbf{0} = (\emptyset, \emptyset)$, $E = (\{0\}, \emptyset)$, $E' = (\emptyset, \{0\})$, $\mathbf{1} = (\{0\}, \{0\})$. The dotted lines indicate the partial order of $\mathbf{A}$, and the arrows show the action of its quantifier. Now $\mathbf{P}\mathcal{F}_{\{0\}}$ and $\mathbf{P}\mathcal{F}^0$ are both basic (Corollary 4.3) and hence belong to **Mod**. But in $\mathbf{A}$ we have $\exists E = E \notin \{\mathbf{0}, \mathbf{1}\}$. Hence by Theorem 2.3, $\mathbf{A}$ is not isomorphic to a functional **MBA**, so does not belong to **FMBA**, let alone **Mod**. Thus neither **Mod** or **FMBA** is closed under direct products, so neither is a variety.

This example marks a difference in behaviour between monadic algebras and **MBA**'s in general, since *every* monadic algebra is isomorphic to a functional one [17, p. 70]. For an example of an algebra in **FMBA** but not **Mod**, take $\mathbf{P}\mathcal{F}_{\{0\}} \times \mathbf{P}\mathcal{F}_{\{0\}}$. This is a direct product of two monadic algebras, hence is a monadic algebra and therefore isomorphic to a functional one. But it is not basic (=simple in the monadic case), hence is not isomorphic to a model, since in fact it satisfies the equation $\exists p = p$, and so has two elements with $p \neq \mathbf{0}$ but $\exists p \neq \mathbf{1}$.

In a sequel paper [8], it will be shown that **FMBA** is precisely the class of **MBA**'s that have $\exists E \in \{\mathbf{0}, \mathbf{1}\}$. Also, every **MBA** is isomorphic to an algebra of $\mathbf{B}$-valued functions on some set X, with $\exists p(y) = \bigvee\{p(x) \wedge E(x) \mid x \in X\}$ holding in this algebra. This looks like the algebra that would result from a function monadic algebra $\mathbf{A}$ by forming the relativised algebra $\mathbf{A}^E$ for some $E \in \mathbf{A}$, as defined in Example 3.1. It gives a weaker kind of representation as there is no distinguished subset X_E involved.

The main result of this section is that every variety of **MBA**'s is generated by its *finite* members, and indeed by its finite *special* members. The heart of the proof is the following technical construction.

Lemma 6.1. *Let $\mathcal{F}$ be a point-generated bounded graph, and $\mathbf{A}$ a subalgebra of $\mathbf{P}\mathcal{F}$. Suppose that a certain equation is falsifiable in $\mathbf{A}$. Then there exists a finite point-generated bounded graph $\bar{\mathcal{F}}$ that is of the same type as $\mathcal{F}$ and is a bounded epimorphic image of $\mathcal{F}$, such that the equation is falsified in $\mathbf{P}\bar{\mathcal{F}}$, and $\mathbf{P}\bar{\mathcal{F}}$ is isomorphic to a subalgebra of $\mathbf{A}$.*

Proof. We invoke the filtration method that has been widely applied in modal logic, and was first introduced by Lemmon [21] as a technique for constructing finite complex algebras.

Let $\mathcal{F} = (W, R, E)$. Now any equation is equivalent to one of the form $t \approx \mathbf{1}$ for some term t, since in general $p = q$ iff $(p + q)' = \mathbf{1}$. So suppose that $\mathbf{A} \not\models t \approx \mathbf{1}$, and let $v_0, \ldots, v_{n-1}$ be the variables of t. Then there are $A_0, \ldots, A_{n-1} \in \mathbf{A}$ such that $t^{\mathbf{A}}(A_0, \ldots, A_{n-1}) \neq \mathbf{1}^{\mathbf{A}} = W$, where $t^{\mathbf{A}}$ is the term function on $\mathbf{A}$ defined by t. Let $\{t_0, \ldots, t_{r-1}\}$ be the set of all subterms of t. Put $B_i = t_i^{\mathbf{A}}(A_0, \ldots, A_{n-1})$ for each $i < r$, and $S = \{B_0, \ldots, B_{r-1}, E\}$. Note that every member of S belongs to $\mathbf{A}$, as $\mathbf{A}$ is a subalgebra of $\mathbf{P}\mathcal{F}$.

Define an equivalence relation $\equiv$ on W by putting $x \equiv y$ iff x and y belong to exactly the same members of S. For every $x \in W$, put $\bar{x} = \{y \in W \mid x \equiv y\}$, and for $B \subseteq W$, let $\bar{B} = \{\bar{x} \mid x \in B\}$. The set $\bar{W}$ of all equivalence classes is finite, with at most 2^{r+1} members, since the map $\bar{x} \mapsto \{B \in S \mid x \in B\}$ is a well defined injection of $\bar{W}$ into $\mathcal{P}(S)$, and S has at most $r + 1$ members. Note that if $B \in S$, then in general $\bar{x} \in \bar{B}$ iff $x \in B$.

Now define the marked graph $\bar{\mathcal{F}} = (\bar{W}, \bar{R}, \bar{E})$, where

$$\bar{x}\bar{R}\bar{y} \text{ iff there exist } x' \in \bar{x} \text{ and } y' \in \bar{y} \text{ such that } x'Ry'. \tag{6.1}$$

This has complex algebra $\mathbf{P}\bar{\mathcal{F}}$. By the techniques of [21, pp. 209-210], it can be shown that

$$\overline{t_i^{\mathbf{A}}(A_0, \ldots, A_{n-1})} = t_i^{\mathbf{P}\bar{\mathcal{F}}}(\,\overline{A_0}, \ldots, \overline{A_{n-1}}\,), \tag{6.2}$$

for all $i < r$. As t is t_i for some i, this implies that $t^{\mathbf{P}\bar{\mathcal{F}}}(\,\overline{A_0}, \ldots, \overline{A_{n-1}}\,) \neq \bar{W}$, since $W \neq t^{\mathbf{A}}(A_0, \ldots, A_{n-1}) \in S$. Hence the equation $t \approx \mathbf{1}$ fails in the finite complex algebra $\mathbf{P}\bar{\mathcal{F}}$. (There are other ways to define an $\bar{R}$ that leads to (6.2), but the one we have given will certainly do.)

Now the function $f(x) = \bar{x}$ maps W onto $\bar{W}$, and induces the injective function $f^{-1} : \mathcal{P}(\bar{W}) \to \mathcal{P}(W)$. Let $\mathbf{B}_S$ be the Boolean subalgebra of $\mathbf{P}\bar{W}$ generated by the set $\{\bar{B} \mid B \in S\}$. Then $\mathbf{B}_S$ is finite, with each of its members constructed from members of $\{\bar{B} \mid B \in S\}$ in finitely many steps by the Boolean set operations $\cap$, $\cup$ and $-$. For each $B \in S$, we have $f^{-1}(\bar{B}) = \{x \in W \mid \bar{x} \in \bar{B}\} = B \in \mathbf{A}$. Thus f^{-1} maps all the generators of $\mathbf{B}_S$ into $\mathbf{A}$ and so, as it preserves the Boolean operations and $\mathbf{A}$ is closed under these operations, f^{-1} maps all of $\mathbf{B}_S$ into $\mathbf{A}$. But every subset of $\bar{W}$ belongs to $\mathbf{B}_S$, so the domain of f^{-1} is the whole powerset $\mathcal{P}(\bar{W})$. To see why, take any $\bar{x} \in \bar{W}$. Then if $\bar{y} \neq \bar{x}$, there is some $B \in S$ with either $x \in B$ and $y \notin B$, hence $\bar{x} \in \bar{B} \in \mathbf{B}_S$ and $\bar{y} \notin \bar{B}$; or else $x \in -B$ and $y \notin -B$, hence $\bar{x} \in -\bar{B} \in \mathbf{B}_S$ and $\bar{y} \notin -\bar{B}$. This shows that $\{\bar{x}\} = \bigcap\{Z \in \mathbf{B}_S \mid \bar{x} \in Z\} \in \mathbf{B}_S$. Thus all singleton subsets of $\bar{W}$ belong to $\mathbf{B}_S$. But any subset of $\bar{W}$ is the union of its finitely many singleton subsets, so belongs to $\mathbf{B}_S$.

We have thus shown that f^{-1} maps $\mathcal{P}(\bar{W})$ into $\mathbf{A}$. It remains to show that $\bar{\mathcal{F}}$ is point-generated of the same type as $\mathcal{F}$, and that f is a bounded morphism from $\mathcal{F}$ to $\bar{\mathcal{F}}$. This will imply that f^{-1} is an injective homomorphism making $\mathbf{P}\bar{\mathcal{F}}$ isomorphic to a subalgebra of $\mathbf{A}$, as required. In all of this there are two cases to consider.

The first case is when the point-generated bounded graph $\mathcal{F}$ is the monadic graph on W, i.e. $E = W$ and $R = W \times W$. Then $\bar{E} = \bar{W}$, and by (6.1) $\bar{R} = \bar{W} \times \bar{W}$. So $\bar{\mathcal{F}}$ is the monadic graph on $\bar{W}$ – of the same type as $\mathcal{F}$ – and f is a bounded epimorphism because any surjective function between monadic bounded graphs is a bounded morphism. Also $\mathcal{F}$ is point-generated by any of its points, and that completes the argument in this case.

In the alternative case, if $\mathcal{F}$ is not monadic it must be either spiked or vacuous, having the form $\mathcal{F}_X^x$, with $E = X$, $W = \{x\} \cup E$, $x \notin E$, and yRz iff $z \in E$. Then $\bar{W} = \{\bar{x}\} \cup \bar{E}$. Since E belongs to S and separates x from the rest of W, we have $\bar{x} \notin \bar{E}$ with $\bar{x} = \{x\}$. Using (6.1) we see that $\bar{y}\bar{R}\bar{z}$ iff $\bar{z} \in \bar{E}$. Thus $\bar{\mathcal{F}}$ is point-generated by $\bar{x}$ and is the bounded graph $\mathcal{F}_{\bar{E}}^{\bar{x}}$. As $E \neq \emptyset$ iff $\bar{E} \neq \emptyset$, $\bar{F}$ of the same type (spiked or vacuous) as $\mathcal{F}$. Also f maps x to $\bar{x}$ and E into $\bar{E}$, i.e. it preserves the generators and the marked points, so is a bounded epimorphism for this type of graph. That completes the argument in this case also, and completes the proof of the Theorem. □

The proof just given showed that if the term t has r subterms, then the algebra $\mathbf{P}\bar{\mathcal{F}}$ has at most $2^{2^{r+1}}$ members. That information can be used to show that the equational theory of any variety of **MBA**'s is decidable. We discuss this at the end of Section 7.

Theorem 6.2. *Every variety of* **MBA***'s is generated by its finite special members.*

Proof. Fix a variety V of **MBA**'s. It suffices to show that if an equation $t \approx \mathbf{1}$ is satisfied by every finite special algebra in V, then it is satisfied by every member of V. Working contrapositively, suppose that there is some $\mathbf{A}_0$ in V with $\mathbf{A}_0 \not\models t \approx \mathbf{1}$. Now $\mathbf{A}_0$ is a subdirect product of a family of subspecial **MBA**'s (Theorem 5.1), and so some member $\mathbf{A}$ of this family must have $\mathbf{A} \not\models t \approx \mathbf{1}$. Since the product is subdirect, $\mathbf{A}$ is a homomorphic image of $\mathbf{A}_0$, so $\mathbf{A}$ also belongs to the variety V. Since $\mathbf{A}$ is subspecial, it is a subalgebra of the complex algebra $\mathbf{P}\mathcal{F}$ of some point-generated bounded graph $\mathcal{F}$. By the Lemma just proved, $t \approx \mathbf{1}$ is falsified in some finite special algebra $\mathbf{P}\bar{\mathcal{F}}$ that is isomorphic to a subalgebra of $\mathbf{A}$. But then as $\mathbf{A}$ belongs to V, so too does $\mathbf{P}\bar{\mathcal{F}}$. This shows that the equation is falsified by a finite special algebra in V. □

7 Fourteen Kinds of Variety

We now define a particular collection of special **MBA**'s that contains two infinite algebras together an isomorphic copy of each finite special **MBA**. The collection can be used to generate every variety of **MBA**'s. From the given analysis it will follow that there are only countably many such varieties. Also, we will associate with each variety an explicit finite set of equations that defines it.

Let $\omega = \{0, 1, 2, \dots\}$ be the set of finite ordinals, with $0 = \emptyset$, and $n = \{0, 1, \dots, n-1\}$ if $1 \leq n < \omega$.

For each $0 \leq n \leq \omega$, let $\mathbf{P}_n$ be the complex algebra $\mathbf{P}\mathcal{F}_n$ of the monadic bounded graph $\mathcal{F}_n$ on the set n. Each $\mathbf{P}_n$ is a simple monadic algebra. In particular, $\mathcal{F}_0$ is the empty graph and $\mathbf{P}_0$ is a one-element monadic algebra.

Let $V(\mathbf{P}_n)$ be the variety generated by $\mathbf{P}_n$. $V(\mathbf{P}_0)$ is the class of all one-element monadic algebras, and $V(\mathbf{P}_\omega)$ the class of all monadic algebras. Monk

[22] observed that

$$V(\mathbf{P}_0) \subsetneq V(\mathbf{P}_1) \subsetneq \cdots \subsetneq V(\mathbf{P}_\omega),$$

and that these are all the varieties of monadic algebras there are. $V(\mathbf{P}_0)$ is also the class of all one-element **MBA**'s and is included in every variety of **MBA**'s.

Now suppose ∞ is an entity not in ω. Let $\mathcal{F}_n^\infty$ be the bounded graph generated by ∞ with n as its set of marked points, and let $\mathbf{P}_n^\infty$ be the complex algebra $\mathbf{P}\mathcal{F}_n^\infty$. For $n \geq 1$, $\mathcal{F}_n^\infty$ is a spiked graph. $\mathcal{F}_0^\infty$ is the vacuous graph on $\{\infty\}$, and $\mathbf{P}_0^\infty$ is a two-element vacuous **MBA**. We will often write $\mathbf{P}_0^\infty$ just as $\mathbf{P}^\infty$.

It is evident than any finite point-generated bounded graph is isomorphic to $\mathcal{F}_n$ or $\mathcal{F}_n^\infty$ for some $0 \leq n < \omega$. Hence the set

$$\mathsf{FinSp} = \{\mathbf{P}_n, \mathbf{P}_n^\infty \mid 0 \leq n < \omega\}$$

contains an isomorphic copy of every finite special **MBA**. It follows by Theorem 6.2 that each variety of **MBA**'s is generated by some subset of FinSp. But there are $2^{\aleph_0}$ such subsets, so this does not establish that there are only countably many such varieties. What we show instead is that every **MBA**-variety is generated by some subset of $\{\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}^\infty\}$ for a suitable choice of n and m. There are countably many such generating sets.

Lemma 7.1.

(1) *For every $1 \leq n \leq m \leq \omega$, $\mathbf{P}_n$ is isomorphic to a subalgebra of $\mathbf{P}_m$, and $\mathbf{P}_n^\infty$ is isomorphic to a subalgebra of $\mathbf{P}_m^\infty$.*

(2) *For every $n \leq \omega$, $\mathbf{P}_n$ is a homomorphic image of $\mathbf{P}_n^\infty$.*

Proof.

(1) If $1 \leq n \leq m$ then $\emptyset \neq n \subseteq m$. Pick any surjection $f : m \twoheadrightarrow n$. Then f is a bounded epimorphism from $\mathcal{F}_m$ onto $\mathcal{F}_n$, inducing the injective homomorphism $f^{-1} : \mathbf{P}\mathcal{F}_n \rightarrowtail \mathbf{P}\mathcal{F}_m$, making $\mathbf{P}\mathcal{F}_n$ isomorphic to a subalgebra of $\mathbf{P}\mathcal{F}_m$.

    Now extend the domain of f to include ∞ by putting $f(\infty) = \infty$. Then f becomes a bounded epimorphism from $\mathcal{F}_m^\infty$ onto $\mathcal{F}_n^\infty$, inducing the injective homomorphism $f^{-1} : \mathbf{P}\mathcal{F}_n^\infty \rightarrowtail \mathbf{P}\mathcal{F}_m^\infty$ giving the desired result.

(2) The inclusion function $n \hookrightarrow n \cup \{\infty\}$ is an injective bounded morphism from $\mathcal{F}_n$ into $\mathcal{F}_n^\infty$, inducing a surjective homomorphism $\mathbf{P}\mathcal{F}_n^\infty \twoheadrightarrow \mathbf{P}\mathcal{F}_n$.

□

Lemma 7.2. *Let V be any variety of* **MBA***'s.*

(1) *For all $m \leq \omega$, if $\mathbf{P}_m \in V$, then $\mathbf{P}_n \in V$ for all $n \leq m$.*

(2) *If $\mathbf{P}_m \in V$ for arbitrarily large $m < \omega$, then $\mathbf{P}_n \in V$ for all $n \leq \omega$.*

(3) *For all $m \leq \omega$, if $\mathbf{P}_m^\infty \in V$, then $\mathbf{P}_n^\infty \in V$ for all $1 \leq n \leq m$.*

(4) *If* $\mathbf{P}_m^\infty \in V$ *for arbitrarily large* $m < \omega$, *then* $\mathbf{P}_n^\infty \in V$ *for all* $1 \leq n \leq \omega$.

Proof.

(1) Let $\mathbf{P}_m \in V$. By (1) of the last Lemma, if $1 \leq n \leq m$, then $\mathbf{P}_n$ is isomorphic to a subalgebra of $\mathbf{P}_m$, so belongs to V. But also the one-element $\mathbf{P}_0$ is a homomorphic image of $\mathbf{P}_m$, so $\mathbf{P}_0 \in V$.

(2) Suppose we have arbitrarily large finite $\mathbf{P}_m$ in V. It suffices to show that $\mathbf{P}_\omega \in V$, for then every $\mathbf{P}_n$ is in V by putting $m = \omega$ in part (1). So, suppose that $\mathbf{P}_\omega \notin V$. Then there is some equation $t \approx \mathbf{1}$ that is satisfied by every member of V, but not by $\mathbf{P}_\omega = \mathbf{P}\mathcal{F}_\omega$. Lemma 6.1 then implies that the equation is false in the complex algebra of some finite monadic graph. Hence there exists some $n < \omega$ such that $\mathbf{P}_n \not\models t \approx \mathbf{1}$. By hypothesis, there is some finite $m \geq n$ with $\mathbf{P}_m \in V$. Hence by (1), $\mathbf{P}_n \in V$, implying that $\mathbf{P}_n \models t \approx \mathbf{1}$. This is a contradiction, forcing us to conclude that $\mathbf{P}_\omega \in V$.

(3) Similarly to (1), using the rest of part (1) of the last Lemma. The case $n = 0$ does not apply here, as the two-element $\mathbf{P}_0^\infty$ is not a homomorphic image of $\mathbf{P}_m^\infty$ if $m \geq 1$.

(4) Similar to (2), using (3). □

We are now ready to show that any variety of **MBA**'s is generated by a set of at most three algebras, with each generator representing a different type (monadic, spiked, vacuous) of special **MBA**.

Theorem 7.3. *Suppose* V *is a variety of* **MBA***'s. Then there exist* $0 \leq n \leq \omega$, $1 \leq m \leq \omega$ *and a subset* S *of* $\{\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}^\infty\}$ *such that* $V = V(S)$.

Proof. Let K be the class of all finite special **MBA**'s in V. Then K is closed under isomorphism, and $V = V(K)$ (Theorem 6.2). We show that there is some set S as described, such $S \subseteq V$ and $K \subseteq V(S)$, so that $V(S) = V$.

First, if K contains some vacuous **MBA**, and hence all of them, put the vacuous $\mathbf{P}^\infty$ into S, ensuring that all vacuous members of K are in $V(S)$. Otherwise, if K contains no vacuous **MBA**, $\mathbf{P}^\infty$ is left out of S.

Next we consider the monadic algebras in K, which are represented isomorphically by finite $\mathbf{P}_n$'s. If there exists a largest finite n such that $\mathbf{P}_n \in V$, put this $\mathbf{P}_n$ into S. In this case, any monadic member of K is isomorphic to $\mathbf{P}_m$ for some $m \leq n$, so part (1) of the last Lemma then ensures that all monadic members of K belong to $V(S)$. If however there is no largest finite $\mathbf{P}_n$ in V, then by part (2) of the last Lemma, $\mathbf{P}_n \in V$ for all $n \leq \omega$. In this case we just put $\mathbf{P}_\omega$ into S. Since $\mathbf{P}_n$ isomorphic to a subalgebra of $\mathbf{P}_\omega$ for all finite $n \geq 1$ (Lemma 7.1(1)), and $\mathbf{P}_0$ is a homomorphic image of $\mathbf{P}_\omega$, this ensures that every finite $\mathbf{P}_n$ is in $V(S)$, including all the monadic members of K.

Finally we consider spiked members of K, represented isomorphically by finite $\mathbf{P}_m^\infty$'s with $m \geq 1$. First of all, there may not be any spiked members of K, e.g. if V consists of monadic algebras. In that case we add nothing more to S. If there is a largest finite $\mathbf{P}_m^\infty$ in K with $m \geq 1$, we put it into S. Otherwise,

S	Equations defining $V(S)$ within **MBA**
$\{\mathbf{P}_\omega^\infty, \mathbf{P}^\infty\}$	$\emptyset$
$\{\mathbf{P}_\omega^\infty\}$	$\{\exists E \approx \mathbf{1}\}$
$\{\mathbf{P}_\omega, \mathbf{P}^\infty\}$	$\{E \vee (\exists E)' \approx \mathbf{1}\}$
$\{\mathbf{P}_\omega\}$	$\{E \approx \mathbf{1}\}$
$\{\mathbf{P}^\infty\}$	$\{E \approx \mathbf{0}\}$
$\{\mathbf{P}_n, \mathbf{P}^\infty\}, 1 \leq n < \omega$	$\{E \vee (\exists E)' \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
$\{\mathbf{P}_n\}, 1 \leq n < \omega$	$\{E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
$\{\mathbf{P}_\omega, \mathbf{P}_n^\infty, \mathbf{P}^\infty\}, 1 \leq n < \omega$	$\{E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$
$\{\mathbf{P}_\omega, \mathbf{P}_n^\infty\}, 1 \leq n < \omega$	$\{\exists E \approx \mathbf{1}, E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}\}$
$\{\mathbf{P}_n^\infty, \mathbf{P}^\infty\}, 1 \leq n < \omega$	$\{\mathsf{Alt}_n \approx \mathbf{0}\}$
$\{\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}^\infty\}, 1 \leq m < n < \omega$	$\{\mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$
$\{\mathbf{P}_n^\infty\}, 1 \leq n < \omega$	$\{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}\}$
$\{\mathbf{P}_n, \mathbf{P}_m^\infty\}, 1 \leq m < n < \omega$	$\{\exists E \approx \mathbf{1}, \mathsf{Alt}_n \approx \mathbf{0}, E \vee (\mathsf{Alt}_m)' \approx \mathbf{1}\}$
$\{\mathbf{P}_0\}$	$\{v_0 \approx v_1\}$

Table 1: Generating sets and defining equations

when there is no largest finite $\mathbf{P}_m^\infty$ in K, then $\mathbf{P}_\omega^\infty \in V$ and we put it into S. Then similar reasoning to the monadic case shows that, whichever of these cases applies, all spiked members of K are in $V(S)$.

Since every member of K is either vacuous, monadic or spiked, the overall definition of S ensures that every member of K is in $V(S)$, hence $V = V(K) \subseteq V(S)$, and that $S \subseteq V$, hence $V(S) \subseteq V$. □

Reflection on the small generating sets S provided by this Theorem shows that there is still redundancy in some cases. For instance, if S contains $\mathbf{P}_0$ and at least one other algebra, then $\mathbf{P}_0$ can be deleted from S, because $\mathbf{P}_0$ is a homomorphic image of any other **MBA**, and so the deletion does not change the variety generated. Also, if $1 \leq n \leq m \leq \omega$, then $\mathbf{P}_n$ is isomorphic to a subalgebra of $\mathbf{P}_m$, which is a homomorphic image of $\mathbf{P}_m^\infty$ (Lemma 7.1), so

$$V(\mathbf{P}_n, \mathbf{P}_m^\infty, \mathbf{P}^\infty) = V(\mathbf{P}_m^\infty, \mathbf{P}^\infty) \quad \text{and} \quad V(\mathbf{P}_n, \mathbf{P}_m^\infty) = V(\mathbf{P}_m^\infty).$$

Thus for example, **MBA** $= V(\mathbf{P}_\omega, \mathbf{P}_\omega^\infty, \mathbf{P}^\infty) = V(\mathbf{P}_\omega^\infty, \mathbf{P}^\infty)$. Such considerations show that there are in fact fourteen distinguishable kinds of generating set S for **MBA**-varieties. These are listed in the first column of Table 1. The second column assigns to each S a set of equations that is satisfied precisely by those **MBA**'s that belong to $V(S)$. Adding (ax1)–(ax6) to this set gives a list of between six and nine equations axiomatising $V(S)$.

The equation $\exists E \approx \mathbf{1}$ holds of all monadic and spiked special algebras, but is false in a vacuous one. Its presence serves to exclude $\mathbf{P}^\infty$ from S. The equation

$E \vee (\exists E)' \approx \mathbf{1}$ asserts that $\exists E \leq E$. This holds of all monadic algebras, which have $E = \mathbf{1}$, and of $\mathbf{P}^\infty$, which has $(\exists E) = \mathbf{0}$, but fails on any spiked algebra $\mathbf{P}\mathcal{F}_E^x$, since this has $x \in \exists E - E$. So this equation excludes $\mathbf{P}_m^\infty$ from S for all $m \geq 1$.

For $1 \leq n < \omega$, the term Alt_n is defined to be

$$\bigwedge_{0 \leq i \leq n} \exists (v_0 \wedge \cdots \wedge v_{i-1} \wedge v_i').$$

The equation $\mathsf{Alt}_n \approx \mathbf{0}$ asserts of a special **MBA** that each image set in its graph has at most n points, so there are at most n marked points[3]. To indicate why, suppose that in $\mathbf{P}_k$ or $\mathbf{P}_k^\infty$ we have

$$\bigcap\nolimits_{i \leq n} \exists (p_0 \cap \cdots \cap p_{i-1} \cap p_i') \neq \emptyset. \tag{7.1}$$

Then there is a point x such that $R[x]$ contains some x_i in $p_0 \cap \cdots \cap p_{i-1} \cap p_i'$ for each $i \leq n$. These points $x_0, \ldots, x_n$ must all be distinct members of $E = k$, so $k > n$. Conversely, if $k > n$, then sets p_i making (7.1) true can be defined by putting $p_i = \{0, \ldots, k - (i+2)\}$ for $i < n$ and $p_n = \emptyset$. In this way it can be shown that

$$\mathbf{P}_k^\infty \models \mathsf{Alt}_n \approx \mathbf{0} \quad \text{iff} \quad \mathbf{P}_k \models \mathsf{Alt}_n \approx \mathbf{0} \quad \text{iff} \quad k \leq n,$$

so this equation excludes $\mathbf{P}_k$ and $\mathbf{P}_k^\infty$ from S when $k > n$.

The equation $E \vee (\mathsf{Alt}_n)' \approx \mathbf{1}$ expresses of a complex algebra that

$$\bigcap\nolimits_{i \leq n} \exists (p_0 \cap \cdots \cap p_{i-1} \cap p_i') \leq E.$$

This requiries that if $R[x]$ has more than n members, then x is a marked point. That condition fails when $x = \infty$ in $\mathcal{F}_k^\infty$ with $k > n$. So the equation excludes $\mathbf{P}_k^\infty$ from S when $k > n$.

An examination of cases shows that the equations given in Table 1 do axiomatise the corresponding varieties of **MBA**'s. Full details of this are provided in [1].

The classification of varieties leads to an algorithmic procedure that will decide whether any given equation is valid in any given variety of **MBA**'s. An equation is valid in $V(S)$ iff it is valid in each of the (at most three) algebras belonging to S. If these algebras are finite, the question can be decided in finite time. But suppose S contains one of the infinite algebras, say $\mathbf{P}_\omega$. If a term t has r subterms, and the equation $t \approx \mathbf{1}$ fails in $\mathbf{P}_\omega$, then by the construction in the proof of Lemma 6.1, it fails in $\mathbf{P}_n$ for some $n \leq 2^{2^{r+1}}$. Hence it fails in $\mathbf{P}_{2^{2^{r+1}}}$, as that algebra has a subalgebra isomorphic to $\mathbf{P}_n$ (Lemma 7.1). Conversely, as $\mathbf{P}_{2^{2^{r+1}}}$ is isomorphic to a subalgebra of $\mathbf{P}_\omega$, any equation failing

[3] The name Alt_n is borrowed from a certain axiom of modal logic, where a y such that xRy is called an *alternative* of x, and validity of the axiom forces each x to have at most n alternatives. See [24, p. 52] or [6, p. 82].

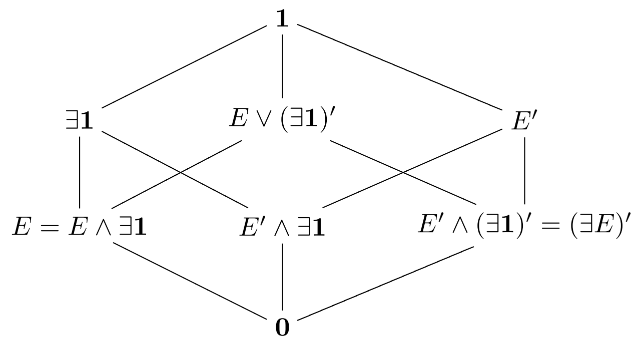


Figure 2: The free **MBA** on 0 generators.

in the former must fail in the latter. So to decide if $t \approx \mathbf{1}$ is valid in $\mathbf{P}_\omega$ it suffices to determine if it is valid in the finite $\mathbf{P}_{2^{2^r+1}}$. Similarly, to decide if an equation is valid in $\mathbf{P}^\infty_\omega$, it suffices to determine if it is valid in $\mathbf{P}^\infty_n$ for some n that is computable from the number of subterms in the equation.

In fact this analysis can be extended from equations to universal sentences, i.e. sentences in prenex normal form that have only universal quantifiers. Validity of such sentences is preserved by subalgebras. There is a general argument showing that a filtration construction, as in Lemma 6.1, that collapses complex algebra countermodels for equations to finite countermodels, will also work for countermodels to universal sentences. This argument can be found in [15, Lemma 2.15]. It can be applied to show that the universal theory of any variety of **MBA**'s is decidable.

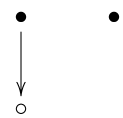
8 Finitely Generated MBA's

Bass [2] showed that every finitely-generated monadic algebra is finite, having at most $2^{2^r \cdot 2^{2^r-1}}$ elements, where r is the number of generators. Moreover, the free monadic algebra on r generators has exactly this many elements.

Here we will demonstrate that the same results hold for **MBA**'s but with the size bound increased to

$$2^{3\cdot 2^r \cdot 2^{2^r-1}}.$$

The additional factor of 3 in the exponent makes for some interesting comparisons. On the empty set of generators ($r = 0$), the free monadic algebra is $\mathbf{P}_2$, i.e. $\mathbf{2}$ with the simple quantifier, whereas the free **MBA** is the eight-element algebra depicted in Figure 2 (the labelling of the atoms will be explained below). This algebra can be described as the direct product $\mathbf{P}^\infty_1 \times \mathbf{P}^\infty_0$, or as the complex algebra $\mathbf{P}\mathcal{F}$ where $\mathcal{F}$ is the three-point graph



(8.1)

formed as the disjoint union of $\mathcal{F}_1^\infty$ and $\mathcal{F}_0^\infty$. An explanation of why this graph gives rise to the 0-generated free **MBA** will emerge at the end of this section.

On one generator ($r = 1$), the free monadic algebra has 16 elements, while the free **MBA** has $2^{12} = 4096$ and is the complex algebra of the 12-point graph depicted in Figure 3 near the end of the paper.

On two generators, the comparative sizes are 4,294,967,296 for the free monadic algebra and

$$79,228,162,514,264,337,593,543,950,336$$

for the free **MBA**.

To study finitely generated **MBA**'s we first recall the situation with Boolean algebras. Let $\mathbf{B}$ be a Boolean algebra, and $\mathbf{B}'$ its subalgebra generated by elements $p_0, \dots, p_{r-1}$. A *minterm* in these elements is an element of the form

$$p_0^\pm \wedge \cdots \wedge p_{r-1}^\pm,$$

where $p_i^+ = p_i$ and $p_i^- = p_i'$. If p_i^+ occurs, we say that p_i *occurs positively* in the minterm. Otherwise, when p_i' occurs, then p_i *occurs negatively.* Note that when $r = 0$, there is only one minterm, the empty meet $\mathbf{1}$.

The subalgebra $\mathbf{B}'$ is finite, and consists of all possible joins of finitely many minterms. The atoms of $\mathbf{B}'$ are precisely those minterms that are non-zero, so there are at most 2^r atoms in $\mathbf{B}'$, and hence at most 2^{2^r} members of $\mathbf{B}'$. In effect, the joins of finite sets of minterms provide a set of "normal forms" for all the elements of $\mathbf{B}$ that can be generated from the p_i's by the Boolean operations [4, §III.5].

We now show that there is a similar set of normal forms for elements generated by **MBA**-operations. Fix an **MBA** $\mathbf{A} = (\mathbf{B}, E, \exists)$ that is generated by elements $p_0, \dots, p_{r-1}$. Let $m = 2^r$, and let $\mu_0, \dots, \mu_{m-1}$ be an enumeration without repetition of all the minterm *expressions* in the p_i, regarding any two such expressions as distinct entitites. Thus for $j \neq l$, there must be some p_i that occurs positively in one of μ_j and μ_l and negatively in the other.

Now let $\mathbf{B}_0$ be the Boolean subalgebra of $\mathbf{B}$ generated by the list of elements

$$p_0, \dots, p_{r-1}, E, \exists\mu_0, \dots, \exists\mu_{m-1}.$$

$\mathbf{B}_0$ is finite, and its atoms are the non-zero minterms in this list, of the form

$$p_0^\pm \wedge \cdots \wedge p_{r-1}^\pm \wedge E^\pm \wedge (\exists\mu_0)^\pm \wedge \cdots \wedge (\exists\mu_{m-1})^\pm.$$

But $p_0^\pm \wedge \cdots \wedge p_{r-1}^\pm$ here is equal to μ_i for some $i < m$, so we see that the atoms of $\mathbf{B}_0$ all have the form

$$\mu_i \wedge E^\pm \wedge (\exists\mu_0)^\pm \wedge \cdots \wedge (\exists\mu_{m-1})^\pm. \qquad (8.2)$$

Let α be an atom in this form. We show that $\exists\alpha$ belongs to $\mathbf{B}_0$. For, by (ax2) and Theorem 3.2(16), we have

$$\exists(p \wedge (\exists q)^{\pm}) = \exists p \wedge (\exists q)^{\pm}$$

in general. Repeated application of this gives

$$\exists\alpha = \exists(\mu_i \wedge E^{\pm}) \wedge (\exists\mu_0)^{\pm} \wedge \cdots \wedge (\exists\mu_{m-1})^{\pm}.$$

Now $\exists(\mu_i \wedge E) = \exists\mu_i$ (Theorem 3.2(3)), so if E occurs positively in (8.2), then

$$\exists\alpha = \exists\mu_i \wedge (\exists\mu_0)^{\pm} \wedge \cdots \wedge (\exists\mu_{m-1})^{\pm},$$

which belongs to $\mathbf{B}_0$ since all of $\exists\mu_0, \ldots, \exists\mu_{m-1}$ do. But $\exists(\mu_i \wedge E') = \mathbf{0}$ (Theorem 3.2(7)), so if E occurs negatively in (8.2), then $\exists\alpha = \mathbf{0} \in \mathbf{B}_0$.

This shows that the set of atoms of $\mathbf{B}_0$ is closed under $\exists$. But then so is $\mathbf{B}_0$ itself, since each member of $\mathbf{B}_0$ is a join $\bigvee_{i<k} \alpha_i$ of finitely many atoms α_i. Since $\exists$ preserves joins (Theorem 3.2(4)), we get

$$\exists(\bigvee_{i<k}\alpha_i) = \bigvee_{i<k}\exists\alpha_i \in \mathbf{B}_0.$$

It follows that $(\mathbf{B}_0, E, \exists)$ is a subalgebra of $\mathbf{A}$. But $p_0, \ldots, p_{r-1}$ all belong to $\mathbf{B}_0$ and generate $\mathbf{A}$, so this implies that $\mathbf{B}_0 = \mathbf{B}$, hence $\mathbf{A}$ is a finite **MBA**.

To compute an upper bound on the size of $\mathbf{A}$, we need to count the number of atoms that there can be in $\mathbf{B}_0$. Consider an element α of the form (8.2). Now if E occurs positively and $\exists\mu_i$ negatively in α, then as $\mu_i \wedge E \wedge (\exists\mu_i)' = \mathbf{0}$ (because $\mu_i \wedge E \leq \exists\mu_i$), we have $\alpha = \mathbf{0}$ and therefore not an atom. This is the only obstacle in principle to α being an atom, and so we will say that α is a *potential atom* if it does not contain E positively and $\exists\mu_i$ negatively.

Now let α be a potential atom. If E occurs positively, then so does $\exists\mu_i$, and since $\mu_i \wedge E \wedge \exists\mu_i = \mu_i \wedge E$, we can leave out $\exists\mu_i$ and conclude that α is

$$\mu_i \wedge E \wedge (\exists\mu_0)^{\pm} \wedge \cdots \wedge (\exists\mu_{i-1})^{\pm} \wedge (\exists\mu_{i+1})^{\pm} \wedge \cdots \wedge (\exists\mu_{m-1})^{\pm}. \tag{8.3}$$

For a fixed μ_i, there are at most 2^{m-1} such atoms. If E occurs negatively, then α is

$$\mu_i \wedge E' \wedge (\exists\mu_0)^{\pm} \wedge \cdots \wedge (\exists\mu_{m-1})^{\pm}, \tag{8.4}$$

and there are at most 2^m such atoms for a fixed μ_i. Altogether, each μ_i occurs in at most $2^m + 2^{m-1} = 3 \cdot 2^{m-1}$ atoms. Hence the total number of atoms in $\mathbf{B}_0$ is at most $m \cdot 3 \cdot 2^{m-1}$. This reasoning has shown

Theorem 8.1. *An* **MBA** *generated by r elements is finite, with at most $3 \cdot 2^r \cdot 2^{2^r-1}$ atoms and at most $2^{3\cdot 2^r \cdot 2^{2^r-1}}$ elements.* □

In the case $r = 0$, where $\mathbf{1}$ is the only minterm, (8.2) reduces to

$$E^{\pm} \wedge (\exists\mathbf{1})^{\pm}.$$

Now $E \wedge (\exists \mathbf{1})' = \mathbf{0}$, which is not an atom, while $E \wedge \exists \mathbf{1} = E$ and $E' \wedge (\exists \mathbf{1})' = (E \vee \exists \mathbf{1})' = (\exists \mathbf{1})'$. So a 0-generated **MBA** has three potential atoms:

$$E,\ E' \wedge \exists \mathbf{1},\ (\exists \mathbf{1})'.$$

Amongst the 0-generated **MBA**'s are

- The monadic algebra $\mathbf{P}_1$ whose only atom is E, and the other two potential atoms are $\mathbf{0}$.
- The vacuous $\mathbf{P}^\infty$, in which the atom is $(\exists \mathbf{1})'$, and the other two potential atoms are $\mathbf{0}$.
- $\mathbf{P}_1^\infty$, in which the atoms are E and $E' \wedge \exists \mathbf{1} = E'$ in this case, while $(\exists \mathbf{1})' = \mathbf{0}$.
- $\mathbf{P}_1 \times \mathbf{P}^\infty$, in which the atoms are E and $(\exists \mathbf{1})' = E'$, while $E' \wedge \exists \mathbf{1} = \mathbf{0}$.

In the algebra of Figure 2, all three potential atoms denote distinct atoms. This uniquely determines the algebra structure, with every element being the value of a constant term. There are no proper subalgebras, so the algebra is generated by the empty set. In fact it is free on 0 generators, since there is a (unique) homomorphism from a free 0-generated algebra to this one, and as the homomorphism must preserve all constant term expressions, it is readily seen to be bijective and hence an isomorphism.

Theorem 8.2. *An* **MBA** *freely generated by r elements has exactly* $3 \cdot 2^r \cdot 2^{2^r-1}$ *atoms and* $2^{3 \cdot 2^r \cdot 2^{2^r-1}}$ *elements.*

Proof. Let $\mathbf{A}$ be an **MBA** that is freely generated by elements $p_0, \dots, p_{r-1}$. We know that $\mathbf{A}$ is finite, hence is of size 2^k where k is its number of atoms. So it suffices to show that $\mathbf{A}$ has $3 \cdot 2^r \cdot 2^{2^r-1}$ atoms. Fix a potential atom α in $\mathbf{A}$ of the form (8.2). We construct a homomorphism f from $\mathbf{A}$ into a special algebra with $f(\alpha) \neq \mathbf{0}$. This will imply that α is non-zero in $\mathbf{A}$, hence is an atom.

Let $m = 2^r$ as before, and let J be the set of numbers $j < m$ such that $\exists \mu_j$ occurs *positively* in α according to (8.2). Consider the point-generated bounded graph $\mathcal{F}_J^\infty$, in which $\infty \notin J$ and $E = J = R[x]$ for all points x. $\mathcal{F}_J^\infty$ is either spiked or vacuous, depending on whether J has any elements. Define subsets $P_0, \dots, P_{r-1}$ of $\mathcal{F}_J^\infty$ as follows. For $j \in J$, put $j \in P_k$ iff p_k occurs positively in μ_j. Hence $j \in P_k'$ iff p_k occurs negatively in μ_j. Also, put $\infty \in P_k$ iff p_k occurs positively in the minterm μ_i at the beginning of α (8.2).

Let $f(p_k) = P_k$ for all $k < r$, and use the freeness property of $\mathbf{A}$ to extend f to a homomorphism from $\mathbf{A}$ to the complex algebra $\mathbf{P}\mathcal{F}_J^\infty$. For each minterm μ_j, let $\bar{\mu}_j$ be the minterm in $\mathbf{P}\mathcal{F}_J$ obtained from μ_j by replacing each p_k by P_k. Likewise, let $\bar{\alpha}$ be result of replacing each μ_j in (8.2) by $\bar{\mu}_j$. Since f preserves all **MBA** operations, we then get $f(\mu_j) = \bar{\mu}_j$ and $f(\alpha) = \bar{\alpha}$.

Now the definition of the P_k's is designed to ensure that $\infty \in \bar{\mu}_i$, and that if $j \in J$ then $j \in \bar{\mu}_j$. Moreover, for any $j < m$, if $l \in J$ with $l \neq j$ then there is some $k < r$ such that p_k occurs positively in one of μ_l and μ_j and negatively in the other. Hence $l \in P_k$ while P'_k occurs in $\bar{\mu}_j$, or $l \in P'_k$ while P_k occurs in $\bar{\mu}_j$, so $l \notin \bar{\mu}_j$. In particular, if $j \notin J$ then this holds for all $l \in J$, so $\bar{\mu}_j \cap J = \emptyset$. Thus if $j \neq i$, then

$$\bar{\mu}_j = \begin{cases} \{j\} & \text{if } j \in J, \\ \emptyset & \text{if } j \notin J; \end{cases}$$

while

$$\bar{\mu}_i = \begin{cases} \{\infty, i\} & \text{if } i \in J, \\ \{\infty\} & \text{if } i \notin J. \end{cases}$$

Now suppose E occurs negatively in α, as in (8.4). Then we have $\infty \in \bar{\mu}_i$ and $\infty \in E'$, and if $j \in J$, then $\infty R j \in \bar{\mu}_j$, so $\infty \in \exists\bar{\mu}_j$ in $\mathbf{P}\mathcal{F}_J^\infty$. Also, if $j \notin J$, then $\infty \in (\exists\bar{\mu}_j)'$ as $\bar{\mu}_j \cap J = \emptyset$. Together these facts show that $\infty \in \bar{\alpha}$, and indeed $\bar{\alpha} = \{\infty\}$, since $\bar{\alpha} \subseteq E' = \{\infty\}$. So $f(\alpha) = \bar{\alpha} \neq \emptyset$.

Alternatively, if E occurs positively in α, then so does $\exists\mu_i$ as α is a potential atom, hence $i \in J$, and in fact α has the form (8.3). Thus $i \in \bar{\mu}_i$. Also $i \in E$. For $j \in J$ we have $iRj \in \bar{\mu}_j$ so $i \in \exists\bar{\mu}_j$. If $j \notin J$, then $i \in (\exists\bar{\mu}_j)'$ as then $\bar{\mu}_j = \emptyset$. This shows that $i \in \bar{\alpha}$, and indeed $\bar{\alpha} = \{i\}$, since $\bar{\alpha} \subseteq \bar{\mu}_i \cap E = \{i\}$.

This construction shows that if α is any potential atom of $\mathbf{A}$, then there is a homomorphism f on $\mathbf{A}$ with $f(\alpha) \neq \emptyset$. Thus $\alpha \neq \mathbf{0}$ as f preserves $\mathbf{0}$, hence α is an actual atom. Any two distinct expressions of the form (8.2) determine distinct elements of $\mathbf{A}$ if they are non-zero, so this finally establishes that $\mathbf{A}$ has $3 \cdot 2^r \cdot 2^{2^r - 1}$ atoms. □

Pursuing these ideas a little further allows us to give an explicit recipe for constructing a free algebra on r generators as the complex algebra of a single bounded graph, thereby giving a complete description of such algebras. If β is an atom got from one of the form (8.4) by replacing μ_i by some other μ_j, leaving the rest of the expression fixed, we can add another unmarked point ∞_j to $\mathcal{F}_J^\infty$ with $R[\infty_j] = J$, and arrange that $\infty_j \in \bar{\beta}$. In fact we can do this for every $j < m$, producing a single graph that has $\bar{\alpha} \neq \emptyset$ for every α associated with J. Then we form the disjoint union of all the graphs thus determined by the different subsets J of m.

To make this more precise, we continue to work with an $\mathbf{A}$ freely generated by elements $p_0, \ldots, p_{r-1}$, and call the expression $(\exists\mu_0)^\pm \wedge \cdots \wedge (\exists\mu_{m-1})^\pm$ the $\exists$-*part* of the atom (8.2) of $\mathbf{A}$. This $\exists$-part determines the set J. Conversely, any set $J \subseteq m$ determines an $\exists$-part of some atoms, namely

$$\exists_J = (\exists\mu_0)^{e_0} \wedge \cdots \wedge (\exists\mu_{m-1})^{e_{m-1}},$$

where e_j is $+$ if $j \in J$ and $-$ otherwise. This gives a convenient notation for the atoms of $\mathbf{A}$. For a given $J \subseteq m$, the atoms with $\exists_J$ as their $\exists$-part consist of the elements

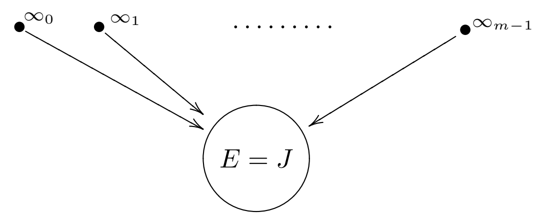
$$\mu_i \wedge E \wedge \exists_J \quad \text{with } i \in J, \tag{8.5}$$

and

$$\mu_i \wedge E' \wedge \exists_J \quad \text{for any } i < m. \tag{8.6}$$

We will call these the J-*atoms* of $\mathbf{A}$.

Now define a marked graph $\mathcal{G}^r_J$ based on a set $G^r_J = \{\infty_0, \dots, \infty_{m-1}\} \cup J$ where $\infty_0, \dots, \infty_{m-1}$ are distinct points[4] not in J, by putting $E = J$ and $R[x] = J$ for all $x \in G^r_J$. This is a bounded graph depicted as



(8.7)

Define subsets $P^J_0, \dots, P^J_{r-1}$ of G^r_J by declaring, for all $j < m$, that $\infty_j \in P^J_k$ iff p_k occurs positively in μ_j; while $j \in P^J_k$ iff $j \in J$ and p_k occurs positively in μ_j. Then for any $j < m$, if $\bar{\mu}_j$ is obtained by replacing the p_k's in μ_j by P^J_k, in $\mathbf{P}\mathcal{G}^r_J$ we get

$$\bar{\mu}_j = \begin{cases} \{\infty_j, j\} & \text{if } j \in J, \\ \{\infty_j\} & \text{if } j \notin J. \end{cases} \tag{8.8}$$

Moreover if α is $\mu_i \wedge E \wedge \exists_J$ with $i \in J$, then $\bar{\alpha} = \{i\}$, while if α is $\mu_i \wedge E' \wedge \exists_J$, then $\bar{\alpha} = \{\infty_i\}$. Thus every singleton subset of G^r_J, i.e. every atom of $\mathbf{P}\mathcal{G}^r_J$, is equal to $\bar{\alpha}$ for some atom α of $\mathbf{A}$. By (8.5) and (8.6), the atoms of $\mathbf{A}$ involved here are precisely those that have $\exists$-part equal to $\exists_J$, i.e. the J-atoms.

Note that the finite algebra $\mathcal{G}^r_J$ is generated by its singletons $\bar{\alpha}$. Hence it is also generated by $P^J_0, \dots, P^J_{r-1}$, since each $\bar{\alpha}$ is constructed from the P^J_k's by **MBA** operations.

The map $p_k \mapsto P^J_k$ on the generators of the free algebra $\mathbf{A}$ extends to a homomorphism $f_J : \mathbf{A} \to \mathbf{P}\mathcal{G}^r_J$. This has $f_J(\alpha) = \bar{\alpha}$ for every atom of $\mathbf{A}$, so the range of f_J includes every singleton subset of $\mathcal{G}^r_J$. Since these singletons generate $\mathbf{P}\mathcal{G}^r_J$, it follows that f_J is *surjective*.

Lemma 8.3. *If α is a J^*-atom of $\mathbf{A}$ with $J^* \neq J$, then $f_J(\alpha) = \emptyset$.*

Proof. If $J^* \neq J$, the $\exists$-part of α differs from $\exists_J$, and there is a $j < m$ such that either (i) $j \in J$ and $(\exists \mu_j)'$ occurs in the expression α, hence $\bar{\alpha} \subseteq (\exists \bar{\mu}_j)'$; or else (ii) $j \notin J$ and $(\exists \mu_j)$ occurs in α. If (i), then for all $x \in G^r_J$ we have $xRj \in \bar{\mu}_j$ (8.8), so $\exists \bar{\mu}_j = G^r_J$, hence $\bar{\alpha} \subseteq (\exists \bar{\mu}_j)' = \emptyset$. If (ii), then using (8.8) again, $\bar{\alpha} \subseteq \exists \bar{\mu}_j = \exists \{\infty_j\} = \emptyset$. □

This provides a clear picture of the action of the surjection f_J. It gives a bijection between the set of J-atoms of $\mathbf{A}$ and the set of all atoms (singletons) of $\mathbf{P}\mathcal{G}^r_J$, and "kills off" all other $\mathbf{A}$-atoms. Each member x of $\mathbf{A}$ is a join of atoms, and f_J kills off those that are not J-atoms, so $f_J(x)$ is the join, i.e. union, of all the singletons that correspond to J-*atoms* below x in $\mathbf{A}$.

Corollary 8.4. $f_J(p_k) = P^J_k$ *for all* $k < r$.

[4]We should really label these points $\infty^J_0, \dots, \infty^J_{m-1}$.

Proof. $f_J(p_k)$ is the union of the singletons $f_J(\alpha) = \bar{\alpha}$ for all J-atoms $\alpha \leq p_k$. Now either $\alpha = \mu_i \wedge E \wedge \exists_J$ with $i \in J$, or $\alpha = \mu_i \wedge E' \wedge \exists_J$ for some i. In both cases p_k must occur positively in μ_i, or else $\alpha \leq p_k'$, hence $\alpha = \mathbf{0}$. So the first case gives $\bar{\alpha} = \{i\}$ and $i \in P_k^J$, while the second gives $\bar{\alpha} = \{\infty_i\}$ and $\infty_i \in P_k^J$. So in any case, $\bar{\alpha} \subseteq P_k^J$. It follows that $f_J(p_k) \subseteq P_k^J$.

Conversely, take any $j \in P_k^J$. Then $j \in J$ and p_k occurs positively in μ_j. Since $j \in J$, $\exists\mu_j$ occurs positively in $\exists_J$, so $\alpha = \mu_j \wedge E \wedge \exists_J$ is a J-atom, with $\alpha \leq \mu_j \leq p_k$, and $\bar{\alpha} = \{j\}$, so $j \in f_J(\alpha) \subseteq f_J(p_k)$. Also, if $\infty_j \in P_k^J$, then again p_k occurs positively in μ_j, and so $\beta = \mu_j \wedge E' \wedge \exists_J$ is a J-atom below p_k with $\bar{\beta} = \{\infty_j\}$, so $\infty_j \in f_J(\beta) \subseteq f_J(p_k)$. Thus every member of P_k^J is in $f_J(p_k)$. □

The homomorphisms $f_J : \mathbf{A} \to \mathbf{P}\mathcal{G}_J^r$ for all $J \subseteq m$ induce a homomorphism

$$f : \mathbf{A} \longrightarrow \textstyle\prod_{J \subseteq m} \mathbf{P}\mathcal{G}_J^r \tag{8.9}$$

from $\mathbf{A}$ into the direct product of the complex algebras $\mathbf{P}\mathcal{G}_J^r$, in the usual way, i.e. $f(x)(J) = f_J(x)$. This f is injective, since any non-zero x in $\mathbf{A}$ has some atom $\alpha \leq x$, and if α is a J-atom, then $f_J(\alpha)$ is non-zero, being an atom of $\mathbf{P}\mathcal{G}_J^r$, so $f_J(x) \geq f_J(\alpha) \neq \mathbf{0}$, which ensures that $f(x) \neq \mathbf{0}$ in the direct product.

Lemma 8.3 implies that f is surjective, and hence altogether is an isomorphism. For, a member of the direct product is a tuple $\langle X_J \mid J \subseteq m \rangle$ with each $X_J \subseteq G_J^r$. Let x_J be the join in $\mathbf{A}$ of all the J-atoms α such that $f_J(\alpha)$ is a singleton subset of X_J. Then $f_J(x_J) = X_J$. Put $x = \bigvee\{x_J \mid J \subseteq m\}$. Now for each J, if $J^* \neq J$, the atoms below x_{J^*} are killed off by f_J, so $f_J(x_{J^*}) = \mathbf{0}$. It follows that $f_J(x) = f(x_J) = X_J$. Since this holds for all $J \subseteq m$, we get that $f(x) = \langle X_J \mid J \subseteq m \rangle$, confirming that f is surjective.

Now we invoke the duality between complex algebras of disjoint unions and direct products. Let $\mathcal{G}^r$ be the *disjoint* union of the collection

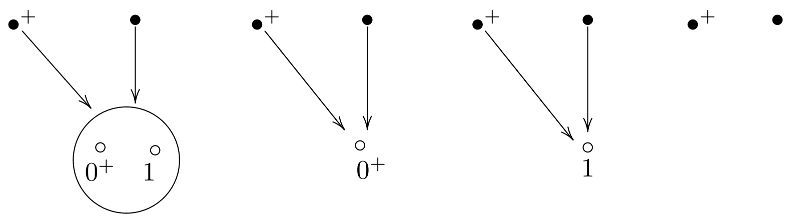
$$\{\mathcal{G}_J^r \mid J \subseteq m\}$$

of bounded graphs. Then $\mathcal{G}^r$ is a bounded graph whose complex algebra $\mathbf{P}\mathcal{G}^r$ is an isomorphic image of the direct product $\prod_{J \subseteq m} \mathbf{P}\mathcal{G}_J^r$. The isomorphism takes each tuple $\langle X_J \mid J \subseteq m \rangle$ from the direct product to the *disjoint* union of the X_J's in $\mathbf{P}\mathcal{G}^r$ [12, Theorem 1.6.5]. Composing this isomorphism with the f of (8.9) gives an isomorphism

$$f^r : \mathbf{A} \longrightarrow \mathbf{P}\mathcal{G}^r$$

taking each $x \in \mathbf{A}$ to the disjoint union of the collection of sets $\{f_J(x) \mid J \subseteq m\}$. For $k < r$, let P_k be the disjoint union of $\{P_k^J \mid J \subseteq m\}$. By Corollary (8.4), this collection is $\{f_J(p_k) \mid J \subseteq m\}$, so $P_k = f^r(p_k)$. We conclude that $\mathbf{P}\mathcal{G}^r$ is freely generated by $P_0, \ldots, P_{r-1}$.

There is a combinatorial way to see that f^r is a bijection, without needing to show that f is surjective via Lemma 8.3. From the injectivity of f we get that f^r is injective, and conclude that it must be a bijection by the pigeonhole principle, since $\mathbf{A}$ and $\mathbf{P}\mathcal{G}^r$ have the same number of elements. To count the



$\mathcal{G}^1_{\{0,1\}}$ $\mathcal{G}^1_{\{0\}}$ $\mathcal{G}^1_{\{1\}}$ $\mathcal{G}^1_{\emptyset}$

Figure 3: The graph $\mathcal{G}^1$ as a disjoint union.

number of elements of $\mathbf{P}\mathcal{G}^r$, consider first the number of marked elements in $\mathcal{G}^r$. The number of marked elements of each $\mathcal{G}^r_J$ is the number of elements of J, so the total number of marked elements in $\mathcal{G}^r$ is equal to the number of pairs (j, J) such that $j \in J \subseteq m$. There are $m \cdot 2^{m-1}$ such pairs, since there are m choices for j, and for each such choice there are 2^{m-1} ways of forming J by adding *other* elements of m to j. For the unmarked elements, there are 2^m graphs $\mathcal{G}^r_J$ whose disjoint union makes up $\mathcal{G}^r$, and each such graph has m unmarked elements ∞_j, so the number of unmarked elements in $\mathcal{G}^r$ is $m \cdot 2^m$. So the total number of elements of $\mathcal{G}^r$ is $m \cdot 2^m + m \cdot 2^{m-1} = 3 \cdot m \cdot 2^{m-1} =$ the number of atoms of $\mathbf{A}$, and hence the number of elements of $\mathbf{P}\mathcal{G}^r$ is $2^{3 \cdot m \cdot 2^{m-1}} =$ the size of $\mathbf{A}$.

This then is our recipe for constructing a free **MBA** on r generators: let $m = 2^r$, construct the bounded graphs $\mathcal{G}^r_J$ for all $J \subseteq m$ as depicted in (8.7), and let $\mathcal{G}^r$ be their disjoint union. Our freely r-generated algebra is $\mathbf{P}\mathcal{G}^r$.

For the case $r = 0$ of the empty set of generators, we have $m = 1$, with subsets 1 and $\emptyset$. $\mathcal{G}^0$ is the disjoint union the two graphs $\mathcal{G}^0_1$ and $\mathcal{G}^0_\emptyset$, with a single unmarked generator each. The first is a copy of $\mathcal{F}^\infty_1$, and the second a copy of $\mathcal{F}^\infty_0$. So $\mathcal{G}^0$ does indeed look as depicted in (8.1).

For the case $r = 1$ of a single generator, $\mathcal{G}^1$ is the 12-point graph depicted in Figure 3. Here $m = 2 = \{0, 1\}$ and $\mathcal{G}^1$ is the disjoint union of the four graphs $\mathcal{G}^1_{\{0,1\}}$, $\mathcal{G}^1_{\{0\}}$, $\mathcal{G}^1_{\{1\}}$, and $\mathcal{G}^1_\emptyset$, each with two irreflexive points, which are ∞_0 on the left and ∞_1 on the right in each case. The 4096-element complex algebra $\mathbf{P}\mathcal{G}^1$ of all subsets of this graph is the free **MBA** on one generator P_0. The definition of P_0 depends on the inital fixed ordering of the minterms μ_j. There are only two such minterms, P_0 and its complement. If these are taken in the order P_0, P_0', then P_0 is the set of points marked with a + in the diagram.

It is noteworthy that by deleting all the unmarked points from this general construction, we are left with a representation of the free monadic algebra on r generators as the direct product of the complex algebras of the monadic graphs based on each of the subsets of 2^r, or as the complex algebra of the disjoint union of all these monadic graphs.

It is possible to take a more syntactic approach to the construction of $\mathcal{G}^r$, by regarding all the potential atoms (8.3) and (8.4) as distinct entities that themselves form the points of a bounded graph. Also, by analysing the properties of maps between generators of MBA's it can be directly shown that $\mathbf{P}\mathcal{G}^r$ is freely generated without invoking the prior existence of a free algebra. Details of these approaches are given in [1].

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