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Grishin Algebras and Cover Systems for Classical Bilinear Logic

Dedicated to Ryszard Wójcicki on his 80th birthday.

Abstract. Grishin algebras are a generalisation of Boolean algebras that provide algebraic models for classical bilinear logic with two mutually cancelling negation connectives. We show how to build complete Grishin algebras as algebras of certain subsets ("propositions") of cover systems that use an orthogonality relation to interpret the negations.

The variety of Grishin algebras is shown to be closed under MacNeille completion, and this is applied to embed an arbitrary Grishin algebra into the algebra of all propositions of some cover system, by a map that preserves all existing joins and meets.

This representation is then used to give a cover system semantics for a version of classical bilinear logic that has first-order quantifiers and infinitary conjunctions and disjunctions.

Keywords: Grishin algebra, bilinear logic, residuated lattice-ordered monoid, quantale, cover system, orthogonality relation, Kripke-Joyal semantics, MacNeille completion.

1. Introduction

Grishin algebras were defined by Lambek [14, 15] as "a generalisation of Boolean algebras which do not obey Gentzen's three structural rules". The motivation was to study algebraic models for *classical bilinear propositional logic*, described as "a non-commutative version of linear logic which allows two negations". Such models were first considered by V. N. Grishin [13].

A Grishin algebra is a residuated lattice-ordered monoid. Its monoid operation $\otimes$ is thought of as a non-commutative conjunction, whose left and right residuals are viewed as implication operations, which we denote by $\Rightarrow_l$ and $\Rightarrow_r$. It has two negation/complementation operations that we denote $-_l$ and $-_r$, and which are mutually cancelling, i.e. satisfy *double-negation elimination*, in the sense that the equations $-_l -_r a = a = -_r -_l a$ are satisfied. There is a distinguished element 0 such that $-_l a = a \Rightarrow_l 0$ and $-_r a = a \Rightarrow_r 0$. Also, there is a definable operation $\oplus$, a kind of De Morgan

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dual of $\otimes$, that has 0 as an identity element, and has

$$a \oplus b = -_l(-_r b \otimes -_r a) = -_r(-_l b \otimes -_l a).$$

In this paper we introduce new, fully representative, examples of the abstract and equationally defined notion of a Grishin algebra. These concrete examples are based on *cover systems*, motivated by topological ideas underlying the Kripke-Joyal intuitionistic semantics from topos theory. A cover system assigns to each point certain sets of points called "covers" in a way that it formally similar to the neighbourhood semantics of modal logics. Covers are used to give a non-classical interpretation of disjunction and existential quantification (see [16, 2]).

Topologically motivated cover systems have a certain algebra of subsets that forms a Heyting algebra, whereas a Grishin algebra need not even be distributive as a lattice. So to construct Grishin algebras we need to adopt weaker properties than those defining the cover systems of Kripke-Joyal semantics (see Remark 3.6). In an earlier paper [10] we used such structures to provide a semantics for the logic of residuated partially-ordered monoids without negation operations. This involved combining cover systems with certain models, based on ordered groupoids $(S, \preccurlyeq, \cdot)$, that have been developed for various connectives in substructural logic [22, 20, 4, 5]. These groupoid models allow a non-commutative conjunction connective &, corresponding to $\otimes$, to be given the semantics

$$x \models \varphi \& \psi \quad \text{iff} \quad \exists y \, \exists z : \; y \cdot z \preccurlyeq x \text{ and } y \models \varphi \text{ and } z \models \psi.$$

Here we extend these ideas to model two negation connectives, denoted $\neg_l$ and $\neg_r$ and corresponding to $-_l$ and $-_r$, by introducing a binary relation $\perp$ on the points of a model, and requiring that

$$\begin{aligned} x \models \neg_l \varphi \quad &\text{iff} \quad \forall y, \; y \models \varphi \text{ implies } x \perp y; \\ x \models \neg_r \varphi \quad &\text{iff} \quad \forall y, \; y \models \varphi \text{ implies } y \perp x. \end{aligned}$$

We can think of $\perp$ as being a relation of *orthogonality* or *incompatibility* between points of S. This kind of modelling of negation was developed by the author for the logic of ortholattices in [8], a context in which $\perp$ is symmetric and the two negations collapse to one. Here we assume that our models have a distinguished subset 0, and define $x \perp y$ to mean that $x \cdot y \in 0$, an approach that is used in the phase space semantics of linear logic [7].[1]

[1] Cover systems with a symmetric orthogonality relation are used in [12, Chapter 6] to

In what follows, we introduce Grishin algebras, consider various alternative definitions, and discuss their relation to quantales. We then develop a notion of a *strong classical residuated* cover system, and prove that certain subsets of such a system, which we call "propositions", form a *complete* Grishin algebra (and a unital quantale). It is then shown, conversely, that any complete Grishin algebra is isomorphic to the algebra of all propositions of some strong classical residuated cover system (Theorem 4.3). This representing cover system is defined by an abstract version of the cover system of a topological space (see Examples 3.1 and Theorem 3.8).

Our main result is that every Grishin algebra has an isomorphic embedding into the algebra of all propositions of some strong classical residuated cover system, by a map that preserves all existing joins and meets (Theorem 6.3). This representation makes use of the celebrated completion of a partially-ordered set due to MacNeille, and involves showing that the variety of Grishin algebras is closed under MacNeille completion (Theorem 6.2).

Preservation of joins and meets is a powerful tool, since these operations can be used to interpret the quantifiers $\exists$ and $\forall$, as well as the standard lattice-modelled disjunctions $\bigvee$ and conjunctions $\bigwedge$. In the final section, we use our representation of Grishin algebras to obtain a sound and complete semantics over cover systems for a version of classical bilinear predicate logic that has these first-order quantifiers as well as infinitary disjunctions and conjunctions.

We will see that the notion of Grishin algebra is equivalent to Ono's notion of *classical FL-algebra* [20]. Thus some observations about Grishin algebras may correspond to, or be implicit in, existing literature on the algebraic logic of residuated lattices [6].

2. Residuated Posets and Grishin Algebras

We begin with some terminology about partially ordered algebras. Given a *poset* $(L, \sqsubseteq)$, comprising a partial ordering $\sqsubseteq$ on a set L, we write $\bigsqcup X$ for the *join* (= least upper bound), and $\bigsqcap X$ for the *meet* (= greatest lower bound), of a set $X \subseteq L$, when these bounds exist. The smaller symbols $\sqcup$ and $\sqcap$ are used for the binary join and meet operations. A poset is *order-complete*, or just *complete*, if every subset has a join, or equivalently if every

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give a new semantics for relevant logic, a logic that has distribution of conjunction over disjunction. For further background and discussion of the cover semantics methodology see also [11], which develops interpretations of intuitionistic modal logic giving the diamond modality $\Diamond$ its standard Kripkean semantics, without validating distribution of $\Diamond$ over disjunction.

subset has a meet.

A *posemigroup* $(L, \sqsubseteq, \otimes)$ has an associative binary operation $\otimes$ that is *monotone* (i.e. order preserving) in each argument, meaning that $b \sqsubseteq c$ implies $a \otimes b \sqsubseteq a \otimes c$ and $b \otimes a \sqsubseteq c \otimes a$. This is a *pomonoid* if in addition $\otimes$ has a *unit* element, i.e. an element 1 such that $a \otimes 1 = 1 \otimes a = a$. An algebra with such a unit is called *unital*.

A posemigroup is *residuated* if there are binary operations $\Rightarrow_l$ and $\Rightarrow_r$ on L,[2] called the *left and right residuals* of $\otimes$, satisfying

$$a \sqsubseteq b \Rightarrow_l c \quad \text{iff} \quad a \otimes b \sqsubseteq c \quad \text{iff} \quad b \sqsubseteq a \Rightarrow_r c. \tag{2.1}$$

Thus $(a \Rightarrow_l b) \otimes a \sqsubseteq b$ and $a \otimes (a \Rightarrow_r b) \sqsubseteq b$. The two residual operations are monotone in their right arguments and antitone in their left, i.e. if $b \sqsubseteq c$, then $a \Rightarrow_l b \sqsubseteq a \Rightarrow_l c$ and $c \Rightarrow_l a \sqsubseteq b \Rightarrow_l a$; and similarly for $\Rightarrow_r$. These residuals are identical precisely when $\otimes$ is commutative.

In any residuated posemigroup, the following equations hold whenever the joins they refer to exist:

$$(\bigsqcup X) \otimes a = \bigsqcup_{x \in X} (x \otimes a) \tag{2.2}$$

$$a \otimes (\bigsqcup X) = \bigsqcup_{x \in X} (a \otimes x). \tag{2.3}$$

A *quantale* is a complete poset with an associative $\otimes$ such that these last equations hold for every set $X \subseteq L$. Thus a complete residuated posemigroup is a quantale. The converse is also true: every quantale is residuated with $a \Rightarrow_l b = \bigsqcup\{x : x \otimes a \leq b\}$ and $a \Rightarrow_r b = \bigsqcup\{x : a \otimes x \leq b\}$.

There is a standard theory of certain closure operators on quantales that we make significant use of. Recall that a *closure operator* on a poset is a unary function j that is *monotone*: $a \sqsubseteq b$ implies $ja \sqsubseteq jb$; *inflationary*: $a \sqsubseteq ja$; and *idempotent*: $jja = ja$. An element a is called *j-closed* if $ja = a$, i.e. if a is a fixed point under j. If L is a complete poset, then the set L^j of j-closed elements is closed under meets $\bigsqcap X$, and so is order-complete under the same partial ordering. The join operation $\bigsqcup^j$ in L^j is given by $\bigsqcup^j X = j(\bigsqcup X)$.

A *quantic nucleus* is a closure operator on a posemigroup that satisfies

$$ja \otimes jb \sqsubseteq j(a \otimes b). \tag{2.4}$$

Such an operator has $j(a \otimes b) = j(a \otimes jb) = j(ja \otimes b) = j(ja \otimes jb)$, as well as $j(a \Rightarrow_l b) \sqsubseteq a \Rightarrow_l jb = ja \Rightarrow_l jb$, and likewise with $\Rightarrow_r$ in place of $\Rightarrow_l$.

[2] Notation: in the literature on residuation, $a \Rightarrow_l b$ is often written as b/a, and $a \Rightarrow_r b$ as $a\backslash b$.

Using such facts, it can be shown that if j is a quantic nucleus on a quantale $(L, \sqsubseteq, \otimes)$, then the complete poset $(L^j, \sqsubseteq)$ of j-closed elements is a quantale under the operation $a \otimes_j b = j(a \otimes b)$ [19, Theorem 2.1]. Moreover L^j is closed under the residuals of $\otimes$, and indeed both $a \Rightarrow_l b$ and $a \Rightarrow_r b$ belong to L^j whenever $b \in L^j$ [23, Prop. 3.1.2]. From this it can be shown that the residuals of $\otimes_j$ on L^j are just the restrictions of the residuals of $\otimes$ on L to L^j. Thus for $a, b, c \in L^j$,

$$a \sqsubseteq b \Rightarrow_l c \quad \text{iff} \quad j(a \otimes b) \sqsubseteq c \quad \text{iff} \quad b \sqsubseteq a \Rightarrow_l c. \tag{2.5}$$

In addition, if 1 is a unit for $\otimes$ in L, then $j1$ is a unit for $\otimes_j$ in L^j, since $j(a \otimes j1) = a = j(j1 \otimes a)$ for all $a \in L^j$.

The quantale $(L^j, \sqsubseteq, \otimes_j)$ is called a *quantic quotient* of $(L, \sqsubseteq, \otimes)$ [23, p. 32]. This construction will be applied in the next section (see Theorem 3.5).

A *Grishin algebra* is defined in [14] to be an algebra of the form

$$\mathbf{L} = (L, \sqsubseteq, \sqcup, \sqcap, \mathsf{T}, \mathsf{F}, \otimes, 1, -_l, -_r, 0),$$

such that:

- $(L, \sqcup, \sqcap, \mathsf{T}, \mathsf{F})$ is a lattice under the partial ordering $\sqsubseteq$, with greatest element T and least element F.
- $(L, \sqsubseteq, \otimes, 1)$ is a pomonoid with identity element 1.
- $-_l$ and $-_r$ are unary operations,[3] and 0 an element of L, satisfying the equations
$$-_l -_r a = a = -_r -_l a, \tag{2.6}$$
and the conditions
$$a \sqsubseteq b \quad \text{iff} \quad a \otimes -_r b \sqsubseteq 0 \quad \text{iff} \quad -_l b \otimes a \sqsubseteq 0. \tag{2.7}$$

We refer to (2.6) as the law of *double-negation elimination* (DNE).

By a *residuated lattice-ordered monoid* we mean a residuated pomonoid that is a lattice under its partial ordering, with greatest and least elements. It can be shown that the notion of a residuated lattice-ordered monoid is definable by equations (see e.g. [6, p. 94]). Lambek gave the following alternative characterisation of Grishin algebras, which shows that they too are equationally definable:

[3]Notation: in [14, 15], $-_r a$ is written $a^{\perp}$ and $-_l a$ is written a^{T}.

Theorem 2.1. [14, Proposition 2.1]
A Grishin algebra may be described as a residuated lattice-ordered monoid with operations $-_l$ *and* $-_r$ *satisfying the equations* (2.6) *and*

$$-_l\, 1 = -_r\, 1, \quad a \Rightarrow_l b = -_l(a \otimes -_r\, b), \quad a \Rightarrow_r b = -_r(-_l\, b \otimes a).$$

Proof. (Sketch.) Any Grishin algebra has $-_l\, 1 = 0$ and $-_r\, 1 = 0$, and is residuated when $a \Rightarrow_l b$ is defined to be $-_l(a \otimes -_r\, b)$ and $a \Rightarrow_r b$ is defined as $-_r(-_l\, b \otimes a)$.

Conversely, given an algebra as described in the statement of the theorem, one defines 0 to be $-_l\, 1$ $(= -_r\, 1)$ and derives the equations (2.7). □

Here we will make use of another simpler equational characterisation:

Theorem 2.2. *A Grishin algebra may be described as a residuated lattice-ordered monoid with a distinguished element* 0 *satisfying*

$$(a \Rightarrow_r 0) \Rightarrow_l 0 = a = (a \Rightarrow_l 0) \Rightarrow_r 0. \tag{2.8}$$

Proof. In any Grishin algebra we have $-_l\, 0 = -_l -_r\, 1 = 1$ by double-negation elimination, and similarly $-_r\, 0 = 1$. Then by Theorem 2.1 we get that the algebra is residuated, has

$$a \Rightarrow_l 0 = -_l(a \otimes -_r\, 0) = -_l(a \otimes 1) = -_l\, a,$$

and similarly $-_r\, a = a \Rightarrow_r 0$. So (2.8) follows by DNE (2.6).

For the converse, given a residuated lattice-ordered monoid with 0 satisfying (2.8), define $-_l\, a$ to be $a \Rightarrow_l 0$, and $-_r\, a$ to be $a \Rightarrow_r 0$. Then DNE is given by (2.8), so we have only to prove (2.7) to show that we have a Grishin algebra. But now

$$a \otimes -_r\, b \sqsubseteq 0 \quad \text{iff} \quad a \sqsubseteq (-_r\, b) \Rightarrow_l 0 = -_l -_r\, b = b,$$

and similarly $-_l\, b \otimes a \sqsubseteq 0$ iff $a \sqsubseteq (-_l\, b) \Rightarrow_r 0 = -_r -_l\, b = b$, as required. □

This result shows that the notion of a Grishin algebra is equivalent to that of a *classical FL-algebra* as defined in [20, p. 264].

3. Residuated Cover Systems

We work now with structures $\mathcal{S} = (S, \preccurlyeq, \lhd, \dots)$ that include the following components:

- A binary relation $\preccurlyeq$ on S that is a *preorder*, i.e. reflexive and transitive. We sometimes write $y \succcurlyeq x$ when $x \preccurlyeq y$, and say that y *refines* x.
- A binary relation $\lhd$ from S to its powerset $\mathcal{P}S$. When $x \lhd C$, where $x \in S$ and $C \subseteq S$, we say that x *is covered by* C, and write this also as $C \rhd x$, saying that C *covers* x or that C is an x*-cover*.

An *up-set* is a subset X of S that is closed upwardly under the preorder: $y \succcurlyeq x \in X$ implies $y \in X$. For an arbitrary $X \subseteq S$,

$$\uparrow X = \{y \in S : (\exists x \in X)\, x \preccurlyeq y\}$$

is the smallest up-set including X. It consists of those points that refine some member of X. Thus X is an up-set iff $\uparrow X = X$. For $x \in S$,

$$\uparrow x := \uparrow\{x\} = \{y : x \preccurlyeq y\}$$

is the smallest up-set containing x. A subset Y of S *refines* a subset X if $Y \subseteq \uparrow X$, i.e. if every member of Y refines some member of X.

We write $Up(\mathcal{S})$ for the collection of all up-sets of $\mathcal{S}$. It is a complete poset under the partial order $\subseteq$ of set inclusion, with the join $\bigsqcup \mathcal{X}$ and meet $\bigsqcap \mathcal{X}$ of any collection $\mathcal{X}$ of up-sets being the set union $\bigcup \mathcal{X}$ and intersection $\bigcap \mathcal{X}$ respectively, while $\mathsf{F} = \emptyset$ and $\mathsf{T} = S$.

For each subset X of S, define

$$jX = \{x \in S : \exists C\, (x \lhd C \subseteq X)\}. \tag{3.1}$$

Now we think of a condition as being *locally true* of x if there is some C such that $x \lhd C$ and each member of C satisfies the condition, i.e. if x is covered by a set of members that have this condition. But the defining condition "$x \lhd C \subseteq X$" for jX in (3.1) states that C is an x-cover that consists of members of X. So x belongs to jX just when the property of *being a member of* X is *locally true* of x, i.e. when x is covered by a set of members of X. Thus we may think of jX as the collection of "local members" of X.

X is called *localised* if $jX \subseteq X$, i.e. if every local member of X is an actual member of X. Now a property whose satisfaction is implied by its own local satisfaction is said to be *of local character*, or more briefly a *local property.* Thus X is a localised set when membership of X is a local property in this sense.

Examples 3.1.

To illustrate these notions, let S be the set of open subsets of some topological space, with $x \preccurlyeq y$ iff $x \supseteq y$ and $x \lhd C$ iff $x \subseteq \bigcup C$. Then $\lhd$ captures the usual notion of open cover; and "Y refines X", i.e. $Y \subseteq {\uparrow}X$, has its usual meaning for open covers that every member of Y is included in a member of X [18, p. 245]. For a different interpretation, think of S as a collection of states that each contain certain information, with $x \preccurlyeq y$ when the information content of y includes that of x. So $\preccurlyeq$ is a relation of refinement in the sense of increase of information. Interpret covering as a relation expressing commonality of information content, so that if $C \rhd x$ then the information content of x consists of that which is common to all the states in C. Further discussion of these and other motivations can be found in [11] and [12, Chapter 6]. □

We call $\mathcal{S}$ a *cover system* if it satisfies the following axioms, for all $x \in S$:

- *Existence*: there exists an x-cover $C \subseteq {\uparrow}x$;
- *Transitivity*: if $x \lhd C$ and for all $y \in C$, $y \lhd C_y$, then $x \lhd \bigcup_{y \in C} C_y$.
- *Refinement*: if $x \preccurlyeq y$, then every x-cover can be refined to a y-cover, i.e. if $C \rhd x$, then there exists a $C' \rhd y$ with $C' \subseteq {\uparrow}C$.

Lemma 3.2. *In any cover system, the function j defined by* (3.1) *is a closure operator on the complete poset* $(\mathit{Up}(\mathcal{S}), \subseteq)$ *of up-sets.*

Proof. This is shown in [10, Theorem 5] and [11, Lemma 3.3]. Briefly: the Refinement axiom ensures that if X is an up-set, then so is jX; and the Existence axiom ensures that j is an inflationary operator on up-sets, i.e. $X \subseteq jX$ for all $X \in \mathit{Up}(\mathcal{S})$. Transitivity ensures that $j(jX) \subseteq jX$ holds for any X. In particular, j is idempotent on up-sets, i.e. $j(jX) = jX$ for all $X \in \mathit{Up}(\mathcal{S})$. That j is monotone, i.e. preserves set inclusion, follows directly from its definition. □

An up-set X in a cover system will be called a *proposition* if it is localised, i.e. if $jX \subseteq X$. Since j is inflationary on $\mathit{Up}(\mathcal{S})$, an up-set X is a proposition iff $jX = X$. In general, a set X is a proposition iff $X = {\uparrow}X = jX$. We write $\mathit{Prop}(\mathcal{S})$ for the set of all localised up-sets of a cover system $\mathcal{S}$. $j{\uparrow}X$ is the smallest proposition that includes an arbitrary X, and $j{\uparrow}x$ is the smallest proposition containing the element x. The smallest proposition including an up-set X is just jX, so in fact j maps $\mathit{Up}(\mathcal{S})$ to $\mathit{Prop}(\mathcal{S})$. Indeed, $\mathit{Prop}(\mathcal{S})$ is precisely the set of j-closed members of $\mathit{Up}(\mathcal{S})$.

We now add some addition structure to $\mathcal{S}$ in order to make $Prop(\mathcal{S})$ into a complete residuated pomonoid, i.e. a unital quantale. Let $\cdot$ be a binary relation on S, which we will call *fusion*. We lift this to an operation on subsets of S by putting, for $X, Y \subseteq S$,

$$X \cdot Y = \{x \cdot y : x \in X \text{ and } y \in Y\}.$$

Then we write $x \cdot Y$ for the set $\{x\} \cdot Y$, and $X \cdot y$ for $X \cdot \{y\}$. Evidently, fusion is $\subseteq$-monotone in each argument.

Define operations $\Rightarrow_l$ and $\Rightarrow_r$ on subsets of S by

$$X \Rightarrow_l Y = \{z \in S : z \cdot X \subseteq Y\}, \quad X \Rightarrow_r Y = \{z \in S : X \cdot z \subseteq Y\}. \tag{3.2}$$

These provide left and right residuals to the fusion operation on the complete poset $(\mathcal{P}S, \subseteq)$, i.e. for all $X, Y, Z \subseteq S$ we have

$$X \subseteq Y \Rightarrow_l Z \quad \text{iff} \quad X \cdot Y \subseteq Z \quad \text{iff} \quad Y \subseteq X \Rightarrow_r Z. \tag{3.3}$$

Consequently, if the lifted operation $X \cdot Y$ on subsets is associative, then $(\mathcal{P}S, \subseteq, \cdot)$ is a quantale, with residuals given by (3.2). In particular, this holds when $\cdot$ is associative as an operation on S.

Now define $X \circ Y$ to be the up-set $\uparrow(X \cdot Y)$ generated by $X \cdot Y$. Then *if Z is an up-set*, we have $X \cdot Y \subseteq Z$ iff $X \circ Y \subseteq Z$, and hence (3.3) implies

$$X \subseteq Y \Rightarrow_l Z \quad \text{iff} \quad X \circ Y \subseteq Z \quad \text{iff} \quad Y \subseteq X \Rightarrow_r Z \tag{3.4}$$

for any X and Y. If the fusion operation $\cdot$ is $\preccurlyeq$-monotone in each argument, then $Y \Rightarrow_l Z$ and $X \Rightarrow_r Z$ are up-sets when Z is an up-set. In particular, this implies that $Up(\mathcal{S})$ is closed under $\Rightarrow_l$ and $\Rightarrow_r$, and so by (3.4), these operations are left and right residuals to $\circ$ on $Up(\mathcal{S})$. Moreover, if the operation $\cdot$ is associative on S, then $\preccurlyeq$-monotonicity implies that $\circ$ is associative too. Altogether this shows:

LEMMA 3.3. *If the fusion operation $\cdot$ is associative and $\preccurlyeq$-monotone in each argument, then $(Up(\mathcal{S}), \subseteq, \circ)$ is a quantale, with residuals given by* (3.2). □

By a *residuated cover system* we will mean a structure

$$\mathcal{S} = (S, \preccurlyeq, \lhd, \cdot, \varepsilon),$$

such that:

- $(S, \preccurlyeq, \lhd)$ is a cover system.

- $(S, \cdot, \varepsilon)$ is a pomonoid, i.e. $\cdot$ is an associative operation on S that is $\preccurlyeq$-monotone in each argument, and has $\varepsilon \in S$ as a unit.
- *Fusion preserves covering*: $x \lhd C$ implies $x \cdot y \lhd C \cdot y$ and $y \cdot x \lhd y \cdot C$.
- *Refinement of ε is local*: $x \lhd C \subseteq \uparrow\varepsilon$ implies $\varepsilon \preccurlyeq x$.

The last condition states that if x locally refines ε, in the sense that it has a cover consisting of points refining ε, then x itself refines ε. This means that the up-set $\uparrow\varepsilon$ of points refining ε is localised, i.e. $j\uparrow\varepsilon \subseteq \uparrow\varepsilon$, and therefore is a proposition.

LEMMA 3.4. *In any residuated cover system $\mathcal{S}$:*

(1) *If $x \lhd C$ and $y \lhd D$, then $x \cdot y \lhd C \cdot D$.*

(2) *If X is an up-set, then $X \circ \uparrow\varepsilon = X = \uparrow\varepsilon \circ X$.*

(3) *The function j defined by* (3.1) *is a quantic nucleus on the quantale $(Up(\mathcal{S}), \subseteq, \circ)$.*

PROOF.

(1) Since fusion preserves covering, from $y \lhd D$ we get $x \cdot y \lhd x \cdot D$; and for each element $x \cdot d$ of $x \cdot D$, from $x \lhd C$ we get $x \cdot d \lhd C \cdot d$. Hence by the Transitivity axiom of cover systems,

$$x \cdot y \lhd \bigcup_{d \in D} C \cdot d = C \cdot D.$$

(2) If $z \in X \circ \uparrow\varepsilon$, then $z \succcurlyeq x \cdot y$ for some $x \in X$ and some $y \succcurlyeq \varepsilon$. Then $\preccurlyeq$-monotonicity of $\cdot$ gives that $z \succcurlyeq x \cdot y \succcurlyeq x \cdot \varepsilon = x \in X$, hence $z \in X$ as X is an up-set. Conversely, if $z \in X$ then $z = z \cdot \varepsilon \in X \circ \uparrow\varepsilon$.

The proof that $X = \uparrow\varepsilon \circ X$ is similar.

(3) Since j is a closure operator by Lemma 3.2, we just have to show that it satisfies 2.4 when $\otimes$ is $\circ$, i.e. that $jX \circ jY \subseteq j(X \circ Y)$. But if $z \in jX \circ jY$, then $z \succcurlyeq x \cdot y$ for some x, y such that there is an x-cover $C \subseteq X$ and a y-cover $D \subseteq Y$. Then by part (1), $x \cdot y \lhd C \cdot D \subseteq X \cdot Y$. Hence by Refinement, $C \cdot D$ can be refined to a z-cover E, giving

$$z \lhd E \subseteq \uparrow(C \cdot D) \subseteq \uparrow(X \cdot Y) = X \circ Y.$$

This shows that $z \in j(X \circ Y)$ as required.

□

THEOREM 3.5. *The set of propositions of a residuated cover system $\mathcal{S}$ forms a quantale $(Prop(\mathcal{S}), \subseteq, \otimes)$ under a monoidal operation $\otimes$ with a unit 1, where*

$$
\begin{aligned}
X \otimes Y &= j(X \circ Y) = j{\uparrow}(X \cdot Y) \\
1 &= {\uparrow}\varepsilon \\
\textstyle\bigsqcap \mathcal{X} &= \textstyle\bigcap \mathcal{X} \\
\textstyle\bigsqcup \mathcal{X} &= j(\textstyle\bigcup \mathcal{X}) \\
X \Rightarrow_l Y &= \{z \in S : z \cdot X \subseteq Y\} \\
X \Rightarrow_r Y &= \{z \in S : X \cdot z \subseteq Y\} \\
\mathsf{T} &= S \\
\mathsf{F} &= j\emptyset = \{x : x \lhd \emptyset\}.
\end{aligned}
$$

PROOF. $Prop(\mathcal{S})$ is the set of all j-closed members of $Up(\mathcal{S})$, and we have just seen that this j is a quantic nucleus on the quantale $(Up(\mathcal{S}), \subseteq, \circ)$. Hence we obtain a quantale $(Prop(\mathcal{S}), \subseteq, \circ_j)$ by the theory of quantic quotients that was described in the previous section. This tells us that the quantic quotient based on $Prop(\mathcal{S})$ has the same partial ordering $\subseteq$ as $Up(\mathcal{S})$, with the same meets $\bigsqcap \mathcal{X}$, which are given by the set intersection $\bigcap \mathcal{X}$, and with joins $\bigsqcup \mathcal{X}$ given by $j(\bigcup \mathcal{X})$ because the joins in $Up(\mathcal{S})$ are given by set union $\bigcup$. Also, the semigroup operation $\otimes$ on $Prop(\mathcal{S})$ is $\circ_j$, i.e. $X \otimes Y = j(X \circ Y)$. Moreover, the residuals of $\circ_j$ on $Prop(\mathcal{S})$ are just the restrictions to $Prop(\mathcal{S})$ of the residuals of $\circ$ on $Up(\mathcal{S})$, so these are indeed the operations $\Rightarrow_l$ and $\Rightarrow_r$ on $Prop(\mathcal{S})$ given by (3.2).

Since ${\uparrow}\varepsilon$ is the unit of $\circ$ on $Up(\mathcal{S})$ (Lemma 3.4(2)), the theory tells us that $j{\uparrow}\varepsilon$ is the unit of $\circ_j$ on $Prop(\mathcal{S})$. But we already observed that, since refinement of ε is local in $\mathcal{S}$, $j{\uparrow}\varepsilon = {\uparrow}\varepsilon$. Hence indeed we have $1 = {\uparrow}\varepsilon$ in the quotient quantale.

Finally, since S is a localised up-set, it is the greatest member T of $Prop(\mathcal{S})$; and since $\emptyset$ is an up-set, $j\emptyset$ is a proposition, and hence is the least member F of $Prop(\mathcal{S})$. □

REMARK 3.6.

The cover systems used in Kripke-Joyal semantics typically have the property that every x-cover is included in the up-set of x, i.e. $x \lhd C$ implies $C \subseteq {\uparrow}x$ (see, e.g. [2]). This is a much stronger constraint than our Existence axiom, which requires only that there be at least one x-cover included in ${\uparrow}x$. The stronger condition is also used in a cover semantics for relevance logic in [12, Chapter 6]. It has the effect of making $Prop(\mathcal{S})$ into a complete Heyting algebra, hence a distributive lattice and a model of intuitionistic logic.

The topological cover systems of Examples 3.1 do not quite have this stronger property: if $x \lhd C$ means that $x \subseteq \bigcup C$, then it does not imply $C \subseteq {\uparrow}x$. However in this situation, we can replace the cover C by $C' = \{x \cap c : c \in C\}$ and have $x \lhd C' \subseteq {\uparrow}x$, with $C' \subseteq {\uparrow}C$. So in these topological cover systems,

every x-cover can be refined to an x-cover that is included in ${\uparrow}x$.

But even that weaker condition is sufficient to make $Prop(\mathcal{S})$ into a complete Heyting algebra, as is shown in [11], and so it too must be abandoned when interpreting non-distributive logics. □

A residuated cover system $\mathcal{S}$ will be called *strong* if $Prop(\mathcal{S})$ is closed under $\circ$, i.e. if $X \circ Y$ is a proposition whenever X and Y are propositions. In that situation, $j(X \circ Y) = (X \circ Y) = {\uparrow}(X \cdot Y)$ for propositions X and Y. Hence when $\mathcal{S}$ is strong, the quantale $Prop(\mathcal{S})$ has $X \otimes Y = {\uparrow}(X \cdot Y)$.

Lemma 3.7. *A residuated cover system $\mathcal{S}$ is strong iff the following condition holds for any propositions X and Y:*

(†) *if there exists a z-cover refining $X \cdot Y$, then there exist x, y with $x \cdot y \preccurlyeq z$; an x-cover $X' \subseteq X$; and a y-cover $Y' \subseteq Y$.*

Proof. Let X and Y be propositions.

Now $X \circ Y$ is an up-set by definition, so is a proposition iff it is localised, i.e. iff $j(X \circ Y) \subseteq (X \circ Y)$. But if $z \in j(X \circ Y)$, then there is some C with $z \lhd C \subseteq {\uparrow}(X \cdot Y)$, i.e. C refines $X \cdot Y$, so if (†) holds we infer that there are $x \in jX = X$ and $y \in jY = Y$ such that $x \cdot y \preccurlyeq z$, hence $x \in X \circ Y$. This shows that if (†) holds then $X \circ Y$ is a proposition.

Conversely, if $X \circ Y$ is localised, then (†) follows readily. □

We now show that every unital quantale is isomorphic to the algebra of propositions of some strong residuated cover system. This is a refinement of a representation of quantales without unit given in [10, Theorem 6]. The construction involves an abstraction of topological cover systems.

Theorem 3.8. *Every order-complete residuated pomonoid $\mathbf{L}$ is isomorphic to the algebra $Prop(\mathcal{S}^{\mathbf{L}})$ of propositions of some strong residuated cover system $\mathcal{S}^{\mathbf{L}}$.*

Proof. Let $\mathbf{L} = (L, \sqsubseteq, \otimes, 1^{\mathbf{L}})$. Define $\mathcal{S}^{\mathbf{L}} = (S, \preccurlyeq, \lhd, \cdot, \varepsilon)$ by putting $S = L$; $x \preccurlyeq y$ iff $y \sqsubseteq x$; $x \lhd C$ iff $x \sqsubseteq \bigsqcup C$; $x \cdot y = x \otimes y$; and $\varepsilon = 1$. Then $\preccurlyeq$ is a preorder, and $\cdot$ is associative and $\preccurlyeq$-monotone with ε as a unit. Note that in $\mathcal{S}^{\mathbf{L}}$, the up-set $\uparrow x = \{y : x \preccurlyeq y\}$ is equal to $\{y : y \sqsubseteq x\}$, which is the *down-set* of x in $(L, \sqsubseteq)$.

Next we show that $\mathcal{S}^{\mathbf{L}}$ satisfies the axioms of a residuated cover system.

- *Existence*: every C with $x \in C \subseteq \uparrow x$ has $x = \bigsqcup C$, and so is an x-cover.
- *Transitivity*: If $x \sqsubseteq \bigsqcup C$, and $(\forall y \in C)(y \sqsubseteq \bigsqcup C_y)$, then $x \sqsubseteq \bigsqcup_{y \in C}(\bigsqcup C_y) = \bigsqcup(\bigcup_{y \in C} C_y)$.
- *Refinement*: If $x \lhd C$ and $x \preccurlyeq y$, then $y \sqsubseteq x \sqsubseteq \bigsqcup C$, so C itself is a y-cover.
- *Fusion preserves covering*: Let $x \lhd C$. Then $x \cdot y = x \otimes y \sqsubseteq (\bigsqcup C) \otimes y = \bigsqcup_{c \in c} c \otimes y$ (by (2.2)) $= \bigsqcup(C \cdot y)$, so $x \cdot y \lhd C \cdot y$. Similarly $y \cdot x \lhd y \cdot C$ by (2.3).
- *Refinement of ε is local*: If $x \lhd C \subseteq \uparrow\varepsilon$, then $x \sqsubseteq \bigsqcup C \sqsubseteq \varepsilon$, so $\varepsilon \preccurlyeq x$ as required.

Thus $\mathcal{S}^{\mathbf{L}}$ is a residuated cover system. To show that it is strong we show (more strongly!) that the condition (†) of Lemma 3.7 holds for arbitrary subsets X, Y of S, not just propositions. For, suppose there exists a z-cover $C \subseteq \uparrow(X \cdot Y)$. Put $X' = \{a \in X : (\exists b \in Y)(\exists c \in C)\, a \cdot b \preccurlyeq c\} \subseteq X$ and $Y' = \{b \in Y : (\exists a \in X)(\exists c \in C)\, a \cdot b \preccurlyeq c\} \subseteq Y$. Let $x' = \bigsqcup X'$ and $y' = \bigsqcup Y'$, so that $x' \lhd X' \subseteq X$ and $y' \lhd Y' \subseteq Y$. It remains to show $x' \cdot y' \preccurlyeq z$. Now if $c \in C$, then $a \cdot b \preccurlyeq c$ for some $a \in X$ and $b \in Y$. Then $a \in X'$ and $b \in Y'$, so $c \sqsubseteq a \cdot b \sqsubseteq x' \cdot y'$. Hence $\bigsqcup C \sqsubseteq x' \cdot y'$. But $z \sqsubseteq \bigsqcup C$, so $z \sqsubseteq x' \cdot y'$ as required.

Now the propositions of $\mathcal{S}^{\mathbf{L}}$ are in fact precisely the up-sets $\uparrow x$ generated by the elements x of L. First, the fact that $\uparrow x$ is localised, hence a proposition, follows by the same argument given for $\uparrow\varepsilon$ above: if $y \lhd C \subseteq \uparrow x$, then $y \sqsubseteq \bigsqcup C \sqsubseteq x$, so $y \in \uparrow x$. But if X is any localised up-set, let $x = \bigsqcup X$. Then $X \subseteq \uparrow x$, since if $y \in X$, then $y \sqsubseteq \bigsqcup X$, i.e. $y \succcurlyeq x$. Also $x \lhd X$, so $x \in jX = X$. Since X is an up-set, this implies $\uparrow x \subseteq X$, so altogether $X = \uparrow x$.

The map $x \mapsto \uparrow x$ is order-invariant: $x \sqsubseteq y$ iff $\uparrow x \subseteq \uparrow y$. Hence this map is an isomorphism between the complete posets $(L, \sqsubseteq)$ and $(\mathit{Prop}(\mathcal{S}^{\mathbf{L}}), \subseteq)$, and preserves all joins and meets. It also satisfies

$$\uparrow(x \otimes y) = (\uparrow x) \circ (\uparrow y) \tag{3.5}$$

for all $x, y \in L$. For if $z \in (\uparrow x) \circ (\uparrow y)$, then $z \succcurlyeq x' \cdot y'$ for some $x' \succcurlyeq x$ and $y' \succcurlyeq y$, hence $x' \cdot y' \succcurlyeq x \cdot y = x \otimes y$ and so $z \in \uparrow(x \otimes y)$. Conversely, if $z \in \uparrow(x \otimes y)$, then $z \succcurlyeq x \cdot y \in (\uparrow x) \cdot (\uparrow y)$, so $z \in (\uparrow x) \circ (\uparrow y)$.

Now $\circ$ is the semigroup operation on $Prop(\mathcal{S}^{\mathbf{L}})$, since $\mathcal{S}^{\mathbf{L}}$ is strong, so (3.5) shows that the map $x \mapsto \uparrow x$ preserves the semigroup operations of our two quantales. Since $\uparrow(1^{\mathbf{L}}) = \uparrow\varepsilon$, it is a full isomorphism of these unital quantales, preserving also their residual operations. $\square$

4. Classical Systems

This section describes the systems whose propositions form a Grishin Algebra. Let

$$\mathcal{S} = (S, \preccurlyeq, \lhd, \cdot, \varepsilon, 0^{\mathcal{S}}) \tag{4.1}$$

be a residuated cover system with a designated member $0^{\mathcal{S}}$ of $Prop(\mathcal{S})$. We use $0^{\mathcal{S}}$ to define unary operations $-_l$ and $-_r$ on subsets of S, by putting $-_l X = X \Rightarrow_l 0^{\mathcal{S}}$ and $-_r X = X \Rightarrow_r 0^{\mathcal{S}}$. Since $0^{\mathcal{S}}$ is a proposition of $\mathcal{S}$, so too are $-_l X$ and $-_r X$ for any $X \subseteq S$. From (3.2) we get

$$-_l X = \{z \in S : z \cdot X \subseteq 0^{\mathcal{S}}\}, \quad -_r X = \{z \in S : X \cdot z \subseteq 0^{\mathcal{S}}\}. \tag{4.2}$$

Lemma 4.1. *For any subsets X and Y of S:*

(1) $X \subseteq Y$ *implies* $-_l Y \subseteq -_l X$ *and* $-_r Y \subseteq -_r X$.

(2) $X \subseteq -_l Y$ *iff* $Y \subseteq -_r X$.

(3) $-_l -_r$ *and* $-_r -_l$ *are closure operators on* $(\mathcal{PS}, \subseteq)$.

(4) $-_l \uparrow X = -_l X$ *and* $-_r \uparrow X = -_r X$.

(5) $-_l X \subseteq -_l jX$ *and* $-_r X \subseteq -_r jX$.

(6) *If* $X \in Up(\mathcal{S})$, *then* $-_l jX = -_l X$ *and* $-_r jX = -_r X$.

Proof. (1) Let $X \subseteq Y$. Then in general $z \cdot Y \subseteq 0^{\mathcal{S}}$ implies $z \cdot X \subseteq 0^{\mathcal{S}}$, i.e. $z \in -_l Y$ implies $z \in -_l X$ (4.2). Similarly $-_r Y \subseteq -_r X$.

(2) By (3.3), $X \subseteq Y \Rightarrow_l 0^{\mathcal{S}}$ iff $Y \subseteq X \Rightarrow_r 0^{\mathcal{S}}$.

(3) *Monotonicity*: By (1), $X \subseteq Y$ implies $-_l -_r X \subseteq -_l -_r Y$ and $-_r -_l X \subseteq -_r -_l Y$.
Inflationarity: Since $-_r X \subseteq -_r X$, (2) gives $X \subseteq -_l -_r X$. Similarly $Y \subseteq -_r -_l Y$.
Idempotence: From $X \subseteq -_l -_r X$ by (1) we get $-_r -_l -_r X \subseteq -_r X$. Hence $-_r -_l -_r X = -_r X$ as $-_r -_l$ is inflationary. It follows that $-_l -_r -_l -_r X = -_l -_r X$. Similarly, $-_r -_l -_r -_l X = -_r -_l X$.

(4) Since $X \subseteq {\uparrow}X$, (1) gives $-_l{\uparrow}X \subseteq -_l X$. Conversely, let $z \in -_l X$. Then for all $y \in {\uparrow}X$ we have $y \succcurlyeq x$ for some $x \in X$, hence $z \cdot y \succcurlyeq z \cdot x$, and $z \cdot x \in z \cdot X \subseteq 0^{\mathcal{S}}$ (4.2), so $z \cdot y \in 0^{\mathcal{S}}$ as $0^{\mathcal{S}}$ is an up-set. This shows that $z \cdot {\uparrow}X \subseteq 0^{\mathcal{S}}$, hence $z \in -_l{\uparrow}X$, as required to prove $-_l{\uparrow}X = -_l X$.
The proof that $-_r{\uparrow}X = -_r X$ is similar.

(5) Let $x \in -_l X$. Then if $y \in jX$, there is some C with $x \lhd C \subseteq X$, so $x \cdot y \lhd x \cdot C \subseteq x \cdot X \subseteq 0^{\mathcal{S}}$. Hence $x \cdot y \in j0^{\mathcal{S}} = 0^{\mathcal{S}}$. This shows that $x \cdot jX \subseteq 0^{\mathcal{S}}$, i.e. that $x \in -_l jX$.
The proof that $-_r X \subseteq -_r jX$ is similar.

(6) j is inflationary on up-sets (Lemma 3.2), so if $X \in \mathit{Up}(\mathcal{S})$, then $X \subseteq jX$, hence $-_l jX \subseteq -_l X$ by (1), implying $-_l jX = -_l X$ by (5).
Similarly $-_r jX = -_r X$.

□

We now define a cover system $\mathcal{S}$ of the form (4.1) to be *classical* if it has

$$j{\uparrow}X = -_l -_r X = -_r -_l X, \quad \text{for all } X \subseteq S. \tag{4.3}$$

Thus in a classical system, the least proposition (localised up-set) containing any given X is equal to both $-_l -_r X$ and $-_r -_l X$.

THEOREM 4.2. *For any residuated cover system $\mathcal{S}$ with distinguished proposition $0^{\mathcal{S}}$, the following are equivalent:*

(1) *$\mathcal{S}$ is classical.*

(2) *$jX = -_l -_r X = -_r -_l X$, for all up-sets X.*

(3) *$\mathit{Prop}(\mathcal{S})$ is a Grishin algebra.*

PROOF. (1) implies (2): if (4.3) holds, then (2) is immediate, as ${\uparrow}X = X$ whenever $X \in \mathit{Up}(\mathcal{S})$.

(2) implies (3): Let $X \in \mathit{Prop}(\mathcal{S})$. Then X is an upset, and $jX = X$, so (2) implies $X = -_l -_r X = -_r -_l X$. Thus the residuated lattice-ordered monoid $\mathit{Prop}(\mathcal{S})$ satisfies the law of double-negation elimination in the form (2.8), so is a Grishin algebra by Theorem 2.2.

(3) implies (1): for any $X \subseteq S$ we have $j{\uparrow}X \in \mathit{Prop}(\mathcal{S})$, so (3) implies

$$j{\uparrow}X = -_l -_r j{\uparrow}X = -_r -_l j{\uparrow}X.$$

But as ${\uparrow}X$ is an up-set, Lemma 4.1(6) gives $-_r j{\uparrow}X = -_r{\uparrow}X$, which is equal to $-_r X$ by Lemma 4.1(4). Hence $-_l -_r j{\uparrow}X = -_l -_r X$. Similarly $-_r -_l j{\uparrow}X = -_r -_l X$. Thus (4.3) holds, making $\mathcal{S}$ classical. □

Thus we see that the propositions of a classical residuated cover system form a complete Grishin algebra. In the converse direction, we have:

THEOREM 4.3. *Every complete Grishin algebra is isomorphic to the algebra of all propositions of some strong classical residuated cover system.*

PROOF. Let $\mathbf{L}$ be a complete Grishin algebra with distinguished element $0^{\mathbf{L}}$. Let $\mathcal{S}^{\mathbf{L}}$ be the associated *strong* residuated cover system of Theorem 3.8, and define $0^{\mathcal{S}^{\mathbf{L}}}$ to be $\uparrow 0^{\mathbf{L}} \in Prop(\mathcal{S}^{\mathbf{L}})$. The map $x \mapsto \uparrow x$ was shown to be an isomorphism between $\mathbf{L}$ and $Prop(\mathcal{S}^{\mathbf{L}})$ as complete residuated pomonoids, and it now preserves the given "0-elements" as well. Since it preserves residuals, it must also preserve the DNE equations (2.8), and so $Prop(\mathcal{S}^{\mathbf{L}})$ is a Grishin algebra. Hence by the previous Theorem, $\mathcal{S}^{\mathbf{L}}$ is classical. □

5. Orthogonality

In a residuated cover system of the form (4.1), an alternative description of the operations $-_l$ and $-_r$ of (4.2) can be developed by introducing a certain binary relation $\perp$ on S. This is defined by

$$z \perp y \quad \text{iff} \quad z \cdot y \in 0^{\mathcal{S}}. \tag{5.1}$$

We think of $\perp$ as a relation of *orthogonality* or *incompatibility* between points of $\mathcal{S}$. It is lifted to a relation between points z and subsets X of S, by putting $z \perp X$ iff $z \perp y$ for all $y \in X$; and $X \perp z$ iff $y \perp z$ for all $y \in X$. Then we get

$$z \perp X \quad \text{iff} \quad z \cdot X \subseteq 0^{\mathcal{S}},$$

and similarly $X \perp z$ iff $X \cdot z \subseteq 0^{\mathcal{S}}$, so the operations $-_l$ and $-_r$ of (4.2) satisfy

$$-_l X = \{z \in S : z \perp X\}, \qquad -_r X = \{z \in S : X \perp z\}. \tag{5.2}$$

Moreover, since ε is a unit for $\cdot$, we have $z \cdot \varepsilon \in 0^{\mathcal{S}}$ iff $z \in 0^{\mathcal{S}}$ iff $\varepsilon \cdot z \in 0^{\mathcal{S}}$, so

$$0^{\mathcal{S}} = \{z : z \perp \varepsilon\} = \{z : \varepsilon \perp z\}. \tag{5.3}$$

This suggests an alternative approach to classical cover systems. Instead of starting with a system $\mathcal{S}$ having a distinguished proposition $0^{\mathcal{S}}$, we begin with a residuated cover system

$$\mathcal{S} = (S, \preccurlyeq, \lhd, \cdot, \varepsilon, \perp) \tag{5.4}$$

having a binary relation $\perp$ satisfying the following conditions:

- $z \perp y$ iff $z \cdot y \perp \varepsilon$.
- *Orthogonality to ε is monotonic*: $y \succcurlyeq z \perp \varepsilon$ implies $y \perp \varepsilon$.
- *Orthogonality to ε is local*: $x \lhd C \perp \varepsilon$ implies $x \perp \varepsilon$.

From the first condition we get that in general $\varepsilon \perp z$ iff $z \perp \varepsilon$, so we can *well-define* a subset $0^{\mathcal{S}}$ by (5.3). The other two conditions ensure that $0^{\mathcal{S}}$ thus defined is an up-set that is localised, i.e. a member of *Prop*($\mathcal{S}$). The first condition then ensures that $z \perp X$ iff $z \cdot X \subseteq 0^{\mathcal{S}}$, and likewise $X \perp z$ iff $X \cdot z \subseteq 0^{\mathcal{S}}$. So operations $-_l$ and$-_r$ defined from $\perp$ by (5.2) agree with those defined by (4.2), where $0^{\mathcal{S}}$ itself is defined from $\perp$ by (5.3).

We leave it to the reader to formulate the precise sense in which these two approaches – starting with $0^{\mathcal{S}}$ or with $\perp$ – are equivalent. In either approach, $\mathcal{S}$ can be specified to be classical when (4.3) holds.

It is noteworthy that, since $0^{\mathcal{S}}$ is an up-set, we have $z \cdot ({\uparrow}\varepsilon) \subseteq 0^{\mathcal{S}}$ iff $z \cdot \varepsilon \in 0^{\mathcal{S}}$ iff $z \in 0^{\mathcal{S}}$; and so $-_l 1 = 0^{\mathcal{S}}$, and likewise $-_r 1 = 0^{\mathcal{S}}$, in any residuated cover system with $0^{\mathcal{S}}$, independently of the classicality condition (4.3).

6. MacNeille Completion

Each poset $\mathbf{L} = (L, \sqsubseteq)$ can be extended to a complete poset $\overline{\mathbf{L}} = (\overline{L}, \sqsubseteq)$ such that any subset of L having a join or meet in $\mathbf{L}$ has the same join or meet in $\overline{\mathbf{L}}$. In particular, if $\mathbf{L}$ is a lattice, then it is a sublattice of $\overline{\mathbf{L}}$. Moreover, each element of $\overline{L}$ is both a join of elements of L and a meet of elements of L. Hence if $a \in \overline{L}$,

$$a = \bigsqcup\{s \in L : s \sqsubseteq a\} = \bigsqcap\{t \in L : a \sqsubseteq t\} \tag{6.1}$$

(we use the letters a, b, c for general members of $\overline{L}$, and reserve s, t for members of L). This property characterises $\overline{\mathbf{L}}$ uniquely up to isomorphism [3, 24].

MacNeille [17] proved the existence of the completion $\overline{\mathbf{L}}$ by generalising the Dedekind completion of the rationals by cuts. Here we will work with the abstract description of $\overline{\mathbf{L}}$ based on (6.1). It implies that for any $a, b \in \overline{L}$,

$$a \sqsubseteq b \quad \text{iff} \quad \forall s, t \in L\, (s \sqsubseteq a \text{ and } b \sqsubseteq t \text{ implies } s \sqsubseteq t). \tag{6.2}$$

The MacNeille completion of a residuated pomonoid

$$\mathbf{L} = (L, \sqsubseteq, \otimes, 1^{\mathbf{L}}, \Rightarrow_l, \Rightarrow_r)$$

is a unital quantale. This was shown in [20] by a generalisation of MacNeille's construction.[4] Here is the abstract version. The operations on $\mathbf{L}$ are lifted to $\overline{L}$ by putting, for $a, b \in \overline{L}$:

$$\begin{aligned} a \otimes b &= \bigsqcup\{s \otimes t : a \sqsupseteq s \in L \text{ and } b \sqsupseteq t \in L\} \\ a \Rightarrow_l b &= \bigsqcap\{s \Rightarrow_l t : a \sqsupseteq s \in L \text{ and } b \sqsubseteq t \in L\} \\ a \Rightarrow_r b &= \bigsqcap\{s \Rightarrow_r t : a \sqsupseteq s \in L \text{ and } b \sqsubseteq t \in L\} \\ 1^{\overline{\mathbf{L}}} &= 1^{\mathbf{L}}. \end{aligned}$$

These operations agree with the corresponding operations of $\mathbf{L}$ (i.e. they are equal to the corresponding operations when restricted to $\mathbf{L}$), and they make $\overline{\mathbf{L}}$ into a residuated lattice-ordered monoid having $\mathbf{L}$ as a subalgebra.[5]

The variety of classical FL-algebras was observed to be closed under MacNeille completion in [20, p. 275] (see also [23, 6.1]). Here we give a proof of the result for Grishin algebras, using the abstract formalism.

Now suppose $\mathbf{L}$ is a Grishin algebra with distinguished element 0. Put $0^{\overline{\mathbf{L}}} = 0$ and use this element to define $-_l$ and $-_r$ on $\overline{\mathbf{L}}$ by $-_l\, a = a \Rightarrow_l 0$, and $-_r\, a = a \Rightarrow_r 0$, thereby extending the operations $-_l$ and $-_r$ of $\mathbf{L}$.

LEMMA 6.1. *For any $a, b \in \overline{\mathbf{L}}$,*

(1) *If $a \sqsubseteq b$, then $-_l\, b \sqsubseteq -_l\, a$ and $-_r\, b \sqsubseteq -_r\, a$.*

(2) *$a \sqsubseteq -_l -_r\, a$ and $a \sqsubseteq -_r -_l\, a$.*

(3) *$-_l\, a = \bigsqcap\{-_l\, s : a \sqsupseteq s \in L\}$, $\quad -_r\, a = \bigsqcap\{-_r\, s : a \sqsupseteq s \in L\}$.*

PROOF. (1) and (2) hold for any residuated pomonoid with 0, given the definitions of $-_l$ and $-_r$.

(1) Residuation is antitonic in the first argument, so $a \sqsubseteq b$ implies $b \Rightarrow_l 0 \sqsubseteq a \Rightarrow_l 0$, i.e. $-_l\, b \sqsubseteq -_l\, a$, and likewise $b \Rightarrow_r 0 \sqsubseteq a \Rightarrow_r 0$.

(2) $a \otimes (a \Rightarrow_r 0) \sqsubseteq 0$, so $a \sqsubseteq (a \Rightarrow_r 0) \Rightarrow_l 0 = -_l -_r 0$. Similarly $a \sqsubseteq (a \Rightarrow_l 0) \Rightarrow_r 0$.

(3) Let $X = \{s \Rightarrow_l t : a \sqsupseteq s \in L \text{ and } 0 \sqsubseteq t \in L\}$. Then $-_l\, a = a \Rightarrow_l 0 = \bigsqcap X$ by definition of $\Rightarrow_l$.
Put $Y = \{-_l\, s : a \sqsupseteq s \in L\}$. Then $Y \subseteq X$, since if $a \sqsupseteq s \in L$, then $-_l\, s = s \Rightarrow_l 0 \in X$. Hence $\bigsqcap X \sqsubseteq \bigsqcap Y$. But if $0 \sqsubseteq t$, then

[4] The result for commutative $\otimes$ is discussed in[25, Chapter 8] and [21]. In that case $\Rightarrow_l$ is equal to $\Rightarrow_r$.

[5] See [24, Prop. 3.17] for a proof that $\Rightarrow_l$ is left residual to $\otimes$ under these definitions.

$s \Rightarrow_l 0 \sqsubseteq s \Rightarrow_l t$, so each member $s \Rightarrow_l t$ of X has the member $s \Rightarrow_l 0$ of Y below it under $\sqsubseteq$, and therefore has $\bigsqcap Y$ below it. This implies that $\bigsqcap Y \sqsubseteq \bigsqcap X$.
Thus $\bigsqcap X = \bigsqcap Y$, i.e. $-_l\, a = \bigsqcap Y$ as required. The equation for $\Rightarrow_r$ is proved similarly.

□

THEOREM 6.2. *The MacNeille completion of a Grishin algebra is also a Grishin algebra.*

PROOF. Let $\mathbf{L}$ be a Grishin algebra, and $\Rightarrow_l$ and $\Rightarrow_r$ the operations defined on $\overline{\mathbf{L}}$ as above. Since $\overline{\mathbf{L}}$ is a residuated lattice-ordered monoid, by Theorem 2.2 it remains only to show that the double-negation elimination law holds in $\overline{\mathbf{L}}$.

Given any $a \in \overline{L}$, we have $a \sqsubseteq -_l -_r a$ by (2) of the above Lemma, so we need to show that $-_l -_r a \sqsubseteq a$. By (6.2), it suffices to take any $s, t \in L$ with $s \sqsubseteq -_l -_r a$ and $a \sqsubseteq t$, and show that $s \sqsubseteq t$. Now if $a \sqsubseteq t \in L$, then $-_l -_r a \sqsubseteq -_l -_r t$ by (1) of the above Lemma. But $-_l -_r t = t$ as $\mathbf{L}$ is a Grishin algebra, so if $s \sqsubseteq -_l -_r a$ then $s \sqsubseteq t$ follows as required.

This proves $a = -_l -_r a$, and the proof that $a = -_r -_l a$ is similar. □

We can now combine our results into a representation theorem for Grishin algebras in general.

THEOREM 6.3. *Every Grishin algebra has an isomorphic embedding into the algebra of all propositions of some strong classical residuated cover system, by a map that preserves all existing joins and meets.*

PROOF. If $\mathbf{L}$ is a Grishin algebra, then $\overline{\mathbf{L}}$ is a complete Grishin algebra, so is isomorphic to $Prop(\mathcal{S}^{\overline{\mathbf{L}}})$, where $\mathcal{S}^{\overline{\mathbf{L}}}$ is the strong classical residuated cover system of Theorem 4.3. The inclusion of $\mathbf{L}$ into $\overline{\mathbf{L}}$ preserves all existing joins and meets, hence so does its composition with the isomorphism from $\overline{\mathbf{L}}$ onto $Prop(\mathcal{S}^{\overline{\mathbf{L}}})$. □

7. Infinitary First-Order Logic

Our representation of Grishin algebras gives rise to a cover system semantics for a version of classical bilinear predicate logic that has quantification of individual variables, and (infinitary) disjunctions and conjunctions of sets of formulas. To describe the formal language for this, we fix a denumerable list $v_0, \ldots, v_n, \ldots$ of individual variables and a set of predicate letters, with typical member P, that are k-ary for various $k < \omega$. These are used to define *atomic* formulas $P(v_{n_1}, \dots, v_{n_k})$. A *preformula* is any expression generated from atomic formulas and the constants T, F, $\mathsf{1}$, $\mathsf{0}$ by using the binary connectives $\&, \to_l, \to_r$ and the quantifiers $\exists v_n$, $\forall v_n$, and by allowing the formation of the disjunction $\bigvee\Phi$ and conjunction $\bigwedge\Phi$ of any *set* Φ of formulas. A *formula* is a preformula that has only finitely many free variables. Binary disjunctions and conjunctions are defined by taking $\varphi \vee \psi$ to be $\bigvee\{\varphi, \psi\}$ and $\varphi \wedge \psi$ to be $\bigwedge\{\varphi, \psi\}$. Left and right negation connectives are introduced by defining $\neg_l\varphi$ to be $\varphi \to_l \mathsf{0}$ and $\neg_r\varphi$ to be $\varphi \to_r \mathsf{0}$.

The semantic analysis builds on the one already given in [10] for this language without the constants T, F, $\mathsf{1}$, $\mathsf{0}$, or the negations $\neg_l, \neg_r$, using quantales and certain cover systems. We take this earlier work as given (see [10, Sections 3.1–3.4, 3.7]), and confine ourselves to explaining what is involved in extending it to the present language.

By a *classical model* we mean a structure $\mathcal{M} = (\mathcal{S}, D, V)$, where $\mathcal{S}$ is a *strong classical* residuated cover system; D is a set of individuals; and for each k-ary predicate letter P, $V(P)$ is a function assigning a proposition of $\mathcal{S}$ to each k-tuple of elements of D. In symbols: $V(P) : D^k \to \mathit{Prop}(\mathcal{S})$. To interpret variables in the model we use *D-valuations*, which are sequences $\sigma = \langle\sigma_0, \dots, \sigma_n, \dots\rangle$ of elements of D, the idea being that σ assigns value σ_n to variable v_n. We write $\sigma(d/n)$ for the valuation obtained from σ by replacing σ_n by d. For each formula φ we specify a proposition $\|\varphi\|^{\mathcal{M}}_{\sigma} \in \mathit{Prop}(\mathcal{S})$ for each valuation σ. This is defined inductively on the formation of φ, using the structure of $\mathit{Prop}(\mathcal{S})$, as follows (cf. Theorem 3.5):

- $\|P(v_{n_1}, \dots, v_{n_k})\|^{\mathcal{M}}_{\sigma} = V(P)(\sigma_{n_1}, \dots, \sigma_{n_k})$.
- $\|\mathsf{T}\|^{\mathcal{M}}_{\sigma} = S$.
- $\|\mathsf{F}\|^{\mathcal{M}}_{\sigma} = j\emptyset$.
- $\|\mathsf{1}\|^{\mathcal{M}}_{\sigma} = \uparrow\varepsilon$.
- $\|\mathsf{0}\|^{\mathcal{M}}_{\sigma} = 0^{\mathcal{S}}$.
- $\|\varphi\&\psi\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma} \otimes \|\psi\|^{\mathcal{M}}_{\sigma}$.
- $\|\varphi \to_l \psi\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma} \Rightarrow_l \|\psi\|^{\mathcal{M}}_{\sigma}$, $\quad \|\varphi \to_r \psi\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma} \Rightarrow_r \|\psi\|^{\mathcal{M}}_{\sigma}$.
- $\|\bigvee\Phi\|^{\mathcal{M}}_{\sigma} = \bigsqcup_{\varphi\in\Phi} \|\varphi\|^{\mathcal{M}}_{\sigma}$.
- $\|\bigwedge\Phi\|^{\mathcal{M}}_{\sigma} = \bigsqcap_{\varphi\in\Phi} \|\varphi\|^{\mathcal{M}}_{\sigma}$.
- $\|\exists v_n\varphi\|^{\mathcal{M}}_{\sigma} = \bigsqcup_{d\in D} \|\varphi\|^{\mathcal{M}}_{\sigma(d/n)}$.
- $\|\forall v_n\varphi\|^{\mathcal{M}}_{\sigma} = \bigsqcap_{d\in D} \|\varphi\|^{\mathcal{M}}_{\sigma(d/n)}$.

Consequently, the negation connectives have

- $\|\neg_l\varphi\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma} \Rightarrow_l 0^{\mathcal{S}} = -_l\,\|\varphi\|^{\mathcal{M}}_{\sigma}$.
- $\|\neg_r\varphi\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma} \Rightarrow_r 0^{\mathcal{S}} = -_r\,\|\varphi\|^{\mathcal{M}}_{\sigma}$.

A *satisfaction relation* is defined by using the notation

$$\mathcal{M}, x \models \varphi[\sigma]$$

to mean that $x \in \|\varphi\|^{\mathcal{M}}_{\sigma}$. This can be read "$\varphi$ is true/satisfied in $\mathcal{M}$ at x under σ". Unravelling the operations in *Prop*($\mathcal{S}$), and using the fact that $X \otimes Y = \uparrow(X \cdot Y)$ in a strong system, this satisfaction relation is characterised as follows (suppressing the symbol $\mathcal{M}$):

$$\begin{array}{lll}
x \models P(v_{n_1}, \dots, v_{n_k})[\sigma] & \text{iff} & x \in V(P)(\sigma_{n_1}, \dots, \sigma_{n_k}). \\
x \models \mathsf{T}[\sigma]. & & \\
x \models \mathsf{F}[\sigma] & \text{iff} & x \lhd \emptyset. \\
x \models \mathsf{1}[\sigma] & \text{iff} & \varepsilon \preccurlyeq x. \\
x \models \mathsf{0}[\sigma] & \text{iff} & x \in 0^{\mathcal{S}} \quad (\text{iff } x \perp \varepsilon, \text{ cf. } (5.3)). \\
x \models \varphi \& \psi[\sigma] & \text{iff} & \text{for some } y \text{ and } z \text{ such that } y \cdot z \preccurlyeq x, \\
 & & y \models \varphi[\sigma] \text{ and } z \models \psi[\sigma]. \\
x \models \varphi \to_l \psi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } x \cdot y \models \psi[\sigma]. \\
x \models \varphi \to_r \psi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } y \cdot x \models \psi[\sigma]. \\
x \models \bigvee \Phi[\sigma] & \text{iff} & \text{there exists } C \rhd x \text{ such that for all } z \in C, \\
 & & z \models \varphi[\sigma] \text{ for some } \varphi \in \Phi. \\
x \models \bigwedge \Phi[\sigma] & \text{iff} & x \models \varphi[\sigma] \text{ for all } \varphi \in \Phi. \\
x \models \exists v_n \varphi[\sigma] & \text{iff} & \text{there exists } C \rhd x \text{ such that for all } z \in C, \\
 & & z \models \varphi[\sigma(d/n)] \text{ for some } d \in D. \\
x \models \forall v_n \varphi[\sigma] & \text{iff} & x \models \varphi[\sigma(d/n)] \text{ for all } d \in D.
\end{array}$$

Furthermore, the negation connectives satisfy:

$$\begin{array}{lll}
x \models \neg_l \varphi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } x \perp y, \\
x \models \neg_r \varphi[\sigma] & \text{iff} & y \models \varphi[\sigma] \text{ implies } y \perp x
\end{array}$$

(cf. (5.2)). This kind of modelling of negation via an orthogonality relation goes back to [8] in the context of orthologic, where $\perp$ is symmetric and the two negations collapse to one. The definition of $\perp$ from a fixed set $0^{\mathcal{S}}$ as in (5.1) comes from [7].

Each model $\mathcal{M}$ gives rise to a semantic implication relation $\models^{\mathcal{M}}$ on formulas, defined by writing

$$\varphi \models^{\mathcal{M}} \psi$$

to mean that $\|\varphi\|_\sigma^{\mathcal{M}} \subseteq \|\psi\|_\sigma^{\mathcal{M}}$ for all D-valuations σ, i.e. that for all x in S and all D-valuations σ, $\mathcal{M}, x \models \varphi[\sigma]$ implies $\mathcal{M}, x \models \psi[\sigma]$. We say that φ *classically implies* ψ, written $\varphi \models^{\mathrm{c}} \psi$, if $\varphi \models^{\mathcal{M}} \psi$ for all classical models $\mathcal{M}$.

To axiomatise the semantic relation $\models^{\mathrm{c}}$, we take a *sequent* to be an expression $\varphi \vdash \psi$ with φ and ψ being formulas. Alternatively, a sequent may be thought of as an ordered pair of formulas, with the symbol $\vdash$ denoting a class of sequents, i.e. a binary relation between formulas. Then we write $\varphi \dashv\vdash \psi$ when both $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

For the logic of residuated posemigroups, we gave in [10, Section 3.3] a list of sequent axiom schemes and a list of rules of inference for generating new sequents. We take those axioms and rules as given here, and add to them the following axiom schemes:

- $\mathsf{F} \vdash \varphi, \quad \varphi \vdash \mathsf{T}$.
- $\varphi \& 1 \dashv\vdash \varphi, \quad 1 \& \varphi \dashv\vdash \varphi$.
- $\neg_l \neg_r \varphi \dashv\vdash \varphi, \quad \neg_r \neg_l \varphi \dashv\vdash \varphi$.

Let $\vdash^{\mathrm{c}}$ be the smallest class of sequents that includes all instances of the axiom schemes and is closed under the rules of inference. To prove *soundness*, i.e.

$$\varphi \vdash^{\mathrm{c}} \psi \quad \text{implies} \quad \varphi \models^{\mathrm{c}} \psi, \tag{7.1}$$

it suffices to observe that for any classical model $\mathcal{M}$, the relation $\models^{\mathcal{M}}$ includes all instances of the axioms and is closed under the rules, so it includes $\vdash^{\mathrm{c}}$. Thus $\varphi \vdash^{\mathrm{c}} \psi$ implies $\varphi \models^{\mathcal{M}} \psi$ for all classical models $\mathcal{M}$.

To prove *completeness*, i.e. the converse of (7.1), we use a Lindenbaum algebra construction to build a Grishin algebra, and then embed it into the algebra of propositions of a model by the representation of Theorem 6.3. But the use of infinitary disjunctions and conjunctions allows a proper class of formulas to be generated, and we have to restrict this to a set, in order for the Lindenbaum algebra to be based on a set. So we define a *fragment* to be a *set* $\mathcal{F}$ of formulas that includes all atomic formulas and the constants T, F, $\mathsf{1}$, $\mathsf{0}$; is closed under the binary connectives $\wedge, \vee, \&, \to_l, \to_r$, and the quantifiers $\exists v_n$, $\forall v_n$; and is closed under subformulas and variable substitution. Hence $\mathcal{F}$ is closed under the negation connectives $\neg_l$ and $\neg_r$.

Let $\vdash$ be a relation satisfying all our axioms and rules, and fix a fragment $\mathcal{F}$. The induced relation $\dashv\vdash$ is an equivalence relation on $\mathcal{F}$. Let $|\varphi| = \{\psi \in$

$\mathcal{F} : \varphi \dashv\vdash \psi\}$ be the equivalence class of $\varphi \in \mathcal{F}$ and $L^{\mathcal{F}} = \{|\varphi| : \varphi \in \mathcal{F}\}$. Put

$$\begin{aligned}
|\varphi| \sqsubseteq |\psi| &\text{ iff } \varphi \vdash \psi, \\
|\varphi| \sqcap |\psi| &= |\varphi \wedge \psi|, \\
|\varphi| \sqcup |\psi| &= |\varphi \vee \psi|, \\
|\varphi| \otimes |\psi| &= |\varphi \& \psi|, \\
|\varphi| \Rightarrow_l |\psi| &= |\varphi \to_l \psi|, \\
|\varphi| \Rightarrow_r |\psi| &= |\varphi \to_r \psi|, \\
-_l\, |\varphi| &= |\neg_l \varphi|, \\
-_r\, |\varphi| &= |\neg_r \varphi|.
\end{aligned}$$

The axioms and rules ensure that this yields a well-defined Grishin algebra $\mathbf{L}^{\mathcal{F}}$ on $L^{\mathcal{F}}$ with greatest element $|\mathsf{T}|$ and least element $|\mathsf{F}|$; with $|\mathsf{1}|$ being the identity element of $\otimes$; and with the distinguished element $|\mathsf{0}|$ having $-_l\, |\varphi| = |\varphi| \to_l |\mathsf{0}|$ and $-_r\, |\varphi| = |\varphi| \to_r |\mathsf{0}|$. Also we have:[6]

$$\begin{aligned}
|\exists v_n \varphi| &= \bigsqcup_{p<\omega} |\varphi(v_p/v_n)|. \\
|\forall v_n \varphi| &= \bigsqcap_{p<\omega} |\varphi(v_p/v_n)|. \\
|\bigvee \Phi| &= \bigsqcup_{\varphi \in \Phi} |\varphi|, \quad \text{when } \bigvee \Phi \in \mathcal{F}. \\
|\bigwedge \Phi| &= \bigsqcap_{\varphi \in \Phi} |\varphi| \quad \text{when } \bigwedge \Phi \in \mathcal{F}.
\end{aligned} \tag{7.2}$$

By Theorem 6.3, there exists a strong classical residuated cover system $\mathcal{S}^{\mathcal{F}}$ and an isomorphic embedding $f : \mathbf{L}^{\mathcal{F}} \to \mathit{Prop}(\mathcal{S}^{\mathcal{F}})$ that preserves all joins and meets that exist in $\mathbf{L}^{\mathcal{F}}$, including those described in (7.2).

Now define the classical model $\mathcal{M}^{\mathcal{F}} = (\mathcal{S}^{\mathcal{F}}, D, V)$, where D is the set of all variables v_n, and $V(P)(v_{n_1}, \ldots, v_{n_k}) = f|P(v_{n_1}, \ldots, v_{n_k})|$. Then if σ is the D-valuation with $\sigma_n = v_n$, we get

$$\|P(v_{n_1}, \ldots, v_{n_k})\|_\sigma^{\mathcal{M}^{\mathcal{F}}} = f|P(v_{n_1}, \ldots, v_{n_k})|.$$

We then extend this to show inductively that

$$\|\varphi\|_\sigma^{\mathcal{M}^{\mathcal{F}}} = f|\varphi| \text{ for all } \varphi \in \mathcal{F}. \tag{7.3}$$

[6] The proofs of the equations for $|\exists v_n \varphi|$ and $|\forall v_n \varphi|$ in (7.2) are as for finitary first-order logic (e.g. [1, Lemma 3.4.1]), and depend on the fact that a formula φ has finitely many free variables.

This uses the the definition of $\|\varphi\|^{\mathcal{M}}_{\sigma}$, results(7.2), the fact that f preserves the Grishin algebra operations and any joins and meets existing in $\mathcal{S}^{\mathcal{F}}$; and the general substitutional result that $\|\varphi(v_p/v_n)\|^{\mathcal{M}}_{\sigma} = \|\varphi\|^{\mathcal{M}}_{\sigma(\sigma_p/n)}$ (which holds of any σ in any model $\mathcal{M}$).

To prove completeness of $\vdash^{\mathrm{c}}$ for $\models^{\mathrm{c}}$, suppose $\varphi \models^{\mathrm{c}} \psi$. Take a fragment $\mathcal{F}$ containing the formulas φ and ψ, and construct the classical model $\mathcal{M}^{\mathcal{F}}$as above using the relation $\vdash^{\mathrm{c}}$ to define $\mathbf{L}^{\mathcal{F}}$ and hence $\mathcal{S}^{\mathcal{F}}$. Then $\varphi \models^{\mathcal{M}^{\mathcal{F}}} \psi$, so $\|\varphi\|^{\mathcal{M}^{\mathcal{F}}}_{\sigma} \subseteq \|\psi\|^{\mathcal{M}^{\mathcal{F}}}_{\sigma}$ in *Prop*$(\mathcal{S}^{\mathcal{F}})$ where $\sigma_n = v_n$. Hence $|\varphi| \sqsubseteq |\psi|$ in $\mathbf{L}^{\mathcal{F}}$ by (7.3) and the fact that f is order-invariant; so $\varphi \vdash^{\mathrm{c}} \psi$. This proves that

$$\varphi \models^{\mathrm{c}} \psi \quad \text{implies} \quad \varphi \vdash^{\mathrm{c}} \psi$$

as required. Together with (7.1), we have shown that the proof-theoretic deducibility relation $\vdash^{\mathrm{c}}$ is both sound and complete for the semantic consequence relation $\models^{\mathrm{c}}$ over models on strong classical residuated covers systems. In other words, these two relations are identical.

References

[1] J. L. Bell and A. B. Slomson. *Models and Ultraproducts*. North-Holland, Amsterdam, 1969.

[2] John L. Bell. Cover schemes, frame-valued sets and their potential uses in spacetime physics. In Albert Reimer, editor, *Spacetime Physics Research Trends, Horizons in World Physics*, volume 248. Nova Science Publishers, 2005. Manuscript at `http://publish.uwo.ca/~jbell`.

[3] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[4] Kosta Došen. Sequent systems and groupoid models II. *Studia Logica*, 48(1):41–65, 1989.

[5] Kosta Došen. A brief survey of frames for the Lambek calculus. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 38:179–187, 1992.

[6] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices : An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 2007.

[7] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[8] Robert Goldblatt. Semantic analysis of orthologic. *Journal of Philosophical Logic*, 3:19–35, 1974. Reprinted in [9].

[9] Robert Goldblatt. *Mathematics of Modality*. CSLI Lecture Notes No. 43. CSLI Publications, Stanford University, 1993.

[10] Robert Goldblatt. A Kripke-Joyal semantics for noncommutative logic in quantales. In Guido Governatori, Ian Hodkinson, and Yde Venema, editors, *Advances in Modal Logic, Volume 6*, pages 209–225. College Publications, London, 2006. `www.aiml.net/volumes/volume6/`.

[11] Robert Goldblatt. Cover semantics for quantified lax logic. *Journal of Logic and Computation*, 2010. `doi:10.1093/logcom/exq029`.

[12] Robert Goldblatt. *Quantifiers, Propositions and Identity: Admissible Semantics for Quantified Modal and Substructural Logics.* Number 38 in Lecture Notes in Logic. Cambridge University Press and the Association for Symbolic Logic, 2011.

[13] V. N. Grishin. On a generalisation of the Ajdukiewicz-Lambek system. In A. I. Mikhailov, editor, *Studies in Non-Classical Logics and Formal Systems*, pages 315–334. Nauka, Moscow, 1983. English translation in V. M. Abrusci and C. Casadio (eds.), *New Perspectives in Logic and Formal Linguistics*, Proceedings 5th Roma Workshop, Bulzoni Editore, Rome, 2002. Corrected version available as pages 1–17 in `http://symcg.pbworks.com/f/esslllinotesnew.pdf`.

[14] J. Lambek. Some lattice models of bilinear logic. *Algebra Universalis*, 34:541–550, 1995.

[15] Joachim Lambek. Bilinear logic and Grishin algebras. In Ewa Orlowska, editor, *Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa*, pages 604–612. Physica-Verlag, 1999.

[16] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer-Verlag, 1992.

[17] H. M. MacNeille. Partially ordered sets. *Transactions of the American Mathematical Society*, 42:416–460, 1937.

[18] James R. Munkres. *Topology.* Prentice Hall, 2000.

[19] Susan B. Niefield and Kimmo I. Rosenthal. Constructing locales from quantales. *Mathematical Proceedings of the Cambridge Philosophical Society*, 104:215–234, 1988.

[20] Hiroakira Ono. Semantics for substructural logics. In Peter Schroeder-Heister and Kosta Došen, editors, *Substructural Logics*, pages 259–291. Oxford University Press, 1993.

[21] Hiroakira Ono. Closure operators and complete embeddings of residuated lattices. *Studia Logica*, 74(3):427–440, 2003.

[22] Hiroakira Ono and Yuichi Komori. Logics without the contraction rule. *The Journal of Symbolic Logic*, 50(1):169–201, 1985.

[23] Kimmo I. Rosenthal. *Quantales and Their Applications*, volume 234 of *Pitman Research Notes in Mathematics.* Longman Scientific & Technical, 1990.

[24] Mark Theunissen and Yde Venema. MacNeille completions of lattice expansions. *Algebra Universalis*, 57:143–193, 2007.

[25] A. S. Troelstra. *Lectures on Linear Logic.* CSLI Lecture Notes No. 29. CSLI Publications, Stanford, California, 1992.

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