Covarieties of Coalgebras: Comonads and Coequations

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Abstract. Coalgebras provide effective models of data structures and state-transition systems. A *virtual covariety* is a class of coalgebras closed under coproducts, images of coalgebraic morphisms, and subcoalgebras defined by split equalisers. A *covariety* has the stronger property of closure under all subcoalgebras, and is *behavioural* if it is closed under domains of morphisms, or equivalently under images of bisimulations. There are many computationally interesting properties that define classes of these kinds.

We identify conditions on the underlying category of a comonad $\mathbb{G}$ which ensure that there is an exact correspondence between (behavioural/virtual) covarieties of $\mathbb{G}$-coalgebras and *subcomonads* of $\mathbb{G}$ defined by comonad morphisms to $\mathbb{G}$ with natural categorical properties. We also relate this analysis to notions of *coequationally defined* classes of coalgebras.

1 Introduction

Coalgebras of functors on the category of sets have proven effective in modelling various computational systems, including data structures (infinite lists, streams, trees), state-based systems (automata, labelled transition systems, process algebras) and classes in object-oriented programming languages [1,2,3,4,5,6]. Consequently, the study of coalgebras has developed as a distinctive theme in the theory of computing over the last decade.

One significant notion is that of a *behavioural covariety*: a class of coalgebras that is closed under coproducts and images of bisimulation relations. A *covariety* is defined by the weaker requirement of closure under coproducts, images of coalgebraic morphisms, and subcoalgebras. Historically, the concept of covariety arose by dualising that of a *variety* as being a class of universal algebras closed under products, subalgebras and homomorphic images. A famous theorem of Garrett Birkhoff [7] characterised varieties as being those classes of algebras that are definable by *equations*.

Behavioural covarieties correspond to computationally significant behaviours. For example, suppose the coalgebras in question are state-transition systems

* The authors acknowledge support from a Logic and Computation programme funded by the New Zealand Institute of Mathematics and its Applications.

D.V. Hung and M. Wirsing (Eds.): ICTAC 2005, LNCS 3722, pp. 288–302, 2005.

having a Hennessy-Milner style logic [8] for specifying their behaviour. Then the class of all models of a logical formula, or set of formulas, will in general be a behavioural covariety.

This paper generalises and extends work of the second author [9] that gave a comonadic characterisation of behavioural covarieties of coalgebras for certain endofunctors $T : \mathbf{Set} \to \mathbf{Set}$ on the category of sets. Under the assumption that the forgetful functor on T-coalgebras has a right adjoint, it was shown that there is a bijective correspondence between behavioural covarieties of T-coalgebras and certain subcomonads of the comonad $\mathbb{G}^T$ induced on **Set** by this adjunction. The subcomonads corresponding to behavioural covarieties were identified by the requirement that the natural transformations on which they are based be *cartesian*, i.e. all their naturality squares are pullbacks.

Here we replace **Set** by an abstract category **C** and seek to analyse the conditions on **C** that are needed for this bijective correspondence to obtain. Moreover we work with classes of $\mathbb{G}$-coalgebras for an arbitrary comonad $\mathbb{G}$ on **C**, rather than classes of T-coalgebras for an endofunctor $T : \mathbf{C} \to \mathbf{C}$. This covers the work of [9] as a special case, since if $\mathbb{G}^T$ is the comonad induced on **Set** as above, then the category of T-coalgebras is isomorphic to the category of $\mathbb{G}^T$-coalgebras.

Furthermore we go beyond the analysis of [9] to give a comonadic characterisation of covarieties themselves. For this we must confront the fact that there has been more than one notion of "covariety" developed in work that dualises Birkhoff's theorem, depending on how "subcoalgebra" is interpreted. Over **Set**, a subcoalgebra is given by an inclusion function that is a coalgebraic morphism. But abstracting this to the concept of subobject, i.e. monomorphism, allows pathological cases, since there can be monomorphisms that are not injective. In fact over **Set** a T-morphism is injective iff it is a *regular* mono, i.e. an equaliser in the category of T-coalgebras [10, 3.4] and experience has shown that it is this concept of *regular subobject* that provides the most suitable notion of subcoalgebra in the abstract.

Awodey and Hughes [11,12,13] showed, in a suitable setting, that covarieties given by this regular notion of subcoalgebra are precisely those classes defined, in a certain way, by *coequations*. They defined a coequation to be a regular mono whose codomain is injective for regular monos. This dualised the analysis of Banaschewski and Herrlich [14], who observed that equations for classical algebras can be identified with regular epis (coequalisers) having a free algebra as domain, and then replaced the free algebras by their more intrinsic property of being an algebra that is projective for regular epis.

Adámek and Porst [15] focused on coequations as regular monos with cofree codomain. Working with an endofunctor T on a suitable category **C**, they showed that classes of T-coalgebras defined by such coequations are precisely those that are closed under coproducts, images of morphisms and *retracts*, where $\mathcal{A}$ is a retract of $\mathcal{B}$ when there is a regular mono $\mathcal{A} \to \mathcal{B}$ whose underlying arrow splits (has a left inverse) in **C**. These closure conditions define what we will call a *virtual* covariety.

It will be shown below that under certain general conditions on a comonad $\mathbb{G}$ on **C**, virtual covarieties of $\mathbb{G}$-coalgebras correspond bijectively to all subcomonads of $\mathbb{G}$, while covarieties (closed under all regular subcoalgebras and not just retracts) correspond bijectively to subcomonads whose underlying transformation is cartesian *for regular monos only.* Behavioural covarieties continue, as in the **Set**-case, to correspond to subcomonads that are fully cartesian-based. We also show in the final Section how these characterisations relate to the work that has been done on coequations.

These ideas are illustrated (in Section 6) by properties of non-deterministic acceptors, represented as coalgebras for a power-object functor on a category that models the starting-state and accepting-state structure of such systems.

2 Coalgebras, Comonads, Covarieties

Given a category **C** with an endofunctor $G : \mathbf{C} \to \mathbf{C}$, a *$G$-coalgebra* $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ consists of an underlying **C**-object A and a **C**-arrow $\alpha_{\mathcal{A}} : A \to GA$ that is sometimes called the *transition (structure)* of the coalgebra. A G-morphism $f : \mathcal{A} \to \mathcal{B} = (B, \alpha_{\mathcal{B}})$ is given by a **C**-arrow $f : A \to B$ that preserves transitions in the sense that $\alpha_{\mathcal{B}} \circ f = Gf \circ \alpha_{\mathcal{A}}$. The G-coalgebras and their morphisms form a category $\mathbf{C}_G$ with a forgetful functor $U_G : \mathbf{C}_G \to \mathbf{C}$ that acts by $U_G\mathcal{A} = A$ on objects and $U_G f = f$ on arrows (so we often write f when we mean $U_G f$). U_G creates and preserves any kind of *colimit* that exists in **C**. Thus if **C** has coproducts, then any set $\{\mathcal{A}_i : i \in I\}$ of G-coalgebras has a coproduct $\Sigma_I \mathcal{A}_i$ in $\mathbf{C}_G$ whose underlying object is the coproduct $\Sigma_I A_i$ in **C**.

We will need to know how U_G treats epi arrows. Since U_G is faithful (injective on hom-sets), it reflects epis, i.e. $U_G f$ epi in **C** implies f epi in $\mathbf{C}_G$. But it seems some condition on **C** is needed for *preservation* of epis. A category is said to *have cokernel pairs* if for each arrow f there is a pair of arrows g, g' making a pushout of f with itself:

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ \bullet & \xrightarrow[g]{} & \bullet \end{array}$$

In particular, f is epi iff it has a cokernel pair with $g = g'$. So if f is a $\mathbf{C}_G$-epi and f has a cokernel pair (g, g') in **C**, then since such **C**-colimits are created by U_G, it follows readily that $g = g'$. This implies

Lemma 1. *If* **C** *has cokernel pairs, then* U_G *preserves epis.* □

A *comonad* $\mathbb{G} = (G, \varepsilon, \delta)$ on **C** consists of a functor $G : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\varepsilon : G \to 1$ and $\delta : G \to GG$ such that the following diagrams commute for each **C**-object A:

$$\begin{array}{ccc} GA & \xrightarrow{\delta_A} & G^2A \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_{GA}} \\ G^2A & \xrightarrow[G\delta_A]{} & G^3A \end{array} \qquad\qquad \begin{array}{ccccc} & & GA & & \\ & {\scriptstyle 1}\swarrow & \downarrow{\scriptstyle \delta_A} & \searrow{\scriptstyle 1} & \\ GA & \xleftarrow[G\varepsilon_A]{} & G^2A & \xrightarrow[\varepsilon_{GA}]{} & GA \end{array} \tag{2.1}$$

A $\mathbb{G}$-*coalgebra* is a G-coalgebra $\mathcal{A}$ for which the following commute:

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathcal{A}}} & GA \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \delta_A} \\ GA & \xrightarrow[G\alpha_{\mathcal{A}}]{} & G^2A \end{array} \qquad\qquad \begin{array}{ccc} A & & \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & \searrow^{1} & \\ GA & \xrightarrow[\varepsilon_A]{} & A \end{array} \tag{2.2}$$

For example, (2.1) implies that $\mathcal{G}A = (GA, \delta_A)$ is a $\mathbb{G}$-coalgebra for any A: it is the *cofree* $\mathbb{G}$-coalgebra over A. The square in (2.2) states that the transition $\alpha_{\mathcal{A}}$ is itself a G-morphism $\alpha_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}A$.

We denote by $\mathbf{C}_{\mathbb{G}}$ the full subcategory of $\mathbf{C}_G$ consisting of the $\mathbb{G}$-coalgebras. Arrows in $\mathbf{C}_{\mathbb{G}}$, i.e. G-morphisms between $\mathbb{G}$-coalgebras, may be referred to as $\mathbb{G}$-*morphisms*. The assignment $A \mapsto \mathcal{G}A$ is the object part of a functor $\mathcal{G} : \mathbf{C} \to \mathbf{C}_{\mathbb{G}}$ that is right adjoint to the forgetful functor $U_{\mathbb{G}} : \mathbf{C}_{\mathbb{G}} \to \mathbf{C}$, which accounts for the cofreeness of $\mathcal{G}A$. The transition-morphisms $\alpha_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}A$ are the components of the unit $1 \to \mathcal{G} \circ U_{\mathbb{G}}$ of this adjunction. Since $U_{\mathbb{G}}$ is a left adjoint it preserves colimits, so altogether a $\mathbb{G}$-morphism is epi in $\mathbf{C}_{\mathbb{G}}$ iff it is epi in $\mathbf{C}$. $U_{\mathbb{G}}$ also creates any colimits that exist in $\mathbf{C}$ [16, dual of Proposition 4.3.1]. This implies that if $\mathbf{C}$ has coproducts, then every set of $\mathbb{G}$-coalgebras has a coproduct in $\mathbf{C}_{\mathbb{G}}$ that is the same as its coproduct in $\mathbf{C}_G$, i.e. $\mathbf{C}_{\mathbb{G}}$ is closed under coproducts in $\mathbf{C}_G$.

Assume from now on that $\mathbb{G}$ is a comonad on a category $\mathbf{C}$ that has cokernel pairs and a coproduct of any set of $\mathbf{C}$-objects. Let $\mathbf{D}$ be any full subcategory of $\mathbf{C}_G$, and K a class of $\mathbf{D}$-objects. K is called a *quasi-covariety in* $\mathbf{D}$ if it is closed under coproducts and under codomains of epis in $\mathbf{D}$. The latter means that for any $\mathbf{D}$-epi $\mathcal{A} \twoheadrightarrow \mathcal{B}$, if $\mathcal{A} \in K$ then $\mathcal{B} \in K$.

A regular mono $m : \mathcal{A} \to \mathcal{B}$ in $\mathbf{D}$ will be called a *subcoalgebra* of $\mathcal{B}$ in $\mathbf{D}$. In that case, $\mathcal{A}$ is a *retract* of $\mathcal{B}$ in $\mathbf{D}$ if Um splits in $\mathbf{C}$, i.e. if there is a $\mathbf{C}$-arrow $g : B \to A$ with $g \circ m = 1_A$. If K is a quasi-covariety in $\mathbf{D}$, then:

- K is a *virtual covariety in* $\mathbf{D}$ if it is closed under retracts in $\mathbf{D}$;
- K is a *covariety in* $\mathbf{D}$ if it is closed under subcoalgebras in $\mathbf{D}$; and
- K is a *behavioural covariety in* $\mathbf{D}$ if it is closed under domains of $\mathbf{D}$-morphisms, i.e. for any $\mathbf{D}$-arrow $\mathcal{A} \to \mathcal{B}$, if $\mathcal{B} \in K$ then $\mathcal{A} \in K$.

It can be shown that a class K is a behavioural covariety in $\mathbf{D}$ iff it is closed under coproducts and under images of bisimulation relations in $\mathbf{D}$ (see [11], [9, 2.1]). The present definition is easier to work with.

Theorem 1. $\mathbf{C}_{\mathbb{G}}$ *is a virtual covariety in* $\mathbf{C}_G$.

Proof. In [9, 5.1] it is shown that if $\mathbf{C} = \mathbf{Set}$, then $\mathbf{C}_{\mathbb{G}}$ is a *covariety* in $\mathbf{C}_G$. Closure of $\mathbf{C}_{\mathbb{G}}$ under coproducts holds as there, as explained above. The proof of closure under codomains of epis depends on a $\mathbf{C}_G$-epi being epi in $\mathbf{C}$, and that is provided here by our Lemma 1. The proof that a $\mathbf{C}_G$-subcoalgebra $m : \mathcal{A} \to \mathcal{B}$ of $\mathcal{B} \in \mathbf{C}_{\mathbb{G}}$ has $\mathcal{A} \in \mathbf{C}_{\mathbb{G}}$ depends on G^2m being mono in $\mathbf{C}$. In $\mathbf{Set}$ we can use the fact that any endofunctor on $\mathbf{Set}$ preserves monos with non-empty domain. Here, if we assume instead that m splits in $\mathbf{C}$, then G^2m will also split in $\mathbf{C}$ and hence be mono. So the proof adapts to show that $\mathbf{C}_{\mathbb{G}}$ is closed under *retracts*. $\square$

3 Coregular Factorisations and Inverse Images

A category has *coregular factorisations* if each arrow f factors as $f = m \circ e$ with m a regular mono and e an epi. Such a factorisation is unique up to a unique isomorphism: if $m' \circ e'$ is a second such factorisation of f, then there is a unique iso arrow $i : \mathrm{dom}\, m \to \mathrm{dom}\, m'$ factoring m through m' and e' through e.

Hughes [12, Section 1.2.3] gives results about the lifting of coregular factorisations from $\mathbf{C}$ to $\mathbf{C}_G$ and their preservation by U_G. Here are the corresponding versions of these results for $\mathbf{C}_{\mathbb{G}}$ and $U_{\mathbb{G}}$.

Theorem 2.

(1) *If* $G : \mathbf{C} \to \mathbf{C}$ *takes regular monos to monos, then* $U_{\mathbb{G}}$ *reflects regular monos.*
(2) *If* $\mathbf{C}$ *has coregular factorisations, and* $G : \mathbf{C} \to \mathbf{C}$ *preserves regular monos, then* $\mathbf{C}_{\mathbb{G}}$ *has coregular factorisations which are preserved and reflected by* $U_{\mathbb{G}}$*. Moreover,* $U_{\mathbb{G}}$ *preserves regular monos.* □

Theorem 3. *If* $\mathbf{C}$ *has coregular factorisations, and* $G : \mathbf{C} \to \mathbf{C}$ *preserves regular monos, then* $\mathbf{C}_{\mathbb{G}}$ *is a covariety in* $\mathbf{C}_G$*.*

Proof. Take a $\mathbf{C}_G$-subcoalgebra $m : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B} \in \mathbf{C}_{\mathbb{G}}$. In the proof of Theorem 1 we noted that $\mathcal{A} \in \mathbf{C}_{\mathbb{G}}$ if $G^2 m$ is a $\mathbf{C}$-mono. By the dual of [12, 1.2.15], U_G preserves regular monos, so m is regular mono in $\mathbf{C}$, hence so is $G^2 m$. Thus $\mathbf{C}_{\mathbb{G}}$ is closed under subcoalgebras. □

Given a regular mono $m : A \to B$ and an arrow $f : C \to B$, an *inverse image* of m with respect to f is a pullback of m along f:

$$\begin{array}{ccc} D & \xrightarrow{f^*} & A \\ {\scriptstyle m^*}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[f]{} & B \end{array}$$

A category *has inverse images* if all such pullbacks exist. A functor H *preserves inverse images* if the H-image of any such pullback is also a pullback and Hm is a regular mono. With the help of Theorem 2(2) we can show:

Theorem 4. *If* $\mathbf{C}$ *has coregular factorisations and inverse images, and* G *preserves regular monos and inverse images, then* $\mathbf{C}_{\mathbb{G}}$ *has inverse images preserved by* $U_{\mathbb{G}}$*.* □

4 Comonad Morphisms

From now on we assume $\mathbf{C}$ has coregular factorisations as well as cokernel pairs and coproducts. A *morphism* from comonad $\mathbb{F} = (F, \varepsilon^F, \delta^F)$ to comonad $\mathbb{G} = (G, \varepsilon^G, \delta^G)$ on $\mathbf{C}$ [17, Section 3.6] is a natural transformation $\sigma : F \to G$ making the diagrams

$$\begin{array}{ccc} FA & \xrightarrow{\sigma_A} & GA \\ & \searrow^{\varepsilon^F_A} & \downarrow{\scriptstyle \varepsilon^G_A} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{\delta^F_A} & FFA \\ {\scriptstyle \sigma_A}\downarrow & & \downarrow{\scriptstyle (\sigma_A)^2} \\ GA & \xrightarrow{\delta^G_A} & GGA \end{array} \tag{4.1}$$

commute for all **C**-objects A, where $(\sigma_A)^2 = G\sigma_A \circ \sigma_{FA} = \sigma_{GA} \circ F\sigma_A$:

$$\begin{array}{ccc} FFA & \xrightarrow{\sigma_{FA}} & GFA \\ {\scriptstyle F\sigma_A}\downarrow & & \downarrow{\scriptstyle G\sigma_A} \\ FGA & \xrightarrow{\sigma_{GA}} & GGA \end{array} \tag{4.2}$$

We will see later that, under some natural assumptions on **C**, any quasi-covariety K in $\mathbf{C}_{\mathbb{G}}$ gives rise to such a comonad morphism to $\mathbb{G}$. This will be constructed from a "coreflection" functor $\mathbf{C}_{\mathbb{G}} \to K$ which takes each $\mathbb{G}$-coalgebra to its largest subcoalgebra that belongs to K.

A comonad morphism $\sigma : \mathbb{F} \to \mathbb{G}$ is *cartesian* if the diagram

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \sigma_A}\downarrow & & \downarrow{\scriptstyle \sigma_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

is a pullback in **C** for any **C**-arrow $f : A \to B$. σ is *cartesian for regular monos* if this square is a pullback whenever f is a regular mono in **C**. σ is *regularly monomorphic* if all of its components σ_A are regular monos in **C**.

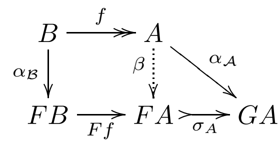
Now any morphism $\sigma : \mathbb{F} \to \mathbb{G}$ induces a mapping $\varphi\sigma$ from $\mathbf{C}_{\mathbb{F}}$ to $\mathbf{C}_{\mathbb{G}}$, taking each $\mathbb{F}$-coalgebra $\mathcal{A}$ to the G-coalgebra $\varphi\sigma\mathcal{A} = (A, \sigma_A \circ \alpha_{\mathcal{A}} : A \to FA \to GA)$ on the same underlying object. We write $\mathrm{Im}\varphi\sigma$ for the full subcategory of $\mathbf{C}_{\mathbb{G}}$ based on the class of all $\mathbb{G}$-coalgebras of the form $\varphi\sigma\mathcal{A}$ for some $\mathbb{F}$-coalgebra $\mathcal{A}$. In [9, Section 6] it is verified that $\varphi\sigma\mathcal{A}$ is a $\mathbb{G}$-coalgebra and that $\varphi\sigma$ becomes a functor $\mathbf{C}_{\mathbb{F}} \to \mathbf{C}_{\mathbb{G}}$ that leaves the underlying **C**-arrow of morphisms unchanged ($U_{\mathbb{G}}\varphi\sigma f = f$), and so is faithful (injective on hom-sets). Furthermore, if σ is regularly monomorphic then $\varphi\sigma$ is also full (surjective on hom-sets) and injective on objects, making $\mathbf{C}_{\mathbb{F}}$ isomorphic to $\mathrm{Im}\varphi\sigma$.

This $\varphi\sigma$ construction is functorial in the sense that $\varphi(\sigma \circ \tau) = \varphi\sigma \circ \varphi\tau$ whenever σ and τ are comonad morphisms whose composition $\sigma \circ \tau$ is defined. The map $\sigma \mapsto \varphi\sigma$ gives a *bijection* between comonad morphisms $\mathbb{F} \to \mathbb{G}$ and those functors $\mathbf{C}_{\mathbb{F}} \to \mathbf{C}_{\mathbb{G}}$ that commute with the forgetful functors to **C** [17, Section 3.6].

Theorem 5. *If $\sigma : \mathbb{F} \to \mathbb{G}$ is regularly monomorphic and G preserves regular monos, then:*

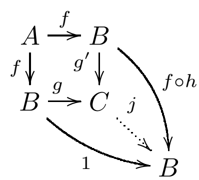
(1) $\mathrm{Im}\varphi\sigma$ *is a virtual covariety in* $\mathbf{C}_{\mathbb{G}}$.
(2) *If σ is cartesian for regular monos, then* $\mathrm{Im}\varphi\sigma$ *is a covariety in* $\mathbf{C}_{\mathbb{G}}$.
(3) *If σ is cartesian, then* $\mathrm{Im}\varphi\sigma$ *is a behavioural covariety in* $\mathbf{C}_{\mathbb{G}}$.

Proof. (1) In [9, 6.3] it is shown that $\mathrm{Im}\varphi\sigma$ is a quasi-variety when $\mathbb{G}$ is any comonad on **Set**. Closure of $\mathrm{Im}\varphi\sigma$ under coproducts continues to hold here, as $\Sigma_I \varphi\sigma\mathcal{A}_i = \varphi\sigma\Sigma_I\mathcal{A}_i$. The proof that any $\mathbf{C}_{\mathbb{G}}$-epi $f : \varphi\sigma\mathcal{B} \to \mathcal{A}$ has $\mathcal{A} \in \mathrm{Im}\varphi\sigma$ requires that f is epi in $\mathbf{C}$, which holds as $U_{\mathbb{G}}$ preserves epis, and that $G\sigma_A$ is mono, which holds now by our assumptions on σ and G. It also requires that there is a $\mathbf{C}$-arrow β making a commuting diagram

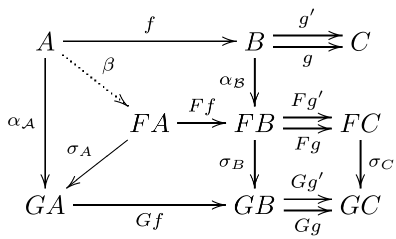


This holds by diagonalization in $\mathbf{C}$ because f is epi and σ_A is regular mono. It can then be shown that (A, β) is an $\mathbb{F}$-coalgebra with $\mathcal{A} = \varphi\sigma(A, \beta) \in \mathrm{Im}\varphi\sigma$, by the argument of [9] and using that $G\sigma_A$ is mono.

Now for something new. Suppose $\mathcal{A}$ is a *retract* of $\varphi\sigma\mathcal{B}$, with a regular mono $f : \mathcal{A} \to \varphi\sigma\mathcal{B}$ that has a left inverse $h : B \to A$ in $\mathbf{C}$. Let $g, g' : \mathcal{B} \to \mathcal{C}$ be a cokernel pair for f in both $\mathbf{C}_{\mathbb{G}}$ and $\mathbf{C}$. By the universal property of pushouts, g has a left inverse j in $\mathbf{C}$ with $j \circ g' = f \circ h$:

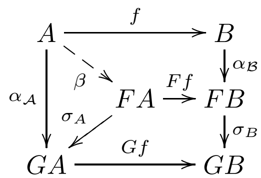


This means that $A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C$ is (the dual of) a *split fork* and so is an absolute equaliser: any functor H on $\mathbf{C}$ makes Hf an equaliser of Hg and Hg' [18, VI.6]. Consider the diagram



From some diagram chasing and the fact that σ_C is mono, we get $Fg \circ (\alpha_{\mathcal{B}} \circ f) = Fg' \circ (\alpha_{\mathcal{B}} \circ f)$, and hence, because Ff equalises Fg and Fg', there is a unique $\beta : A \to FA$ as shown making $\alpha_{\mathcal{B}} \circ f = Ff \circ \alpha$. Then $Gf \circ \alpha_{\mathcal{A}} = Gf \circ \sigma_A \circ \beta$, so as G preserves the regular mono f, $\alpha_{\mathcal{A}} = \sigma_A \circ \beta$ and $\mathcal{A} = \varphi\sigma(A, \beta)$. It can be checked that (A, β) is an $\mathbb{F}$-coalgebra, and we conclude that $\mathrm{Im}\varphi\sigma$ is closed under retracts.

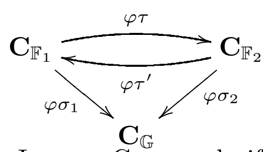
(2) Given a regular $\mathbf{C}_{\mathbb{G}}$-mono $f : \mathcal{A} \to \varphi\sigma\mathcal{B}$, f is a regular $\mathbf{C}$-mono by Theorem 2(2), so the lower quadrangle of the diagram



is a pullback as σ is cartesian for regular monos. Hence the arrow β exists as shown to make the whole diagram commute. It can be checked that (A, β) is an $\mathbb{F}$-coalgebra. Then $\mathcal{A} = \varphi\sigma(A, \beta)$, so we conclude that Im$\varphi\sigma$ is closed under subcoalgebras in $\mathbf{C}_{\mathbb{G}}$.

(3) If σ is cartesian, then the lower quadrangle of the diagram in (2) is a pullback for any $\mathbf{C}_{\mathbb{G}}$-morphism $f : \mathcal{A} \to \varphi\sigma\mathcal{B}$, so we get that Im$\varphi\sigma$ is closed under domains of all such morphisms. □

There is an equivalence $\sigma_1 \simeq \sigma_2$ between regularly monomorphic $\sigma_i : \mathbb{F}_i \to \mathbb{G}$ that holds when there exists a morphism $\tau : \mathbb{F}_1 \to \mathbb{F}_2$ that factors σ_1 through σ_2, i.e. $\sigma_1 = \sigma_2 \circ \tau$, and likewise a morphism $\tau' : \mathbb{F}_2 \to \mathbb{F}_1$ factoring σ_2 through σ_1. If such τ, τ' exist then they are unique, because the components of the σ_i are mono, and are mutually inverse, giving a natural isomorphism between the underlying functors of $\mathbb{F}_1$ and $\mathbb{F}_2$. The functoriality of the $\varphi\sigma$ construction then gives a commuting functor diagram



which implies that Im$\varphi\sigma_1$ = Im$\varphi\sigma_2$. Conversely if Im$\varphi\sigma_1$ = Im$\varphi\sigma_2$, then from $\varphi\sigma_i : \mathbf{C}_{\mathbb{F}_i} \cong \text{Im}\varphi\sigma_i$ we get isomorphisms between $\mathbf{C}_{\mathbb{F}_1}$ and $\mathbf{C}_{\mathbb{F}_2}$ that are of the form $\varphi\tau$ and $\varphi\tau'$ for some τ, τ' that establish $\sigma_1 \simeq \sigma_2$.

Now two equivalent regularly monomorphic $\sigma_i : \mathbb{F}_i \to \mathbb{G}$ can be regarded as representing the same *subcomonad* of $\mathbb{G}$, so in this sense the map $\sigma \mapsto \text{Im}\varphi\sigma$ injectively assigns virtual covarieties in $\mathbf{C}_{\mathbb{G}}$ to regularly monomorphic comonad morphisms with codomain $\mathbb{G}$. In the next sections we will identify further conditions on $\mathbf{C}$ and $\mathbb{G}$ that ensure this map is surjective and gives a bijective correspondence between virtual covarieties in $\mathbf{C}_{\mathbb{G}}$ and subcomonads of $\mathbb{G}$. We will also show that it restricts to give a bijection between covarieties and subcomonads whose morphism σ is cartesian for regular monos, as well as a bijection between behavioural covarieties and subcomonads with fully cartesian σ.

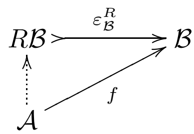
5 Coreflective Subcategories

We assume from now on that

- $\mathbf{C}$ has coregular factorisations, cokernel pairs, coproducts and inverse images.
- $\mathbf{C}$ is *regularly well-powered*, i.e. for each object A there is a *set* of representatives of all the isomorphism classes of regular subobjects of A.
- $\mathbb{G} = (G, \varepsilon, \delta)$ is a comonad on $\mathbf{C}$ with G preserving regular monos and inverse images.

Recall that the adjunction $U_{\mathbb{G}} \dashv \mathcal{G} : \mathbf{C}_{\mathbb{G}} \to \mathbf{C}$ has counit ε, and unit η with components $\eta_{\mathcal{A}} = \mathcal{A} \xrightarrow{\alpha_{\mathcal{A}}} \mathcal{G}A$ for all $\mathbb{G}$-coalgebras $\mathcal{A}$.

Let K be a *quasi-covariety* of $\mathbb{G}$-coalgebras, regarded as a full subcategory of $\mathbf{C}_{\mathbb{G}}$. Then K is a *regular-mono-coreflective subcategory of* $\mathbf{C}_{\mathbb{G}}$, which means that the inclusion functor $I : K \hookrightarrow \mathbf{C}_{\mathbb{G}}$ has a right adjoint (coreflector) $R : \mathbf{C}_{\mathbb{G}} \to K$ whose counit ε^R has regular mono components (coreflections) $\varepsilon^R_{\mathcal{B}} : R\mathcal{B} \rightarrowtail \mathcal{B}$ for all $\mathbb{G}$-coalgebras $\mathcal{B}$. This is a well-known result (essentially the dual of [19, 37.1]). Briefly, R is constructed by taking a set $\{\mathcal{A}_j \xrightarrow{m_j} \mathcal{B} : j \in J\}$ representing all the subcoalgebras of $\mathcal{B}$ with domain $\mathcal{A}_j$ in K, and taking $\varepsilon^R_{\mathcal{B}}$ to be the regular-mono part of the coregular factorisation of the coproduct arrow $\Sigma_J \mathcal{A}_j \xrightarrow{\Sigma_J m_j} \mathcal{B}$. The important point for us is that if $\mathcal{A} \in K$, then any $\mathbb{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ factors uniquely through $\varepsilon^R_{\mathcal{B}}$:



In particular, $\varepsilon^R_{\mathcal{B}}$ is the *largest* subcoalgebra of $\mathcal{B}$ with domain in K.

By composing the adjunctions $K \underset{I}{\overset{R}{\rightleftarrows}} \mathbf{C}_{\mathbb{G}} \underset{U_{\mathbb{G}}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{C}$ we obtain the functor $\mathcal{G}^K = R \circ \mathcal{G} : \mathbf{C} \to K$ right adjoint to the forgetful functor $U_K = U_{\mathbb{G}} \circ I : K \to \mathbf{C}$. For each $\mathbf{C}$-object A, let σ^K_A be the coreflection morphism $\varepsilon^R_{\mathcal{G}A}$. This associates with K a natural transformation $\sigma^K : \mathcal{G}^K \to \mathcal{G}$, with the regular-mono component σ^K_A giving the largest K-subobject of $\mathcal{G}A$.

Now let $\mathbb{G}^K = (G^K, \varepsilon^K, \delta^K)$ be the comonad on $\mathbf{C}$ induced by the adjunction $U_K \dashv \mathcal{G}^K$. Thus $G^K = U_K \circ \mathcal{G}^K : \mathbf{C} \to \mathbf{C}$, and ε^K is the counit of this adjunction. We write η^K for its unit, with components $\eta^K_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^K A$. Applying the forgetful functor to the components of σ^K defines a natural transformation $G^K \to G$ which we will also denote by σ^K. Standard calculations for composition of adjunctions [18, IV.8] give the formulas

$$\varepsilon_A \circ \sigma^K_A = \varepsilon^K_A \tag{5.1}$$

$$\sigma^K_A \circ \eta^K_{\mathcal{A}} = \eta_{\mathcal{A}} = \alpha_{\mathcal{A}} \tag{5.2}$$

The transformation δ^K has components $\delta^K_A = \eta^K_{\mathcal{G}^K A} : G^K A \to G^K G^K A$, which is a $\mathbb{G}$-morphism $\mathcal{G}^K A \to \mathcal{G}^K G^K A$ for each A, the unique one factoring the identity on $G^K A$ through $\varepsilon^K_{G^K A}$, so

$$\varepsilon^K_{G^K A} \circ \delta^K_A = 1_{G^K A}, \tag{5.3}$$

and thus by (5.2),

$$\sigma^K_{G^K A} \circ \delta^K_A = \alpha_{\mathcal{G}^K A}. \tag{5.4}$$

By the reasoning of [9, 6.1], $\sigma^K : G^K \to G$ is a (regularly monomorphic) comonad morphism from $\mathbb{G}^K$ to $\mathbb{G}$. Hence, by the work of Section 4, σ^K induces a functor

$\varphi\sigma^K : \mathbf{C}_{\mathbb{G}^K} \to \mathbf{C}_{\mathbb{G}}$ making $\mathbf{C}_{\mathbb{G}^K}$ isomorphic to $\mathrm{Im}\varphi\sigma^K$. The theory of comonads also provides the quasi-covariety K with a *comparison functor* $\chi^K : K \to \mathbf{C}_{\mathbb{G}^K}$ that acts on objects by $\chi^K \mathcal{A} = (A, \eta^K_{\mathcal{A}})$, and leaves the underlying $\mathbf{C}$-arrow of morphisms unchanged.

Theorem 6. *If K is a virtual covariety in $\mathbf{C}_{\mathbb{G}}$, then χ^K is an isomorphism of categories with inverse $\varphi\sigma^K$, and so $K = \mathrm{Im}\varphi\sigma^K$:*

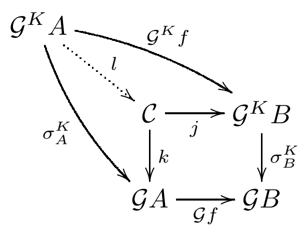
$$\mathbf{C}_{\mathbb{G}^K} \underset{\varphi\sigma^K}{\overset{\chi^K}{\rightleftarrows}} K \hookrightarrow \mathbf{C}_{\mathbb{G}}$$

□

By this Theorem, the correspondence $\sigma \mapsto \mathrm{Im}\varphi\sigma$ from subcomonads of $\mathbb{G}$ to virtual covarieties in $\mathbf{C}_{\mathbb{G}}$ is surjective: every virtual covariety is of the form $\mathrm{Im}\varphi\sigma$. Together with our earlier work it follows that the correspondence is bijective. It remains to show that it also gives a bijection between covarieties and subcomonads whose morphism σ is cartesian for regular monos, as well as a bijection between behavioural covarieties and subcomonads with cartesian σ. This is provided by the following results together with parts 2 and 3 of Theorem 5.

Theorem 7. *If K is a covariety, then σ^K is cartesian for regular monos. If K is a behavioural covariety, then σ^K is cartesian.*

Proof. Given a regular mono $\mathbf{C}$-arrow $f : A \to B$, consider the diagram



The inner square is a pullback giving an inverse image k of σ^K_B along $\mathcal{G}f$ in $\mathbf{C}_{\mathbb{G}}$. This exists by Theorem 4. k is a subcoalgebra of $\mathcal{G}A$, being a pullback of a regular mono. The outer perimeter of the diagram commutes by naturality of σ^K, so a unique $\mathbb{G}$-morphism l exists as shown to make the subcoalgebra σ^K_A factor through k. We will show that l is an isomorphism.

Now $\mathcal{G}$ preserves limits, being a right adjoint, so $\mathcal{G}f$ is a regular mono, hence its pullback j is a subcoalgebra of $\mathcal{G}^K B \in K$. As K is a covariety, the domain $\mathcal{C}$ of the $\mathcal{G}A$-subcoalgebra k belongs to K. But σ^K_A is the largest such subcoalgebra of $\mathcal{G}A$, so k in turn factors through σ^K_A, making these two subcoalgebras equivalent. Hence l is an iso, and therefore the perimeter is also a pullback, making σ^K_A an inverse image of σ^K_B along $\mathcal{G}f$. But $U_{\mathbb{G}}$ preserves inverse images (Theorem 4) so

$$\begin{array}{ccc} G^K A & \xrightarrow{G^K f} & G^K B \\ {\scriptstyle \sigma^K A}\downarrow & & \downarrow{\scriptstyle \sigma^K B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

is a pullback in $\mathbf{C}$, proving σ^K is cartesian for regular monos.

Finally, if K is a behavioural covariety, then the domain $\mathcal{C}$ of j will be in K for any $\mathbb{G}$-morphism j to to $\mathcal{G}^K B$, regardless of whether $\mathcal{G}f$ is regular mono, so the last diagram will be a pullback for every **C**-arrow f, i.e. σ^K is cartesian. □

6 Acceptors

To illustrate some of these ideas we define an *acceptor* space to be a triple $\mathbb{A} = (A, A^{\mathsf{st}}, A^{\mathsf{ac}})$ consisting of two distinguished subsets $A^{\mathsf{st}}, A^{\mathsf{ac}}$ of a set A which is itself thought of as a set of *states*. A^{st} comprises the *starting* states and A^{ac} the *accepting* states. An *acceptor space morphism* $f : \mathbb{A} \to \mathbb{B} = (B, B^{\mathsf{st}}, B^{\mathsf{ac}})$ is a function $f : A \to B$ preserving the subsets, i.e. $f(A^{\mathsf{x}}) \subseteq B^{\mathsf{x}}$ for $\mathsf{x} = \mathsf{st}, \mathsf{ac}$. This defines a category **Asp** of acceptor spaces, in which monos are injective and the *regular* monos are those for which $f(A)^{\mathsf{x}} = f(A) \cap B^{\mathsf{x}}$. This category is complete and cocomplete, and has all the properties required of the category **C** at the start of Section 5. Actually **Asp** is a quasi-topos (see [20, 31.7] for a description of a quasi-topos of sets with a single distinguished subset).

A nondeterministic acceptor within input set I has an I-labelled state-transition relation $x \overset{i}{\mapsto} y$, meaning that y is a possible next state on input of $i \in I$ to state x. Letting $\alpha(x)$ be the map assigning $\{y : x \overset{i}{\mapsto} y\}$ to each $i \in I$ gives a function $\alpha : A \to (\mathcal{P}A)^I$, where $\mathcal{P}A$ is the powerset of A. *Finitely branching* nondeterminism can be modelled by using the finitary powerset operation $\mathcal{P}_\omega$, where $\mathcal{P}_\omega A$ is the set of all finite subsets of A. This determines a functor $\mathcal{P}_\omega$: **Asp** $\to$ **Asp** that has $\mathcal{P}_\omega \mathbb{A} = (\mathcal{P}_\omega A, \mathcal{P}_\omega A, \mathcal{P}_\omega A)$ and takes each morphism $f : \mathbb{A} \to \mathbb{B}$ to the function $\mathcal{P}_\omega f$ assigning to each $X \subseteq A$ its image $fX \subseteq B$. A functor $T = (\mathcal{P}_\omega -)^I$: **Asp** $\to$ **Asp** is then defined on objects by $T\mathbb{A} = (\mathcal{P}_\omega \mathbb{A})^I = ((\mathcal{P}_\omega A)^I, (\mathcal{P}_\omega A)^I, (\mathcal{P}_\omega A)^I)$ and on arrows by $Tf(g) = (\mathcal{P}_\omega f) \circ g$.

We write **Finac**$_I$ for the category of $(\mathcal{P}_\omega -)^I$-coalgebras over **Asp**. Its objects $\mathcal{A} = (\mathbb{A}, \alpha_{\mathcal{A}})$ can be identified with the nondeterministic acceptors with input set I that are *image-finite*, i.e. the set $\{y : x \overset{i}{\mapsto} y\}$ of possible next states is finite for all pairs (x, i). Its arrows $f : \mathcal{A} \to \mathcal{B}$ can be characterised as those acceptor space morphisms $f : \mathbb{A} \to \mathbb{B}$ for which $f(x) \overset{i}{\mapsto} z$ iff $\exists y \in A(x \overset{i}{\mapsto} y$ and $f(y) = z)$.

It can be shown that the forgetful functor on **Finac**$_I$ has a right adjoint, and that **Finac**$_I$ is isomorphic to the category of $\mathbb{G}$-coalgebras for the associated adjunction. So there is an exact correspondence between (quasi/virtual/behavioural) covarieties in **Finac**$_I$ and the same kinds of subcategory of the category of $\mathbb{G}$-coalgebras. $(\mathcal{P}_\omega -)^I$ preserves regular monos, so the forgetful functor on **Finac**$_I$ preserves and reflects regular monos. $(\mathcal{P}_\omega -)^I$ preserves inverse images, and so **Finac**$_I$ has inverse images preserved by this forgetful functor.

There are many properties of acceptors that define covarieties in **Finac**$_I$, such as the following:

- Every state x is recurrent, in the sense that there is a transition path $x \overset{i}{\mapsto} y \overset{i'}{\mapsto} \cdots \mapsto x$ returning to x.
- There are no deadlocked states, where x is deadlocked if there is no transition $x \overset{i}{\mapsto} y$ starting from x.

– Every non-deadlocked state can reach a deadlocked one in finitely many transitions.
– Every transition path $x \overset{i}{\mapsto} y \overset{i'}{\mapsto} y \mapsto \cdots$ is finite (i.e. eventually reaches a deadlocked state).

The last three in fact define behavioural covarieties. Virtual covarieties can be defined by considering existential properties of starting and accepting states. For instance, let K be the class $\{\mathcal{A} : A^{\mathsf{ac}} \neq \emptyset\}$ of all coalgebras having at least one accepting state. This property is evidently preserved by coproducts (disjoint unions) and codomains of epis (indeed by codomains of all morphisms). It is also preserved by retracts, for if the underlying **Asp**-arrow of some $\mathbf{Finac}_I$-arrow $\mathcal{B} \to \mathcal{A}$ has a left inverse in **Asp**, and $\mathcal{A}$ has an accepting state, then this state will be preserved by the left-inverse, i.e. carried to an accepting state in $\mathcal{B}$. However K is not a covariety: let $\mathcal{A} \in K$ be any one-state acceptor whose one state is both starting and accepting, while $\mathcal{B}$ is the empty acceptor. Then the inclusion function is a regular mono $\mathcal{B} \to \mathcal{A}$ giving a subcoalgebra of $\mathcal{A}$ with $\mathcal{B} \notin K$.

Some other properties defining virtual covarieties that are not closed under subcoalgebras are:

– There is a state that is both starting and accepting.
– There exist starting states and all of them can reach an accepting state.
– There exist accepting states and they include all the deadlocked states.

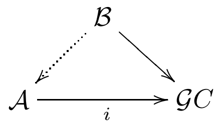
Further examples can be given by various combinations of the properties listed, e.g. “there exists an accepting state but no deadlocked states”, and so on.

7 Coequations

Birkhoff’s theorem [7] states that the *varieties* of universal algebras are precisely those classes that are definable by *equations*. This has been dualised to a notion of *coequation* giving characterisations of covarieties of coalgebras. In this final Section we briefly review this theory and indicate how it relates to our work with cartesian comonad morphisms.

An algebraic equation $t_1 = t_2$ with variables from a set X is given by a pair of terms that can be taken as elements of the free algebra $\mathbf{F}X$ generated by X. If E is the smallest congruence on $\mathbf{F}X$ containing (t_1, t_2), then the quotient map $e : \mathbf{F}X \twoheadrightarrow \mathbf{F}X/E$ is a regular epi (coequaliser) identifying t_1 and t_2. Moreover, an algebra $\mathcal{A}$ satisfies $t_1 = t_2$ iff every homomorphism $\mathbf{F}X \to \mathcal{A}$ factors as e followed by a homomorphism $\mathbf{F}X/E \to \mathcal{A}$, a property that is expressed by saying that $\mathcal{A}$ is *injective for* e. Abstracting, the notion of an equation became in [14] the notion of a regular epi with free domain, and the class defined by an equation became the class of algebras injective for it.

Dually, a *coequation* associated with a comonad $\mathbb{G}$ can be defined as a regular mono $i : \mathcal{A} \rightarrowtail \mathcal{G}C$ in $\mathbf{C}_{\mathbb{G}}$ with cofree codomain. Another regular mono $i' : \mathcal{A}' \to \mathcal{G}C$ will be taken to represent the same coequation as i if it represents the same subobject of $\mathcal{G}C$. A coalgebra $\mathcal{B}$ is *projective for* coequation i if every morphism $\mathcal{B} \to \mathcal{G}C$ factors through i:



The class $(i)_\perp$ of objects projective for i is invariably a *virtual* covariety [15, 6.16]. If i and i' represent the same coequation, then $(i)_\perp = (i')_\perp$. If i is a *coequation over* 1 (i.e. $C = 1$ is terminal in $\mathbf{C}$), then $(i)_\perp$ is a *behavioural* covariety. This is because $\mathcal{G}1$ is terminal in $\mathbf{C}_\mathbb{G}$, so if $\mathcal{B} \in (i)_\perp$, then the unique morphism $\mathcal{B} \to \mathcal{G}1$ is factored through i by some $g : \mathcal{B} \to \mathcal{A}$, so given any $\mathbb{G}$-morphism $f : \mathcal{B}' \to \mathcal{B}$, the unique $\mathcal{B}' \to \mathcal{G}1$ is factored through i by $g \circ f$. Hence $\mathcal{B}' \in (i)_\perp$, and $(i)_\perp$ is closed under domains of $\mathbb{G}$-morphisms.

Now it was shown in [11, 4.2] that if $\mathbf{C}$ has a terminal 1, then any behavioural covariety K in $\mathbf{C}_\mathbb{G}$ is equal to $(\varepsilon^R_{\mathcal{G}1})_\perp$, where ε^R is the co-unit of the coreflector $R : \mathbf{C}_\mathbb{G} \to K$ described in Section 5. But by definition $\varepsilon^R_{\mathcal{G}1} = \sigma^K_1$, where σ^K is the comonad morphism to $\mathbb{G}$ determined by K. The maps $i \mapsto (i)_\perp$ and $K \mapsto \sigma^K_1$ provide a bijection between (equivalence classes of) coequations over 1 in $\mathbf{C}_\mathbb{G}$ and behavioural covarieties of $\mathbb{G}$-coalgebras. In fact for any regularly monomorphic cartesian $\sigma : \mathbb{F} \to \mathbb{G}$, there is a $\mathbb{G}$-coalgebra $(F1, \alpha)$ based on $F1$ such that σ_1 is a coequation $(F1, \alpha) \to \mathcal{G}1$. This can be shown from the fact that $\sigma_1 \simeq \sigma^K_1$ where $K = \mathrm{Im}\varphi\sigma$, and the fact that the components of σ^K are $\mathbb{G}$-morphisms. We thus have the picture of correspondences shown in Figure 1. Any virtual covariety K of $\mathbb{G}$-coalgebras is *coequational* in the sense that there is a class $\mathcal{E}$ of coequations $\mathcal{A} \rightarrowtail \mathcal{G}C$ such that $K = \mathcal{E}_\perp$ = the class of all coalgebras that are projective for every member of $\mathcal{E}$ [15, 6.16]. In fact $\mathcal{E}$ can be taken to be the class of all coreflection morphisms $\varepsilon^R_\mathcal{B}$ with $\mathcal{B}$ a cofree coalgebra $\mathcal{G}C$. Since $\varepsilon^R_{\mathcal{G}C} = \sigma^K_C$, we see that the class of coequations defining a virtual covariety K is just the class

$$\{\sigma^K_C : C \text{ is any } \mathbf{C}\text{-object}\} \tag{7.1}$$

of all components of the comonad morphism σ^K.

Now as we saw in Section 6, there are coequational classes (virtual covarieties) that are not covarieties, so to obtain a coequational characterisation of covarieties

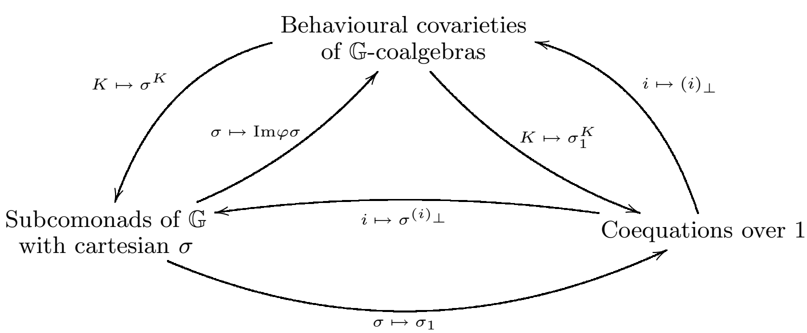


Fig. 1.

themselves we need some refinement of the notion of coequation. For equations, such a refinement was given in [14] by replacing free domains by domains that are *regular-projective*, i.e. projective for every regular epi, this being a property enjoyed by classical free universal algebras. Dually we contemplate coequations as regular monos i whose codomain is *regular-injective*, i.e. injective for every regular mono. Then the class $(i)_\perp$ will be an abstract covariety, i.e. closed under coproducts, codomains of epis and regular subobjects.

For the converse of this to work it is required that the ambient category *has enough injectives*, i.e. each object is a regular subobject of some regular-injective object. Then it can be shown that each abstract covariety is equal to $\mathcal{E}_\perp$ where $\mathcal{E}$ is some class of regular monos with regular-injective codomains [11,12,13,21]. For categories of coalgebras, if $\mathbf{C}$ has enough injectives, then so does $\mathbf{C}_\mathbb{G}$: in fact $\mathcal{G}C$ is regular-injective in $\mathbf{C}_\mathbb{G}$ whenever C is regular-injective in $\mathbf{C}$, and from this it can be shown that $\mathbf{C}_\mathbb{G}$ has enough regular-injectives that are *cofree*. Then each covariety K of $\mathbb{G}$-coalgebras is the coequational class $(\mathcal{E}^K)_\perp$ for some class $\mathcal{E}^K$ of regular monos with cofree regular-injective codomains [11,13]. Indeed, in our present terminology, we can take $\mathcal{E}^K = \{\sigma_C^K : C \text{ is regular-injective in } \mathbf{C}\}$, giving a direct comparison via (7.1) with the case that K is a *virtual* covariety.

Note that in the presence of enough injectives there can still be coequational classes of coalgebras that are not covarieties. For example, this happens when $\mathbf{C}$ is the category **Asp** of acceptor spaces of Section 6. The regular-injective objects of **Asp** are just those acceptor spaces that have at least one starting state that is also accepting ($A^{\mathsf{st}} \cap A^{\mathsf{ac}} \neq \emptyset$). We can always expand a space by adding such a state if there is none, which implies that **Asp** has enough injectives.

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