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Commutativity of Quantifiers in Varying-Domain Kripke Models

Abstract. A possible-worlds semantics is defined that validates the main axioms of Kripke's original system for first-order modal logic over varying-domain structures. The novelty of this semantics is that it does not validate the commutative quantification schema $\forall x \forall y \varphi \to \forall y \forall x \varphi$, as we show by constructing a counter-model.

Keywords: possible-worlds semantics, commutative quantification, premodel, model, Kripkean model.

Introduction and Overview

Kripke's model theory for first-order modal logic [4] assigns to each world w a set Dw thought of as the domain of individuals that exist in w. The quantifier $\forall x$ is interpreted at a world as meaning "for all existing x". This semantics does not validate the Universal Instantiation schema

UI $\forall x \varphi \to \varphi(y/x)$, where y is free for x in φ,[1]

because the value of variable y may not exist in a particular world. It does however validate the variant

UI° $\forall y(\forall x \varphi \to \varphi(y/x))$, where y is free for x in φ,

along with the schemata

UD $\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$,

VQ $\varphi \to \forall x \varphi$, where x is not free in φ,

of Universal Distribution, and Vacuous Quantification, as well as being sound for the Universal Generalisation rule

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[1] $\varphi(\tau/x)$ is the formula obtained by uniform substitution of term τ in place of free x in φ; the side condition is the usual proviso that no variable of τ becomes bound in $\varphi(\tau/x)$ as a result.

UG from φ infer $\forall x\varphi$.

In addition this semantics validates the schema

CQ $\forall x\forall y\varphi \to \forall y\forall x\varphi$

of Commutative Quantification, which was shown by Fine [1] not to be derivable from UI°, UD and VQ by using UG and valid Boolean reasoning. Fine's method involves a syntactic transformation that provides a nonstandard definition of the substitution of a constant for free x in φ. Recently Grant Reaber (personal communication) has noted that reading "$\forall x$" as "for cofinitely many values of x" in the structure $(\omega, <)$ gives an interpretation that falsifies CQ while verifying UI°, UD, VQ and UG.

These observations raise the question of whether there is some plausible, "possible-worlds style", structural model theory for systems that have the axioms UI°, UD and VQ, but perhaps not CQ.[2] In this paper such a semantics is presented, and a model constructed that falsifies CQ while validating the other three quantificational axioms, along with the axioms for any specified normal propositional modal logic. The approach has been used previously in [6] and [3] to give a complete semantics for the quantified relevant logic RQ and for a range of first-order modal logics that are incomplete for their standard possible-worlds models.

There are two basic ideas involved. The first, already long exploited in propositional modal logic, is that not every set of worlds need count as a proposition. Instead we take a collection *Prop* of sets of worlds, the *admissible propositions*, that forms a Boolean set algebra closed also under the operation that interprets the modality $\Box$. The "truth value" of any formula must then be a member of *Prop*.

The second notion has long been exploited in algebraic logic: the universal quantifier $\forall x$ is interpreted as a *greatest lower bound* in the lattice of propositions, this being the natural interpretation of arbitrary *conjunctions.* To illustrate this, suppose we have the set W of worlds, and a universe U of individuals that serves as the range of the quantifier $\forall x$. If φ is a formula in which x is the only free variable, let $\varphi(a)$ be the result of replacing free x in φ by the individual a, viewed as a constant. Let $|\forall x\varphi|$ and $|\varphi(a)|$ be the sets of worlds (subsets of W) at which these sentences are true, respectively.

[2] The axiomatisation of [4] took as axioms the *closures* of all instances of UI°, UD, VQ, tautologies and appropriate modal schemata, with detachment for material implication as the only inference rule. UG and Necessitation (from φ infer $\Box\varphi$) are then derivable rules. Here a closure of φ is any sentence obtained by prefixing universal quantifiers and copies of $\Box$ to φ in any order.

Intuitively, $\forall x\varphi$ is semantically equivalent to the conjunction of the $\varphi(a)$'s for all $a \in U$. So

$$|\forall x\varphi| = \bigcap_{a\in U} |\varphi(a)|,$$

where $\bigcap$ is set-theoretic intersection. This makes $|\forall x\varphi|$ the greatest lower bound of the $|\varphi(a)|$'s in the lattice of *all* subsets of W, i.e. the largest/weakest proposition that implies all of the propositions $|\varphi(a)|$. But if we are constrained to use the set *Prop* of *admissible* propositions, which may not be the full powerset $\wp W$ of W, then instead we should take

$$|\forall x\varphi| = \bigsqcap_{a\in U} |\varphi(a)|,$$

where $\bigsqcap$ is the greatest lower bound operation in the ordered set $(\mathit{Prop}, \subseteq)$. The definition of "model" should require that $\bigsqcap_{a\in U} |\varphi(a)|$ always exists in *Prop*. It will be the weakest *admissible* proposition that implies all of the $|\varphi(a)|$'s. *But it may not be equal to* $\bigcap_{a\in U} |\varphi(a)|$!

This interpretation, as developed in [3], has the quantifiers ranging over a fixed domain of possible individuals. But here we have the varying domains $Dw \subseteq U$ of existing individuals, with $\forall x\varphi$ being equivalent to the conjunction of the assertions "if a exists then $\varphi(a)$" for all $a \in U$. To formalise this, let $Ea = \{w \in W : a \in Dw\}$, so that Ea represents the proposition "a exists". Then we want

$$|\forall x\varphi| = \bigsqcap_{a\in U} Ea \Rightarrow |\varphi(a)|, \tag{0.1}$$

where $\Rightarrow$ is the Boolean set implication operation: $X \Rightarrow Y = (W \setminus X) \cup Y$. When $\bigsqcap = \bigcap$, equation (0.1) reproduces the Kripkean semantics of [4] for the quantifier $\forall x$.

In working with greatest lower bounds we put

$$\bigsqcap S = \bigcup\{X \in \mathit{Prop} : X \subseteq \bigcap S\},$$

so that $\bigsqcap S$ is defined for an arbitrary $S \subseteq \wp W$. When $S \subseteq \mathit{Prop}$ and $\bigsqcap S \in \mathit{Prop}$, then $\bigsqcap S$ is indeed the greatest lower bound of S in *Prop*. Also, if $\bigcap S \in \mathit{Prop}$, then $\bigsqcap S = \bigcap S$. But by making $\bigsqcap$ a totally defined operation we ensure that $|\forall x\varphi|$ is always defined, regardless of whether it is admissible. We will see that admissibility of $|\forall x\varphi|$ is not required for the validity of a number of principles, including UI°, UD and UG, but is required for VQ.

We will show that if all of the Ea's are admissible (i.e. $Ea \in \mathit{Prop}$), then the definition (0.1) of $|\forall x\varphi|$ validates CQ. The same conclusion holds if U is finite, or if the Boolean algebra *Prop* is atomic, hence if *Prop* is finite, and hence if W is finite. Moreover, validity of CQ follows if equality is definable in the model in the sense that there is a formula "$x \approx y$" such that when instantiated with any two elements a, b in the domain we obtain

$$|a \approx b| = \begin{cases} W, & \text{if } a = b, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus the construction of a falsifying model for CQ is not a simple matter.

In Sections 1–3 we define *model structures, premodels* (in which $|\forall x\varphi|$ need not be admissible) and *models* (in which it is), and prove several soundness results. Section 4 gives sufficient criteria for validity of CQ, and Section 5 constructs its falsifying model. The final Section 6 briefly states completeness results for various logics relative to the given semantics, and points out some interesting relationships between CQ and the Barcan formula.

1. Model Structures

A *model structure* is a system $\mathcal{S} = (W, R, \mathit{Prop}, U, D)$ such that

- W is a set, and R is a binary relation on W;
- *Prop* is a Boolean subalgebra of the powerset algebra $\wp W$;
- *Prop* is closed under the operation $[R]$ defined by

$$[R]X = \{w \in W : \forall v \in W (wRv \text{ implies } v \in X)\};$$

- U is a set, and D is a function assigning to each $w \in W$ a subset $Dw \subseteq U$.

Members of *Prop* are called the *admissible* sets of $\mathcal{S}$. For each $a \in U$ we define $Ea = \{w \in W : a \in Dw\}$. Sets of the form Ea may be referred to as "existence sets". They are not required to be admissible.

Using *Prop* we define, for each $X \subseteq W$,

$$X{\downarrow} = \bigcup\{Y \in \mathit{Prop} : Y \subseteq X\},$$
$$X{\uparrow} = \bigcap\{Y \in \mathit{Prop} : X \subseteq Y\},$$

giving $X{\downarrow} \subseteq X \subseteq X{\uparrow}$. The sets $X{\downarrow}$ and $X{\uparrow}$ need not belong to *Prop*, but if they do, then $X{\downarrow}$ is the largest admissible subset of X, and $X{\uparrow}$ the smallest admissible superset. So if $X \in \mathit{Prop}$, then $X{\downarrow} = X{\uparrow} = X$. Operations $\sqcap$ and $\sqcup$ on $\wp\wp W$ are defined by putting, for all $S \subseteq \wp W$,

$$\sqcap S = (\bigcap S){\downarrow}, \qquad \sqcup S = (\bigcup S){\uparrow}.$$

Then any *admissible* X has $X \subseteq \sqcap S$ iff $X \subseteq \bigcap S$. If $S \subseteq \mathit{Prop}$ and $\sqcap S \in \mathit{Prop}$, then $\sqcap S$ is the *greatest lower bound* of S in the partially-ordered set $(\mathit{Prop}, \subseteq)$, i.e. the largest admissible set included in every member of S. Dual statements hold concerning the role of $\sqcup S$ as the *least upper bound* of $S \subseteq \mathit{Prop}$.

It is quite possible that $\sqcap S$ is admissible while $\bigcap S$ is not. However, if $\bigcap S \in \mathit{Prop}$ then $\sqcap S = \bigcap S$.

We now record some useful facts about $\sqcap$, some of which involve the Boolean set "implication" operation $\Rightarrow$, defined by $X \Rightarrow Y = (W \setminus X) \cup Y$. Its main property is that $Z \subseteq X \Rightarrow Y$ iff $Z \cap X \subseteq Y$.

In the following Lemma, X_i, Y_i, X_{ij} are subsets of W, S is a subset of $\wp W$, and $\sqcap_{i \in I} X_i$ is $\sqcap\{X_i : i \in I\}$.

Lemma 1.1.

(1) *If $X_i \subseteq Y_i$ for all $i \in I$, then $\sqcap_{i\in I} X_i \subseteq \sqcap_{i \in I} Y_i$.*

(2) $\sqcap_{i\in I} \sqcap_{j \in J} X_{ij} = \sqcap_{j \in J} \sqcap_{i \in I} X_{ij}$.

(3) *If $X \in \mathit{Prop}$, then $X \Rightarrow \sqcap S = \sqcap_{Y \in S}(X \Rightarrow Y)$.*

(4) *If $\{Y_i : i \in I\} \subseteq \mathit{Prop}$, then $\sqcap_{i \in I}(X_i \Rightarrow Y_i) = \sqcap_{i\in I}(X_i{\uparrow} \Rightarrow Y_i)$.*

Proof.

(1) $\bigcap_{i \in I} X_i \subseteq \bigcap_{i \in I} Y_i$, and the operation $\downarrow$ is $\subseteq$-monotonic.

(2) (N.B: the X_{ij}'s need not be admissible here.)
Let X be an admissible subset of $\sqcap_{i\in I} \sqcap_{j\in J} X_{ij}$. Then $X \subseteq X_{ij}$ for all $(i, j) \in I \times J$. So, for a given $j_0 \in J$ we have $X \subseteq X_{ij_0}$ for all $i \in I$, hence $X \subseteq \sqcap_{i\in I} X_{ij_0}$ because $X \in \mathit{Prop}$. Since this holds for every $j_0 \in J$, $X \subseteq \sqcap_{j\in J}\sqcap_{i \in I} X_{ij}$, again as X is admissible. Symmetrically, each admissible subset of $\sqcap_{j\in J}\sqcap_{i\in I} X_{ij}$ is a subset of $\sqcap_{i\in I}\sqcap_{j\in J} X_{ij}$. Hence $\sqcap_{i\in I}\sqcap_{j\in J} X_{ij} = \sqcap_{j\in J}\sqcap_{i\in I} X_{ij}$, since both are unions of admissible subsets.

(3) (N.B: the members of S need not be admissible.)
Since $Y \subseteq (X \Rightarrow Y)$, $\sqcap S \subseteq \sqcap_{Y\in S}(X \Rightarrow Y)$ by (1). Also, as $W \setminus X \subseteq (X \Rightarrow Y)$, and $W \setminus X \in \mathit{Prop}$ because $X \in \mathit{Prop}$, we have $W \setminus X \subseteq \sqcap_{Y \in S}(X \Rightarrow Y)$. Altogether then,

$$X \Rightarrow \sqcap S = W \setminus X \cup \sqcap S \subseteq \sqcap_{Y\in S}(X \Rightarrow Y).$$

For the converse inclusion it is enough to show that any admissible subset of $\bigcap_{Y\in S}(X \Rightarrow Y)$ is a subset of $X \Rightarrow \bigsqcap S$. But if $Z \in \mathit{Prop}$ has $Z \subseteq \bigcap_{Y\in S}(X \Rightarrow Y)$, then for all $Y \in S$, $Z \subseteq (X \Rightarrow Y)$, so $Z \cap X \subseteq Y$. Hence $Z \cap X \subseteq \bigsqcap S$ as $Z \cap X \in \mathit{Prop}$. Therefore $Z \subseteq X \Rightarrow \bigsqcap S$.

(4) (N.B: the X_i need not be admissible.)
First, since $X_i \subseteq X_i{\uparrow}$, we have $(X_i{\uparrow} \Rightarrow Y_i) \subseteq (X_i \Rightarrow Y_i)$, for all $i \in I$. Hence $\bigsqcap_{i\in I}(X_i{\uparrow} \Rightarrow Y_i) \subseteq \bigsqcap_{i\in I}(X_i \Rightarrow Y_i)$ by (1).
For the converse inclusion, let Z be any admissible subset of $\bigsqcap_{i\in I}(X_i \Rightarrow Y_i)$. Then for all $i \in I$, $Z \subseteq X_i \Rightarrow Y_i$, hence $X_i \subseteq Z \Rightarrow Y_i$. But $Z \Rightarrow Y_i$ is admissible (by admissibility of Z and Y_i), and so $X_i{\uparrow} \subseteq Z \Rightarrow Y_i$, implying that $Z \subseteq X_i{\uparrow} \Rightarrow Y_i$. Hence $Z \subseteq \bigsqcap_{i\in I}(X_i{\uparrow} \Rightarrow Y_i)$.

■

2. Premodels and Models

Let $\mathcal{L}$ be a set of relation and function symbols and individual constants. A *premodel* $\mathcal{M} = (\mathcal{S}, |\cdot|^{\mathcal{M}})$ for $\mathcal{L}$, based on a model structure $\mathcal{S}$, is given by an interpretation function $|\cdot|^{\mathcal{M}}$ on $\mathcal{L}$ that assigns

- to each n-ary relation symbol P a function $|P|^{\mathcal{M}} : U^n \to \mathit{Prop}$,
- to each individual constant c an element $|\mathsf{c}|^{\mathcal{M}} \in U$, and
- to each n-ary function symbol F a function $|F|^{\mathcal{M}} : U^n \to U$.

We emphasize that the language is not assumed to have an equality symbol, by which we would mean a binary relation symbol P interpreted in $\mathcal{M}$ by

$$|P|^{\mathcal{M}}(a,b) = \begin{cases} W, & \text{if } a = b, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As indicated in the Introduction, and as will be proved in Corollary 4.5, the presence of a relation symbol thus interpreted, or even the definability of the equality relation at all in the model, entails the validity of CQ.

We deal with first-order modal $\mathcal{L}$-formulas, without equality, generated using a set $\{x_n : n < \omega\}$ of first-order variables, but often regard this set simply as ω by identifying x_n with n. A variable-assignment is then a map $f \in {}^{\omega}U$. Any $\mathcal{L}$-term τ can be interpreted via f as an element $\tau^{\mathcal{M}} f \in U$ in the usual way. We use the letters $x, y, z, \cdots$ for variables, and define $f[a/x]$ to be the function that "updates" f by assigning the value $a \in U$ to x and otherwise acting as f.

A premodel gives an interpretation $|\varphi|^{\mathcal{M}} : {}^{\omega}U \to \wp W$ to each $\mathcal{L}$-formula. For each assignment f, $|\varphi|^{\mathcal{M}} f$ is thought of as the set of worlds at which φ is true under f. This is defined by induction on the formation of φ:

- $|P\tau_1 \cdots \tau_n|^{\mathcal{M}} f = |P|^{\mathcal{M}}(\tau_1^{\mathcal{M}} f, \ldots, \tau_n^{\mathcal{M}} f) \in \mathit{Prop}$,
- $|\top|^{\mathcal{M}} f = W$ and $|\bot|^{\mathcal{M}} f = \emptyset$,
- $|\neg\varphi|^{\mathcal{M}} f = W \setminus |\varphi|^{\mathcal{M}} f$, and $|\varphi \wedge \psi|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f \cap |\psi|^{\mathcal{M}} f$,
- $|\Box\varphi|^{\mathcal{M}} f = [R]|\varphi|^{\mathcal{M}} f$,
- $|\forall x\varphi|^{\mathcal{M}} f = \bigsqcap_{a \in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

Thus if $X \in \mathit{Prop}$, then $X \subseteq |\forall x\varphi|^{\mathcal{M}} f$ iff $X \subseteq Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]$ for all $a \in U$. We have

$$\begin{aligned} |\forall x\varphi|^{\mathcal{M}} f &= \Big[\bigcap_{a \in U} Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\Big]{\downarrow}. \\ &= \Big[\bigcap_{a \in U} (W \setminus Ea) \cup |\varphi|^{\mathcal{M}} f[a/x]\Big]{\downarrow}. \end{aligned}$$

Identifying $\exists$ with $\neg\forall\neg$ gives

$$\begin{aligned} |\exists x\varphi|^{\mathcal{M}} f &= \bigsqcup_{a \in U} Ea \cap |\varphi|^{\mathcal{M}} f[a/x] \\ &= \Big[\bigcup_{a \in U} Ea \cap |\varphi|^{\mathcal{M}} f[a/x]\Big]{\uparrow}. \end{aligned}$$

Remark 2.1. The semantics of [4] interprets an n-ary relation symbol P as a function

$$\Phi(P, \cdot) : W \to \wp(U^n)$$

assigning to each world w an n-ary relation $\Phi(P, w) \subseteq U^n$. From such a Φ we can define $|P| : U^n \to \wp W$ by

$$w \in |P|(a_1, \ldots, a_n) \quad \text{iff} \quad \langle a_1, \ldots, a_n\rangle \in \Phi(P, w).$$

Alternatively, this can be viewed as a definition of Φ, given $|P|$, so the two methods are equivalent. We find that use of the "proposition-valued" functions $|\varphi|$ provides a convenient way of handling the restriction to admissible propositions.

It is worth emphasising that this kind of model theory allows relations and properties to hold of non-existent objects (e.g. Pegasus has wings). Thus it is not required that $\Phi(P, w) \subseteq (Dw)^n$; equivalently, it is not required that

$$|P|(a_1, \ldots, a_n) \subseteq Ea_1 \cap \cdots \cap Ea_n.$$

In fact there are numerous ways to set up a model theory for the language of first-order modal logic, depending on a whole range of potential requirements like this, including whether terms are allowed to be non-rigid (i.e. world-dependent), whether they are interpreted locally at a world (i.e. as a member of the domain of that world), whether predicates are taken to be extensional or intensional, whether domains are fixed or variable or nested, etc. The "quantified modal logic roadmap" of [2, Figure 1] gives some impression of the complexity of this range of possibilities. ■

Writing $\mathcal{M}, w, f \models \varphi$ to mean that $w \in |\varphi|^{\mathcal{M}} f$, we get the following clauses for this satisfaction relation $\models$, with all except that for $\forall$ being familiar:

- $\mathcal{M}, w, f \models P\tau_1 \cdots \tau_n$ iff $w \in |P\tau_1 \dots \tau_n|^{\mathcal{M}} f$,
- $\mathcal{M}, w, f \models \top$ and $\mathcal{M}, w, f \not\models \bot$,
- $\mathcal{M}, w, f \models \neg\varphi$ iff $\mathcal{M}, w, f \not\models \varphi$,
- $\mathcal{M}, w, f \models \varphi \wedge \psi$ iff $\mathcal{M}, w, f \models \varphi$ and $\mathcal{M}, w, f \models \psi$,
- $\mathcal{M}, w, f \models \Box\varphi$ iff for all $v \in W(wRv$ implies $\mathcal{M}, v, f \models \varphi)$.
- $\mathcal{M}, w, f \models \forall x\varphi$ iff there is an $X \in \mathit{Prop}$ such that $w \in X$ and $X \subseteq \bigcap_{a \in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.

A formula φ is *valid in* premodel $\mathcal{M}$, written $\mathcal{M} \models \varphi$, if $|\varphi|^{\mathcal{M}} f = W$ for all f, i.e. if $\mathcal{M}, w, f \models \varphi$ for all $w \in W$ and $f \in {}^{\omega}U$.

As with standard semantics, satisfaction of a formula depends only on value-assignment to *free* variables:

LEMMA 2.2. *In any premodel $\mathcal{M}$, for any formula φ, if assignments $f, g \in {}^{\omega}U$ agree on all free variables of φ, then $|\varphi|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} g$.*

PROOF. The only departure from the standard proof is the inductive case that φ is $\forall x\psi$. Then if f and g agree on all free variables of φ, then for each $a \in U$, $f[a/x]$ and $g[a/x]$ agree on all free variables of ψ, so $|\psi|^{\mathcal{M}} f[a/x] = |\psi|^{\mathcal{M}} g[a/x]$ by induction hypothesis. Hence

$$|\varphi|^{\mathcal{M}} f = \prod_{a \in U} \big(Ea \Rightarrow |\psi|^{\mathcal{M}} f[a/x]\big) = \prod_{a \in U} \big(Ea \Rightarrow |\psi|^{\mathcal{M}} g[a/x]\big) = |\varphi|^{\mathcal{M}} g.$$

■

This result can be used to establish the usual relationship between syntactic substitution of terms for variables and updating of evaluations:

LEMMA 2.3. *Let φ be any formula, and τ a term that is free for x in φ. Then in any premodel $\mathcal{M}$, for any $f \in {}^{\omega}U$, $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x]$.*

PROOF. Again the only nonstandard case is when φ is of the form $\forall y\psi$. First, when x is not free in φ then f and $f[\tau^{\mathcal{M}} f/x]$ agree on all free variables of φ, and $\varphi(\tau/x)$ is just φ, so the result is given by Lemma 2.2.

Otherwise, x is free in φ, so $x \neq y$ and $\varphi(\tau/x) = \forall y(\psi(\tau/x))$ with τ free for x in ψ, so y does not occur in τ. Then

$$|\varphi(\tau/x)|^{\mathcal{M}} f = \bigsqcap_{a \in U} Ea \Rightarrow |\psi(\tau/x)|^{\mathcal{M}} f[a/y], \quad \text{and}$$

$$|\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x] = \bigsqcap_{a \in U} Ea \Rightarrow |\psi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x][a/y].$$

But for any $a \in U$, the induction hypothesis on ψ gives

$$|\psi(\tau/x)|^{\mathcal{M}} f[a/y] = |\psi|^{\mathcal{M}} f[a/y][\tau^{\mathcal{M}} f[a/y]/x],$$

and $\tau^{\mathcal{M}} f[a/y] = \tau^{\mathcal{M}} f$ because y is not in τ, while

$$f[a/y][\tau^{\mathcal{M}} f/x] = f[\tau^{\mathcal{M}} f/x][a/y]$$

as $y \neq x$. So altogether

$$|\psi(\tau/x)|^{\mathcal{M}} f[a/y] = |\psi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x][a/y],$$

and hence $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x]$ in this case. ■

COROLLARY 2.4. *If $\mathcal{M} \models \varphi$, then $\mathcal{M} \models \varphi(\tau/x)$ whenever τ is free for x in φ.*

PROOF. If $\mathcal{M} \models \varphi$, then for any f, $|\varphi(\tau/x)|^{\mathcal{M}} f = |\varphi|^{\mathcal{M}} f[\tau^{\mathcal{M}} f/x] = W$. ■

We will say that a formula φ is *admissible in* $\mathcal{M}$ if the function $|\varphi|^{\mathcal{M}}$ has the form ${}^{\omega}U \to \mathit{Prop}$, i.e. $|\varphi|^{\mathcal{M}} f \in \mathit{Prop}$ for all $f \in {}^{\omega}U$. Every atomic formula $P\tau_1 \cdots \tau_n$ is admissible. Given the closure properties of *Prop* it is evident that the set of admissible formulas is closed under the Boolean connectives and $\Box$. In particular, every *quantifier-free* formula is admissible.

A *model* for $\mathcal{L}$ is a premodel in which every $\mathcal{L}$-formula is admissible.

LEMMA 2.5. *In any model $\mathcal{M}$, $|\forall x\varphi|^{\mathcal{M}} f = \bigsqcap_{a \in U} \big(Ea{\uparrow} \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big)$.*

PROOF. As φ is admissible in $\mathcal{M}$, $\{|\varphi|^{\mathcal{M}} f[a/x] : a \in U\} \subseteq \mathit{Prop}$. Hence by Lemma 1.1(4),

$$\bigsqcap_{a \in U} \big(Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big) = \bigsqcap_{a \in U} \big(Ea{\uparrow} \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\big).$$

■

3. Soundness and $\mathcal{M}$-Equivalence

We now fix a premodel $\mathcal{M}$, and examine the validity of various principles in it, identifying some whose validity requires $\mathcal{M}$ to be a model. From now on, the $\mathcal{M}$-superscript will often be dropped from the notation $|\varphi|^{\mathcal{M}}f$.

PROPOSITION 3.1. *The schemata* UI° *and* UD *are valid in* $\mathcal{M}$*, and the rule* UG *is sound for validity in* $\mathcal{M}$*.*

PROOF. UG is dealt with first, as it is simplest. If $\mathcal{M} \models \varphi$, then for any f and a, $Ea \Rightarrow |\varphi|f[a/x] = Ea \Rightarrow W = W$, so $|\forall x\varphi|f = \bigsqcap\{W\} = W$. Hence $\mathcal{M} \models \forall x\varphi$.

For UD, suppose that $\mathcal{M}, w, f \models \forall x(\varphi \to \psi)$ and $\mathcal{M}, w, f \models \forall x\varphi$. Then there exist $X, Y \in \mathit{Prop}$ such that

$$w \in X \subseteq \bigcap_{a \in U} Ea \Rightarrow |\varphi \to \psi|f[a/x], \quad \text{and}$$
$$w \in Y \subseteq \bigcap_{a \in U} Ea \Rightarrow |\varphi|f[a/x].$$

Then $w \in X \cap Y \in \mathit{Prop}$, and for all a,

$$X \cap Y \cap Ea \subseteq |\varphi \to \psi|f[a/x] \cap |\varphi|f[a/x] \subseteq |\psi|f[a/x],$$

hence $X \cap Y \subseteq Ea \Rightarrow |\psi|f[a/x]$. This shows $\mathcal{M}, w, f \models \forall x\psi$.

For UI°, let y be free for x in φ. It suffices to show that for any f and a,

$$Ea \subseteq |\forall x\varphi \to \varphi(y/x)|f[a/y]. \tag{3.1}$$

For then $Ea \Rightarrow |\forall x\varphi \to \varphi(y/x)|f[a/y] = W$ for all $a \in U$, so

$$|\forall y(\forall x\varphi \to \varphi(y/x))|f = \bigsqcap\{W\} = W,$$

and hence $\mathcal{M} \models \forall y(\forall x\varphi \to \varphi(y/x))$.

To prove (3.1), let $w \in Ea$. Then if $w \in |\forall x\varphi|f[a/y]$, there exists $X \in \mathit{Prop}$ with

$$w \in X \subseteq \bigcap_{b \in U} Eb \Rightarrow |\varphi|f[a/y][b/x].$$

In particular, when $b = a$, since $w \in Ea$ we get $w \in |\varphi|f[a/y][a/x]$. But by Lemma 2.3, $|\varphi|f[a/y][a/x] = |\varphi(y/x)|f[a/y]$ because $y^{\mathcal{M}}f[a/y] = a$. Thus

$$w \in |\forall x\varphi|f[a/y] \Rightarrow |\varphi(y/x)|f[a/y] = |\forall x\varphi \to \varphi(y/x)|f[a/y].$$

■

Next we consider the validity of VQ:

Proposition 3.2. *Suppose that x has no free occurrence in φ. If φ is admissible in $\mathcal{M}$, then $\mathcal{M} \models \varphi \to \forall x\varphi$.*

Proof. For any $f \in {}^{\omega}U$ and $a \in U$, the assignments f and $f[a/x]$ agree on all free variables of φ, so by Lemma 2.2,

$$|\varphi|f = |\varphi|f[a/x] \subseteq Ea \Rightarrow |\varphi|f[a/x].$$

But $|\varphi|f \in \mathit{Prop}$ by $\mathcal{M}$-admissibility of φ, so

$$|\varphi|f \subseteq \prod_{a \in U} \big(Ea \Rightarrow |\varphi|f[a/x]\big) = |\forall x\varphi|f.$$

Hence $|\varphi|f \Rightarrow |\forall x\varphi|f = W$ for all f. ■

Corollary 3.3. *Every model validates VQ.*

Proof. In a model, every φ is admissible. ■

We say that formulas φ and ψ are *$\mathcal{M}$-equivalent* if $|\varphi|^{\mathcal{M}} = |\psi|^{\mathcal{M}}$. The following properties of this equivalence relation are left to the reader to check.

Proposition 3.4. *In any premodel $\mathcal{M}$:*

(1) *φ is $\mathcal{M}$-equivalent to ψ iff $\mathcal{M} \models \varphi \leftrightarrow \psi$.*

(2) *If φ is tautologically equivalent to ψ (i.e. $\varphi \leftrightarrow \psi$ is a tautology), then φ and ψ are $\mathcal{M}$-equivalent.*

(3) *$\mathcal{M}$-equivalence is a congruence on the algebra of $\mathcal{L}$-formulas, i.e. if the pair φ, ψ are $\mathcal{M}$-equivalent, then so are the pairs $\neg\varphi, \neg\psi$ and $\varphi \wedge \theta, \psi \wedge \theta$ and $\Box\varphi, \Box\psi$ and $\forall x\varphi, \forall x\psi$ and $\exists x\varphi, \exists x\psi$ etc.*

(4) *If ψ is obtained from φ by replacing some subformula by an $\mathcal{M}$-equivalent formula, then ψ is $\mathcal{M}$-equivalent to φ.* ■

The next result will be used in a model construction in Section 5.

Proposition 3.5. *In any premodel $\mathcal{M}$:*

(1) *$\exists x(\varphi \vee \psi)$ and $\exists x\varphi \vee \exists x\psi$ are $\mathcal{M}$-equivalent.*

(2) *$\exists x(\varphi \wedge \psi)$ and $\varphi \wedge \exists x\psi$ are $\mathcal{M}$-equivalent if φ is admissible in $\mathcal{M}$ and has no free occurrences of x.*

Proof. (1) It is enough to show that the formula

$$\exists x(\varphi \vee \psi) \leftrightarrow \exists x\varphi \vee \exists x\psi$$

is valid in $\mathcal{M}$. But, as the reader can check, this formula is derivable from tautologies and instances of UD using the rule UG and valid Boolean reasoning. Hence it is valid in $\mathcal{M}$ by Proposition 3.1.

(2) If φ is $\mathcal{M}$-admissible and without free x, then $\neg\varphi$ is $\mathcal{M}$-admissible and without free x, so by Lemma 3.2 the formulas $\varphi \to \forall x\varphi$ and $\neg\varphi \to \forall x\neg\varphi$ are valid in $\mathcal{M}$. But from these two, using tautologies, UD, UG and valid Boolean reasoning we can derive

$$\exists x(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \exists x\psi,$$

which is therefore valid in $\mathcal{M}$.

■

4. Validating CQ

We now give some conditions under which the formulas $\forall x\forall y\varphi$ and $\forall y\forall x\varphi$ are $\mathcal{M}$-equivalent in a model. Of course we can assume $x \neq y$ here, for otherwise there is no work to do. Then assignments $f[a/x][b/y]$ and $f[b/y][a/x]$ are identical, and may be written $f[a/x, b/y]$ or $f[b/y, a/x]$.

Lemma 4.1. *In a premodel $\mathcal{M}$, let $f \in {}^{\omega}U$ and let $\mathcal{B}$ be any Boolean subalgebra of Prop that contains $|\varphi|^{\mathcal{M}} f[a/x, b/y]$, $|\forall x\varphi| f[b/y]$, and $|\forall y\varphi| f[a/x]$ for all $a, b \in U$. Then exactly the same atoms of $\mathcal{B}$ are included in the sets $|\forall x\forall y\varphi|^{\mathcal{M}} f$ and $|\forall y\forall x\varphi|^{\mathcal{M}} f$.*

Proof. Let X be an atom of $\mathcal{B}$ with $X \not\subseteq |\forall x\forall y\varphi| f$. Then as $X \in Prop$, there exists $a_0 \in U$ such that

$$X \not\subseteq Ea_0 \Rightarrow |\forall y\varphi| f[a_0/x]. \tag{4.1}$$

Hence $X \not\subseteq |\forall y\varphi| f[a_0/x]$, so again as $X \in Prop$ there exists $b_0 \in U$ such that

$$X \not\subseteq Eb_0 \Rightarrow |\varphi| f[a_0/x, b_0/y]. \tag{4.2}$$

Hence $X \not\subseteq |\varphi| f[a_0/x, b_0/y]$. But X is a $\mathcal{B}$-atom and $|\varphi| f[a_0/x, b_0/y] \in \mathcal{B}$ as given, so X must be *disjoint* from $|\varphi| f[a_0/x, b_0/y] = |\varphi| f[b_0/y, a_0/x]$. Since $X \cap Ea_0 \neq \emptyset$ by (4.1), this implies

$$X \not\subseteq Ea_0 \Rightarrow |\varphi| f[b_0/y, a_0/x].$$

Hence

$$X \not\subseteq \bigsqcap_{a\in U} Ea \Rightarrow |\varphi|f[b_0/y, a/x] = |\forall x\varphi|f[b_0/y].$$

Again the atomicity of X then makes X disjoint from $|\forall x\varphi|f[b_0/y] \in \mathcal{B}$. Since $X \cap Eb_0 \neq \emptyset$ by (4.2),

$$X \not\subseteq Eb_0 \Rightarrow |\forall x\varphi|f[b_0/y].$$

Hence

$$X \not\subseteq \bigsqcap_{b\in U} Eb \Rightarrow |\varphi|f[b/y] = |\forall y\forall x\varphi|f.$$

Conversely, interchanging x and y in this argument shows that if $X \not\subseteq |\forall y\forall x\varphi|f$, then $X \not\subseteq |\forall x\forall y\varphi|f$. ■

PROPOSITION 4.2. *A* ***model*** *validates CQ if any of the following hold:*

(1) *Prop is an atomic Boolean algebra.*

(2) *Prop is finite.*

(3) *The universe U is finite.*

PROOF. (1) Put $\mathcal{B} = \mathit{Prop}$. For any f, all sets $|\varphi|f[a/x, b/y]$, $|\forall x\varphi|f[b/y]$, $|\forall y\varphi|f[a/x]$ are in $\mathcal{B}$ by admissibility. But likewise the sets $|\forall x\forall y\varphi|f$ and $|\forall y\forall x\varphi|f$ are in $\mathcal{B}$, and include the same atoms of $\mathcal{B}$ by Lemma 4.1, hence as $\mathcal{B}$ is atomic this makes $|\forall x\forall y\varphi|f = |\forall y\forall x\varphi|f$.

(2) By (1), as any finite Boolean algebra is atomic.

(3) If U is finite, then for any f,

$$\{|\forall x\forall y\varphi|f, |\forall y\forall x\varphi|f\}$$
$$\cup \{|\varphi|f[a/x, b/y], |\forall x\varphi|f[b/y], |\forall y\varphi|f[a/x] : a, b \in U\}$$

is a finite subset of *Prop*, so it generates a Boolean subalgebra $\mathcal{B}$ of *Prop* that is finite, hence atomic. The proof that $|\forall x\forall y\varphi|f = |\forall y\forall x\varphi|f$ in $\mathcal{B}$ then follows by the argument of (1).

■

Next we consider consequences of admissibility of the "existence sets" Ea and $Ea{\uparrow}$.

PROPOSITION 4.3. *If a model has $Ea{\uparrow} \in \mathit{Prop}$ for all $a \in U$, then it validates CQ.*

Proof. Since we are working in a model, we can use Lemma 2.5 to replace Ea by $Ea{\uparrow}$ in the definition of $|\forall x\varphi|$. Thus

$$\begin{aligned}
&|\forall x\forall y\varphi|f \\
&= \textstyle\bigsqcap_{a\in U} \big(Ea{\uparrow} \Rightarrow \bigsqcap_{b\in U}(Eb{\uparrow} \Rightarrow |\varphi|f[a/x,b/y])\big) \\
&= \textstyle\bigsqcap_{a\in U}\bigsqcap_{b\in U} \big(Ea{\uparrow} \Rightarrow (Eb{\uparrow} \Rightarrow |\varphi|f[a/x,b/y])\big) \quad \text{by Lemma 1.1(3) as } Ea{\uparrow}\in \mathit{Prop}, \\
&= \textstyle\bigsqcap_{a\in U}\bigsqcap_{b\in U} \big(Ea{\uparrow}\cap Eb{\uparrow} \Rightarrow |\varphi|f[a/x,b/y]\big) \quad \text{by set theory.}
\end{aligned}$$

Similarly, $|\forall y\forall x\varphi|f = \bigsqcap_{b\in U}\bigsqcap_{a\in U} \big(Eb{\uparrow}\cap Ea{\uparrow} \Rightarrow |\varphi|f[b/y,a/x]\big)$.

But $Eb{\uparrow}\cap Ea{\uparrow} \Rightarrow |\varphi|f[b/y,a/x] = Ea{\uparrow}\cap Eb{\uparrow} \Rightarrow |\varphi|f[a/x,b/y]$, so the $\bigsqcap$-commutation result of Lemma 1.1(2) applies to give $|\forall x\forall y\varphi|f = |\forall y\forall x\varphi|f$. ∎

Corollary 4.4. *If a model has $Ea \in \mathit{Prop}$ for all $a\in U$, then it validates CQ.*

Proof. If $Ea\in \mathit{Prop}$, then $Ea = Ea{\uparrow}$. ∎

We say that *equality is definable in $\mathcal{M}$* if for any distinct variables x, y, there is an $\mathcal{L}$-formula "$x\approx y$" such that

$$|x\approx y|^{\mathcal{M}}f = \begin{cases} W, & \text{if } fx = fy,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Corollary 4.5. *If equality is definable in a model, then it validates CQ.*

Proof. Let $a\in U$ be arbitrary, and suppose $f\in {}^{\omega}U$ satisfies $fx = a$. Then $|\exists y(x\approx y)|f = [\bigcup_{b\in U} Eb\cap |x\approx y|f[b/y]]{\uparrow} = Ea{\uparrow}$. Hence $Ea{\uparrow}\in \mathit{Prop}$ as every formula is admissible in $\mathcal{M}$. By Proposition 4.3, CQ is valid in $\mathcal{M}$.[3] ∎

A premodel $\mathcal{M}$ will be called *Kripkean* if it always has

$$|\forall x\varphi|^{\mathcal{M}}f = \bigcap_{a\in U}\big(Ea \Rightarrow |\varphi|^{\mathcal{M}}f[a/x]\big).$$

This means that $\forall$ gets the varying-domain semantics of Kripke [4]:

$$\mathcal{M}, w, f \models \forall x\varphi \ \text{ iff } \ \text{for all } a\in Dw,\ \mathcal{M}, w, f[a/x] \models \varphi. \tag{4.3}$$

[3]For this proof to work it suffices in fact that $|x\approx y|^{\mathcal{M}}f \supseteq Efx$ when $fx = fy$, and $|x\approx y|^{\mathcal{M}}f = \emptyset$ otherwise.

A Kripkean *model* has

$$\Big[\bigcap_{a\in U} Ea \Rightarrow |\varphi|^{\mathcal{M}} f[a/x]\Big] \in \mathit{Prop}$$

by admissibility of formula $\forall x\varphi$, and conversely this last condition implies that a model is Kripkean.

PROPOSITION 4.6. *Every Kripkean premodel validates CQ.*

PROOF. This is straightforward, essentially because the quantifiers *for all existing ...* commute in the metalanguage. A more formal proof can be given by repeating the proof of Proposition 4.3 with $\bigcap$ in place of $\prod$ (and Ea in place of $Ea\!\uparrow$). Instead of parts (2) and (3) of Lemma 1.1, the results

$$\bigcap_{i\in I}\bigcap_{j\in J} X_{ij} = \bigcap_{j\in J}\bigcap_{i\in I} X_{ij}, \qquad X \Rightarrow \bigcap S = \bigcap_{Y\in S}(X \Rightarrow Y),$$

are used. These are laws of set theory that hold independently of any admissibility constraints. ■

REMARK 4.7. If a model structure $\mathcal{S}$ has *all* sets admissible, i.e. $\mathit{Prop} = \wp W$, then in general $\prod S = \bigcap S$, so all models on $\mathcal{S}$ are Kripkean and must validate CQ by Proposition 4.6. This shows that a falsifying model for CQ must restrict the admissible sets.

Increasing the expressivity of the language may force more sets to be admissible. This is simply because all formulas have to be interpreted in models as admissible sets. In particular, if equality is definable, all sets $Ea\!\uparrow$ are of the form $|\exists y(x \approx y)|f$ and so become admissible, which is enough to entail CQ (Corollary 4.5).

5. A Countermodel to CQ

This section exhibits a model that falsifies an instance of CQ. It is not so hard to construct a premodel that does this, but we wish to ensure that every formula is admissible in $\mathcal{M}$, so that it validates VQ as well as UI° and UD. From what has been shown in the last Section, our model must have infinite sets for U and *Prop*, and hence for W. Also *Prop* cannot be atomic, and cannot contain every Ea, or every $Ea\!\uparrow$. Moreover, the model cannot be Kripkean, or permit the definability of equality.

Let $\sim$ denote a fixed (but arbitrary) equivalence relation on $\mathbb{Q}$ (the rationals) with infinitely many equivalence classes, each of which is dense in $\mathbb{Q}$:

so each interval (a, b) for $a < b$ in $\mathbb{Q}$ contains a point from each equivalence class. Such a relation is easy to construct. Let $b/{\sim}$ denote the $\sim$-equivalence class containing b.

We define a model structure $\mathcal{S} = (W, R, \mathit{Prop}, U, D)$, where

- $W = U = \mathbb{Q}$;
- either $R = \emptyset$, or $R = \{(a, a) : a \in \mathbb{Q}\}$;
- *Prop* is the Boolean subalgebra of $\wp(\mathbb{Q})$ generated by the set of all half-open intervals $[a, b) = \{x \in \mathbb{Q} : a \leq x < b\}$, where $a, b \in \mathbb{Q}$ and $a < b$;
- $Da = \{a\}$ for each $a \in \mathbb{Q}$. Hence $Ea = \{a\}$.

We have actually defined two model structures, depending on the choice of R. In the first case with $R = \emptyset$, $[R]X = W$ for all $X \subseteq W$. In the second case with R the identity relation, $[R]X = X$. Hence in both cases *Prop* is $[R]$-closed. In the first case (W, R) (and hence (W, R, Prop)) validates the smallest normal propositional modal logic containing $\Box\bot$, while in the second case it validates the smallest normal logic containing the schema $\Box\varphi \leftrightarrow \varphi$. But each normal propositional modal logic is a sublogic of one of these two [5], so is validated by one of these structures. We will make use of that fact in Section 6.

Each non-empty $X \in \mathit{Prop}$ is a finite union of intervals of the form $(-\infty, a)$, $[b, c)$, and $[d, +\infty)$. *Prop* is atomless, and $Ea{\uparrow} = Ea = \{a\} \notin \mathit{Prop}$ for all $a \in \mathbb{Q}$.

Lemma 5.1. *Write $\mathbb{Q}/{\sim}$ for the set of all $\sim$-classes, and let $\mathcal{E} \subseteq \mathbb{Q}/{\sim}$. Then $(\bigcup \mathcal{E}){\uparrow}$ and $(\bigcup \mathcal{E}){\downarrow}$ are admissible, with*

$$(\textstyle\bigcup \mathcal{E}){\uparrow} = \begin{cases} \emptyset, & \text{if } \mathcal{E} = \emptyset, \\ \mathbb{Q}, & \text{otherwise,} \end{cases} \qquad (\textstyle\bigcup \mathcal{E}){\downarrow} = \begin{cases} \mathbb{Q}, & \text{if } \mathcal{E} = \mathbb{Q}/{\sim}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. If $\mathcal{E} = \emptyset$ then $\bigcup \mathcal{E} = \emptyset$, and clearly $\emptyset{\uparrow} = \emptyset$. Otherwise, by density, any non-empty $X \in \mathit{Prop}$ intersects $\bigcup \mathcal{E}$, and so $(\bigcup \mathcal{E}){\uparrow} = \mathbb{Q}$. The case of $\downarrow$ is similar (or it can be derived from the $\uparrow$ case, using the equation $S{\downarrow} = \mathbb{Q} \setminus ((\mathbb{Q} \setminus S){\uparrow})$ for $S \subseteq \mathbb{Q}$). ∎

Now let $\mathcal{L}$ consist of two binary relation symbols, P and $\sim$. (The two uses of $\sim$ will be distinguished by context.) We define an $\mathcal{L}$-premodel on $\mathcal{S}$ by putting, for each $a, b \in \mathbb{Q}$,

- $|{\sim}|^{\mathcal{M}}(a, b) = \begin{cases} \mathbb{Q}, & \text{if } a \sim b, \\ \emptyset, & \text{otherwise;} \end{cases}$

- $|P|^{\mathcal{M}}(a,b) = \begin{cases} \mathbb{Q}, & \text{if } a \sim b, \\ \text{some non-empty interval} & \\ [b,c) \text{ not containing } a, & \text{otherwise.} \end{cases}$

Note that *Prop* contains $|{\sim}|^{\mathcal{M}}(a,b)$ and $|P|^{\mathcal{M}}(a,b)$ for all $a,b \in \mathbb{Q}$, as required. The definition ensures that $b \in |P|^{\mathcal{M}}(a,b)$ for all b, while $a \in |P|^{\mathcal{M}}(a,b)$ iff $a \sim b$.

PROPOSITION 5.2. *$\mathcal{M}$ does not validate $\forall x \forall y Pxy \to \forall y \forall x Pxy$.*

PROOF. We show that for any $f \in {}^{\omega}U$,

$$|\forall x \forall y Pxy| f = \mathbb{Q} \quad \text{while} \quad |\forall y \forall x Pxy| f = \emptyset.$$

Now $|\forall y Pxy| f = \big[\bigcap_{b \in \mathbb{Q}} Eb \Rightarrow |P|(fx,b)\big]\!\downarrow$. But for any b,

$$Eb \Rightarrow |P|(fx,b) = \{b\} \Rightarrow |P|(fx,b) = \mathbb{Q},$$

since $b \in |P|(fx,b)$. Hence $|\forall y Pxy| f = \mathbb{Q}\!\downarrow = \mathbb{Q}$. It follows that for any f, $|\forall x \forall y Pxy| f = [\bigcap_{a \in \mathbb{Q}} Ea \Rightarrow \mathbb{Q}]\!\downarrow = \mathbb{Q}$ as well.

On the other hand, $|\forall x Pxy| f = \big[\bigcap_{a \in \mathbb{Q}} Ea \Rightarrow |P|(a,fy)\big]\!\downarrow$. But

$$Ea \Rightarrow |P|(a,fy) = \mathbb{Q} \setminus \{a\} \cup |P|(a,fy) = \begin{cases} \mathbb{Q}, & \text{if } a \sim fy, \\ \mathbb{Q} \setminus \{a\}, & \text{otherwise }, \end{cases}$$

so $|\forall x Pxy| f = \big[\bigcap_{a \not\sim fy} \mathbb{Q} \setminus \{a\}\big]\!\downarrow = (fy/{\sim})\!\downarrow = \emptyset$ by Lemma 5.1.

It follows that for any f, $|\forall y \forall x Pxy| f = [\bigcap_{b \in \mathbb{Q}} \mathbb{Q} \setminus \{b\} \cup \emptyset]\!\downarrow = \emptyset\!\downarrow = \emptyset$ as well. ■

Notice that this proof shows that $\mathcal{M}$ is *non-Kripkean*: since $\emptyset \neq fy/{\sim}$, we have

$$|\forall x Pxy| f \neq \bigcap_{a \in \mathbb{Q}} Ea \Rightarrow |P|(a,fy).$$

We now have to show that the premodel $\mathcal{M}$ is actually a *model*, i.e. $|\varphi|^{\mathcal{M}} f$ is always an admissible set. For this we recall that formulas φ, ψ are $\mathcal{M}$-*equivalent* if $|\varphi| = |\psi|$ in this $\mathcal{M}$. The above proof of Proposition 5.2 shows that the formulas $\forall x \forall y Pxy$ and $\forall y \forall x Pxy$ are not only admissible, they are $\mathcal{M}$-equivalent to the formulas $\top$ and $\bot$, respectively. It turns out that every formula φ is $\mathcal{M}$-equivalent to some formula ψ that lacks quantifiers. But as observed earlier, all quantifier-free formulas are admissible, so we get $|\varphi| = |\psi| \in \mathit{Prop}$.

The proof of this "quantifier-eliminability" will be given next. A naive approach would require us to express some basic equality types of tuples, by which we mean conjunctions of formulas of the form $x \approx y$ and $x \not\approx y$. This has to be handled carefully as we also need equality not to be definable. So in the example, $\sim$ plays the role of equality. It is a "weak equality", with only $|\exists y(x \sim y)|^{\mathcal{M}} f = \mathbb{Q}$, rather than $|\exists y(x \sim y)|^{\mathcal{M}} f = Efx\!\uparrow$, so it does not add to the admissible sets (cf. Remark 4.7). But it is still expressive enough to allow quantifier elimination, as we now show.

Proposition 5.3. *Every formula is $\mathcal{M}$-equivalent to a quantifier-free formula, and hence is admissible in $\mathcal{M}$.*

Proof. Let us say that a formula φ is *reducible* if it is $\mathcal{M}$-equivalent to a quantifier-free formula. We show that every φ is reducible, by induction on φ. In the proof, we write '$\mathcal{M}$-equivalent' simply as 'equivalent'.

Note that *any formula that is equivalent to a reducible one is itself reducible*, a fact that will be used repeatedly. To begin with, any formula is equivalent to one formed from atomic formulas by the propositional connectives (including $\Box$) and the quantifier $\exists$, so we can suppose without loss of generality that φ has this form.

If φ is atomic, we are given the reducibility. The set of reducible formulas is clearly closed under the Boolean connectives. It is also closed under $\Box$, since $\Box\varphi$ is equivalent to the reducible $\top$ when $R = \emptyset$, and equivalent to φ itself when R is the identity relation.

Assume that φ is reducible. We will prove that $\exists x\varphi$ is reducible. Inductively, there is a quantifier-free formula ψ equivalent to φ, and so $\exists x\varphi$ is reducible if the equivalent $\exists x\psi$ is reducible. Thus we can suppose that φ is quantifier-free. But then there is a quantifier-free ψ in disjunctive normal form that is tautologically equivalent to φ, and hence equivalent to φ in $\mathcal{M}$. Again, $\exists x\varphi$ will be reducible if the equivalent $\exists x\psi$ is. Thus we can suppose that φ is in disjunctive normal form.

So, suppose that φ is $\varphi_1 \vee \cdots \vee \varphi_n$, where each φ_i is a conjunction of *literals*, i.e. atomic and negated-atomic formulas. If each $\exists x\varphi_i$ is reducible, then so is $\exists x\varphi_1 \vee \cdots \vee \exists x\varphi_n$, which is equivalent to $\exists x(\varphi_1 \vee \cdots \vee \varphi_n)$ by Lemma 3.5(1), so $\exists x\varphi$ will be reducible. Hence we can suppose that φ is a conjunction of literals.

Next we can split off the conjuncts of φ in which x does not occur. For, if φ is equivalent to $\psi \wedge \theta$ with ψ a literal not containing x, and $\exists x\theta$ is reducible, then so is $\psi \wedge \exists x\theta$, which is equivalent to $\exists x(\psi \wedge \theta)$ by admissibility of ψ and Lemma 3.5(2), hence equivalent to $\exists x\varphi$. So we can suppose that x occurs in each conjunct of φ.

Similarly, we can delete $P(x,x)$ and $x \sim x$ if they occur as conjuncts of φ, since each is equivalent to $\top$ by the definitions of $|{\sim}|^{\mathcal{M}}$ and $|P|^{\mathcal{M}}$, and $\exists x(\top \wedge \theta)$ is equivalent to $\exists x\theta$. Moreover, if the negation of $P(x,x)$ or $x \sim x$ occurs in φ then we are done, since $\exists x(\bot \wedge \theta)$ is equivalent to the reducible $\bot$. Finally, $y \sim x$ with y different to x can be replaced by the equivalent $x \sim y$. So altogether we can suppose that we are dealing with a formula of the form $\exists x\varphi$, where

$$\begin{aligned}\varphi \quad = \quad & \bigwedge_i P(x,y_i) \wedge \bigwedge_j P(z_j,x) \wedge \bigwedge_k \neg P(x,u_k) \wedge \bigwedge_l \neg P(v_l,x) \\ & \wedge \bigwedge_m (x \sim s_m) \wedge \bigwedge_n \neg(x \sim t_n),\end{aligned}$$

all variables y_i, z_j, etc are distinct from x, and each $\bigwedge$ could be empty. Now for any $f \in {}^{\omega}U$, we have

$$\begin{aligned}|\exists x\varphi|f \quad = \quad & \Big[\bigcup_{a\in\mathbb{Q}} \Big(Ea \cap \bigcap_i |P|(a,fy_i) \cap \bigcap_j |P|(fz_j,a) \\ & \cap \bigcap_k \big(\mathbb{Q} \setminus |P|(a,fu_k)\big) \cap \bigcap_l \big(\mathbb{Q} \setminus |P|(fv_l,a)\big) \\ & \cap \bigcap_m |{\sim}|(a,fs_m) \cap \bigcap_n \big(\mathbb{Q} \setminus |{\sim}|(a,ft_n)\big)\Big)\Big]{\uparrow}.\end{aligned}$$

Any empty intersection here is interpreted as $\mathbb{Q}$. Now $Ea = \{a\}$ for any $a \in \mathbb{Q}$. So

$$\begin{aligned}|\exists x\varphi|f = \Big\{a \in \mathbb{Q} : \quad & a \in \bigcap_i |P|(a,fy_i) \cap \bigcap_j |P|(fz_j,a) \\ & \cap \bigcap_k \big(\mathbb{Q} \setminus |P|(a,fu_k)\big) \cap \bigcap_l \big(\mathbb{Q} \setminus |P|(fv_l,a)\big) \\ & \cap \bigcap_m |{\sim}|(a,fs_m) \cap \bigcap_n \big(\mathbb{Q} \setminus |{\sim}|(a,ft_n)\big)\Big\}{\uparrow}.\end{aligned}$$

Observe now that

- $\{a \in \mathbb{Q} : a \in |P|^{\mathcal{M}}(a,b)\} = \{a \in \mathbb{Q} : a \in |{\sim}|^{\mathcal{M}}(a,b)\} = b/{\sim}$ for any $b \in \mathbb{Q}$,
- $\{b \in \mathbb{Q} : b \in |P|^{\mathcal{M}}(a,b)\} = \mathbb{Q}$ for any $a \in \mathbb{Q}$.

So the set $|\exists x\varphi|f$ above is

$$\begin{aligned}\Big[\bigcap_i (fy_i/{\sim}) \ \cap \bigcap_j \mathbb{Q} \ \cap \bigcap_k (\mathbb{Q} \setminus (fu_k/{\sim})) \ \cap \bigcap_l \emptyset \\ \cap \bigcap_m (fs_m/{\sim}) \ \cap \bigcap_n (\mathbb{Q} \setminus (ft_n/{\sim}))\Big]{\uparrow}.\end{aligned}$$

If the l-conjunction is non-empty — a condition determined by φ and independent of f — this set is $\emptyset$, and so $\exists x\varphi$ is equivalent to $\bot$. We are done. Otherwise, write Y for the set of all variables y_i, s_m above, and write Z for the set of all variables u_k, t_n. Then

$$\begin{aligned}|\exists x\varphi|f &= \Big[\bigcap_{y\in Y}(fy/{\sim})\ \cap \bigcap_{z\in Z}(\mathbb{Q}\setminus(fz/{\sim}))\Big]\!\uparrow\\ &= \Big[\bigcap_{y\in Y}(fy/{\sim})\ \setminus \bigcup_{z\in Z}(fz/{\sim})\Big]\!\uparrow\end{aligned}$$

The set in square brackets here is a Boolean combination of $\sim$-equivalence classes. It is therefore of the form $\bigcup\mathcal{E}$ for some set $\mathcal{E}$ of $\sim$-classes. So by Lemma 5.1, the $\uparrow$ of the set belongs to *Prop*. This shows that $\exists x\varphi$ is admissible. The proof that it is reducible involves two cases, syntactically determined by φ:

- If $Y=\emptyset$, then $|\exists x\varphi|f=\mathbb{Q}$ for all f, because there are infinitely many $\sim$-classes in $\mathbb{Q}$ and only finitely many of them are eliminated by the Z-term. So $\exists x\varphi$ is equivalent to $\top$ in this case.
- if $Y\neq\emptyset$, then $|\exists x\varphi|f$ is $\mathbb{Q}$ if all the fy are $\sim$-equivalent and no fz is $\sim$-equivalent to them: for then, the set inside the square brackets is a single $\sim$-equivalence class, so its $\uparrow$ is $\mathbb{Q}$. Otherwise, $|\exists x\varphi|f$ is $\emptyset$. Thus, for any $f\in{}^{\omega}U$,

  $$|\exists x\varphi|f=\Big|\bigwedge_{y,y'\in Y} y\sim y' \wedge \bigwedge_{y\in Y,z\in Z}\neg(y\sim z)\Big|f.$$

  So $\exists x\varphi$ is equivalent to this quantifier-free formula if $Y\neq\emptyset$ (and, as one can see, if $Y=\emptyset$ as well).

This completes the proof of Proposition 5.3, and hence the proof that $\mathcal{M}$ is a model. ∎

6. Completeness and the Barcan Formulas

Let L be any (consistent) normal propositional modal logic. For a given signature $\mathcal{L}$, let $\mathrm{Q}^{-}\mathrm{L}$ be the smallest set of $\mathcal{L}$-formulas that includes

- all tautologies,
- all $\mathcal{L}$-substitution-instances of L-theorems,

- the schemata UI°, UD and VQ,

and is closed under

- detachment for material implication,
- the rule of Necessitation: from φ infer $\Box\varphi$, and
- the rule UG.

Now in the last section we defined two models for $\mathcal{L} = \{P, \sim\}$, call them $\mathcal{M}_0$ and $\mathcal{M}_1$, with $R = \emptyset$ and $R =$ the identity relation, respectively. We noted that the underlying propositional frame (W, R) of one of these models validates L, by the result of [5]. But then this model itself validates all $\mathcal{L}$-substitution-instances of L-theorems, by an argument given in the proof of [3, Theorem 2]. From the soundness results we have proved, and the evident soundness of Necessitation in any premodel, it then follows that this model validates Q$^-$L, while falsifying CQ.

It is notable that both the "Barcan formula"

BF $\forall x\Box\varphi \to \Box\forall x\varphi$

and its converse

CBF $\Box\forall x\varphi \to \forall x\Box\varphi$

are valid in $\mathcal{M}_0$ and $\mathcal{M}_1$. This follows from the fact that $\Box\psi$ is equivalent to $\top$ in $\mathcal{M}_0$, and to ψ in $\mathcal{M}_1$.

It turns out that for any $\mathcal{L}$, the logic Q$^-$L is complete for the class of all $\mathcal{L}$-models validating L (i.e. validating all $\mathcal{L}$-substitution-instances of L-theorems). This can be shown by a Henkin-model construction which reveals that the axioms UI°, UD and VQ, together with the rule UG, exactly capture the $\forall$-semantics

$$|\forall x\varphi| = \prod_{a\in U} Ea \Rightarrow |\varphi(a)|$$

of the $\mathcal{L}$-models we have used.

The converse Barcan formula is valid in any $\mathcal{L}$-model satisfying the *expanding domains* condition

$$wRv \quad \text{implies} \quad Dw \subseteq Dv, \tag{6.1}$$

equivalent to the requirement that $Ea \subseteq [R]Ea$ for all $a \in U$.

The logic Q$^-$L+CBF is complete for the class of its expanding domain models. But it is also complete for the class of its models that have *constant domains*:

$$wRv \quad \text{implies} \quad Dw = Dv. \tag{6.2}$$

This last claim may raise the eyebrows of some readers who are used to thinking of (6.2) as a condition that also validates the Barcan formula, which is typically not derivable in $Q^{-}L$+CBF. But the point is that BF can only be shown to be valid in the presence of (6.2) when the model is *Kripkean* in the sense of (4.3), in which case it also validates CQ.

The schema CQ is not a theorem of $Q^{-}L$+CBF+BF, as the models $\mathcal{M}_0$ and $\mathcal{M}_1$ show. The logic $Q^{-}L$+CBF+BF+CQ can be shown to be complete for its class of constant-domain *Kripkean* models. These results indicate that the main role of the Barcan formula in possible-worlds model theory is not to provide models that have constant domains, but rather to ensure that in a Henkin-style construction, the quantifier $\forall$ can be given the Kripkean interpretation via $\bigcap$.

Justification of all these claims will be presented elsewhere.

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