Algebraic Polymodal Logic: A Survey

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Abstract

This is a review of those aspects of the theory of varieties of Boolean algebras with operators (BAO's) that emphasise connections with modal logic and structural properties that are related to natural properties of logical systems.

It begins with a survey of the duality that exists between BAO's and relational structures, focusing on the notions of bounded morphisms, inner substructures, disjoint and bounded unions, and canonical extensions of structures that originate in the study of validity-preserving operations on Kripke frames. This duality is then applied to *polymodal* propositional logics having finitary intensional connectives that generalise the Box and Diamond connectives of unary modal logic. Issues discussed include validity in canonical structures, completeness and incompleteness under the relational semantics, and characterisations of logics by elementary classes of structures and by finite structures.

It turns out that a logic is *strongly* complete for the relational semantics iff the variety of algebras it defines is *complex*, which means that every algebra in the variety is embeddable into a full powerset algebra that is also in the variety. A hitherto unpublished formulation and proof of this is given (Theorem 5.6.1) that applies to *quasi*-varieties. This is followed by an algebraic demonstration that the temporal logic of Dedekind complete linear orderings defines a complex variety, adapting Gabbay's model-theoretic proof that this logic is strongly complete.

1 Introduction

This article provides an introduction to the study of varieties of Boolean algebras with operators, emphasising their connections with modal logic, and focusing on structural properties (canonicity, completeness, complexity, elementary generation) that are related to natural properties of logical systems.

Now an *operator* on a Boolean algebra $\mathfrak{B}$ is a finitary function $\mathfrak{B}^n \to \mathfrak{B}$ that is join preserving in each of its arguments. Functions of this type that are unary ($n = 1$) provide natural interpretations of modal connectives, and there is an intimate relationship between their algebraic theory and the Kripke semantics for modal logics. Standard algebraic constructions (subalgebras, homomorphisms, direct products) correspond to certain truth-preserving constructions on Kripke models (bounded morphisms, inner submodels, disjoint unions). This correspondence is a *duality* in category-theoretic terms, and can be developed for arbitrary operators. Thus we may refer to the situation of $n = 1$ as being the *modal case* of a general theory of finitary operators on Boolean algebras, and when $n > 1$ such operators may be called *polymodal*.

There appear to be two traditions of algebraic logic in this area. The *algebraic* tradition, founded on the seminal work of Jónsson and Tarski [32, 33], has focused on the study of relation algebras [23, 36, 30] and cylindric algebras and their relativised versions [26, 28, 27, 45, 46] and their connections with first-order logic and set theory. The *logical* tradition has emphasised the use of unary operators in the study of modal and temporal logics, with highlights including the early work of McKinsey and Tarski [40, 44] on Lewis modal systems and intuitionistic logic; the pioneering use of algebra by Bull [8] in proving that all normal extensions of the modal logic S4.3 have the finite model property and in obtaining the first axiomatisations of the temporal logics of discrete and continuous time [9]; Thomason's incomplete temporal logic [55]; and Blok's demonstration [5] of the pervasiveness of the incompleteness phenomenon. More recently there have been applications focusing on the connections between weak versions of first-order quantificational logic and modal logics [47, 3, 4].

Algebraic methods can be effectively employed to obtain model-theoretic results. An example is the application of duality in [22] to characterise those classes of Kripke frames that are defined by a set of modal formulae (Theorem 5.9 below). On the other hand, the vigorous development of model-theoretic studies of modal logics under the Kripke semantics has produced notions and results that translate into significant observations about varieties of Boolean algebras with operators (BAO's). Here many examples come to mind:

- The study of model-theoretic conditions under which a modal formula is valid (Correspondence Theory) can be viewed as the analysis of those conditions on a relational structure which ensure that its algebra of subsets (*complex* algebra) satisfies certain equations. It is also concerned which the connection between properties of a BAO $\mathfrak{A}$ and those of its *canonical structure* $\mathbb{C}\mathsf{st}\,\mathfrak{A}$, which is a relational structure defined on the Stone-type representation of $\mathfrak{A}$. The powerset algebra of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is the *canonical extension* of $\mathfrak{A}$, and contains a subalgebra isomorphic to $\mathfrak{A}$.
- The work of Sahlqvist [49] giving a general completeness theorem for a large syntactically defined class of modal axioms extends the class of properties known to be preserved by canonical extensions of BAO's, and can be given an elegant algebraic treatment [31].
- The *canonical frames* widely used to prove completeness theorems for modal logics are essentially the same thing as the canonical structures of the Lindenbaum-Tarski algebras of these logics, which are themselves the free algebras in the varieties that the logics define.
- The discovery of Fine [12] that an elementary class of Kripke frames determines a logic validated by its canonical frames generalises to the result [14, 15] that the powerset algebras of an ultraproduct-closed class of structures generate a variety of BAO's closed under canonical extensions.
- The question of whether a logic is *complete* with respect to some class of Kripke frames corresponds to the question of whether a variety of algebras is generated by its powerset algebras.
- The property of a logic being *strongly complete* with respect to a class of Kripke frames (i.e. every consistent set of formulae is satisfiable in a model on a frame in the class) proves to be equivalent to that of a variety $\mathcal{V}$ of BAO's being *complex*, meaning that each member of $\mathcal{V}$ can be embedded into a powerset algebra that belongs to $\mathcal{V}$ (see Theorem 5.14 below for a hitherto unpublished formulation of this relationship that applies to *quasi*-varieties).

The purpose of this article is to survey these matters, not in an encyclopaedic fashion, but with a view to explaining the major ideas and their interconnections, including indications of proofs for the more substantial results, and providing references to the literature for details. There are numerous overlapping (and conflicting) uses of terminology and notation in this literature, and more may be perpetrated here, but an attempt will be made to offer some guidance as to these various conventions. The intention is to exhibit the two fundamental facets of algebraic logic: on the one hand the investigation of mathematical structures that arise by abstraction from the properties of logical systems, and on the other hand the use of algebra to establish significant results about such logical systems.

The reader is assumed to be familiar with the theory of Boolean algebras and their subalgebras, homomorphisms, representation by ultrafilters etc., and with the basic formalisms of universal algebra. In particular, the standard symbols $\mathsf{H}, \mathsf{S}, \mathsf{P}$ will be used to denote the operations of closure of a class of algebras under (isomorphic copies of) homomorphic images, subalgebras, and direct products, respectively. An *equational class* or *variety* is a class $\mathcal{V}$ of algebras defined by some set of equations. $\mathsf{Var}\,\mathcal{W}$ denotes the variety generated by a class $\mathcal{W}$ of algebras (i.e. the smallest variety containing $\mathcal{W}$). Repeated use is made of the following facts.

- (Birkhoff's Theorem) $\mathcal{V}$ is a variety iff it is closed under homomorphic images, subalgebras, and direct products: $\mathsf{H}\mathcal{V} \subseteq \mathcal{V}$, $\mathsf{S}\mathcal{V} \subseteq \mathcal{V}$, and $\mathsf{P}\mathcal{V} \subseteq \mathcal{V}$.
- (Tarski) $\mathsf{Var}\,\mathcal{W}$ is equal to $\mathsf{HSP}\,\mathcal{W}$.
- $\mathsf{Var}\,\mathcal{W}$ is the class of all models of the equational theory in infinitely many variables of $\mathcal{W}$, i.e. $\mathfrak{A} \in \mathsf{Var}\,\mathcal{W}$ if, and only if, $\mathfrak{A}$ satisfies every equation that holds of all members of $\mathcal{W}$.

2 Modal Algebras

We begin with a discussion of the modal case, as preparation for the general polymodal situation.

A Boolean algebra (BA) will be presented in the form $\mathfrak{B} = \langle B, +, \cdot, ^-, 0, 1\rangle$. For any set S, the associated *powerset algebra* is

$$\mathsf{Sb}\,S = \langle Sb\,S, \cup, \cap, -, \emptyset, S\rangle,$$

where $Sb\,S$ is the collection $\{T : T \subseteq S\}$ of all subsets of S.

2.1 Operators

A function $m : B \to B$ is called an *operator* on a Boolean algebra $\mathfrak{B}$ if it is *additive*: $m(x + y) = mx + my$ for all $x, y \in B$. m is *normal* if $m0 = 0$. Any operator has

$$m(x_1 + \cdots + x_n) = mx_1 + \cdots + mx_n$$

for any $n \geq 2$. Since 0 is the join of the empty set, a normal operator can alternatively be specified as a function satsfying

$$m(\textstyle\sum C) = \textstyle\sum m(C)$$

for any finite $C \subseteq B$, including $C = \emptyset$. All of the operators we will discuss are normal.

The *dual* of an operator m is the function $l : B \to B$ having

$$lx = (m(x^-))^-.$$

l is multiplicative ($l(x \cdot y) = lx \cdot ly$), and has $l1 = 1$ if m is normal. Thus the dual of a normal operator preserves the lattice meet of any finite subset of B. A notation such as m^d is sometimes used for the dual to indicate the dependence on m. Our use of the letters m and l derives from the common use of M and L to denote the modal "possibility" and "necessity" connectives.

If α is an ordinal, a *normal modal algebra* (MA) of *type* α is an algebra

$$\mathfrak{A} = \langle \mathfrak{B}, m_\beta \rangle_{\beta < \alpha}$$

with each m_β a normal operator on the BA $\mathfrak{B}$. Most studied have been type 1 algebras $\langle \mathfrak{B}, m \rangle$ and *temporal algebras* $\langle \mathfrak{B}, m_0, m_1 \rangle$ which are type 2 algebras whose pair of operators are *conjugate*, meaning that for all $x, y \in B$,

$$m_0 x \cdot y = 0 \quad \text{iff} \quad m_1 y \cdot x = 0.$$

This is equivalent to the equationally expressible condition that for all $x \in B$,

$$x \leq l_0 m_1 x \cdot l_1 m_0 x.$$

Some important equationally defined classes of type 1 algebras are the following.

- *Closure algebras.* These are MA's $\langle \mathfrak{B}, m \rangle$ in which

  $$x + mmx \leq mx,$$

  and are sometimes known as topological Boolean algebras [42, 43]. They include the algebras $\langle \mathsf{Sb}\, S, m \rangle$ with S a topological space and mT the closure of the set T in S. The dual operator lT gives the topological interior of T. In a general closure algebra an element x is thus called *closed* if $mx = x$, and *open* if $lx = x$ (i.e. if x^- is closed). Closure algebras model the modal logic S4 (which is defined in Section 5.2).
- *Monadic algebras.* These are the closure algebras in which $x \leq lmx$, which is equivalent to requiring that elements are closed iff they are open, or that the closure operator m is conjugate to itself. Monadic algebras can also be described as the one-dimensional cylindric algebras [26] and polyadic algebras [24]. They model the logic S5 (again see Section 5.2).
- *Diagonalisable algebras.* These satisfy

  $$mx \leq m(x - mx).$$

  They model the *provability interpretation* of modality, in which "necessarily A" means "it is provable in Peano arithmetic that A" [6, 54]. The equational class of diagonalisable algebras is generated by powerset algebras of certain well-founded relations, as will be explained shortly.

2.2 Complex Algebras of Kripke Frames

Let R be a binary relation on a set S. On the powerset algebra $\mathsf{Sb}\,S$ there are two normal operators naturally associated with R, taking each $T \subseteq S$ to its direct image

$$R(T) = \{s \in S : \exists t \in T(tRs)\},$$

and to its inverse image

$$R^{-1}(T) = \{s \in S : \exists t \in T(sRt)\},$$

respectively. The algebraic tradition has worked with direct images, for reasons that will be clarified in Section 3.1, while the logical tradition has used inverse images because of conventions associated with Kripke semantics (Section 5.3). The choice really is a matter of convention, since the inverse image of T under R is the same thing as the direct image of T under the inverse relation R^{-1}. We will follow the logical tradition here, and also will use the notation m_R for the inverse image operator:

$$m_R(T) = \{s \in S : \exists t \in T(sRt)\}.$$

The dual operator to m_R is $l_R : Sb\,S \to Sb\,S$, where

$$l_R(T) = -m_R(-T) = \{s \in S : \forall t\,(sRt \text{ implies } t \in T)\}.$$

These descriptions display the role of m_R and l_R as quantifiers, existential and universal, relative to, or *bounded by*, the relation R.

The pair $\langle S, R\rangle$ is known in modal logic as a *Kripke frame*, or *K-frame*. More generally we define a *K-frame of type* α to be a relational structure

$$\mathfrak{S} = \langle S, R_\beta\rangle_{\beta<\alpha}$$

with each R_β being a binary relation on S. The *full complex algebra* of $\mathfrak{S}$ is

$$\mathsf{Cm}\,\mathfrak{S} = \langle \mathsf{Sb}\,S, m_{R_\beta}\rangle_{\beta<\alpha},$$

which is a modal algebra of type α. Any algebra that is (isomorphic to) a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ is a *complex algebra of type* α. The terminology derives from group theory of the Nineteenth Century: before set theory became the lingua franca of mathematicians the word "complex" was used to denote a collection of elements (subset) of a group.

In the case of a K-frame $\mathfrak{S} = \langle S, R_0, R_1\rangle$ of type 2, $\mathsf{Cm}\,\mathfrak{S}$ is a tense algebra, i.e. m_{R_0} and m_{R_1} are conjugate, iff R_0 and R_1 are mutually inverse: $R_1 = R_0^{-1}$. Important examples are the frames $\langle S, <, >\rangle$ where S is one of the number systems $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, representing a discrete, dense or continuous flow of time respectively.

Our observations about conjugacy indicate that for a type 1 frame $\mathfrak{S} = \langle S, R\rangle$, m_R is self conjugate iff $R = R^{-1}$, i.e. iff R is symmetric. Analogously, if R is reflexive then $T \subseteq m_R(T)$ for all $T \in Sb\,S$, while conversely it suffices to have $\{s\} \subseteq m_R(\{s\})$ for all $s \in S$ to make R reflexive.

These examples illustrate the fact that there is an extensive catalogue of conditions on R that are equivalent to various equational properties of $\mathsf{Cm}\,\mathfrak{S}$. This was first demonstrated by Jónsson and Tarski [32, Theorem 3.5], several of whose observations are included in the following table.

Equational Property of $\mathsf{Cm}\,\mathfrak{S}$	**Equivalent condition on** R
$x \leq mx$	reflexive
$mmx \leq mx$	transitive
closure algebra	quasi-order (reflexive and transitive)
$x \leq lmx$	symmetric
monadic algebra	equivalence relation
$mx \leq lx$	functional
$mx = lx$	total function
diagonalisable	transitive with R^{-1} well-founded

(cf. [6] for a proof of the last entry.)

Some potent questions arise. Given a modal algebra $\mathfrak{A}$ satisfying a condition from the left column, is $\mathfrak{A}$ isomorphic to a complex algebra based on a frame satisfying the corresponding condition from the right column? To address such issues requires the representation theory of the next section.

Does every MA-equation correspond to a "natural" condition on frames? Note first that any equational assertion about $\mathsf{Cm}\,\mathfrak{S}$ can be translated via the definitions of m_R and l_R into a sentence in the universal monadic second-order logic of $\mathfrak{S}$, i.e. a sentence that quantifies universally over subsets of S. For instance, the condition

$$\forall T(T \subseteq m_R(T))$$

is equivalent to

$$\forall T\, \forall s\, (s \in T \to \exists t\, (t \in T \wedge sRt))$$

when T ranges over $Sb\, S$. But this itself proves to be equivalent to the simple *first-order* condition $\forall s(sRs)$ of reflexivity. Indeed all entries in the right column except the last are expressible in the first-order language (with equality) of $\mathfrak{S}$. But the class

$$\{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \text{ is diagonalisable}\}$$

is not elementary, i.e. not definable by any set of sentences in first-order logic, since the condition "R^{-1} is well-founded" is not preserved by elementary equivalence. In particular, an ultrapower of a frame satisfying this condition will not in general satisfy it. It transpires that for any equational class $\mathcal{V}$ of modal algebras, closure of the class

$$\{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in \mathcal{V}\}$$

under ultrapowers is necessary and sufficient for it to be an elementary class (cf. Corollary 4.12).

Do all first-order conditions on R characterise an equational property of $\mathsf{Cm}\,\mathfrak{S}$? In fact not: irreflexivity ($\forall s \neg(sRs)$) and antisymmetry are two counterexamples, as can be shown by using the notion of *bounded morphism* between frames (and is so shown in Section 4.2). In that case, which first-order conditions are equational? This question can be formulated in the following way: if $\mathcal{K}$ is an elementary class of frames, when is there an equational class of algebras $\mathcal{V}$ for which

$$\mathsf{Cm}\,\mathfrak{S} \in \mathcal{V} \quad \text{iff} \quad \mathfrak{S} \in \mathcal{K}?$$

There is an answer to this in terms of the closure of $\mathcal{K}$ under certain model-theoretic constructions (Theorem 5.9). The proof applies duality theory (Section 4) to Birkhoff's characterisation of equational classes of algebras as those closed under homomorphisms, subalgebras, and direct products.

These various questions provoked by the phenomena exhibited in the above table will be taken up below and answered as generally as possible for polymodal operators.

3 BAO's

3.1 Polymodal Operators

An operator of *rank*, or *arity*, n on a BA $\mathfrak{B}$ is a function $m : B^n \to B$ that is additive in each of its arguments. This means that for each $i < n$ and any elements $a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-1}$, the unary function

$$x \mapsto m(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{n-1})$$

is additive. m is *normal* if for each $i < n$ it satisfies the equation

$$m(a_0, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_{n-1}) = 0.$$

Note that the definition allows that $n = 0$: a nullary operation (i.e. a constant $m \in \mathfrak{B}$) is a normal operator. If $n > 1$, m is *polymodal.*

A *type* in this context is a pair $\tau = \langle \alpha_\tau, \rho_\tau \rangle$ with α_τ an ordinal and $\rho_\tau : \alpha_\tau \to \omega$ a *rank function* assigning a natural number $\rho_\tau(\beta) \geq 0$ to each $\beta < \alpha$. A *Boolean algebra with operators* (BAO) *of type* τ is an algebra

$$\mathfrak{A} = \langle \mathfrak{B}, m_\beta \rangle_{\beta < \alpha_\tau}$$

with each m_β an operator of rank $\rho_\tau(\beta)$ on the BA $\mathfrak{B}$. $\mathfrak{A}$ is *normal* if each m_β is normal. If $\rho_\tau(\beta) = 1$ for all β we have the earlier notion of an MA of type α_τ.

A *relational structure of type* τ has the form

$$\mathfrak{S} = \langle S, R_\beta \rangle_{\beta < \alpha_\tau}$$

with each R_β being an $(\rho_\tau(\beta)+1)$-ary relation on S, i.e. $R_\beta \subseteq S^{\rho_\tau(\beta)+1}$ (if $\rho_\tau(\beta) = 1$ for all β then $\mathfrak{S}$ is a K-frame of type α_τ). To build a complex algebra out of $\mathfrak{S}$ we have to explain how to obtain an operator on $\mathsf{Sb}\, S$ of rank n from a relation $R \subseteq S^{n+1}$. In the original case of the complex algebra of a group $(S, \cdot)$ the group operation lifts to a binary operation on the powerset of S by putting

$$\begin{aligned} T_0 \cdot T_1 &= \{t_0 \cdot t_1 : t_0 \in T_0 \text{ and } t_1 \in T_1\} \\ &= \text{image of } T_0 \times T_1 \text{ under the group operation.} \end{aligned}$$

Generalising, an n-ary operation $f : S^n \to S$ lifts to the n-ary operation

$$m_f : (Sb\, S)^n \to (Sb\, S)$$

having

$$\begin{aligned} m_f(T_0, \dots, T_{n-1}) &= f\text{-image of } T_0 \times \cdots \times T_{n-1} \\ &= \{f(t_0, \dots, t_{n-1}) : t_i \in T_i \text{ all } i < n\}. \end{aligned}$$

Now if $R = \{\langle t_0, \ldots, t_{n-1}, s\rangle : f(t_0, \ldots, t_{n-1}) = s\}$ is the $n+1$-ary graph of f, then the right-side of the definition of m_f can be described as the

$$\begin{aligned}&R\text{-image of } T_0 \times \cdots \times T_{n-1}\\ &\quad = \{s \in S : \exists t_0 \cdots \exists t_{n-1}(R(t_0, \ldots, t_{n-1}, s) \text{ and } t_i \in T_i \text{ all } i < n)\}.\end{aligned}$$

But now this is a description that makes sense for any $n+1$-ary relation R, and we have the promised explanation of why the algebraic tradition has found it mathematically natural to work with direct images of relations. On the other hand we can "permute" the definition by choosing any of the $n+1$ arguments of R to fill the role of the unquantified variable s. Thus an arbitrary relation $R \subseteq S^{n+1}$ defines $n+1$ normal operators on $\mathsf{Sb}\, S$. We will opt to maintain contact with the logical tradition here and single out the first argument of R, thereby defining $m_R : (Sb\, S)^n \to (Sb\, S)$ by extension of the $n = 1$ case of Section 2.2:

$$\begin{aligned}&m_R(T_0, \ldots, T_{n-1})\\ &\quad = \{s \in S : \exists t_0 \cdots \exists t_{n-1}(R(s, t_0, \ldots, t_{n-1}) \text{ and } t_i \in T_i \text{ all } i < n)\}\end{aligned}$$

(when $n = 0$, i.e. $R \subseteq S$, m_R is just the constant $R \in Sb\, S$). Then the (full) complex algebra of the structure $\mathfrak{S} = \langle S, R_\beta\rangle_{\beta<\alpha_\tau}$ can be defined as

$$\mathsf{Cm}\,\mathfrak{S} = \langle \mathsf{Sb}\, S, m_{R_\beta}\rangle_{\beta<\alpha_\tau}.$$

3.2 Canonical Entities

Associated with a Boolean algebra $\mathfrak{B}$ is the set $S_\mathfrak{B}$ of ultrafilters of $\mathfrak{B}$ and the injective BA-homomorphism $\eta_\mathfrak{B} : \mathfrak{B} \rightarrowtail \mathsf{Sb}\, S_\mathfrak{B}$ having $\eta_B(x) = \{s \in S_\mathfrak{B} : x \in s\}$. This is the fundamental Stone representation of $\mathfrak{B}$ as the isomorphic algebra of sets $\eta_\mathfrak{B}(\mathfrak{B})$, which is in general a proper subalgebra of $\mathsf{Sb}\, S_\mathfrak{B}$.

Given an n-ary function $m : B^n \to B$, an $n+1$-ary relation $R_m \subseteq (S_\mathfrak{B})^{n+1}$ is defined by

$$\begin{aligned}R_m(s, t_0, \ldots, t_{n-1}) \quad &\text{iff} \quad m(t_0 \times \cdots \times t_{n-1}) \subseteq s\\ &\text{iff} \quad (\forall i < n\, (x_i \in t_i)) \text{ implies } m(x_0, \ldots, x_{n-1}) \in s.\end{aligned}$$

When $n = 0$ this entails $R_m = \{s \in S_\mathfrak{B} : m \in s\} = \eta_\mathfrak{B}(m)$. When $n = 1$ we have, using the infix notation for binary relations,

$$sR_m t \quad \text{iff} \quad \{mx : x \in t\} \subseteq s \quad \text{iff} \quad \{x : lx \in s\} \subseteq t.$$

The relation R_m induces the normal operator m_{R_m} on $\mathsf{Sb}\, S_\mathfrak{B}$. In order for $\eta_\mathfrak{B}$ to preserve the operations m and m_{R_m}, i.e.

$$\eta_\mathfrak{B}(m(x_0, \ldots, x_{n-1})) = m_{R_m}(\eta_\mathfrak{B}(x_0), \ldots, \eta_\mathfrak{B}(x_{n-1})),$$

it must be the case that for each $s \in S_\mathfrak{B}$ and $x_0, \ldots, x_{n-1} \in B$, the condition

$$m(x_0, \ldots, x_{n-1}) \in s$$

is equivalent to

$$\exists t_0, \ldots, t_{n-1} \in S_\mathfrak{B}(R_m(s, t_0, \ldots, t_{n-1}) \text{ and } (\forall i < n)(x_i \in t_i)).$$

The implication from top to bottom holds by the definition of R_m, and the whole equivalence holds when $n = 0$, again from the definitions. The difficult part is to prove it from bottom to top when $n \geq 1$, and this requires m to be a normal operator. One way to proceed is to construct the ultrafilters t_i by induction on i in such a way that the following two clauses hold:

(i) $x_i \in t_i$,

(ii) if $y_k \in t_k$ for all $k \leq i$, then $m(y_0, \dots, y_i, x_{i+1}, \dots, x_{n-1}) \in s$.

Then when $i = n-1$, clause (ii) immediately yields $R_m(s, t_0, \dots, t_{n-1})$ by definition of R_m. Together with (i), this will complete the proof.

The inductive argument is to fix a $j \leq n-1$, and suppose that for each $i < j$, t_i has been defined to satisfy (i) and (ii). Let

$$u_j = \{z : \forall i < j \, \exists y_i \in t_i (m(y_0, \dots, y_{j-1}, z, x_{j+1}, \dots, x_{n-1}) \notin s)\}.$$

Using the fact that m is additive and normal it can be shown that u_j is an ideal of $\mathfrak{B}$ that is disjoint from the principal filter generated by x_j. But then $\mathfrak{B}$ must contain an ultrafilter t_j that includes x_j and is disjoint from u_j. This is enough to ensure that (i) and (ii) hold with j in place of i.

The full details of this argument may be found in Theorem 2.2.1 of [14], where the proof is shown to work for any normal operator on a distributive lattice.

Now to each BAO $\mathfrak{A} = \langle \mathfrak{B}, m_\beta \rangle_{\beta < \alpha_\tau}$ of type τ we can associate the type-τ relational structure

$$\mathbb{C}\mathsf{st}\, \mathfrak{A} = \langle S_{\mathfrak{B}}, R_{m_\beta} \rangle_{\beta < \alpha_\tau}$$

which we call the *canonical structure* of $\mathfrak{A}$. Its complex algebra will be denoted [1] $\mathsf{Em}\,\mathfrak{A}$ and is the *canonical embedding algebra* of $\mathfrak{A}$:

$$\mathsf{Em}\,\mathfrak{A} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}.$$

Writing $\eta_{\mathfrak{A}}$ for the function $\eta_{\mathfrak{B}}$ determined by the underlying BA $\mathfrak{B}$ of $\mathfrak{A}$, we have:

Theorem 3.1 *If $\mathfrak{A}$ is a normal BAO, the function $\eta_{\mathfrak{A}} : \mathfrak{A} \to \mathsf{Em}\,\mathfrak{A}$ is an injective BAO-homomorphism, representing $\mathfrak{A}$, by its isomorphic image in $\mathsf{Em}\,\mathfrak{A}$, as a complex algebra.*

This result is due to Jónsson and Tarski [32] who developed it from a more abstract standpoint, in two stages, using the notion of perfect extension. If Boolean algebra $\mathfrak{B}$ is a subalgebra of Boolean algebra $\mathfrak{B}^\sigma$, then $\mathfrak{B}^\sigma$ is a *perfect extension* of $\mathfrak{B}$, and $\mathfrak{B}$ is a *regular subalgebra* of $\mathfrak{B}^\sigma$, if $\mathfrak{B}^\sigma$ is complete and atomic and satisfies

(I) if x and y are distinct atoms of $\mathfrak{B}^\sigma$, there is an element b of $\mathfrak{B}$ with $x \leq b$ and $y \cdot b = 0$,

(II) if D is a subset of $\mathfrak{B}$ whose join in $\mathfrak{B}^\sigma$ is 1, then D has a finite subset whose join in $\mathfrak{B}^\sigma$ is 1.

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[1] In general, "sans serif" capitals E, H, P, S ... will be used as the first letter in symbolic names for operations on *algebras*, while "blackboard bold" letters $\mathbb{C}$, $\mathbb{E}$, $\mathbb{H}$, $\mathbb{S}$, $\mathbb{U}$ ... occur likewise in names of operations on *structures* (cf. especially Section 4.4).

These conditions characterise $\mathfrak{B}^\sigma$ uniquely up to isomorphism. But if $\mathfrak{B}$ is identified with $\eta_{\mathfrak{B}}(\mathfrak{B}) \subseteq \mathsf{Sb}\, S_{\mathfrak{B}}$, then $\mathsf{Sb}\, S_{\mathfrak{B}}$ fulfills these conditions, and so every BA has a perfect extension. In the topological version of the Stone representation, the members of $\eta_{\mathfrak{B}}(B)$ form a clopen base for a topology on $S_{\mathfrak{B}}$. Conditions (I) and (II) express the fact that the resulting space is Hausdorff and compact.

By analogy with the topological case, an element of a perfect extension $\mathfrak{B}^\sigma$ is called *closed* if it is the meet of a set of elements from $\mathfrak{B}$. Let C be the set of closed elements of $\mathfrak{B}^\sigma$. A function $m : B^n \to B$ induces the n-ary function m^σ on $\mathfrak{B}^\sigma$ defined by the formula

$$m^\sigma(x) = \sum_{x \geq y \in C^n} \prod_{y \leq z \in B^n} m(z) \qquad \text{for all } x \in (B^\sigma)^n \qquad (\dagger)$$

(here $\leq$ and $\geq$ are the product orderings). If m is an operator then m^σ is an extension of m that is completely additive (preserves arbitrary joins) in each of its arguments and is the largest such extension of m (in the pointwise ordering of functions).

There is also an abstract approach to m^σ in the style of (I) and (II). If At denotes the set of atoms of $\mathfrak{B}^\sigma$, consider the statement

(III) $m^\sigma(x) = \prod_{x \leq z \in B^n} m(z)$ for all $x \in (At)^n$,

which is implied by the formula ($\dagger$). A BAO

$$\mathfrak{A}^\sigma = \langle \mathfrak{B}^\sigma, m^\sigma_\beta \rangle_{\beta < \alpha_\tau}$$

is a perfect extension of $\mathfrak{A} = \langle \mathfrak{B}, m_\beta \rangle_{\beta < \alpha_\tau}$, and $\mathfrak{A}$ a regular subalgebra of $\mathfrak{A}^\sigma$, if (I) and (II) hold, each m^σ_β is completely additive, and (III) holds with m_β in place of m for all $\beta < \alpha_\tau$. These axioms characterise $\mathfrak{A}^\sigma$ uniquely up to isomorphism, and the construction of m^σ by ($\dagger$) establishes the Extension Theorem of [32, Th. 2.15] that every BAO $\mathfrak{A}$ has a perfect extension $\mathfrak{A}^\sigma$.

Now if $\mathfrak{A}$ is normal then $\mathfrak{A}^\sigma$ is normal and is isomorphic to the complex algebra $\mathsf{Cm}\,\mathfrak{S}$ of some relational structure $\mathfrak{S}$. Here the underlying set of $\mathfrak{S}$ can be taken as the set of atoms of $\mathfrak{A}^\sigma$, and the relation R_β of $\mathfrak{S}$ as

$$\{\langle s, t_0, \ldots, t_{\rho_\tau(\beta)-1} \rangle : s \leq m^\sigma_\beta(t_0, \ldots, t_{\rho_\tau(\beta)-1})\}.$$

In this way we arrive at the Representation Theorem of [32, Th. 3.10] that every normal BAO of type τ is isomorphic to a regular subalgebra of the complex algebra of a relational structure of type τ.

Thus we may say that the definitions of canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ and canonical embedding algebra $\mathsf{Em}\,\mathfrak{A}$ gives a particular realisation of the abstract notion of perfect extension. It is also noteworthy that in terms of the topological representation based on $S_{\mathfrak{B}}$, axiom (III) for m_β is equivalent to the requirement that the relation R_{m_β} be a closed subset of $(S_{\mathfrak{B}})^{\rho_\tau(\beta)+1}$ in the product topology.

If $\mathfrak{A}$ is finite then $\mathfrak{A} \cong \mathfrak{A}^\sigma$, each ultrafilter of $\mathfrak{A}$ is principal and can be identified with its generating element (an atom), and we get:

Theorem 3.2 *If $\mathfrak{A}$ is a finite BAO, then $\mathfrak{A}$ is isomorphic to the full complex algebra of its canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$.*

The word "canonical" is used extensively in this subject, and we will extend it even further by following the practice of [31] of referring to a perfect extension of $\mathfrak{A}$ as its *canonical extension*. In addition, the *canonical extension* $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ of a relational structure $\mathfrak{S}$ is the canonical structure of the full complex algebra of $\mathfrak{S}$. $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ is a new structure built out of $\mathfrak{S}$ and is called by some authors the *ultrafilter extension* of $\mathfrak{S}$, since its points are the ultrafilters on the underlying set S of $\mathfrak{S}$.

Thus both algebras and structures now have canonical extensions, and the reader will need to identify which is intended from the context. They are most readily compared by the equations

$$\begin{aligned} \mathbb{E}\mathsf{x}\,\mathfrak{S} &= \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S} \\ \mathsf{Em}\,\mathfrak{A} &= \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}. \end{aligned}$$

Note also that

$$\mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S}.$$

3.3 Canonical, Complex and Complete Varieties

As a first application of the representation theory just described, consider the variety $\mathcal{V}_{cl}$ of closure algebras. If $\mathfrak{A} = \langle \mathfrak{B}, m\rangle$ is a closure algebra, then its canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A} = \langle S_{\mathfrak{B}}, R_m\rangle$ is a quasi-ordering. The fact that $x \leq mx$ in $\mathfrak{A}$ ensures that $\{mx : x \in s\} \subseteq s$, and hence $sR_m s$ for any $s \in S_{\mathfrak{B}}$, so R_m is reflexive. Also, if $sR_m tR_m u$ and $x \in u$ then $mmx \in s$ and hence $mx \in s$ as $mmx \leq mx$, showing that $sR_m u$. Thus R_m is transitive. A number of observations about $\mathcal{V}_{cl}$ then follow:

- every member of $\mathcal{V}_{cl}$ is isomorphic to a regular subalgebra of the full complex algebra of a quasi-ordering;
- since the complex algebra of a quasi-ordering is a closure algebra (cf. the table of Section 2.2), the canonical extension $\mathsf{Em}\,\mathfrak{A} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$ belongs to $\mathcal{V}_{cl}$. Hence $\mathcal{V}_{cl}$ is closed under the operation $\mathfrak{A} \mapsto \mathsf{Em}\,\mathfrak{A}$.
- $\mathcal{V}_{cl}$ is generated as a variety by its full complex algebras.

If $\mathfrak{A}$ is a monadic algebra, then the condition $x \leq lmx$ forces R_m to be symmetric. Therefore these three observations hold if $\mathcal{V}_{cl}$ is replaced by the variety $\mathcal{V}_{mn}$ of monadic algebras, "quasi-ordering" is replaced by "equivalence relation", and "closure algebra" is replaced by "monadic algebra".

However, the situation is different for the variety $\mathcal{V}_{dg}$ of diagonalisable algebras. If $\mathfrak{S} = \langle \omega, R\rangle$ with mRn iff $m > n$, then R is transitive and R^{-1} is well-founded, so that $\mathsf{Cm}\,\mathfrak{S}$ belongs to $\mathcal{V}_{dg}$. Now let $\mathfrak{A}$ be the subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ consisting of the finite and the cofinite subsets of ω. We also have $\mathfrak{A} \in \mathcal{V}_{dg}$. However $\mathsf{Em}\,\mathfrak{A}$ is not in $\mathcal{V}_{dg}$. To see this, let s be the set of all cofinite sets. Then s is an ultrafilter of $\mathfrak{A}$, so it is a member of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. If $T \in s$ then T is a non-empty subset of ω, so $m_R(T)$ is cofinite – indeed $m_R(T) = \{m : m > n\}$ where n is the least member of T – and hence $m_R(T) \in s$. This shows that $sR_{m_R}s$ in $\mathbb{C}\mathsf{st}\,\mathfrak{A}$, which is enough to violate the defining condition

$$mx \leq m(x - mx)$$

of $\mathcal{V}_{dg}$ in $\mathsf{Em}\,\mathfrak{A}$ when $x = \{s\}$, since then $0 \neq x \leq mx$ while $m(x - mx) = m0 = 0$.

This example shows that $\mathcal{V}_{dg}$ is not closed under canonical extensions. An example has been given in [14, Th. 3.7.1] of a diagonalisable algebra that is not isomorphic to a subalgebra of *any* diagonalisable algebra of the form $\mathsf{Cm}\,\mathfrak{S}$ (we will give another demonstration of this fact in Section 5.6). Nonetheless $\mathcal{V}_{dg}$ is generated by its full complex algebras, since it is known that it is generated by its finite members, and every finite algebra is isomorphic to one of the form $\mathsf{Cm}\,\mathfrak{S}$ (Theorem 3.2).

If $\mathcal{K}$ is a class of structures, we define

$$\mathsf{Cm}\,\mathcal{K} = \{\mathfrak{A} : \mathfrak{A} \cong \mathsf{Cm}\,\mathfrak{S} \text{ for some } \mathfrak{S} \in \mathcal{K}\}.$$

Since a complex algebra per se is one that is isomorphic to a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ for some $\mathfrak{S}$, the class of *complex algebras of* $\mathcal{K}$ is $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$. The *variety* $\mathsf{Var}\,\mathcal{K}$ *generated by* $\mathcal{K}$ is the smallest variety containing $\mathsf{Cm}\,\mathcal{K}$, i.e. $\mathsf{Var}\,\mathcal{K} = \mathsf{Var}\,\mathsf{Cm}\,\mathcal{K} = \mathsf{HSP}\,\mathsf{Cm}\,\mathcal{K}$.

For a class $\mathcal{W}$ of BAO's, the class of *structures in* $\mathcal{W}$ is defined to be

$$\mathbb{S}\mathsf{tr}\mathcal{W} = \{\mathfrak{S} : \mathsf{Cm}\,\mathfrak{S} \in \mathcal{W}\}.$$

If $\mathcal{W}$ is closed under isomorphism then so is $\mathbb{S}\mathsf{tr}\mathcal{W}$, and $\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathcal{W} \subseteq \mathcal{W}$. Hence if $\mathcal{V}$ is a variety, it contains the variety generated by its own structures: $\mathsf{Var}\,\mathbb{S}\mathsf{tr}\mathcal{V} \subseteq \mathcal{V}$.

Armed with these concepts, we now introduce three fundamental definitions concerning a variety $\mathcal{V}$.

- $\mathcal{V}$ is *canonical* if it is closed under canonical extensions: $\mathfrak{A} \in \mathcal{V}$ implies $\mathsf{Em}\,\mathfrak{A} \in \mathcal{V}$.
- $\mathcal{V}$ is *complex* if every member of $\mathcal{V}$ is isomorphically embeddable into the full complex algebra of some structure in $\mathcal{V}$, i.e. if $\mathcal{V}$ is equal to $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ for some class $\mathcal{K}$ of structures, and hence consists entirely of complex algebras. Equivalently, a complex variety is one satisfying $\mathcal{V} = \mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathcal{V}$.
- $\mathcal{V}$ is *complete* if it is generated by a class of full complex algebras, i.e. $\mathcal{V} = \mathsf{Var}\,\mathcal{K}$ for some class of structures $\mathcal{K}$, or equivalently $\mathcal{V} = \mathsf{Var}\,\mathbb{S}\mathsf{tr}\mathcal{V}$.

It is immediate that if $\mathcal{V} = \mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ then $\mathcal{V} = \mathsf{Var}\,\mathcal{K}$, so every complex variety is complete. The diagonalisable algebras $\mathcal{V}_{dg}$ form a complete variety that is not complex.

Every canonical variety is complex, as $\mathfrak{A}$ is embeddable in $\mathsf{Em}\,\mathfrak{A} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$, and if $\mathsf{Em}\,\mathfrak{A} \in \mathcal{V}$ then $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\mathcal{V}$. Thus $\mathsf{Em}\,\mathcal{V} \subseteq \mathcal{V}$ implies $\mathcal{V} = \mathsf{S}\,\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathcal{V}$. An instance of a non-canonical complex variety is the one generated by the type 2 real-number frame $\langle \mathbb{R}, <, > \rangle$. This example will be discussed in detail in Section 5.6.

While each of these three properties of a variety are in general distinct, it turns out that when $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultrapowers they become equivalent (cf. Corollary 4.14). We will see in Sections 5.4–5.6 that each of them corresponds to a significant property of modal logics.

The question of which varieties are canonical comes down to the question of which equations are preserved by canonical extensions of algebras. The first general result about this was given by Jónsson and Tarski in Theorem 2.18 of [32] which established that any equation holding in a BAO $\mathfrak{A}$ and not involving the Boolean complementation operation must continue to hold in any perfect extension of $\mathfrak{A}$. Later work in modal logic, culminating in [49], greatly extended the class of such preserved properties. This will be discussed further in Section 5.5.

4 Duality

Each of the fundamental algebraic operations H (homomorphic images) S (subalgebras) and P (direct products) that characterise varieties has a corresponding operation on relational structures. We discuss them in turn.

4.1 Bounded Morphisms

Let $\mathfrak{S}_1 = \langle S_1, R^1_\beta\rangle_{\beta<\alpha_\tau}$ and $\mathfrak{S}_2 = \langle S_2, R^2_\beta\rangle_{\beta<\alpha_\tau}$ be structures of type τ. A *bounded morphism* $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ is a function $f : S_1 \to S_2$ satisfying, for each $\beta < \alpha_\tau$,

$$\begin{array}{rll} R^1_\beta(s_0, \dots, s_{\rho_\tau(\beta)}) & \text{implies} & R^2_\beta(f(s_0), \dots, f(s_{\rho_\tau(\beta)})), \qquad \text{and} \\ R^2_\beta(f(s), u_1, \dots, u_{\rho_\tau(\beta)}) & \text{implies} & \text{there exist } t_1 \dots, t_{\rho_\tau(\beta)} \in S_1 \text{ such that} \\ & & f(t_k) = u_k \text{ for } 1 \le k \le \rho_\tau(\beta), \text{ and} \\ & & R^1_\beta(s, t_1, \dots, t_{\rho_\tau(\beta)}). \end{array}$$

For K-frames with binary relations, this takes the form

$$\begin{array}{rll} s_0 R^1_\beta\, s_1 & \text{implies} & f(s_0) R^2_\beta\, f(s_1) \qquad \text{and} \\ f(s) R^2_\beta\, u & \text{implies} & \text{there exists } t \in S_1 \text{ such that } f(t) = u \text{ and } s R^1_\beta\, t, \end{array}$$

which can be expressed even more succinctly as

$$f(s) R^2_\beta\, u \quad \text{iff} \quad \exists t \in S_1 (f(t) = u \text{ and } s R^1_\beta\, t).$$

In modal logic such functions are often called *p-morphisms* for reasons that are obscure, or *zig-zag morphisms* in view of their "back-and-forth" character. Our choice of the adjective "bounded" reflects the use of bounded existential quantification in expressing the second part of the definition. There is a model-theoretic preservation theorem showing that a first-order sentence preserved by surjective bounded morphisms is equivalent to a positive sentence in which quantifiers only occur in the "R-bounded" forms

$$\forall v_0 \cdots \forall v_{n-1}(R(v, v_0, \dots, v_{n-1}) \to \varphi), \quad \exists v_0 \cdots \exists v_{n-1}(R(v, v_0, \dots, v_{n-1}) \wedge \varphi)$$

(cf. e.g. [14, Th. 4.2.5], and Section 4.6 below).

There is another way of explaining what a bounded morphism is that may appeal to some mathematical tastes. This is based on the observation that a relation $R \subseteq S^{n+1}$ can be identified with the function $R[\text{-}] : S \to Sb\, S^n$ having

$$R[s] = \{\langle t_1, , \dots , , t_n\rangle : R(s, t_1, \dots , t_n)\}.$$

The definition of bounded morphism is equivalent to the requirement that for all $s \in S_1$,

$$f(R^1_\beta[s]) = R^2_\beta[f(s)],$$

which states that the following diagram commutes (where $n = \rho_\tau(\beta)$ and f^n is the function induced coordinate-wise by f).

$$\begin{array}{ccc} S_1 & \stackrel{R^1_\beta[\text{-}]}{\longrightarrow} & Sb\, S_1^n \\ \downarrow f & & \downarrow f^n \\ S_2 & \stackrel{R^2_\beta[\text{-}]}{\longrightarrow} & Sb\, S_2^n \end{array}$$

A bounded morphism $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ induces the function $f^+ : Sb\, S_2 \to Sb\, S_1$ that pulls subsets of S_2 back along f to their inverse images, i.e. $f^+(T) = f^{-1}(T)$. Then f^+ proves to be a homomorphism from $\mathsf{Cm}\,\mathfrak{S}_2$ to $\mathsf{Cm}\,\mathfrak{S}_1$, and indeed the reader may check that the conditions defining a bounded morphism are exactly what is required to make f^+ preserve the polymodal operators:

$$f^+(m_{R^2_\beta}(T_0, \dots, T_{\rho_\tau(\beta)-1})) = m_{R^1_\beta}(f^+(T_0), \dots, f^+(T_{\rho_\tau(\beta)-1})),$$

which is the ultimate explanation of why bounded morphisms are the natural maps to deal with in this context.

Assigning the complex algebra $\mathsf{Cm}\,\mathfrak{S}$ to $\mathfrak{S}$, and f^+ to f gives a contravariant functor from the category of relational structures of type τ with bounded morphisms as arrows to the category of BAO's of type τ with BAO homomorphisms as arrows. This functor is part of a dual equivalence between the former category and the subcategory of the latter category consisting of complete and atomic BAO's with complete homomorphisms. For modal algebras and K-frames, this equivalence is discussed in detail in [57].

The standard symbols $\rightarrowtail$ and $\twoheadrightarrow$ will be used to denote functions that are injective and surjective, respectively. The notations $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ and $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$ indicate that there exists a bounded morphism from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ that is injective or surjective, respectively. Similarly, $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ and $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ indicate the existence of injective and surjective homomorphisms between algebras.

The dual correspondence interchanges injections and surjections: if $f : \mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ then $f^+ : \mathsf{Cm}\,\mathfrak{S}_2 \twoheadrightarrow \mathsf{Cm}\,\mathfrak{S}_1$, and if $f : \mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$ then $f^+ : \mathsf{Cm}\,\mathfrak{S}_2 \rightarrowtail \mathsf{Cm}\,\mathfrak{S}_1$.

A surjective bounded morphism will be called a *bounded epimorphism*, and if $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, then $\mathfrak{S}_2$ is a *bounded epimorphic image* of $\mathfrak{S}_1$. When this happens, the injection $\mathsf{Cm}\,\mathfrak{S}_2 \rightarrowtail \mathsf{Cm}\,\mathfrak{S}_1$ makes $\mathsf{Cm}\,\mathfrak{S}_2$ isomorphic to a subalgebra of $\mathsf{Cm}\,\mathfrak{S}_1$. Hence

Lemma 4.1 *If $\mathfrak{S}_2$ is a bounded epimorphic image of $\mathfrak{S}_1$, then $\mathsf{Cm}\,\mathfrak{S}_2 \rightarrowtail \mathsf{Cm}\,\mathfrak{S}_1$ and every equation satisfied by $\mathsf{Cm}\,\mathfrak{S}_1$ is satisfied by $\mathsf{Cm}\,\mathfrak{S}_2$.*

A homomorphism $g : \mathfrak{A}_1 \to \mathfrak{A}_2$ of BAO's gives rise to a bounded morphism $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \to \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ of their associated canonical structures. g_+ assigns to each ultrafilter s of $\mathfrak{A}_2$ its inverse image $\{x \in A_1 : g(x) \in s\}$, which is an ultrafilter of $\mathfrak{A}_1$. The proof that g_+ is a bounded morphism is elaborate, and similar in strategy to the proof described in Section 3.2 that the canonical embedding function $\eta_\mathfrak{B}$ preserves polymodal operators. Full details may be found in Theorem 2.3.2 of [14].

The correspondence $g \mapsto g_+$ also interchanges injections and surjections: if $g : \mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ then $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \twoheadrightarrow \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$, and if $g : \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ then $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$. Thus if $\mathfrak{A}_1$ is (isomorphic to) a subalgebra of $\mathfrak{A}_2$, then $\mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ is a bounded epimorphic image of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2$. (The reader should be aware that the prefix "epi" is sometimes used for homomorphisms between algebras to indicate a category-theoretic property weaker than surjectivity. In the present article however the word "epimorphism" will only be applied to bounded morphisms of structures, and will be used precisely to indicate their surjectivity.)

Note that $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \to \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ in its turn induces the homomorphism $(g_+)^+$ from $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ to $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}_2$, i.e. from $\mathsf{Em}\,\mathfrak{A}_1$ to $\mathsf{Em}\,\mathfrak{A}_2$. Thus if $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ or $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$, then $\mathsf{Em}\,\mathfrak{A}_1 \rightarrowtail \mathsf{Em}\,\mathfrak{A}_2$ or $\mathsf{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathsf{Em}\,\mathfrak{A}_2$, respectively.

Likewise, from a bounded morphism $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ we get the bounded morphism $(f^+)_+$ from $\mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S}_1$ to $\mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S}_2$, i.e. $(f^+)_+ : \mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \to \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$. Thus if $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ or $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, then $\mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \rightarrowtail \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$ or $\mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \twoheadrightarrow \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$, respectively.

As an application of this duality, we have

Theorem 4.2 *A variety $\mathcal{V}$ is canonical if, and only if, it contains the canonical extensions of all its infinitely-generated free algebras.*

PROOF. If $\mathfrak{A} \in \mathcal{V}$, then there is an infinitely-generated free $\mathfrak{A}_1$ in $\mathcal{V}$ with $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}$. Then $\mathsf{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathsf{Em}\,\mathfrak{A}$, so if $\mathcal{V}$ contains $\mathsf{Em}\,\mathfrak{A}_1$, it will contain $\mathsf{Em}\,\mathfrak{A}$ by closure under homomorphic images. ∎

There is a useful generalisation of the notion of bounded morphism which we will call a *bounded ultrafilter map* from a structure $\mathfrak{S}$ to a BAO $\mathfrak{A}$ of the same type τ. Let $f : \mathfrak{S} \to \mathbb{C}\mathsf{st}\,\mathfrak{A}$ be a function assigning to each point s in $\mathfrak{S}$ an ultrafilter $f(s)$ of $\mathfrak{A}$. Then f induces the function $f^+ : \mathfrak{A} \to \mathsf{Cm}\,\mathfrak{S}$, defined for each element a of $\mathfrak{A}$ by

$$f^+(a) = \{s \in S : a \in f(s)\}.$$

The properties of ultrafilters ensure that f^+ is a Boolean algebra homomorphism. f will be called a *bounded ultrafilter map from $\mathfrak{S}$ to $\mathfrak{A}$* if it satisfies, for all $\beta < \alpha_\tau$, $s \in S$, and $a_0, \dots, a_{\rho_\tau(\beta)-1} \in \mathfrak{A}$,

$$m_\beta(a_0, \dots, a_{\rho_\tau(\beta)-1}) \in f(s) \quad \text{iff} \quad \begin{array}{l}\text{there exist } t_0, \dots, t_{\rho_\tau(\beta)-1} \in S \text{ such} \\ \text{that } R_\beta(s, t_0, \dots, t_{\rho_\tau(\beta)-1}) \text{ and} \\ a_i \in f(t_i) \text{ for all } i < \rho_\tau(\beta).\end{array}$$

This is the condition that ensures that f^+ preserves the polymodal operators m_β of $\mathfrak{A}$ and m_{R_β} of $\mathsf{Cm}\,\mathfrak{S}$, and hence is a BAO homomorphism.

f will be said to *cover* $\mathfrak{A}$ if for each non-zero element a of $\mathfrak{A}$ there is some $s \in S$ such that $a \in f(s)$. This ensures that

$$a \neq 0 \quad \text{implies} \quad f^+(a) \neq \emptyset,$$

so that f^+ is an injection of $\mathfrak{A}$ into $\mathsf{Cm}\,\mathfrak{S}$. To summarize:

Theorem 4.3 *If there exists a bounded ultrafilter map from $\mathfrak{S}$ to $\mathfrak{A}$ that covers $\mathfrak{A}$, then the induced homomorphism $\mathfrak{A} \to \mathsf{Cm}\,\mathfrak{S}$ is injective and makes $\mathfrak{A}$ isomorphic to a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$.* □

A bounded ultrafilter map f from $\mathfrak{S}$ to $\mathsf{Cm}\,\mathfrak{T}$ may be thought of as a bounded ultrafilter map from $\mathfrak{S}$ to the structure $\mathfrak{T}$. Such a map covers $\mathsf{Cm}\,\mathfrak{T}$ precisely when its range includes all principal ultrafilters of $\mathsf{Cm}\,\mathfrak{T}$.

A special case of this arises from a bounded morphism $f : \mathfrak{S} \to \mathfrak{T}$, which can be identified with the ultrafilter map $f_\# : \mathfrak{S} \to \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{T} = \mathbb{E}\mathsf{x}\,\mathfrak{T}$ for which $f_\#(s)$ is the principal ultrafilter

$$\{U \subseteq T : f(s) \in U\}$$

of $\mathsf{Cm}\,\mathfrak{T}$ generated by $\{f(s)\}$. $f_{\#}$ covers $\mathsf{Cm}\,\mathfrak{T}$ precisely when f is surjective.

Ultrafilter maps were introduced for type 1 Kripke frames in [57], where it was shown that the category of frames with bounded ultrafilter maps is dually equivalent to the category of complete and atomic modal algebras with ordinary algebraic homomorphisms as arrows.

Bounded ultrafilter maps will be used in Section 5.6 in proving that the variety generated by Dedekind-complete linear orderings is complex.

4.2 Non-Equational Properties

Bounded morphisms can be used to show that there is no equation that is satisfied by the complex algebra of a type 1 frame precisely when its binary relation is irreflexive, i.e. $\forall s \neg(sRs)$.

Let $\mathfrak{S}_1$ be the frame $\langle \omega, < \rangle$ and $\mathfrak{S}_2$ the one-element frame $\langle \{0\}, R \rangle$ with $0R0$. The unique map $\omega \to \{0\}$ is a bounded epimorphism, so every equation satisfied by $\mathsf{Cm}\,\mathfrak{S}_1$ is satisfied by $\mathsf{Cm}\,\mathfrak{S}_2$ (4.1). But $\mathfrak{S}_1$ is irreflexive while $\mathfrak{S}_2$ is not.

Notice that another property enjoyed by $\mathfrak{S}_1$ but not $\mathfrak{S}_2$ is asymmetry, i.e. if sRt then not tRs, so this is not equationally definable either.

Similarly, there is no equation that we can add to the definition of "closure algebra" to characterise those quasi-ordered frames that are *partially* ordered, meaning that they are antisymmetric: sRt and tRs implies $s = t$. If $\mathfrak{S}'_1$ is the partial order $\langle \omega, \leq \rangle$ and $\mathfrak{S}'_2 = \langle \{0,1\}, R \rangle$ with R the universal relation, then putting $f(m) = 1$ iff m is even gives a bounded epimorphism $\mathfrak{S}'_1 \to \mathfrak{S}'_2$ showing that equations are preserved in passing from $\mathsf{Cm}\,\mathfrak{S}'_1$ to $\mathsf{Cm}\,\mathfrak{S}'_2$. But $\mathfrak{S}'_2$ is not antisymmetric.

By the same token, we can use bounded morphisms to *impose* conditions like irreflexivity and antisymmetry when representing certain algebras as complex algebras. For example, if a frame $\mathfrak{S} = \langle S, R \rangle$ contains a point s that is reflexive, i.e. sRs, we remove s and replace it by a copy $\{\langle n, s \rangle : n < \omega\}$ of the frame $\langle \omega, < \rangle$. Each new point $\langle n, s \rangle$ bears the same relation to the old points that s did, the old points are unaltered in their relation to each other, and finally

$$\langle n, s \rangle R \langle m, s \rangle \quad \text{iff} \quad n < m.$$

Thus none of the new points are reflexive, and the new frame $\mathfrak{S}'$ has a bounded morphism f onto $\mathfrak{S}$, that acts by $f(\langle n, s \rangle) = s$ and otherwise is the identity function. It follows that $\mathsf{Cm}\,\mathfrak{S}$ is isomorphic to a subalgebra of $\mathsf{Cm}\,\mathfrak{S}'$. By removing all reflexive points in this way, it can be shown that any modal algebra can be embedded into the complex algebra of a K-frame whose relations are irreflexive.

This technique, which is sometimes called "bulldozing" in modal logic, has been most effectively used for modifying frames $\mathfrak{S} = \langle S, R \rangle$ with R a *transitive* binary relation. On such a frame an equivalence relation $\sim$ is given by

$$s \sim t \quad \text{iff} \quad s = t \text{ or } (sRt \text{ and } tRs).$$

The equivalence class $C_s = \{t : s \sim t\}$ is called the *cluster* of s. Putting

$$C_s \leq C_t \quad \text{iff} \quad sRt$$

gives a well-defined relation between clusters that is transitive and antisymmetric. Hence putting

$$\begin{aligned} C_s < C_t \quad &\text{iff} \quad C_s \leq C_t \text{ and } C_s \neq C_t \\ &\text{iff} \quad sRt \text{ and not } tRs \end{aligned}$$

defines $<$ to be a *strict ordering*, i.e. transitive and irreflexive, hence asymmetric.

There are three types of cluster. A *degenerate* cluster consists of a single irreflexive point, a *simple* one consists of a single reflexive point, and a *proper* cluster contains at least two points, which must be reflexive because the relation R is universal on a proper cluster. Thus a partial ordering is itself a transitive frame in which all clusters are simple, and a strict ordering is one in which all clusters are degenerate.

A partial ordering is called *linear* if it is *connected*, i.e. one of sRt and tRs holds for all distinct s, t. If C is a proper cluster in $\mathfrak{S}$, we "flatten" C to a linear ordering by first taking an arbitrary linear ordering $\leq_C$ of C and then replacing C by ω copies of $\leq_C$, i.e. one for each natural number. The new frame $\mathfrak{S}'$ has $\omega \times C$ in place of C, with the new points being ordered by putting

$$\langle n, s\rangle R' \langle m, t\rangle \quad \text{iff} \quad n < m \text{ or else } n = m \text{ and } s \leq_C t.$$

Then similarly to the above case we can show that $\mathfrak{S}$ is a bounded epimorphic image of $\mathfrak{S}'$, with $\mathfrak{S}'$ having a sequence of simple clusters $\{\langle n, s\rangle\}$ in place of C.

This construction leads to the following conclusions:

- every quasi-ordering is a bounded epimorphic image of a partial ordering;
- every connected quasi-ordering is a bounded epimorphic image of a linear ordering;
- every closure algebra is isomorphic to a subalgebra of the complex algebra of a partial ordering, and hence
- the variety $\mathcal{V}_{cl}$ of closure algebras is generated by the complex algebras of partial orderings.

If, instead of $\leq_C$, we take a *strict* linear ordering $<_C$ of C and put

$$\langle n, s\rangle R' \langle m, t\rangle \quad \text{iff} \quad n < m \text{ or else } n = m \text{ and } s <_C t,$$

the result is to bulldoze C into a strict linear ordering. By doing this to all non-degenerate clusters we show that every transitive frame is a bounded epimorphic image of a strict ordering, and every connected transitive frame is a bounded epimorphic image of a strict linear ordering.

The study of linear temporal logic is based on *connected time-frames*, which are type 2 frames of the form $\mathfrak{S} = \langle S, R, R^{-1}\rangle$ with R (and R^{-1}) being transitive and connected. Bounded morphisms for such structures have to respect both R and R^{-1}, and so in bulldozing a cluster C we use $\mathbb{Z} \times C$, i.e. replace C by one copy of $\leq_C$ or $<_C$ for each integer, giving a strict linear ordering that is endless in both directions. In this way it is shown that every connected time-frame is a bounded epimorphic image of a strict linear time-frame, and every reflexive connected time-frame is a bounded epimorphic image of a linearly ordering. In certain circumstances we can then carry this even further by replacing each member of $\mathbb{Z} \times C$ by a copy of the rationals $\mathbb{Q}$ to obtain a dense linear ordering having the original frame as a bounded epimorphic image.

4.3 Inner Substructures

A structure $\mathfrak{S}_1 = \langle S_1, R^1_\beta\rangle_{\beta<\alpha_\tau}$ is an *inner substructure* of $\mathfrak{S}_2 = \langle S_2, R^2_\beta\rangle_{\beta<\alpha_\tau}$ if $S_1 \subseteq S_2$, and the inclusion $S_1 \hookrightarrow S_2$ is a bounded morphism from $\mathfrak{S}_1$ to $\mathfrak{S}_2$. This is equivalent to requiring that $\mathfrak{S}_1$ be a *substructure* of $\mathfrak{S}_2$ in the standard sense, i.e.

$$R^1_\beta = R^2_\beta \cap S_1^{\rho_\tau(\beta)+1},$$

and that

$$R^2_\beta(s, t_1, \dots, t_{\rho_\tau(\beta)}) \text{ and } s \in S_1 \quad \text{implies} \quad t_1, \dots, t_{\rho_\tau(\beta)} \in S_1$$

(cf. [14, Lemma 3.2.2]).

The image of any bounded morphism is always an inner substructure of the codomain. In particular, if $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ is an injective bounded morphism then the image of f is an inner substructure of $\mathfrak{S}_2$ isomorphic to $\mathfrak{S}_1$ under f, and conversely. For instance, if a BAO $\mathfrak{A}_2$ is a homomorphic image of $\mathfrak{A}_1$, then the epimorphism $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ induces an injective bounded morphism $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ making the canonical structure of $\mathfrak{A}_2$ isomorphic to an inner substructure of the canonical structure of $\mathfrak{A}_1$.

Observe also that from a bounded injection $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ we get a surjective homomorphism $\mathsf{Cm}\,\mathfrak{S}_2 \twoheadrightarrow \mathsf{Cm}\,\mathfrak{S}_1$ which preserves equations. Hence

Lemma 4.4 *If $\mathfrak{S}_1$ is isomorphic to an inner substructure of $\mathfrak{S}_2$, then $\mathsf{Cm}\,\mathfrak{S}_1$ is a homomorphic image of $\mathsf{Cm}\,\mathfrak{S}_2$ and every equation satisfied by $\mathsf{Cm}\,\mathfrak{S}_2$ is satisfied by $\mathsf{Cm}\,\mathfrak{S}_1$.*

For type 1 frames, the definition of $\mathfrak{S}_1 = \langle S_1, R_1\rangle$ being an inner substructure of $\mathfrak{S}_2 = \langle S_2, R_2\rangle$ is particularly direct: S_1 is a subset of S_2, R_1 is the restriction of R_2 to S_1, and S_1 is closed under R_2 in the sense that

if sR_2t and $s \in S_1$, then $t \in S_1$.

In modal logic some authors refer here to $\mathfrak{S}_1$ being a *generated* subframe of $\mathfrak{S}_2$, the name originating from the emphasis there is in modal model theory on subframes that are generated by a single element. To consider this notion, let s be a point in a type 1 frame $\mathfrak{S} = \langle S, R\rangle$. Then the subframe *generated by* s is the substructure $\mathfrak{S}_s$ of $\mathfrak{S}$ whose underlying set S_s is the intersection of all inner substructures of $\mathfrak{S}$ that contain s. $\mathfrak{S}_s$ is itself an inner substructure of $\mathfrak{S}$, with

$$S_s = \{t \in S : sR^*t\},$$

where R^* is the reflexive transitive closure of R. Thus $t \in S_s$ iff there exists a sequence $t_0, \dots, t_n$ of members of S (for some $n \geq 0$) such that

$$s = t_0Rt_1R\cdots Rt_n = t.$$

The importance of this notion derives from the fact that in a modal model based on $\mathfrak{S}$ (cf. Section 5.3), truth-values of formulae at s depend only on the truth-values at points in $\mathfrak{S}_s$.

For a general structure $\mathfrak{S} = \langle S, R_\beta\rangle_{\beta<\alpha_\tau}$ of type τ, the characterisation of the smallest inner substructure $\mathfrak{S}_s$ of $\mathfrak{S}$ containing the point s is rather more elaborate, but in similar vein. First, define a binary relation $R_{\mathfrak{S}}$ on S by putting

$$tR_{\mathfrak{S}}u \quad \text{iff} \quad \exists\beta < \alpha_\tau \text{ and } \exists u_1, \dots, u_{\rho_\tau(\beta)} \in S \text{ such that } R_\beta(t, u_1, \dots, u_{\rho_\tau(\beta)}) \text{ and } u = u_i \text{ for some } i \leq \rho_\tau(\beta).$$

Then $\mathfrak{S}_s$ is the inner substructure of the frame $\langle S, R_{\mathfrak{S}}\rangle$ that is generated by s, i.e.

$$S_s = \{t \in S : sR^*_{\mathfrak{S}}t\}.$$

Now the bounded inclusion $\mathfrak{S}_s \hookrightarrow \mathfrak{S}$ induces the homomorphism $g_s : \mathsf{Cm}\,\mathfrak{S} \twoheadrightarrow \mathsf{Cm}\,\mathfrak{S}_s$ having $g_s(T) = T \cap S_s$. Since $g_s(T) \neq \emptyset$ if $s \in T$, it follows that the product map $\langle g_s : s \in S\rangle$ is an injection of $Sb\,S$ into the product of the algebras $\mathsf{Cm}\,\mathfrak{S}_s$, and since the g_s's are surjective, this give a subdirect-product representation

$$\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \prod_{s \in S} \mathsf{Cm}\,\mathfrak{S}_s$$

of $\mathsf{Cm}\,\mathfrak{S}$ in terms of the complex algebras of point-generated structures. In fact this is a representation by subdirectly irreducibles: the algebra $\mathsf{Cm}\,\mathfrak{S}_s$ is always subdirectly irreducible, as shown in [14, Th. 3.3.2].

If $\mathfrak{A}$ is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$, then taking $\mathfrak{A}_s$ to be the subalgebra of $\mathsf{Cm}\,\mathfrak{S}_s$ that is the image of $\mathfrak{A}$ under g_s, by similar reasoning we get the subdirect representation

$$\mathfrak{A} \rightarrowtail \prod_{s \in S} \mathfrak{A}_s.$$

Combined with the representation underlying Theorem 3.1, this yields

Theorem 4.5 *Every normal BAO has a subdirect representation by complex algebras based on point-generated structures.*

An important case of the notion of inner substructure arises in the context of the study of *cylindric algebras*, specifically in the concept of a *weak Cartesian structure*. If U is a set then ${}^\alpha U$ is the set of all sequences $x = \langle x_\lambda : \lambda < \alpha\rangle$ of length α whose terms x_λ all belong to U. ${}^\alpha U$ is known as the *α-dimensional Cartesian space with base U*. Each subset S of ${}^\alpha U$ determines the structure

$$\mathfrak{S}(S) = \langle S, R^S_\lambda, E^S_{\lambda\mu}\rangle_{\lambda,\mu<\alpha},$$

where

$$\begin{aligned} R^S_\lambda &= \{\langle x, y\rangle : x, y \in S \text{ and } x_\mu = y_\mu \text{ for all } \mu < \alpha \text{ with } \mu \neq \lambda\}, \\ E^S_{\lambda\mu} &= \{x \in S : x_\lambda = x_\mu\}. \end{aligned}$$

When $S = {}^\alpha U$, structures of the form $\mathfrak{S}({}^\alpha U)$ are called *(full) Cartesian structures of dimension α*, and the structures isomorphic to these form the class $\mathfrak{Fct}_\alpha$.

Complex algebras that are based on Cartesian structures $\mathfrak{S}({}^\alpha U)$ are known as *cylindric set algebras of dimension α* and form the class $\mathbf{Cs}_\alpha$ [HMTII, Definition

3.1.1]. (Note that if $S \neq {}^\alpha U$, $\mathsf{Cm}\,\mathfrak{S}(S)$ may not be a cylindric algebra at all.) Thus if $\mathbf{I}$ is the isomorphism closure operator, then

$$\mathbf{ICs}_\alpha = \mathsf{S}\,\mathsf{Cm}\,\mathfrak{Fct}_\alpha.$$

A *representable cylindric algebra of dimension* α is an algebra that is isomorphic to a direct product of cylindric set algebras of dimension α. Thus the class $\mathbf{RCA}_\alpha$ of representable cylindric algebras is given by

$$\mathbf{RCA}_\alpha = \mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathfrak{Fct}_\alpha.$$

In [19, Lemma 3.4] we show that

$$\mathbf{RCA}_\alpha = \mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathbb{S}\,\mathfrak{Fct}_\alpha,$$

where $\mathbb{S}$ denotes the operation of forming the class of inner substructures of the members of a given class of structures (cf. Section 4.4). This fact, together with results described in Section 4.6 below, can be used to give a new proof that $\mathbf{RCA}_\alpha$ is a canonical variety.

Now if $x \in {}^\alpha U$ then the *weak Cartesian space with base* U *and dimension* α *determined by* x is the set

$${}^\alpha U^{(x)} = \{y \in {}^\alpha U : \{\lambda < \alpha : y_\lambda \neq x_\lambda\} \text{ is finite}\},$$

and $\mathfrak{S}({}^\alpha U^{(x)})$ is a *weak Cartesian structure of dimension* α. The class $\mathfrak{Wct}_\alpha$ consists of all structures isomorphic to those of the form $\mathfrak{S}({}^\alpha U^{(x)})$.

In the case that α is finite, then by definition ${}^\alpha U^{(x)} = {}^\alpha U$, and so $\mathfrak{S}({}^\alpha U^{(x)})$ is just $\mathfrak{S}({}^\alpha U)$ itself, i.e. in this case all weak Cartesian structures are full. But in any case we have that $\mathfrak{S}({}^\alpha U^{(x)})$ *is the inner substructure* $\mathfrak{S}({}^\alpha U)_x$ *of* $\mathfrak{S}({}^\alpha U)$ *point-generated by* x, as described above. This follows because the relation $R_{\mathfrak{S}({}^\alpha U)}$ used to define $\mathfrak{S}({}^\alpha U)_x$ satisfies

$$xR_{\mathfrak{S}({}^\alpha U)}y \quad \text{iff} \quad \{\lambda < \alpha : y_\lambda \neq x_\lambda\} \text{ is finite.}$$

$R_{\mathfrak{S}({}^\alpha U)}$ is in fact the smallest equivalence relation on ${}^\alpha U$ that contains all the relations R^S_λ, and the point-generated structure $\mathfrak{S}({}^\alpha U)_x$ is based on the $R_{\mathfrak{S}({}^\alpha U)}$-equivalence class of the point x. Thus distinct weak Cartesian substructures of $\mathfrak{S}({}^\alpha U)$ are disjoint, and if $\mathfrak{T}$ is any inner substructure of $\mathfrak{S}({}^\alpha U)$ then $\mathfrak{T}$ will be the disjoint union of those weak Cartesian substructures generated by points of $\mathfrak{T}$. It follows that

$$\mathsf{Cm}\,\mathfrak{T} \cong \prod_{x \in \mathfrak{T}} \mathsf{Cm}\,\mathfrak{S}({}^\alpha U^{(x)}),$$

and this establishes the relationship

$$\mathsf{Cm}\,\mathbb{S}\,\mathfrak{Fct}_\alpha \subseteq \mathsf{P}\,\mathsf{Cm}\,\mathfrak{Wct}_\alpha.$$

Further characterisations of representable cylindric algebras obtained in [19] include

$$\mathbf{RCA}_\alpha = \mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathbb{S}\,\mathfrak{Wct}_\alpha = \mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathfrak{Wct}_\alpha.$$

4.4 The Calculus of Class Operations

First, here is a summary of the main features of the duality between BAO's and relational structures thus far developed:

- A bounded morphism $f : \mathfrak{S}_1 \to \mathfrak{S}_2$ induces a homomorphism $f^+ : \mathsf{Cm}\,\mathfrak{S}_2 \to \mathsf{Cm}\,\mathfrak{S}_1$, such that if $f : \mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ then $f^+ : \mathsf{Cm}\,\mathfrak{S}_2 \twoheadrightarrow \mathsf{Cm}\,\mathfrak{S}_1$, and if $f : \mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$ then $f^+ : \mathsf{Cm}\,\mathfrak{S}_2 \rightarrowtail \mathsf{Cm}\,\mathfrak{S}_1$.
- A homomorphism $g : \mathfrak{A}_1 \to \mathfrak{A}_2$ induces a bounded morphism $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \to \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ such that if $g : \mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ then $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \twoheadrightarrow \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$, and if $g : \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ then $g_+ : \mathbb{C}\mathsf{st}\,\mathfrak{A}_2 \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}_1$.
- If $\mathfrak{S}_1$ is (isomorphic to) an inner substructure of $\mathfrak{S}_2$, then $\mathsf{Cm}\,\mathfrak{S}_1$ is a homomorphic image of $\mathsf{Cm}\,\mathfrak{S}_2$.
- If $\mathfrak{S}_2$ is a bounded epimorphic image of $\mathfrak{S}_1$, then $\mathsf{Cm}\,\mathfrak{S}_2$ is isomorphic to a subalgebra of $\mathsf{Cm}\,\mathfrak{S}_1$.
- If $\mathfrak{A}_1$ is (isomorphic to) a subalgebra of $\mathfrak{A}_2$, then $\mathbb{C}\mathsf{st}\,\mathfrak{A}_1$ is a bounded epimorphic image of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2$.
- If $\mathfrak{A}_2$ is a homomorphic image of $\mathfrak{A}_1$, then $\mathbb{C}\mathsf{st}\,\mathfrak{A}_2$ is isomorphic to an inner substructure of $\mathbb{C}\mathsf{st}\,\mathfrak{A}_1$.
- If $\mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2$ or $\mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$, respectively, then $\mathsf{Em}\,\mathfrak{A}_1 \rightarrowtail \mathsf{Em}\,\mathfrak{A}_2$ or $\mathsf{Em}\,\mathfrak{A}_1 \twoheadrightarrow \mathsf{Em}\,\mathfrak{A}_2$, respectively.
- If $\mathfrak{S}_1 \rightarrowtail \mathfrak{S}_2$ or $\mathfrak{S}_1 \twoheadrightarrow \mathfrak{S}_2$, respectively, then $\mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \rightarrowtail \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$ or $\mathbb{E}\mathsf{x}\,\mathfrak{S}_1 \twoheadrightarrow \mathbb{E}\mathsf{x}\,\mathfrak{S}_2$, respectively.

Now for a class $\mathcal{K}$ of structures, let $\mathbb{H}\mathcal{K}$ be the class of all bounded epimorphic images of members of $\mathcal{K}$, and $\mathbb{S}\,\mathcal{K}$ the class of all structures that are isomorphic to an inner substructure of some member of $\mathcal{K}$. We can combine these operations to form $\mathbb{S}\,\mathbb{H}\mathcal{K}$, $\mathbb{H}\mathbb{H}\mathcal{K}$ etc., and also combine them with other operations on classes of structures or algebras, as in $\mathsf{S}\,\mathsf{Cm}\,\mathbb{H}\mathcal{K}$ etc. To compare such class operations $\mathbb{X}, \mathbb{Y}$, the partial ordering $\mathbb{X} \leq \mathbb{Y}$ is defined to mean that $\mathbb{X}\mathcal{K} \subseteq \mathbb{Y}\mathcal{K}$ for all classes $\mathcal{K}$. Thus the first duality statement above entails that

$$\mathsf{Cm}\,\mathbb{S} \leq \mathsf{H}\,\mathsf{Cm} \quad \text{and} \quad \mathsf{Cm}\,\mathbb{H} \leq \mathsf{S}\,\mathsf{Cm}.$$

Theorem 4.6

(1) $\mathbb{H}\mathbb{H} = \mathbb{H}$, $\mathbb{S}\,\mathbb{S} = \mathbb{S}$.

(2) $\mathbb{S}\,\mathbb{H} \leq \mathbb{H}\mathbb{S}$.

(3) *If $\mathcal{V}$ is closed under subalgebras and homomorphic images, then $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under $\mathbb{H}$ and $\mathbb{S}$ and reflects $\mathbb{E}\mathsf{x}$, i.e. $\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$ implies $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$.*

(4) $\mathsf{Em}\,\mathsf{S}\,\mathsf{Cm} \leq \mathsf{S}\,\mathsf{Cm}\,\mathbb{E}\mathsf{x}$.

(5) $\mathsf{H}\,\mathsf{Cm}\,\mathbb{S} = \mathsf{H}\,\mathsf{Cm}$ *and* $\mathsf{S}\,\mathsf{Cm}\,\mathbb{H} = \mathsf{S}\,\mathsf{Cm}$.

(6) $\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S} \leq \mathbb{S}\,\mathbb{H}\mathbb{C}\mathsf{st}$.

(7) $\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S}\,\mathsf{Cm} \leq \mathbb{S}\,\mathbb{H}\mathbb{E}\mathsf{x}$.

Proof. (1) and (2) are fairly routine and left to the reader.

(3) Closure of $\mathbb{S}\mathsf{tr}\mathcal{V}$ under $\mathbb{H}$ and $\mathbb{S}$ is given by Lemmas 4.1 and 4.4. For reflection of $\mathbb{E}\mathsf{x}$, observe that from the canonical embedding

$$\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S}$$

it follows that if $\mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathcal{V}$ then $\mathsf{Cm}\,\mathfrak{S} \in \mathcal{V}$, as desired.

(4) Let $\mathfrak{A} \in \mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$. Then $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{S}$ for some $\mathfrak{S} \in \mathcal{K}$. Hence

$$\mathsf{Em}\,\mathfrak{A} \rightarrowtail \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S},$$

showing $\mathsf{Em}\,\mathfrak{A} \in \mathsf{S}\,\mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathcal{K}$.

(5) We noted above that $\mathsf{Cm}\,\mathbb{S} \leq \mathsf{H}\,\mathsf{Cm}$. Therefore

$$\mathsf{H}\,\mathsf{Cm}\,\mathbb{S} \leq \mathsf{H}\,\mathsf{H}\,\mathsf{Cm} = \mathsf{H}\,\mathsf{Cm} \leq \mathsf{H}\,\mathsf{Cm}\,\mathbb{S},$$

giving $\mathsf{H}\,\mathsf{Cm}\,\mathbb{S} = \mathsf{H}\,\mathsf{Cm}$. Similarly $\mathsf{S}\,\mathsf{Cm}\,\mathbb{H} = \mathsf{S}\,\mathsf{Cm}$ follows from $\mathsf{Cm}\,\mathbb{H} \leq \mathsf{S}\,\mathsf{Cm}$.

(6) If $\mathfrak{A} \in \mathsf{H}\,\mathsf{S}\,\mathcal{W}$, for $\mathcal{W}$ a class of BAO's, then there are algebras $\mathfrak{A}_1, \mathfrak{A}_2$ with $\mathfrak{A}_2 \in \mathcal{W}$ and

$$\mathfrak{A} \twoheadleftarrow \mathfrak{A}_1 \rightarrowtail \mathfrak{A}_2.$$

Hence by duality,

$$\mathbb{C}\mathsf{st}\,\mathfrak{A} \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}_1 \twoheadleftarrow \mathbb{C}\mathsf{st}\,\mathfrak{A}_2,$$

showing $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\,\mathbb{H}\mathbb{C}\mathsf{st}\,\mathcal{W}$, as desired.

(7) By (6) $\mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathcal{K}$. But $\mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathcal{K} = \mathbb{E}\mathsf{x}\,\mathcal{K}$ by definition of $\mathbb{E}\mathsf{x}$.

∎

The results listed in 4.4.1 provide an effective calculus for reasoning about the closure properties of various classes (cf. the proof of 4.5.3 and 4.6.6 below for example). We may view 4.4.1(4) as saying that the operator Em can pass to the right of the combination $\mathsf{S}\,\mathsf{Cm}$ to become $\mathbb{E}\mathsf{x}$, while 4.4.1(5) says that $\mathsf{H}\,\mathsf{Cm}$ absorbs $\mathbb{S}$ on the right, etc.

4.5 Disjoint and Bounded Unions

The dual to the algebraic construction of direct products is the structural operation of formation of disjoint unions. If $\{\mathfrak{S}_j : j \in J\}$ is a collection of τ-structures $\mathfrak{S}_j = \langle S_j, R^j_\beta\rangle_{\beta<\alpha_\tau}$, then their *disjoint union* is the τ-structure

$$\coprod_J \mathfrak{S}_j = \langle \textstyle\bigcup_J (X_j \times \{j\}), R_\beta\rangle_{\beta<\alpha_\tau},$$

where

$$R_\beta = \{\langle\langle s_0, j\rangle, \dots, \langle s_{\rho_\tau(\beta)}, j\rangle\rangle : j \in J \text{ and } R^j_\beta(s_0, \dots, s_{\rho_\tau(\beta)})\}.$$

Essentially then, $\coprod_J \mathfrak{S}_j$ is the union of a collection of pairwise disjoint copies $\mathfrak{S}_j \times \{j\}$ of the structures $\mathfrak{S}_j$.

For each $i \in J$, the correspondence $s \mapsto \langle s, i\rangle$ gives an injective bounded morphism $\mathfrak{S}_i \rightarrowtail \coprod_J \mathfrak{S}_j$, whose image $\mathfrak{S}_i \times \{i\}$ is an inner substructure of $\coprod_J \mathfrak{S}_j$ isomorphic to

$\mathfrak{S}_i$. In practice it is often convenient to identify this image with $\mathfrak{S}_i$, i.e. to regard the $\mathfrak{S}_j$'s as being pairwise disjoint, and $\coprod_J \mathfrak{S}_j$ as simply being their union. Then each $\mathfrak{S}_j$ is itself an inner substructure of the disjoint union.

The duality between direct products and disjoint unions is provided by an isomorphism

$$\prod_J \mathsf{Cm}\,\mathfrak{S}_j \cong \mathsf{Cm} \coprod_J \mathfrak{S}_j$$

associating to each member $\langle T_j : j \in J\rangle$ of the direct product of the $\mathsf{Cm}\,\mathfrak{S}_j$'s, the disjoint union of the T_j's ([14, Lemma 3.4.1]). Hence in general

$$\mathsf{P}\,\mathsf{Cm}\,\mathcal{K} = \mathsf{Cm}\,\mathbb{U}\mathsf{d}\,\mathcal{K},$$

where $\mathbb{U}\mathsf{d}\,\mathcal{K}$ is the class of disjoint unions of structures isomorphic to members of $\mathcal{K}$.

A given family $\{\mathfrak{S}_j \xrightarrow{f_j} \mathfrak{S} : j \in J\}$ of functions with the same codomain $\mathfrak{S}$ induces naturally the function

$$\coprod_J \mathfrak{S}_j \xrightarrow{f} \mathfrak{S},$$

where $f(\langle s, j\rangle) = f_j(s)$. It is readily seen that if each f_j is a bounded morphism, then f is also a bounded morphism.

A structure $\mathfrak{S}$ is the *bounded union* of $\{\mathfrak{S}_j : j \in J\}$ if it is the union of the $\mathfrak{S}_j$'s as *inner* substructures, i.e. if

(1) each $\mathfrak{S}_j$ is an inner substructure of $\mathfrak{S}$, and
(2) $S = \bigcup\{S_j : j \in J\}$.

In this case, if $\mathfrak{S}_j \xrightarrow{f_j} \mathfrak{S}$ is the inclusion $S_j \hookrightarrow S$, then the function $f : \coprod_J \mathfrak{S}_j \to \mathfrak{S}$ of the previous paragraph is a bounded epimorphism. Thus *a bounded union of structures is a bounded epimorphic image of their disjoint union.* A weak converse of this is also true: a bounded epimorphic image of a disjoint union $\coprod_J \mathfrak{S}_j$ is the bounded union of the images of the $\mathfrak{S}_j$'s.

Notice also that a disjoint union $\coprod_J \mathfrak{S}_j$ is itself the bounded union of the isomorphic copies $\mathfrak{S}_j \times \{j\}$ of the $\mathfrak{S}_j$'s.

We use the notation $\mathbb{U}\mathsf{b}$ for the operation of forming bounded unions, analogously to $\mathbb{U}\mathsf{d}$.

Observe that if $\mathfrak{S}$ is the bounded union of $\{\mathfrak{S}_j : j \in J\}$, then from (1) by duality we get a surjective homomorphism $\mathsf{Cm}\,\mathfrak{S} \twoheadrightarrow \mathsf{Cm}\,\mathfrak{S}_j$ for each $j \in J$, and these surjections give rise to the product map from $\mathsf{Cm}\,\mathfrak{S}$ to $\prod_J \mathsf{Cm}\,\mathfrak{S}_j$ taking each $T \subseteq S$ to the element $\langle T \cap S_j : j \in J\rangle$ of the direct product of the $\mathsf{Cm}\,\mathfrak{S}_j$'s. But then it follows from (2) that this product map is injective, so we have a *subdirect* embedding

$$\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \prod_J \mathsf{Cm}\,\mathfrak{S}_j.$$

Moreover, if the $\mathfrak{S}_j$'s happen to be pairwise disjoint then this embedding is surjective and reproduces the isomorphism between $\mathsf{Cm} \coprod_J \mathfrak{S}_j$ and $\prod_J \mathsf{Cm}\,\mathfrak{S}_j$ described above. The upshot of this discussion is that

- the notion of bounded union is dual to that of *subdirect product.*

Theorem 4.7

(1) $\mathsf{P}\,\mathsf{Cm} = \mathsf{Cm}\,\mathbb{U}\mathsf{d}$.

(2) *If* $\mathcal{V}$ *is closed under direct products, then* $\mathbb{S}\mathsf{tr}\mathcal{V}$ *is closed under disjoint unions.*

(3) $\mathbb{U}\mathsf{d}\,\mathbb{U}\mathsf{d} = \mathbb{U}\mathsf{d}$, $\mathbb{U}\mathsf{b}\,\mathbb{U}\mathsf{b} = \mathbb{U}\mathsf{b}$.

(4) $\mathbb{U}\mathsf{d} \leq \mathbb{U}\mathsf{b} \leq \mathbb{H}\mathbb{U}\mathsf{d} = \mathbb{H}\mathbb{U}\mathsf{b} = \mathbb{U}\mathsf{b}\,\mathbb{H}$.

(5) $\mathbb{U}\mathsf{d}\,\mathbb{H} \leq \mathbb{H}\mathbb{U}\mathsf{d}$.

(6) $\mathbb{S}\,\mathbb{U}\mathsf{d} = \mathbb{U}\mathsf{d}\,\mathbb{S}$.

(7) $\mathbb{S}\,\mathbb{U}\mathsf{b} \leq \mathbb{U}\mathsf{b}\,\mathbb{S} \leq \mathbb{H}\mathbb{S}\,\mathbb{U}\mathsf{d} = \mathbb{H}\mathbb{S}\,\mathbb{U}\mathsf{b}$.

PROOF. (1) was observed above, and (2) follows from it. The remainder are left to the reader. ∎

We are now in a position to establish some characterisations of canonicity.

Theorem 4.8 *The variety* $\mathsf{Var}\mathcal{K}$ *generated by a class of structures* $\mathcal{K}$ *is canonical if, and only if, the class* $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ *of structures in* $\mathsf{Var}\mathcal{K}$ *is closed under canonical extensions.*

PROOF. If $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ then $\mathsf{Cm}\,\mathfrak{S} \in \mathsf{Var}\mathcal{K}$ so if $\mathsf{Var}\mathcal{K}$ is canonical then $\mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} \in \mathsf{Var}\mathcal{K}$. But $\mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S}$, so this makes $\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ as desired.

For the converse, if $\mathfrak{A}$ belongs to $\mathsf{Var}\mathcal{K} = \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{K}$ there exists an algebra $\mathfrak{A}^*$ and a subfamily $\{\mathfrak{S}_j : j \in J\}$ of $\mathcal{K}$ such that

$$\mathfrak{A} \twoheadleftarrow \mathfrak{A}^* \rightarrowtail \prod_J \mathsf{Cm}\,\mathfrak{S}_j \cong \mathsf{Cm}\,(\coprod_J \mathfrak{S}_j).$$

Putting $\mathfrak{S} = \coprod_J \mathfrak{S}_j$, we then get

$$\mathsf{Em}\,\mathfrak{A} \twoheadleftarrow \mathsf{Em}\,\mathfrak{A}^* \rightarrowtail \mathsf{Em}\,\mathsf{Cm}\,\mathfrak{S} = \mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S}.$$

But $\mathsf{Cm}\,\mathfrak{S}$ is in $\mathsf{Var}\mathcal{K}$, by closure under products and isomorphism, so $\mathsf{Cm}\,\mathbb{E}\mathsf{x}\,\mathfrak{S}$ is in $\mathsf{Var}\mathcal{K}$ if $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ is assumed closed under $\mathbb{E}\mathsf{x}$. Closure of $\mathsf{Var}\mathcal{K}$ under subalgebras and homomorphic images then implies $\mathsf{Em}\,\mathfrak{A} \in \mathsf{Var}\mathcal{K}$. Hence $\mathsf{Var}\mathcal{K}$ is canonical. ∎

Theorem 4.9 *A variety* $\mathcal{V}$ *of BAO's is canonical if, and only if,*

(1) $\mathcal{V}$ *is complete, and*

(2) *the class* $\mathbb{S}\mathsf{tr}\mathcal{V}$ *of structures in* $\mathcal{V}$ *is closed under canonical extensions.*

PROOF. A canonical variety is complete, while closure of $\mathbb{S}\mathsf{tr}\mathcal{V}$ under canonical extensions is a special case of canonicity, as the first part of the previous proof shows.

Now suppose that (1) and (2) hold. Since $\mathcal{V}$ is complete, it is generated by its class of structures $\mathbb{S}\mathsf{tr}\mathcal{V}$, so

$$\begin{aligned} \mathcal{V} &= \mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathcal{V} \\ &= \mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathbb{U}\mathsf{d}\,\mathbb{S}\mathsf{tr}\mathcal{V} \quad \text{as } \mathsf{P}\,\mathsf{Cm} = \mathsf{Cm}\,\mathbb{U}\mathsf{d} \ (4.7(1)) \\ &= \mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathcal{V}, \end{aligned}$$

the last step being because $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under disjoint unions (4.7(2)).

Thus if $\mathfrak{A}$ belongs to $\mathcal{V}$ then $\mathfrak{A} \in \mathsf{HSCm}\,\mathbb{S}\mathsf{tr}\mathcal{V}$, so the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is in $\mathbb{C}\mathsf{st}\,\mathsf{HSCm}\,\mathbb{S}\mathsf{tr}\mathcal{V}$. Now by Theorem 4.6(7),

$$\mathbb{C}\mathsf{st}\,\mathsf{HSCm}\,\mathbb{S}\mathsf{tr}\mathcal{V} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{E}\mathsf{x}\,\mathbb{S}\mathsf{tr}\mathcal{V},$$

and by our hypothesis (2) $\mathbb{E}\mathsf{x}\,\mathbb{S}\mathsf{tr}\mathcal{V} \subseteq \mathbb{S}\mathsf{tr}\mathcal{V}$. But $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under $\mathbb{H}$ and $\mathbb{S}$ by 4.6(3), so altogether $\mathbb{S}\,\mathbb{H}\mathbb{E}\mathsf{x}\,\mathbb{S}\mathsf{tr}\mathcal{V} \subseteq \mathbb{S}\mathsf{tr}\mathcal{V}$ and therefore $\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathbb{S}\mathsf{tr}\mathcal{V}$. Hence $\mathsf{Em}\,\mathfrak{A} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A} \in \mathcal{V}$.

This proves that if $\mathfrak{A}$ is in $\mathcal{V}$ then so is $\mathsf{Em}\,\mathfrak{A}$, i.e. $\mathcal{V}$ is canonical. ∎

A more detailed analysis of the relationship between properties of $\mathbb{S}\mathsf{tr}\mathcal{V}$ and canonicity of $\mathcal{V}$ is given in [14, Sections 3.5, 3.7].

4.6 Ultrapowers and Ultraproducts

$\mathbb{P}\mathsf{u}\,\mathcal{K}$ is the class of all structures that are isomorphic to an ultraproduct of members of $\mathcal{K}$. $\mathbb{P}\mathsf{w}\,\mathcal{K}$ is likewise defined as the closure of $\mathcal{K}$ under ultrapowers. The symbol $\mathbb{R}\mathsf{u}$ denotes the inverse $\mathbb{P}\mathsf{w}^{-1}$ to the operation $\mathbb{P}\mathsf{w}$: $\mathbb{R}\mathsf{u}\,\mathcal{K}$ is the class of *ultraroots* of $\mathcal{K}$, comprising those structures $\mathfrak{S}$ having some ultrapower $\mathfrak{S}^J/F$ isomorphic to a member of $\mathcal{K}$.

$\mathcal{K}$ is defined to be an *elementary class* of relational structures if it is the class of all models of some set of sentences in the first-order language of its type. Elementary classes are characterised as those closed under $\mathbb{P}\mathsf{u}$ and $\mathbb{R}\mathsf{u}$.

There are a number of fundamental results about ultrapowers and ultraproducts that bear on the relationship between elementary logic and the equational logic of complex algebras. The first we consider is

Theorem 4.10 *The class $\mathbb{S}\mathsf{tr}\mathcal{V}$ of structures in a variety $\mathcal{V}$ is closed under ultraroots.*

Proof. This follow from the fact that for any ultraproduct $(\prod_J \mathfrak{S}_j)/F$ there is an injective homomorphism

$$(\prod_J \mathsf{Cm}\,\mathfrak{S}_j)/F \rightarrowtail \mathsf{Cm}\,(\prod_J \mathfrak{S}_j/F),$$

as described in detail in [14, 3.6.5]. In the case of an ultrapower this takes the form

$$\theta : (\mathsf{Cm}\,\mathfrak{S})^J/F \rightarrowtail \mathsf{Cm}\,(\mathfrak{S}^J/F),$$

where, for $T \in (\mathsf{Cm}\,\mathfrak{S})^J$, θ maps T/F to the set

$$\theta(T/F) = \{f/F \in \mathfrak{S}^J/F : \{j : f(j) \in T(j)\} \in F\}.$$

Composing θ with the (elementary) embedding of algebras $\mathsf{Cm}\,\mathfrak{S} \rightarrowtail (\mathsf{Cm}\,\mathfrak{S})^J/F$ yields

$$\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \mathsf{Cm}\,(\mathfrak{S}^J/F).$$

By closure of $\mathcal{V}$ under subalgebras, it follows that $\mathfrak{S}^J/F \in \mathbb{S}\mathsf{tr}\mathcal{V}$ implies $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathcal{V}$, i.e. $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under $\mathbb{R}\mathsf{u}$. ∎

The symbols Pu and Pw will be used for the operations of forming ultraproducts and ultrapowers of *algebras*. Thus the first two sentences of the proof just given assert that

$$\mathsf{Pu}\,\mathsf{Cm} \leq \mathsf{S}\,\mathsf{Cm}\,\mathbb{P}\mathsf{u} \quad \text{and} \quad \mathsf{Pw}\,\mathsf{Cm} \leq \mathsf{S}\,\mathsf{Cm}\,\mathbb{P}\mathsf{w},$$

while the embedding $\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \mathsf{Cm}\,(\mathfrak{S}^J/F)$ establishes

$$\mathsf{Cm}\,\mathbb{R}\mathsf{u} \leq \mathsf{S}\,\mathsf{Cm},$$

and hence $\mathsf{S}\,\mathsf{Cm}\,\mathbb{R}\mathsf{u} = \mathsf{S}\,\mathsf{Cm}$.

Theorem 4.10 implies that $\mathbb{S}\mathsf{tr}\mathcal{V}$ is elementary iff it is closed under ultraproducts. But in fact classes of the form $\mathbb{S}\mathsf{tr}\mathcal{V}$ are even more constrained than this: it is enough for them to be closed under ultra*powers* for it to follow that they are elementary. The proof of this result is based on the following observation.

Theorem 4.11 *An ultraproduct of a collection of structures* $\{\mathfrak{S}_j : j \in J\}$ *is isomorphic to an inner substructure of an ultrapower of their disjoint union. Hence* $\mathbb{P}\mathsf{u} \leq \mathbb{S}\,\mathbb{P}\mathsf{w}\mathbb{U}\mathsf{d}$.

Proof. There is a natural bounded injection

$$(\prod_J \mathfrak{S}_j)/F \rightarrowtail (\coprod_J \mathfrak{S}_j)^J/F$$

taking f/F to g/F, where $g(j) = \langle f(j), j\rangle$. Cf. [14, 3.8.3] for details. ∎

Corollary 4.12 *For any variety of BAO's* $\mathcal{V}$, *the following are equivalent.*

(1) $\mathbb{S}\mathsf{tr}\mathcal{V}$ *is an elementary class.*

(2) $\mathbb{S}\mathsf{tr}\mathcal{V}$ *is closed under elementary equivalence.*

(3) $\mathbb{S}\mathsf{tr}\mathcal{V}$ *is closed under ultrapowers.*

(4) $\mathbb{S}\mathsf{tr}\mathcal{V}$ *is closed under ultraproducts.*

Proof. That (1) implies (2) and (2) implies (3) is standard. That (3) implies (4) follows from Theorem 4.11 and the fact that $\mathbb{S}\mathsf{tr}\mathcal{V}$ is always closed under disjoint unions, inner substructures and isomorphism. Finally, as already noted, (4) implies (1) as a consequence of 4.6.1 and the characterisation of elementary classes as those closed under $\mathbb{P}\mathsf{u}$ and $\mathbb{R}\mathsf{u}$. ∎

Corollary 4.6.3 is the algebraic generalisation of a result that Johan van Benthem originally proved for the class of Kripke frames validating a modal formula. His approach used a model-theoretic compactness argument. A discussion of that proof is given in [20].

The next result is essentially an ultrapowers version of an application of saturated models to modal logic that first appeared in [12].

Theorem 4.13 *For any structure* $\mathfrak{S}$, *the canonical extension* $\mathbb{E}\mathsf{x}\,\mathfrak{S}$ *is a bounded epimorphic image of some ultrapower of* $\mathfrak{S}$. *Hence* $\mathbb{E}\mathsf{x} \leq \mathbb{H}\mathbb{P}\mathsf{w}$.

PROOF. Given an ultrapower $\mathfrak{S}^J/F$, a map

$$\mathfrak{S}^J/F \to \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S}$$

of the desired form is obtained by assigning to each element f/F of $\mathfrak{S}^J/F$ the set

$$\{T \subseteq S : \{j \in J : f(j) \in T\} \in F\},$$

which is indeed an ultrafilter of $\mathsf{Cm}\,\mathfrak{S}$ and hence a member of the canonical structure of $\mathsf{Cm}\,\mathfrak{S}$. If the ultrapower $\mathfrak{S}^J/F$ is ω-saturated, this map is a bounded epimorphism, as shown in detail in Section 3.6 of [14]. ∎

Corollary 4.14 *For any variety of BAO's $\mathcal{V}$, if $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultrapowers, then the following are equivalent.*

(1) *$\mathcal{V}$ is canonical.*

(2) *$\mathcal{V}$ is complex.*

(3) *$\mathcal{V}$ is complete.*

PROOF. We have already observed that (1) implies (2) and (2) implies (3) in general. But if $\mathbb{S}\mathsf{tr}\mathcal{V}$ is closed under ultrapowers, then since it is always closed under bounded epimorphic images (4.6(3)), 4.6.4 implies that it must also be closed under canonical extensions. Hence by Theorem 4.9, if it is complete then it is canonical. ∎

Theorem 4.15 *If a variety of BAO's is generated by an elementary class of structures, then it is canonical.*

PROOF. We give the main features of a proof that has been discussed in detail in the papers [14, 15, 19]. There are two main additional ingredients. First, the fact that an ultraproduct of bounded unions of structures can be represented as a bounded union of ultraproducts of those structures: $\mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{b} \le \mathbb{U}\mathsf{b}\,\mathbb{P}\mathsf{u}$ (cf. Theorem 2.4 of [19] for the proof). To be precise, we need a special case of this fact, namely

(i) $\quad \mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d} \le \mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}.$

Second, a result that shows how the canonical structures of members of $\mathsf{Var}\mathcal{K}$ can be constructed out of members of $\mathcal{K}$:

(ii) $\quad \mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\mathcal{K}.$

The proof of (ii), which holds for any class $\mathcal{K}$, is as follows.

$$\begin{array}{rcll}
\mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} & = & \mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\,\mathcal{K} & \text{by definition of } \mathsf{Var} \\
& = & \mathbb{C}\mathsf{st}\,\mathsf{H}\,\mathsf{S}\,\mathsf{Cm}\,\mathbb{U}\mathsf{d}\,\mathcal{K} & \text{as } \mathsf{P}\,\mathsf{Cm} = \mathsf{Cm}\,\mathbb{U}\mathsf{d}\ \ (4.7(1)) \\
& \subseteq & \mathbb{S}\,\mathbb{H}\mathbb{E}\mathsf{x}\,\mathbb{U}\mathsf{d}\,\mathcal{K} & \text{by } 4.6(7) \\
& \subseteq & \mathbb{S}\,\mathbb{H}\mathbb{H}\,\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K} & \text{by } 4.13 \\
& \subseteq & \mathbb{S}\,\mathbb{H}\mathbb{H}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\,\mathcal{K} & \text{by (i)} \\
& = & \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\,\mathcal{K} & \text{by } 4.6(1).
\end{array}$$

Now suppose that our variety is $\mathsf{Var}\mathcal{K}$ and $\mathcal{K}$ is elementary. Then $\mathbb{P}\mathsf{u}\mathcal{K} = \mathcal{K}$, so as $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ contains $\mathcal{K}$ and is closed under $\mathbb{S}$, $\mathbb{H}$, and $\mathbb{U}\mathsf{d}$, from (ii) we then get

$$\mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}.$$

Therefore
$$\mathsf{Em}\,\mathsf{Var}\mathcal{K} = \mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq \mathsf{Cm}\,\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K} \subseteq \mathsf{Var}\mathcal{K},$$
showing that $\mathsf{Var}\mathcal{K}$ is canonical. ∎

In the proof just given, we only used the fact that $\mathbb{P}\mathsf{u}\mathcal{K} = \mathcal{K}$ and not the stronger assumption that $\mathcal{K}$ is elementary (i.e. $\mathbb{R}\mathsf{u}$-closed as well). However, for an arbitrary $\mathcal{K}$ we have $\mathsf{Var}\mathcal{K} = \mathsf{Var}\,\mathbb{R}\mathsf{u}\,\mathcal{K}$, since $\mathbb{R}\mathsf{u}\,\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ (4.10), and when $\mathbb{P}\mathsf{u}\mathcal{K} = \mathcal{K}$ we have that $\mathbb{R}\mathsf{u}\,\mathcal{K}$ is an elementary class, the smallest one containing $\mathcal{K}$. In this sense we may always assume we are dealing with an elementary generating class for the variety in question, rather than just a $\mathbb{P}\mathsf{u}$-closed one.

Now when $\mathsf{Var}\mathcal{K}$ is canonical, it consists of complex algebras and so can be described as $\mathsf{S}\,\mathsf{Cm}\mathcal{N}$ for some class of structures $\mathcal{N}$. This $\mathcal{N}$ is by no means unique, and can be taken to be elementary when $\mathcal{K}$ is, as shown by the following result from [19, 4.10–4.12].

Theorem 4.16 *If* $\mathbb{P}\mathsf{u}\mathcal{K} = \mathcal{K}$, *and* $\mathcal{N}$ *is any class satisfying*
$$(\dagger) \qquad \mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq \mathcal{N} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K},$$
then $\mathsf{Em}\mathsf{Var}\mathcal{K} \subseteq \mathsf{Cm}\mathcal{N} \subseteq \mathsf{Var}\mathcal{K}$, *and so* $\mathsf{Var}\mathcal{K} = \mathsf{S}\,\mathsf{Cm}\mathcal{N}$. *In particular if* $\mathcal{M}$ *is any class satisfying*
$$\mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq \mathcal{M} = \mathbb{P}\mathsf{u}\,\mathcal{M} \subseteq \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K},$$
then $\mathcal{N} = \mathbb{R}\mathsf{u}\,\mathcal{M}$ *is an elementary class fulfilling* $(\dagger)$.

In some cases, an assumption weaker than $\mathbb{P}\mathsf{u}$-closure can be used to show that a class of complex algebras forms a variety closed under canonical extensions. The most general statement of this kind known to the author is

- *If* $\mathbb{P}\mathsf{u}\mathcal{K} \subseteq \mathbb{H}\mathbb{S}\,\mathbb{U}\mathsf{d}\mathcal{K}$, *then* $\mathsf{S}\,\mathsf{Cm}\mathbb{S}\,\mathbb{U}\mathsf{d}\mathcal{K}$ *is a canonical variety equal to* $\mathsf{H}\,\mathsf{S}\,\mathsf{P}\,\mathsf{Cm}\mathcal{K}$.

A proof of this is given in [19], where the result is applied to give another proof that the class of *representable cylindric algebras* of a given dimension form a canonical variety. This application uses our characterisation of $\mathbf{RCA}_\alpha$ as $\mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\,\mathbb{U}\mathsf{d}\,\mathfrak{F}\mathsf{ct}_\alpha$ (Section 4.3 above), together with the following results about ultraproducts of Cartesian and weak Cartesian structures:

$$\begin{aligned} \mathbb{P}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathfrak{F}\mathsf{ct}_\alpha &\subseteq \mathbb{U}\mathsf{b}\,\mathfrak{F}\mathsf{ct}_\alpha; \\ \mathsf{Cm}\,\mathcal{W}\mathsf{ct}_\alpha &\subseteq \mathsf{S}\,\mathsf{Cm}\,\mathbb{P}\mathsf{w}\,\mathfrak{F}\mathsf{ct}_\alpha. \end{aligned}$$

The proof method can also be applied to other kind of algebras whose elements are α-ary relations, including the cylindric-relativised set algebras that are involved in recent studied of fragments of first-order logic [47, 3, 39] and representable quasi-polyadic algebras [46].

An unresolved issue in this subject is whether the converse of 4.15 is true, i.e. whether every canonical variety $\mathcal{V}$ must be of the form $\mathsf{Var}\mathcal{K}$ for some elementary class $\mathcal{K}$. All known canonical varieties are of this form (including examples involving cylindric algebras and relation algebras), and experience from modal logic suggests that a natural way to approach the problem is to focus on the free $\mathcal{V}$-algebra $\mathfrak{A}^{\mathcal{V}}_\omega$ on denumerably many generators and the first-order theory of its canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_\omega$. Theorem 4.15 of [19] provides the following justification of this approach:

- *If a variety of BAO's $\mathcal{V}$ is generated by some elementary class of structures, then it is generated by the elementary class of those structures that satisfy the same first-order sentences as the structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\omega}$.*

This result can in turn be strengthened by limiting the class of first-order sentences involved. A first-order sentence will be called *quasi-modal* if it is of the form $\forall v\varphi$ with φ being constructed from amongst atomic formulae and the constants $\perp$ and $\top$ using at most $\wedge$ (conjunction), $\vee$ (disjunction), and *bounded* universal and existential quantifiers

$$\forall v_0 \cdots \forall v_{n-1}(R(v, v_0, \dots, v_{n-1}) \to \psi)$$
$$\exists v_0 \cdots \exists v_{n-1}(R(v, v_0, \dots, v_{n-1}) \wedge \psi)$$

with v distinct from $v_0, \dots, v_{n-1}$. Any quasi-modal sentence is preserved by $\mathbb{S}$, $\mathbb{H}$, and $\mathbb{U}\mathsf{d}$ while conversely, if a set of first-order sentences is preserved by these three operations, then it is logically equivalent to a set of quasi-modal sentences. This was proven in [59] for the language of a binary predicate, and in [14, Section 4] for languages of arbitrary type.

This preservation theorem was analysed further in [19, Section 7] (where quasi-modal sentences were called "pseudo-equational"). The analysis showed that if $\Psi_{\mathcal{K}}$ is the set of all quasi-modal sentences true of a class $\mathcal{K}$ of structures, and $Mod\,\Psi_{\mathcal{K}}$ is the class of all models of $\Psi_{\mathcal{K}}$, then

$$Mod\,\Psi_{\mathcal{K}} = \mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{U}\mathsf{b}\,\mathbb{R}\mathsf{u}\,\mathbb{H}\mathbb{S}\,\mathcal{K}.$$

Since $\mathbb{S}\mathsf{trVar}\mathcal{K}$ is closed under the operations $\mathbb{R}\mathsf{u}$, $\mathbb{U}\mathsf{b}$, $\mathbb{H}$, $\mathbb{S}$ it follows that

$$Mod\,\Psi_{\mathcal{K}} \subseteq \mathbb{S}\mathsf{trVar}\mathcal{K}.$$

Moreover we have

$$\mathbb{S}\,\mathbb{H}\mathbb{U}\mathsf{d}\,\mathbb{P}\mathsf{u}\mathcal{K} \subseteq Mod\,\Psi_{\mathcal{K}}$$

since $\Psi_{\mathcal{K}}$ is preserved by $\mathbb{S}$, $\mathbb{H}$, $\mathbb{U}\mathsf{d}$, and $\mathbb{P}\mathsf{u}$, so result (ii) in the proof of 4.6.6 yields

$$\mathbb{C}\mathsf{st}\,\mathsf{Var}\mathcal{K} \subseteq Mod\,\Psi_{\mathcal{K}}.$$

Thus we can apply 4.16 with $\mathcal{N} = Mod\,\Psi_{\mathcal{K}}$ to infer that

if $\mathbb{P}\mathsf{u}\mathcal{K} = \mathcal{K}$ then $\mathsf{Var}\mathcal{K} = \mathsf{S}\,\mathsf{Cm}\,Mod\,\Psi_{\mathcal{K}}$.

Now for a variety $\mathcal{V}$, if $\Psi_{\mathcal{V}}$ is the quasi-modal theory of the structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\omega}$, then it is shown in [19] that when $\mathcal{V} = \mathsf{Var}\mathcal{K}$ for some $\mathbb{P}\mathsf{u}$-closed $\mathcal{K}$, then

$$\mathsf{S}\,\mathsf{Cm}\,Mod\,\Psi_{\mathcal{K}} = \mathsf{S}\,\mathsf{Cm}\,Mod\,\Psi_{\mathcal{V}}$$

(although possibly $\Psi_{\mathcal{K}} \neq \Psi_{\mathcal{V}}$). Combined with the above results, this yields

Theorem 4.17 *If a variety of BAO's $\mathcal{V}$ is generated by some elementary class of structures, then $\mathcal{V} = \mathsf{S}\,\mathsf{Cm}\,Mod\,\Psi_{\mathcal{V}}$, where $\Psi_{\mathcal{V}}$ is the quasi-modal theory of the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathcal{V}}_{\omega}$.*

5 Polymodal Logic

5.1 Languages and Logics

A *modality* is a linguistic construction that takes a statement φ and forms a new statement that asserts something about the way in which φ is true. There are many words and phrases of ordinary language that function as modalities, and some of these form interdefinable pairs, like *possibly/necessarily*, *eventually/henceforth*, and *it is permissible that/it ought to be*. In formal languages, the symbols $\Diamond$ and $\Box$ are often used for for a pair of modalities of this type, with the interdefinability given by

$$\Box = \neg\Diamond\neg, \qquad \Diamond = \neg\Box\neg.$$

We will be discussing languages with several (possibly infinitely many) such modal connectives, so we use ordinals to index them and present them in the form $\langle\beta\rangle, [\beta]$ with

$$[\beta] = \neg\langle\beta\rangle\neg, \qquad \langle\beta\rangle = \neg[\beta]\neg.$$

Most studies of modal logic are based on a language with denumerably many propositional variables. Here we will find it useful to consider languages with larger sets of variables, so from the outset we suppose we have a distinct variable p_λ for each ordinal λ, and for each *infinite* cardinal number κ define

$$\Phi_\kappa = \{p_\lambda : \lambda < \kappa\}.$$

Then for each ordinal α, a modal language $\mathcal{L}_\kappa(\alpha)$ is generated from Φ_κ, the usual Boolean connectives, and a collection $\{\langle\beta\rangle : \beta < \alpha\}$ of "diamond" modalities. The set of formulae φ of $\mathcal{L}_\kappa(\alpha)$ is given by the definition

$$\varphi ::= p_\lambda \mid \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \langle\beta\rangle\varphi$$

where λ ranges over ordinals less than κ and β over ordinals less than α. Other connectives are given by the usual abbreviations

$$\begin{array}{lll} \varphi\wedge\psi & \text{for} & \neg(\neg\varphi\vee\neg\psi) \\ \varphi\to\psi & \text{for} & \neg\varphi\vee\psi \\ \varphi\leftrightarrow\psi & \text{for} & (\varphi\to\psi)\wedge(\psi\to\varphi) \\ [\beta]\varphi & \text{for} & \neg\langle\beta\rangle\neg\varphi. \end{array}$$

Thus the standard language for type 1 logic is $\mathcal{L}_\omega(1)$ and that for type 2 logic, including temporal logic, is $\mathcal{L}_\omega(2)$. Languages of the kind $\mathcal{L}_\kappa(\alpha)$ may be called *unary* since they involve only one-placed modal connectives. More generally, given a *type* $\tau = \langle\alpha_\tau, \rho_\tau\rangle$ as defined in Section 3.1, an associated language $\mathcal{L}_\kappa(\tau)$ is defined for each infinite cardinal κ by using connectives $\langle\beta\rangle$ of rank $\rho_\tau(\beta)$ for $\beta < \alpha_\tau$. The formulae of $\mathcal{L}_\kappa(\tau)$ are specified by

$$\varphi ::= p_\lambda \mid \bot \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \langle\beta\rangle(\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1}),$$

and now the $\rho_\tau(\beta)$-ary "box" operator associated with $\langle\beta\rangle$ has

$$[\beta](\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1}) = \neg\langle\beta\rangle(\neg\varphi_0, \dots, \neg\varphi_{\rho_\tau(\beta)-1}).$$

A *logic* in the language $\mathcal{L}_\kappa(\tau)$ is defined to be any set Λ of $\mathcal{L}_\kappa(\tau)$-formulae such that

- Λ includes all $\mathcal{L}_\kappa(\tau)$-formulae that are instances of tautologies, and
- Λ is closed under the inference rule of *Detachment*, i.e.
  if φ, $\varphi \to \psi \in \Lambda$, then $\psi \in \Lambda$.

Λ is *uniform* if it is closed under the rule of uniform substitution of $\mathcal{L}_\kappa(\tau)$-formulae for propositional variables. Λ is a *normal* logic if it contains the schemata

(K) $\langle \beta \rangle(\varphi_0, \dots, \psi \vee \chi, \dots, \varphi_{\rho_\tau(\beta)-1}) \to$
$\langle \beta \rangle(\varphi_0, \dots, \psi, \dots, \varphi_{\rho_\tau(\beta)-1}) \vee \langle \beta \rangle(\varphi_0, \dots, \chi, \dots, \varphi_{\rho_\tau(\beta)-1})$,
(N) $\neg\langle \beta \rangle(\varphi_0, \dots, \bot, \dots, \varphi_{\rho_\tau(\beta)-1})$,

and satisfies the *Monotonicity* rule

if $\psi \to \chi \in \Lambda$, then
$\langle \beta \rangle(\varphi_0, \dots, \psi, \dots, \varphi_{\rho_\tau(\beta)-1}) \to \langle \beta \rangle(\varphi_0, \dots, \chi, \dots, \varphi_{\rho_\tau(\beta)-1}) \in \Lambda$.

The members of a logic are called its *theorems*, and we write $\vdash_\Lambda \varphi$ to mean that φ is a Λ-theorem, i.e.

$$\vdash_\Lambda \varphi \quad \text{iff} \quad \varphi \in \Lambda.$$

If $\Gamma \cup \{\varphi\}$ is a set of formulae, then φ is Λ-*deducible from* Γ, denoted $\Gamma \vdash_\Lambda \varphi$, if there exist finitely many $\psi_0, \dots, \psi_{n-1} \in \Gamma$ such that

$$\vdash_\Lambda \psi_0 \to (\psi_1 \to (\cdots \to (\psi_{n-1} \to \varphi) \cdots))$$

(in the case $n = 0$, this means that $\vdash_\Lambda \varphi$). We write $\Gamma \nvdash_\Lambda \varphi$ when φ is not Λ-deducible from Γ.

Γ is a Λ-*consistent* set of formulae if $\Gamma \nvdash_\Lambda \bot$, and is Λ-*maximal* if it is Λ-consistent and for each $\mathcal{L}_\kappa(\tau)$-formula φ,

$$\text{either } \varphi \in \Gamma \text{ or } \neg\varphi \in \Gamma.$$

Put

$$S_\kappa^\Lambda = \{\Gamma : \Gamma \text{ is a } \Lambda\text{-maximal set of } \mathcal{L}_\kappa(\tau)\text{-formulae}\}.$$

By a result usually known as *Lindenbaum's Lemma*, every Λ-consistent set is extendible to a Λ-maximal set of $\mathcal{L}_\kappa(\tau)$-formulae. Hence if $\nvdash_\Lambda \bot$, so that there do exist Λ-consistent sets, then $S_\kappa^\Lambda \neq \emptyset$. The *canonical* Λ-*structure* is then the type τ structure

$$\mathfrak{S}_\kappa^\Lambda = \langle S_\kappa^\Lambda, R_\beta^\Lambda \rangle_{\beta < \alpha_\tau},$$

where

$$R_\beta^\Lambda(\Gamma, \Delta_0, \dots, \Delta_{\rho_\tau(\beta)-1})$$
$$\text{iff } \{\langle \beta \rangle(\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1}) : \varphi_i \in \Delta_i \text{ all } i < \rho_\tau(\beta)\} \subseteq \Gamma.$$

For unary languages $\mathfrak{S}_\kappa^\Lambda$ is known as the canonical Λ-*frame*.

Associated with any *normal* logic Λ in a language $\mathcal{L}_\kappa(\tau)$ is an algebra $\mathfrak{A}_\kappa^\Lambda$, a BAO of type τ, called the *Lindenbaum-Tarski algebra* of Λ. The collection of all $\mathcal{L}_\kappa(\tau)$-formulae forms an absolutely free algebra of type τ under the operations on formulae induced by the connectives $\vee, \wedge, \neg, \bot, \top, \langle\, \beta\,\rangle$, and $\mathfrak{A}^\Lambda_\kappa$ is the quotient of this algebra by the congruence $\cong_\Lambda$, where

$$\varphi \cong_\Lambda \psi \quad \text{iff} \quad \vdash_\Lambda \varphi \leftrightarrow \psi.$$

Thus the elements of $\mathfrak{A}^\Lambda_\kappa$ are the equivalence classes

$$\|\varphi\| = \{\psi : \vdash_\Lambda \varphi \leftrightarrow \psi\},$$

with the operations

$$\begin{aligned} \|\varphi\| + \|\psi\| &= \|\varphi \vee \psi\| \\ \|\varphi\| \cdot \|\psi\| &= \|\varphi \wedge \psi\| \\ \|\varphi\|^- &= \|\neg\varphi\| \\ 0 &= \|\bot\| \\ 1 &= \|\top\| \\ m_\beta(\|\varphi_0\|, \dots, \|\varphi_{\rho_\tau(\beta)-1}\|) &= \|\langle\, \beta\,\rangle(\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1})\|. \end{aligned}$$

The axiom schemata (K) and (N) and the Monotonicity rule are needed to show that m_β is a well-defined normal additive operator. In $\mathfrak{A}^\Lambda_\kappa$ we have

$$\begin{aligned} \|\varphi\| \leq \|\psi\| \quad &\text{iff} \quad \vdash_\Lambda \varphi \to \psi, \\ \|\varphi\| = 1 \quad &\text{iff} \quad \vdash_\Lambda \varphi. \end{aligned}$$

If Γ is a Λ-maximal set of formulae, then

$$x_\Gamma = \{\|\varphi\| : \varphi \in \Gamma\}$$

is an ultrafilter of $\mathfrak{A}^\Lambda_\kappa$. The correspondence $\Gamma \mapsto x_\Gamma$ proves to be a bijection between S^Λ_κ and the set of ultrafilters of $\mathfrak{A}^\Lambda_\kappa$ which respects the relations R^Λ_β of $\mathfrak{S}^\Lambda_\kappa$ and R_{m_β} of the canonical structure of $\mathfrak{A}^\Lambda_\kappa$ (Section 3.2). In other words:

- *the canonical* Λ*-structure* $\mathfrak{S}^\Lambda_\kappa$ *is isomorphic to the canonical structure* $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$ *of the Lindenbaum-Tarski algebra* $\mathfrak{A}^\Lambda_\kappa$ *of* Λ.

We will see shortly that $\mathfrak{A}^\Lambda_\kappa$ is the free algebra on κ generators in a variety of BAO's determined by the normal logic Λ.

5.2 Algebraic Semantics

Let $\mathfrak{A} = \langle \mathfrak{B}, m_\beta \rangle_{\beta < \alpha_\tau}$ be a BAO of type τ and φ an $\mathcal{L}_\kappa(\tau)$-formula whose variables are among $p_{\lambda_0}, \dots, p_{\lambda_{n-1}}$ with $\lambda_0 < \cdots < \lambda_{n-1}$. Then φ induces an n-ary operation $\mathfrak{A}(\varphi)$ on $\mathfrak{A}$ which is defined by induction on the formation of φ as follows.

$$\begin{aligned} \mathfrak{A}(p_{\lambda_i})(a_0, \dots, a_{n-1}) &= a_i \\ \mathfrak{A}(\bot)(a_0, \dots, a_{n-1}) &= 0 \\ \mathfrak{A}(\neg\varphi)(a_0, \dots, a_{n-1}) &= \mathfrak{A}(\varphi)(a_0, \dots, a_{n-1})^- \\ \mathfrak{A}(\varphi_1 \vee \varphi_2)(a_0, \dots, a_{n-1}) &= \mathfrak{A}(\varphi_1)(a_0, \dots, a_{n-1}) + \mathfrak{A}(\varphi_2)(a_0, \dots, a_{n-1}) \end{aligned}$$

and

$$\mathfrak{A}(\langle \beta \rangle(\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1}))(a_0, \dots, a_{n-1}) = \\ m_\beta(\mathfrak{A}(\varphi_0)(a_0, \dots, a_{n-1}), \dots, \mathfrak{A}(\varphi_{\rho_\tau(\beta)-1})(a_0, \dots, a_{n-1})).$$

φ is *valid* in $\mathfrak{A}$, $\mathfrak{A} \models \varphi$, if the function $\mathfrak{A}(\varphi)$ is constantly equal to 1. If $\mathcal{V}$ is a class of BAO's, then $\mathcal{V} \models \varphi$ if $\mathfrak{A} \models \varphi$ for all $\mathfrak{A} \in \mathcal{V}$.

If Δ is a set of formulae, then $\mathfrak{A} \models \Delta$ if $\mathfrak{A} \models \varphi$ for all $\varphi \in \Delta$. It is readily seen that

$$\mathfrak{A}(\varphi) = \mathfrak{A}(\psi) \quad \text{iff} \quad \mathfrak{A} \models \varphi \leftrightarrow \psi \quad \text{iff} \quad \mathfrak{A}(\varphi \leftrightarrow \psi) = 1 \text{ constantly.}$$

Now a formula φ may be regarded as a *term* in the language of a BAO $\mathfrak{A}$, with the propositional variables of φ treated as variables ranging over the elements of $\mathfrak{A}$, and the symbols $\vee, \wedge, \neg, \bot, \top, \langle \beta \rangle$ naming the $\mathfrak{A}$-operations $+, \cdot, ^-, 0, 1, m_\beta$. Then $\mathfrak{A}(\varphi)$ is just the term operation on $\mathfrak{A}$ induced by φ as a term. Every term for $\mathfrak{A}$ corresponds to a formula, and every term function is of the form $\mathfrak{A}(\varphi)$ for some formula φ. From this there follows an equivalence between formulae and BAO equations. Formula φ is valid in $\mathfrak{A}$ if, and only if, $\mathfrak{A}$ satisfies the equation "$\varphi = 1$". Each equation is of the form "$\varphi = \psi$" for some formulae, and is satisfied in $\mathfrak{A}$ iff the formula $\varphi \leftrightarrow \psi$ is valid in $\mathfrak{A}$. Thus for a set of formulae Δ, the class of algebras

$$\{\mathfrak{A} : \mathfrak{A} \models \Delta\}$$

is an equational class, which we denote $\mathsf{Var}\,\Delta$, and every equational class is of this form. $\mathsf{Var}\,\Delta$ is closed under the operations $\mathsf{H}, \mathsf{S}, \mathsf{P}$, i.e. these operations preserve validity of formulae.

For any class $\mathcal{V}$ of BAO's, the set

$$\Lambda_{\mathcal{V}} = \{\varphi : \mathcal{V} \models \varphi\}$$

is a normal uniform logic. In particular,

$$\Lambda_{\mathfrak{A}} = \{\varphi : \mathfrak{A} \models \varphi\}$$

is a normal uniform logic for each algebra $\mathfrak{A}$. Thus if $\mathfrak{A} \models \Delta$, then $\Delta \subseteq \Lambda_{\mathfrak{A}}$ and $\Lambda_{\mathfrak{A}}$ contains the normal uniform logic $\Lambda(\Delta)$ *generated by* Δ, which is defined as the intersection of all such logics that contain Δ. Consequently,

$$\mathfrak{A} \models \Delta \quad \text{iff} \quad \mathfrak{A} \models \Lambda(\Delta),$$

and $\mathsf{Var}\,\Delta = \mathsf{Var}\,\Lambda(\Delta)$: every variety of algebras is the class of all algebraic models of some logic of the form $\Lambda(\Delta)$.

We say that a logic Λ *is characterised by* a class $\mathcal{V}$ of BAO's, or that Λ *axiomatises* $\mathcal{V}$, if for any formula φ,

$$\vdash_\Lambda \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi \text{ for all } \mathfrak{A} \in \mathcal{V}.$$

In other words, Λ is characterised by $\mathcal{V}$ if, and only if, $\Lambda = \Lambda_{\mathcal{V}}$.

Every normal uniform logic Λ turns out to be of the form $\Lambda_{\mathfrak{A}}$, because such a Λ is characterised by its Lindenbaum-Tarski algebra:

$$(\dagger) \qquad \vdash_\Lambda \varphi \quad \text{iff} \quad \mathfrak{A}^\Lambda_\kappa \models \varphi.$$

Hence the equational theory of $\mathfrak{A}_\kappa^\Lambda$ is just the set of equations defined by Λ, and $\mathsf{Var}\,\Lambda$ is the variety generated by $\mathfrak{A}_\kappa^\Lambda$. From this it follows that $\Lambda \mapsto \mathsf{Var}\,\Lambda$ is a bijective correspondence between normal uniform logics and varieties.

The proof of (†) follows from the fact that in general

$$\mathfrak{A}_\kappa^\Lambda(\varphi)(\|\psi_0\|, \dots, \|\psi_{n-1}\|) = \|\varphi[p_{\lambda_i}/\psi_i]\|,$$

where the formula $\varphi[p_{\lambda_i}/\psi_i]$ is the result of uniformly substituting ψ_i for the variable p_{λ_i} in φ for all $i < n$. Then if $\vdash_\Lambda \varphi$, by uniform substitution we have $\vdash_\Lambda \varphi[p_{\lambda_i}/\psi_i]$ and hence

$$\mathfrak{A}_\kappa^\Lambda(\varphi)(\|\psi_0\|, \dots, \|\psi_{n-1}\|) = 1,$$

for any ψ_i, showing that $\mathfrak{A}_\kappa^\Lambda(\varphi) = 1$ constantly. But since

$$(\ddagger) \qquad \mathfrak{A}_\kappa^\Lambda(\varphi)(\|p_{\lambda_0}\|, \dots, \|p_{\lambda_{n-1}}\|) = \|\varphi[p_{\lambda_i}/p_{\lambda_i}]\| = \|\varphi\|,$$

if $\not\vdash_\Lambda \varphi$, then $\mathfrak{A}_\kappa^\Lambda(\varphi)(\|p_{\lambda_0}\|, \dots, \|p_{\lambda_{n-1}}\|) \neq 1$, so $\mathfrak{A}_\kappa^\Lambda \not\models \varphi$.

Theorem 5.1 *In the variety* $\mathsf{Var}\,\Lambda$, $\mathfrak{A}_\kappa^\Lambda$ *is a free algebra on the set of generators*

$$\|\Phi_\kappa\| = \{\|p_\lambda\| : \lambda < \kappa\}.$$

PROOF. It is evident from the definition of $\mathfrak{A}_\kappa^\Lambda$ that it is generated as a BAO by $\|\Phi_\kappa\|$. Given a function $f : \|\Phi_\kappa\| \to \mathfrak{A}$, since homomorphisms preserve term operations it follows from (‡) that the only possible lifting of f to a homomorphism $f : \mathfrak{A}_\kappa^\Lambda \to \mathfrak{A}$ would be to take

$$f(\|\varphi\|) = \mathfrak{A}(\varphi)(f(\|p_{\lambda_0}\|), \dots, f(\|p_{\lambda_{n-1}}\|)).$$

This does indeed give a homomorphism of BAO's, provided that it is well-defined. But if $\|\varphi\| = \|\psi\|$ then $\vdash_\Lambda \varphi \leftrightarrow \psi$, so if $\mathfrak{A}$ belongs to $\mathsf{Var}\,\Lambda$ then $\mathfrak{A} \models \varphi \leftrightarrow \psi$ and hence $\mathfrak{A}(\varphi) = \mathfrak{A}(\psi)$ as desired. ∎

Theorem 5.2 *The smallest normal logic in* $\mathcal{L}_\kappa(\tau)$ *is characterised by the class of all BAO's.*

PROOF. By "the smallest" is meant the intersection of all normal logics. Let Λ be this intersection. Then Λ is contained in $\Lambda_{\mathfrak{A}}$ for any BAO $\mathfrak{A}$, which shows that the Λ-theorems are valid in all BAO's. This is the *Soundness* part of the characterisation.

Conversely, for the *Completeness* part, if a formula is valid in all BAO's, then it is valid in the Lindenbaum-Tarski algebra $\mathfrak{A}_\kappa^\Lambda$, and so is a Λ-theorem, as above. ∎

Algebraic characterisations of many logics can be obtained by this method. For instance, in the language $\mathcal{L}_\omega(1)$ with modality $\Diamond$, the logic S4 is defined as the smallest normal logic containing the schemata

(T) $\varphi \to \Diamond\varphi$, and
(4) $\Diamond\Diamond\varphi \to \Diamond\varphi$.

These schemata are valid in any closure algebra, so if $\mathfrak{A}$ is a closure algebra then $\mathrm{S4} \subseteq \Lambda_{\mathfrak{A}}$. But (T) and (4) force the Lindenbaum-Tarski algebra $\mathfrak{A}_\omega^{\mathrm{S4}}$ for S4 to be a closure algebra, so if φ is valid in all closure algebras then it is valid in $\mathfrak{A}_\omega^{\mathrm{S4}}$, and hence $\vdash_{\mathrm{S4}} \varphi$. This shows

- *S4 is characterised by the variety $\mathcal{V}_{cl}$ of all closure algebras, which is generated by* $\mathfrak{A}^{S4}_\omega$.

For the logic S5, defined as the smallest normal logic containing (T), (4) and the schema

$$\varphi \to \Box\Diamond\varphi,$$

we show by similar reasoning that

- *S5 is characterised by the variety $\mathcal{V}_{mn}$ of all monadic algebras, which is generated by* $\mathfrak{A}^{S5}_\omega$.

To conclude this section, we briefly discuss the question of how the notion of a logic can be made language independent. Since any formula or equation has only finitely many variables, to define a logic or an equational class we really need only n variables for arbitrary finite n. Hence the languages $\mathcal{L}_\omega(\tau)$ suffice for this purpose. At the same time the definition of a particular logic in many cases should be independent of the size of the set of variables. For instance, in any modal language $\mathcal{L}_\kappa(1)$, whatever the cardinal κ may be, we should be able to say that "S4" means the smallest normal logic containing the schemata (T) and (4). If a logic is defined as a set of formulae (theorems), rather than as a system of axioms and inference rules, then we need to say something about how the different instantiations of this logic are related as the set Φ_κ of propositional variables varies with κ.

If Λ is a normal uniform logic in a language $\mathcal{L}_\kappa(\tau)$, and μ is any cardinal greater than κ, then by "Λ in $\mathcal{L}_\mu(\tau)$" we mean the set Λ_μ of $\mathcal{L}_\mu(\tau)$-formulae that are obtained by uniform substitution in $\mathcal{L}_\mu(\tau)$ from Λ-theorems in $\mathcal{L}_\kappa(\tau)$. This is the smallest normal uniform logic in $\mathcal{L}_\mu(\tau)$ containing the original Λ. On the other hand, if $\kappa > \mu \geq \omega$, there is a unique logic Λ_μ in $\mathcal{L}_\mu(\tau)$ such that Λ arises in this way from Λ_μ by substitution. Λ_μ is simply the set of $\mathcal{L}_\mu(\tau)$-formulae that belong to Λ.

A natural way to approach this issue from the point of view of algebra is to observe that Λ defines the variety $\mathsf{Var}\,\Lambda$ which in turn, for each μ, specifies the logic

$$\{\varphi \text{ in } \mathcal{L}_\mu(\tau) : \mathsf{Var}\,\Lambda \models \varphi\}.$$

With the help of the Lindenbaum-Tarski algebra construction, it can be shown that this is the same as the logic Λ_μ just defined.

5.3 Kripke Semantics

We turn now to the relational semantics attributed to Kripke, and motivate this by reviewing the interpretation of some unary modalities. The distinction between an interdefinable pair $\langle\beta\rangle, [\beta]$ can be accounted for logically by observing that $\langle\beta\rangle$ distributes across a disjunction, in the sense that

$$\vdash_\Lambda \langle\beta\rangle(\varphi \vee \psi) \leftrightarrow \langle\beta\rangle\varphi \vee \langle\beta\rangle\psi$$

for a normal logic Λ, while $[\beta]$ correspondingly respects conjunction:

$$\vdash_\Lambda [\beta](\varphi \wedge \psi) \leftrightarrow [\beta]\varphi \wedge [\beta]\psi.$$

¿From another perspective, in terms of intended interpretations, "diamond" modalities function like existential quantifiers over states/ worlds/ situations, while "box" modalities are like universal quantifiers. Here are some illustrations:

Modality	**Interpretation**
possibly	in some possible world
necessarily	in all possible worlds
eventually	at some future time
henceforth	at all future times
it is consistent that	in some model
it is provable that	in all models
after the program finishes	after all terminating executions
the program enables	there is a terminating execution such that

A *model* for a unary language $\mathcal{L}_\kappa(\alpha)$ is a pair $\mathfrak{M} = \langle \mathfrak{S}, V \rangle$ where $\mathfrak{S} = \langle S, R_\beta \rangle_{\beta<\alpha}$ is a K-frame, consisting of binary relations R_β on S, and

$$V : \Phi_\kappa \to Sb\, S$$

is a *valuation* function assigning a subset $V(p_\lambda)$ of S to the variable p_λ for each $\lambda < \kappa$. $V(p_\lambda)$ is to be thought of as the set of points at which p_λ is true. The satisfaction relation "φ is true at point s in $\mathfrak{M}$", denoted

$$\mathfrak{M} \models_s \varphi,$$

is defined inductively by the clauses

$$\begin{array}{lll} \mathfrak{M} \models_s p_\lambda & \text{iff} & s \in V(p_\lambda) \\ \mathfrak{M} \not\models_s \bot & & (\text{i.e. not } \mathfrak{M} \models \bot) \\ \mathfrak{M} \models_s \varphi \vee \psi & \text{iff} & \mathfrak{M} \models_s \varphi \text{ or } \mathfrak{M} \models_s \psi \\ \mathfrak{M} \models_s \langle \beta \rangle \varphi & \text{iff} & \text{for some } t \in S,\ sR_\beta t \text{ and } \mathfrak{M} \models_t \varphi, \end{array}$$

and hence

$$\mathfrak{M} \models_s [\beta]\varphi \quad \text{iff} \quad \text{for all } t \in S,\ sR_\beta t \text{ implies } \mathfrak{M} \models_t \varphi.$$

These last two clauses formally express the character of $\langle \beta \rangle$ and $[\beta]$ as bounded existential and universal quantifiers.

For a polymodal language $\mathcal{L}_\kappa(\tau)$ of type τ, a model takes the form $\mathfrak{M} = \langle \mathfrak{S}, V \rangle$ where now $\mathfrak{S}$ is a relational structure of type τ. The definition of satisfaction is modified to read

$$\begin{array}{lll} \mathfrak{M} \models_s \langle \beta \rangle(\varphi_0, \dots, \varphi_{\rho_\tau(\beta)-1}) & \text{iff} & \text{for some } t_0, \dots, t_{\rho_\tau(\beta)-1} \in S, \\ & & R_\beta(s, t_0, \dots, t_{\rho_\tau(\beta)-1}) \text{ and} \\ & & \mathfrak{M} \models_{t_i} \varphi_i \text{ for all } i < \rho_\tau(\beta). \end{array}$$

Formula φ is *true in model* $\mathfrak{M}$, $\mathfrak{M} \models \varphi$, if it is true at all points in $\mathfrak{M}$, i.e. if

$$\mathfrak{M} \models_s \varphi \text{ for all } s \in S.$$

φ is *valid* in the structure $\mathfrak{S}$, $\mathfrak{S} \models \varphi$, if

$$\mathfrak{M} \models \varphi \text{ for all models } \mathfrak{M} = \langle \mathfrak{S}, V \rangle \text{ based on } \mathfrak{S}.$$

A logic Λ is *characterised* by a class $\mathcal{C}$ of models, or structures, if each formula is a Λ-theorem precisely when it is true, or valid respectively, in all members of $\mathcal{C}$:

$$\vdash_\Lambda \varphi \quad \text{iff} \quad \mathcal{C} \models \varphi.$$

For any model $\mathfrak{M}$, the set

$$\Lambda_{\mathfrak{M}} = \{\varphi : \mathfrak{M} \models \varphi\}$$

is a normal logic, while for any structure $\mathfrak{S}$,

$$\Lambda_{\mathfrak{S}} = \{\varphi : \mathfrak{S} \models \varphi\}$$

is a normal and uniform logic.

If Λ is a normal logic in a language $\mathcal{L}_\kappa(\tau)$, then Λ has a single characteristic model $\mathfrak{M}_\kappa^\Lambda = \langle \mathfrak{S}_\kappa^\Lambda, V^\Lambda \rangle$, called the *canonical* Λ-*model*, where $\mathfrak{S}_\kappa^\Lambda$ is the canonical Λ-structure defined in Section 5.1, and

$$V^\Lambda(p_\lambda) = \{\Gamma \in S_\kappa^\Lambda : p_\lambda \in \Gamma\}.$$

A fundamental result, which uses the proof theory of normal logics, is that

$$\mathfrak{M}_\kappa^\Lambda \models_\Gamma \varphi \quad \text{iff} \quad \varphi \in \Gamma$$

for all formulae φ and all $\Gamma \in S_\kappa^\Lambda$ (this is a model-theoretic analogue of the algebraic argument showing that the canonical embedding function $\eta_{\mathfrak{B}}$ of Section 3.2 is a BAO-homomorphism). Since the only formulae that belong to all Λ-maximal sets are the Λ-theorems, this entails that

$$\mathfrak{M}_\kappa^\Lambda \models \varphi \quad \text{iff} \quad \vdash_\Lambda \varphi,$$

which establishes that $\mathfrak{M}_\kappa^\Lambda$ characterises Λ.

It follows immediately that

$$\mathfrak{S}_\kappa^\Lambda \models \varphi \quad \text{implies} \quad \vdash_\Lambda \varphi,$$

but the converse need not hold. There are logics that are not validated by their canonical structure, as will be explained further below (Theorem 5.7).

To relate Kripke semantics to the algebraic semantics, we reformulate the definition of the satisfaction relation in models. A given model $\mathfrak{M}$ associates with each formula φ the "truth-set"

$$\mathfrak{M}(\varphi) = \{s : \mathfrak{M} \models_s \varphi\}$$

of all points in $\mathfrak{M}$ at which φ is true. The clauses specifying satisfaction amount to the following properties of truth sets.

$$\begin{aligned}
\mathfrak{M}(p_\lambda) &= V(p_\lambda) \\
\mathfrak{M}(\bot) &= \emptyset \\
\mathfrak{M}(\neg\varphi) &= S - \mathfrak{M}(\varphi) \\
\mathfrak{M}(\varphi \vee \psi) &= \mathfrak{M}(\varphi) \cup \mathfrak{M}(\psi) \\
\mathfrak{M}(\langle \beta \rangle(\varphi_0, \ldots, \varphi_{\rho_\tau(\beta)-1})) &= m_{R_\beta}(\mathfrak{M}(\varphi_0), \ldots, \mathfrak{M}(\varphi_{\rho_\tau(\beta)-1})).
\end{aligned}$$

This shows that the truth of φ is obtained from the term function $\mathsf{Cm}\,\mathfrak{S}(\varphi)$ induced by φ on the complex algebra of $\mathfrak{S}$. Precisely, the following can be proven by induction on the formation of φ:

Theorem 5.3 *If φ is a formula whose variables are among $p_{\lambda_0}, \dots, p_{\lambda_{n-1}}$ with $\lambda_0 < \dots < \lambda_{n-1}$, then for any model $\mathfrak{M}$ on $\mathfrak{S}$,*

$$\mathsf{Cm}\,\mathfrak{S}(\varphi)(\mathfrak{M}(p_{\lambda_0}), \dots, \mathfrak{M}(p_{\lambda_{n-1}})) = \mathfrak{M}(\varphi).$$

□

Corollary 5.4 $\mathfrak{S} \models \varphi$ *iff* $\mathsf{Cm}\,\mathfrak{S} \models \varphi$.

PROOF. If $\mathfrak{S} \not\models \varphi$ then $\mathfrak{M} \not\models \varphi$ for some model $\mathfrak{M}$ on $\mathfrak{S}$, so $\mathfrak{M}(\varphi) \neq S$. By 5.3.1 it is follows directly that $\mathsf{Cm}\,\mathfrak{S}(\varphi)$ is not the "constantly 1" function, so $\mathsf{Cm}\,\mathfrak{S} \not\models \varphi$.

Conversely, if φ is not valid in $\mathsf{Cm}\,\mathfrak{S}$, then

$$\mathsf{Cm}\,\mathfrak{S}(\varphi)(T_0, \dots, T_{\lambda_{n-1}})) \neq S$$

for some $T_i \in Sb\,S$. Letting $\mathfrak{M}$ be any model on $\mathfrak{S}$ having $\mathfrak{M}(p_{\lambda_i}) = T_{\lambda_i}$ for $i < n$, 5.3.1 again implies $\mathfrak{M}(\varphi) \neq S$, so $\mathfrak{S} \not\models \varphi$. ■

Now if a formula is valid in all BAO's, then it is valid in all complex algebras and so, by 5.3.2 is valid in all structures. Conversely, if φ is valid in all structures, then it is valid in the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ of any BAO, and so by 5.3.2 is valid in the algebra $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A} = \mathsf{Em}\,\mathfrak{A}$. In view of the embedding $\mathfrak{A} \rightarrowtail \mathsf{Em}\,\mathfrak{A}$ and the fact that validity is preserved by subalgebras and isomorphism, it follows that $\mathfrak{A} \models \varphi$. This shows

Theorem 5.5 *A formula is valid in all structures of type τ if, and only if, it is valid in all BAO's of type τ. Hence the smallest normal logic in $\mathcal{L}_\kappa(\tau)$ is characterised by the class of all τ-structures.* □

In order to obtain relational characterisations of other logics, we can combine the algebraic completeness theorems of Section 5.2 with various representation theorems from Sections 3 and 4. Here are some typical results.

Theorem 5.6

(1) *The logic S4 is characterised by the class of all quasi-orderings, as well as by the class of all partial orderings.*

(2) *The logic S5 is characterised by the class of all equivalence relations, as well as by the class of K-frames $\mathfrak{S} = \langle S, R \rangle$ in which the relation R is universal.*

PROOF. (1) For the Soundness part, it is readily seen that if $\mathfrak{S}$ is a quasi-ordering then $S4 \subseteq \Lambda_{\mathfrak{S}}$. For the converse, if $\mathfrak{A}$ is a closure algebra, then as shown in Section 3.3 $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is a quasi-order, so if φ is valid in all quasi-orders it is valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ and hence as in the proof of 5.3.3 is valid in $\mathfrak{A}$. This shows that a formula valid in all quasi-orders is valid in all closure algebras, and so is an S4-theorem by the work of Section 5.2.

For the case of partial orderings, we similarly use the result of Section 4.2 that a closure algebra can be embedded into the complex algebra of a partial ordering.

(2) The fact that S5 is characterised by equivalence relations is shown by extending the analysis of S4, using the fact that if $\mathfrak{A}$ is a monadic algebra then $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ is an equivalence relation. But for any K-frame $\mathfrak{S}$ there is an embedding

$$\mathsf{Cm}\,\mathfrak{S} \rightarrowtail \prod_{s\in S} \mathsf{Cm}\,\mathfrak{S}_s$$

where $\mathfrak{S}_s$ is the inner substructure of $\mathfrak{S}$ generated by the point s (cf. Section 4.3). Now if $\mathfrak{S}$ is an equivalence relation then $\mathfrak{S}_s$ is the equivalence class of s, on which the equivalence relation is universal. Thus if a formula is valid in all universal frames, then it is valid in each $\mathsf{Cm}\,\mathfrak{S}_s$ and so by preservation of validity under P and S is valid in $\mathsf{Cm}\,\mathfrak{S}$ for any equivalence relation $\mathfrak{S}$, and therefore is an S5-theorem.

∎

Instead of working with canonical structures $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ of algebras, an alternative but equivalent approach to these results is to directly use the axioms of a logic like S4 to prove that the canonical frame $\mathfrak{S}^{\Lambda}_{\kappa}$ has the desired properties, like reflexiveness and transitivity, that ensure that it validates the logic. Although there are numerous axioms for which this method works, it does not apply to all. A counter-example is the $\mathcal{L}_\omega(1)$-logic KW, where W is the schema

$$\Diamond\varphi \to \Diamond(\varphi \wedge \neg\Diamond\varphi).$$

VarKW is the variety $\mathcal{V}_{dg}$ of all diagonalisable algebras.

Theorem 5.7 *The schema W is not valid in the canonical KW-frame* $\mathfrak{S}^{\mathrm{KW}}_{\omega}$.

PROOF. Let $\mathfrak{A}$ be the algebra of all finite or cofinite subsets of the frame $\mathfrak{S} = \langle \omega, > \rangle$. It was shown in Section 3.3 that $\mathfrak{A} \in \mathcal{V}_{dg}$ but $\mathsf{Em}\,\mathfrak{A} \notin \mathcal{V}_{dg}$, hence W is not valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}$.

Since the Lindenbaum-Tarski algebra $\mathfrak{A}^{\mathrm{KW}}_{\omega}$ for KW is free in $\mathcal{V}_{dg}$ on denumerably many generators (Theorem 5.1), there is a surjective homomorphism $\mathfrak{A}^{\mathrm{KW}}_{\omega} \twoheadrightarrow \mathfrak{A}$, and hence by duality an injective bounded $\mathbb{C}\mathsf{st}\,\mathfrak{A} \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathrm{KW}}_{\omega}$. It follows that W cannot be valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathrm{KW}}_{\omega}$, or else it would be valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. But $\mathfrak{S}^{\mathrm{KW}}_{\omega}$ is isomorphic to $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\mathrm{KW}}_{\omega}$ (Section 5.1). ∎

5.4 Completeness and Incompleteness

Each normal uniform logic Λ is characterised by the variety $\mathsf{Var}\Lambda$ of all algebras that validate Λ. Correspondingly for the relational semantics we may ask: is Λ characterised by the class

$$\mathbb{S}\mathsf{tr}\Lambda = \{\mathfrak{S} : \mathfrak{S} \models \Lambda\}$$

of all structures that validate Λ? For this to hold it suffices that every formula valid in $\mathbb{S}\mathsf{tr}\Lambda$ be a Λ-theorem. (Note that $\mathbb{S}\mathsf{tr}\Lambda$ is the same as the class $\mathbb{S}\mathsf{tr}\mathsf{Var}\Lambda$ of all structures in the variety $\mathsf{Var}\Lambda$.)

We will say that a logic Λ is *complete* if it is characterised by some class $\mathcal{K}$ of structures. Such a $\mathcal{K}$ is contained in $\mathbb{S}\mathsf{tr}\Lambda$, from which it follows that Λ is complete if, and only if, it is characterised by $\mathbb{S}\mathsf{tr}\Lambda$.

The property of completeness does not depend on the cardinality of the language. If the statement ($\vdash_\Lambda \varphi$ iff $\mathcal{K} \models \varphi$) holds for all $\mathcal{L}_\omega(\tau)$-formulae φ, then it can be shown to hold for all $\mathcal{L}_\kappa(\tau)$-formulae φ^* when $k > \omega$, using the fact for any such φ^* there is an $\mathcal{L}_\omega(\tau)$-formula φ such that φ and φ^* are substitution instances of each other.

Now if $\mathcal{K}$ characterises Λ, then $\mathsf{Cm}\mathcal{K} \subseteq \mathsf{Var}\Lambda$ and, invoking the equivalence of formulae and algebraic equations, each equation valid in $\mathsf{Cm}\mathcal{K}$ is equivalent to a Λ-theorem and hence hold in $\mathsf{Var}\Lambda$. This implies that $\mathsf{Var}\Lambda$ is the variety $\mathsf{Var}\mathcal{K} = \mathsf{HSPCm}\mathcal{K}$ generated by $\mathcal{K}$. Consequently

- Λ is a complete logic iff $\mathsf{Var}\Lambda$ is generated by $\mathbb{S}\mathsf{tr}\Lambda$, i.e. iff $\mathsf{Var}\Lambda$ is a complete variety in the sense of Section 3.3.

It was discovered by Thomason [55] that there exist *incomplete* logics, ones for which $\mathsf{Var}\Lambda$ is a non-trivial variety that is not generated by $\mathbb{S}\mathsf{tr}\Lambda$. This first example was a temporal logic for which there are no validating frames at all: $\mathbb{S}\mathsf{tr}\Lambda = \emptyset$! Later examples of incomplete modal logics were found by Thomason [56] and Fine [11]. The simplest example now known [7] is the smallest normal logic containing the schema

$$\Diamond\varphi \to \Diamond\neg(\Diamond\varphi \leftrightarrow \varphi).$$

The full possibility of the phenomenon of incompleteness was established by Blok [5]. He showed that for any variety $\mathcal{V}$ of type 1 modal algebras satisfying $x \leq mx$ there are uncountably many other varieties $\mathcal{W}$ with $\mathbb{S}\mathsf{tr}\mathcal{W} = \mathbb{S}\mathsf{tr}\mathcal{V}$, so that $\mathcal{W}$ has exactly the same powerset algebras as $\mathcal{V}$. All of these varieties contain $\mathsf{Var}\mathbb{S}\mathsf{tr}\mathcal{V}$, which is the only one of them that is complete.

The question as to when a class of algebras is defined by a set of equations was answered by Birkhoff's theorem about closure under the operations $\mathsf{H}, \mathsf{S}, \mathsf{P}$. The dual of this question for relational structures is to ask when a class $\mathcal{K}$ of structures is equal to the class $\mathbb{S}\mathsf{tr}\Delta$ of all structures validating some set of formulae Δ. Classes of the form $\mathbb{S}\mathsf{tr}\Delta$ will be called *polymodal axiomatic* classes, since they are defined by a set of polymodal formulae. For such classes satisfying certain natural properties (e.g. $\mathbb{P}\mathsf{w}\mathcal{K} = \mathcal{K}$) there is a characterisation involving the dual operations to $\mathsf{H}, \mathsf{S}, \mathsf{P}$. Before demonstrating this (in 5.4.2) we note that the property of being polymodal axiomatic may be viewed as being dual to the property of completeness. This is because a variety $\mathcal{V}$ is complete if and only if

$$\mathcal{V} = \mathsf{Var}\mathbb{S}\mathsf{tr}\mathcal{V},$$

an equation whose dual for a class $\mathcal{K}$ of structures is

$$\mathcal{K} = \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}.$$

We have

Lemma 5.8 *A class $\mathcal{K}$ of relational structures is polymodal axiomatic if, and only if, $\mathcal{K} = \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$.*

PROOF. In general the variety $\mathsf{Var}\mathcal{K}$ generated by $\mathcal{K}$ is equal to the class $\mathsf{Var}\Lambda_\mathcal{K}$ of all algebras validating the logic $\Lambda_\mathcal{K} = \{\varphi : \mathcal{K} \models \varphi\}$ characterised by $\mathcal{K}$. Thus the class $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$ of structures in $\mathsf{Var}\mathcal{K}$ is equal to the polymodal axiomatic class $\mathbb{S}\mathsf{tr}\Lambda_{\mathcal{K}}$ of structures validating $\Lambda_{\mathcal{K}}$. Hence if $\mathcal{K} = \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$, then $\mathcal{K}$ is polymodal axiomatic.

Conversely, suppose $\mathcal{K} = \mathbb{S}\mathsf{tr}\Delta$ for some Δ. Then if $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$, the algebra $\mathsf{Cm}\,\mathfrak{S}$ belongs to $\mathsf{Var}\mathcal{K}$ and so validates any formulae that are valid in $\mathcal{K}$. In particular $\mathsf{Cm}\,\mathfrak{S} \models \Delta$, so $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\Delta = \mathcal{K}$. This establishes $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K} = \mathcal{K}$ as desired. ∎

Theorem 5.9 *Let $\mathcal{K}$ be a class of structures that is closed under ultrapowers. Then $\mathcal{K}$ is polymodal axiomatic if, and only if,*

(1) *$\mathcal{K}$ is closed under bounded epimorphic images, inner substructures and disjoint unions; and*

(2) *$\mathcal{K}$ reflects canonical extensions, i.e. $\mathbb{E}\mathsf{x}\,\mathfrak{S} \in \mathcal{K}$ implies $\mathfrak{S} \in \mathcal{K}$.*

Proof. Every polymodal axiomatic class satisfies (1) and (2). For the converse we use the fact, shown in the proof of Theorem 4.15, that for arbitrary $\mathcal{K}$,

$$\mathbb{C}\mathsf{st}\,\mathsf{Var}\,\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{E}\mathsf{x}\,\mathbb{U}\mathsf{d}\,\mathcal{K} \subseteq \mathbb{S}\,\mathbb{H}\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K}.$$

Now if $\mathbb{P}\mathsf{w}\mathcal{K} = \mathcal{K}$ and $\mathcal{K}$ satisfies (1) then $\mathbb{S}\,\mathbb{H}\mathbb{P}\mathsf{w}\,\mathbb{U}\mathsf{d}\,\mathcal{K} = \mathcal{K}$. But then if $\mathfrak{S} \in \mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K}$, we have $\mathsf{Cm}\,\mathfrak{S} \in \mathsf{Var}\mathcal{K}$, so

$$\mathbb{E}\mathsf{x}\,\mathfrak{S} = \mathbb{C}\mathsf{st}\,\mathsf{Cm}\,\mathfrak{S} \in \mathbb{C}\mathsf{st}\,\mathsf{Var}\,\mathcal{K} \subseteq \mathcal{K},$$

and hence $\mathfrak{S} \in \mathcal{K}$ if $\mathcal{K}$ reflects $\mathbb{E}\mathsf{x}$ (2). This shows that under the given hypotheses $\mathbb{S}\mathsf{tr}\mathsf{Var}\mathcal{K} = \mathcal{K}$, implying that $\mathcal{K}$ is polymodal axiomatic by Lemma 5.4.1. ∎

Theorem 5.4.2 was first presented in [22] under the hypothesis (for type 1 frames) that $\mathcal{K}$ is closed under elementary equivalence. Inspection of the proof just given reveals that an alternative sufficient hypothesis would be that $\mathcal{K}$ is closed under canonical extensions. More importantly, in view of the discussion in Section 2.2 about correspondences between definable properties of $\mathfrak{S}$ and equational properties of $\mathsf{Cm}\,\mathfrak{S}$, the Theorem gives as a special case a characterisation of those *elementary* classes that are polymodal axiomatic. A syntactic characterisation of the elementary classes that are closed under $\mathbb{H}, \mathbb{S}, \mathbb{U}\mathsf{d}$ (5.4.2(1)) is provided by the notion of a *quasi-modal* first-order sentence as described at the end of Section 4.6. There is currently no such "preservation theorem" known for elementary classes satisfying both 5.9(1) and 5.9(2). That this is a non-trivial question is shown by the fact that the quasi-modal sentence

$$\forall v \exists w\, (vRw \wedge wRw)$$

is preserved by $\mathbb{H}$, $\mathbb{S}$, and $\mathbb{U}\mathsf{d}$, but is not reflected by $\mathbb{E}\mathsf{x}$ since it holds in the structure $\mathbb{E}\mathsf{x}\,\langle \omega, < \rangle$.

The converse question of when a polymodal axiomatic class is elementary is already answered by the analysis of Corollary 4.12. The class $\mathbb{S}\mathsf{tr}\Delta$ of structures validating Δ is the same as the class $\mathbb{S}\mathsf{tr}\mathsf{Var}\Delta$ of structures in the variety $\mathsf{Var}\,\Delta$ of algebras validating Δ. So putting $\mathcal{V} = \mathsf{Var}\,\Delta$ in 4.6.3 immediately gives:

Theorem 5.10 *A polymodal axiomatic class is elementary if, and only if, it is closed under ultrapowers.* □

5.5 Canonical Logics

It is a standard practice in modal logic to say that a normal logic Λ in the language $\mathcal{L}_\omega(1)$ is canonical if it is validated by the canonical Λ-structure $\mathfrak{S}^\Lambda_\omega$. However Λ has manifestations in the languages $\mathcal{L}_\kappa(1)$ for $\kappa \geq \omega$ and hence canonical frames in all of these languages. A convenient abstract way of dealing with these special structures is provided by the observation from Section 5.1 that $\mathfrak{S}^\Lambda_\kappa$ is isomorphic to the canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$ of the Lindenbaum-Tarski algebra $\mathfrak{A}^\Lambda_\kappa$, and that the latter is the free algebra on κ generators in the variety $\mathsf{Var}\,\Lambda$ defined by Λ (Theorem 5.1). Moreover $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$ validates Λ if, and only if, it belongs to the class $\mathbb{S}\mathsf{tr}\mathsf{Var}\,\Lambda$ of structures in $\mathsf{Var}\,\Lambda$, i.e. iff its complex algebra $\mathsf{Em}\,\mathfrak{A}^\Lambda_\kappa$ belongs to $\mathsf{Var}\,\Lambda$.

We will say that a logic Λ in a language of arbitrary type is *κ-canonical* if Λ is valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$. Note that if $\mu < \kappa$ then by freeness $\mathfrak{A}^\Lambda_\kappa \twoheadrightarrow \mathfrak{A}^\Lambda_\mu$ and hence $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\mu \rightarrowtail \mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa$, so that $\mathfrak{A}^\Lambda_\kappa \models \Lambda$ implies $\mathfrak{A}^\Lambda_\mu \models \Lambda$.

Since there are free algebras on finitely many generators, we can use this approach to consider κ-canonicity for finite κ. In fact there exist logics that are κ-canonical for all $\kappa < \omega$ but are not ω-canonical. One example, analysed in detail in [19, Section 6], is the smallest type 1 logic containing $\neg\Diamond\Diamond\Diamond\top$, and the schemata (4) and

$$\Diamond\Diamond\varphi \wedge \Diamond\Diamond\neg\varphi \to \Diamond(\Diamond\varphi \wedge \Diamond\neg\varphi).$$

For this logic $\mathsf{Var}\,\Lambda$ is locally-finite, i.e. all finitely generated members are finite. Hence for $\kappa < \omega$, the free algebra $\mathfrak{A}^\Lambda_\kappa$ in $\mathsf{Var}\,\Lambda$ on κ generators is finite and so $\mathsf{Em}\,\mathfrak{A}^\Lambda_\kappa \cong \mathfrak{A}^\Lambda_\kappa \in \mathsf{Var}\,\Lambda$. But $\mathsf{Em}\,\mathfrak{A}^\Lambda_\omega \notin \mathsf{Var}\,\Lambda$.

A logic Λ will be defined to be *canonical* if it is κ-canonical for all $\kappa \geq \omega$. We have

Theorem 5.11 *A logic Λ is canonical if, and only if, the variety $\mathsf{Var}\,\Lambda$ is canonical in the sense that $\mathsf{Em}\,\mathsf{Var}\,\Lambda \subseteq \mathsf{Var}\,\Lambda$.*

PROOF. Theorem 4.2 established that $\mathsf{Var}\,\Lambda$ is canonical iff it contains $\mathsf{Em}\,\mathfrak{A}$ for all infinitely-generated free $\mathfrak{A}$ in $\mathsf{Var}\,\Lambda$, which we now see means that $\mathbb{C}\mathsf{st}\,\mathfrak{A}^\Lambda_\kappa \in \mathbb{S}\mathsf{tr}\mathsf{Var}\,\Lambda$ for all cardinals $\kappa \geq \omega$. ∎

It is immediate that canonicity implies completeness: if Λ is a logic in $\mathcal{L}_\kappa(\tau)$ that is valid in $\mathfrak{S}^\Lambda_\kappa$ then it is characterised by $\mathfrak{S}^\Lambda_\kappa$. This provides a methodology that has been used to obtain completeness theorems for numerous logics by the following procedure.

1. Find some condition π_Λ on structures with respect to which the logic Λ is sound, i.e. every structure satisfying π_Λ validates Λ.
2. Prove that the canonical Λ-structure $\mathfrak{S}^\Lambda_\omega$ satisfies π_Λ, and hence validates Λ.
3. Since $\mathfrak{S}^\Lambda_\omega$ invalidates all non-theorems of Λ via its canonical model $\mathfrak{M}^\Lambda_\omega$, conclude that Λ is characterised by $\mathfrak{S}^\Lambda_\omega$, as well as by the class of all structures satisfying π_Λ.

In all known examples π_Λ is a first-order condition on structures, defining a subclass $\mathbb{S}\mathsf{tr}\pi_\Lambda$ of $\mathbb{S}\mathsf{tr}\Lambda$. It is not necessary for π_Λ to exactly characterise Λ (i.e. $\mathbb{S}\mathsf{tr}\pi_\Lambda = \mathbb{S}\mathsf{tr}\Lambda$)

for the method to apply. For example take Λ as the smallest normal modal logic containing the schemata

$$\Diamond((\varphi_1 \to \Diamond\varphi_1) \wedge \cdots \wedge (\varphi_n \to \Diamond\varphi_n))$$

for all $n < \omega$, and π_Λ as the quasi-modal condition

$$\forall v \exists w\,(vRw \wedge wRw)$$

mentioned in the previous section. Then Λ is sound for π_Λ and $\mathfrak{S}_\omega^\Lambda$ satisfies π_Λ, but Λ is also valid in $\langle \omega, < \rangle$, a structure in which π_Λ is clearly false (cf. [29]).

A very general situation in which π_Λ does exactly define $\mathbb{S}\mathsf{tr}\Lambda$ is provided by the work of Sahlqvist [49], which is generalised to arbitrary types in [10]. This gives the broadest known syntactic definition of a class of formulae to which the canonical structure methodology applies. To describe this, define a formula φ to be *positive*, or *negative* respectively, if every variable of φ occurs within the scope of an even, or odd respectively, number of negations. A *box string* is a formula of the form

$$[\,\beta_0\,] \cdots [\,\beta_{n-1}\,]p$$

where p is a variable and each $[\,\beta_i\,]$ is a unary box modality. A *Sahlqvist antecedent* is a formula constructed from the constants $\bot, \top$, box strings and negative formulae using only $\wedge, \vee$ and diamond poly-modalities. A *Sahlqvist formula* is one constructed out of implications $\varphi \to \psi$ in which φ is a Sahlqvist antecedent and ψ is any positive formula by using only $\wedge$ and formation of box polymodalities $[\,\beta\,](\varphi_0, \ldots, \varphi_{n-1})$ in which none of the arguments φ_i have any variables in common.

For each Sahlqvist formula φ, let Λ_φ be the smallest normal uniform logic containing φ. There is an effective procedure associating with such φ a first-order sentence π_φ that holds exactly in the members of $\mathbb{S}\mathsf{tr}\Lambda_\varphi$ (cf. [10, Section 3]), so $\mathbb{S}\mathsf{tr}\Lambda_\varphi$ is an elementary class. The fact that Λ_φ is validated by its canonical frame $\mathfrak{S}_\omega^{\Lambda_\varphi}$ was demonstrated model-theoretically in [49], but there is now an elegant algebraic approach [31] for showing that a variety characterised by Sahlqvist formulae is canonical.

The simplest type 1 formula that is not a Sahlqvist formula is the well-known *McKinsey axiom*

$$\Box\Diamond\varphi \to \Diamond\Box\varphi.$$

This was shown not to be canonical in [16], indicating that there is no natural way to extend the class of Sahlqvist formulae to a larger class of canonical formulae.

The following fundamental result was also first shown model-theoretically for modal logic, by Fine in [12].

Theorem 5.12 *If a logic is characterised by an elementary class of structures, then it is canonical.*

PROOF. Let Λ be characterised by the class $\mathcal{K}$. Then an algebra $\mathfrak{A}$ is in the variety $\mathsf{Var}\mathcal{K}$ generated by $\mathcal{K}$ iff it validates all formulae validated by $\mathcal{K}$, i.e. iff $\mathfrak{A} \models \Lambda$. Thus $\mathsf{Var}\mathcal{K} = \mathsf{Var}\Lambda$. But if $\mathcal{K}$ is elementary, by Theorem 4.15 $\mathsf{Var}\mathcal{K}$ is canonical, and so Λ is canonical by Theorem 5.11. ∎

Corollary 5.13 *If a logic Λ is complete, and the class $\mathbb{S}\mathsf{tr}\Lambda$ of structures is closed under ultrapowers, then Λ is canonical.*

PROOF. Suppose $\mathbb{S}\mathsf{tr}\Lambda$ ($= \mathbb{S}\mathsf{trVar}\Lambda$) is closed under ultrapowers. Then by Corollary 4.12 it is an elementary class. But if Λ is complete it is characterised by $\mathbb{S}\mathsf{tr}\Lambda$, so then canonicity follows from 5.12. ∎

It is important to recognise that the fact that the class of structures validating a logic is elementary does not by itself guarantee canonicity. An additional hypothesis about completeness is necessary. For instance, there are logics Λ that are *incomplete* and therefore not canonical, but for which $\mathbb{S}\mathsf{tr}\Lambda$ is an elementary class. One example is the incomplete temporal logic of [55] having $\mathbb{S}\mathsf{tr}\Lambda = \emptyset$. Another is the incomplete modal logic of [56] for which $\mathbb{S}\mathsf{tr}\Lambda$ is the class of all quasi-orderings, i.e. the logic is distinct from S4 but is valid in exactly the same structures as S4.

The converse of Theorem 5.5.2 – that *every canonical logic is characterised by an elementary class of structures* – is one of the main unsolved conjectures of this subject. A related conjecture is this:

- *if Λ is ω-canonical, then it is canonical,*

or equivalently

- *if $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\omega} \models \Lambda$, then $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\kappa} \models \Lambda$ for all $\kappa > \omega$.*

The intuition behind this is that if some Λ-theorem is falsifiable in $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\kappa}$ for an infinite κ, then it should be falsifiable in the canonical structure of the denumerably generated algebra $\mathfrak{A}^{\Lambda}_{\omega}$.

To prove the conjecture it would be enough (by 5.12) to prove that if Λ is ω-canonical then it is characterised by an elementary class. Now the discussion following Theorem 4.16 indicates that Λ is characterised by an elementary class iff it is characterised by the class of models of the first-order theory of $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\omega}$. Thus a natural approach to setting this conjecture about the sufficiency of ω-canonicity would be to show that

> if Λ is valid in $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\omega}$, then it is valid in every structure elementarily equivalent to $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\omega}$.

Theorem 4.6.8 gives further information about the syntactic form of first-order sentences involved in elementary characterisations of logics:

> if Λ is characterised by an elementary class then it is characterised by the class of all models of the *quasi-modal* theory of $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\omega}$.

(See [18, Section 11.4] for details. Quasi-modal sentences were defined at the end of Section 4.6 above.)

In a recent article [21], the author has investigated the quasi-modal theories of the canonical structures $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\kappa}$ for all $\kappa \geq \omega$. They turn out to be the same, and indeed to be the same as the quasi-modal theories of the canonical structures of two important sub-logics of Λ. The results of [21] can be summarized as follows.

- All of the canonical structures $\mathbb{C}\mathsf{st}\,\mathfrak{A}^{\Lambda}_{\kappa}$ of a given logic Λ have the same quasi-modal first-order theory Ψ^{Λ}.

- The models of Ψ^{Λ} characterise a logic Λ^e which is the largest sublogic of Λ to be characterised by some elementary class.
- The canonical structures of Λ^e also have Ψ^{Λ} as their quasi-modal theory.
- There is a largest sublogic Λ^c of Λ that is characterised by its own canonical structures. Since Λ^e is canonical (5.5.2), $\Lambda^e \subseteq \Lambda^c$.
- The canonical structures of Λ^c also have Ψ^{Λ} as their quasi-modal theory. Thus $\Psi^{\Lambda} = \Psi^{\Lambda^c} = \Psi^{\Lambda^e}$.
- All finite structures validating Λ are models of Ψ^{Λ}. If Λ is characterised by its finite structures (see Section 6), then Ψ^{Λ} is equal to the quasi-modal theory of these structures.

Of course if all canonical logics are elementarily characterised, then $\Lambda^e = \Lambda^c$. But that is the unresolved question.

5.6 *Strong Completeness and Complex Varieties*

Let Λ be a normal logic in a language $\mathcal{L}_\kappa(\tau)$. Λ is called *strongly κ-complete* if there exists a class $\mathcal{K}$ of τ-structures such that the following hold:

- every member of $\mathcal{K}$ validates Λ, i.e. $\mathcal{K} \subseteq \mathbb{S}\mathsf{tr}\Lambda$; and
- if Δ is any Λ-consistent set of $\mathcal{L}_\kappa(\tau)$-formulae, then Δ is satisfiable at some point of some model based on a structure that belongs to $\mathcal{K}$.

If such a $\mathcal{K}$ exists, then $\mathcal{K}$ characterises Λ, so Λ is complete. It also follows directly that κ-canonicity implies strong κ-completeness, since if $\mathfrak{S}^{\Lambda}_{\kappa}$ validates Λ then putting $\mathcal{K} = \{\mathfrak{S}^{\Lambda}_{\kappa}\}$ fulfills the above definition. This is because if Δ is Λ-consistent it can be extended to a Λ-maximal set Γ, and then

$$\mathfrak{M}^{\Lambda}_{\kappa} \models_{\Gamma} \Delta,$$

where $\mathfrak{M}^{\Lambda}_{\kappa}$ is the canonical Λ-model on $\mathfrak{S}^{\Lambda}_{\kappa}$.

An example of a complete logic for which strong completeness fails is the modal logic KW, discussed at the end of Section 5.3, which is characterised by the variety $\mathcal{V}_{dg}$ of diagonalisable algebras, as well as by $\mathbb{S}\mathsf{tr}$KW which is the class of frames $\mathfrak{S} = \langle S, R \rangle$ in which R is transitive and R^{-1} is well-founded, i.e. there are no infinite "R-sequences"

$$s_0 R s_1 R \cdots R s_n R s_{n+1} \cdots\cdots$$

Put

$$\begin{aligned} \varphi_1 &= \Diamond p_1, \\ \varphi_{n+1} &= \Box(p_n \to \Diamond p_{n+1}), \\ \Delta &= \{\varphi_n : 1 \leq n < \omega\}. \end{aligned}$$

Then Δ is KW-consistent, but cannot be satisfied in any model based on a a KW-frame. For, if $\mathfrak{M}_n$ is any model on the KW-frame $\langle \{0, \ldots, n\}, < \rangle$ that has $\mathfrak{M}_n(p_i) = \{i\}$ for $1 \leq i \leq n$, then $\mathfrak{M}_n \models_0 \varphi_i$ for all $1 \leq i \leq n$. This shows that every finite subset of Δ is satisfiable in a model on a KW-frame and so must be KW-consistent.

Since the proof theory of KW is finitary, this entails that Δ itself is KW-consistent. However if $\mathfrak{M} \models_{s_0} \Delta$ and the frame $\mathfrak{S}$ of $\mathfrak{M}$ is transitive, then there must be an R-sequence as above with $\mathfrak{M} \models_{s_n} p_n$ for all $n \geq 1$, hence $\mathfrak{S}$ is not a KW-frame.

An algebraic version of this argument was used in [14, Theorem 3.7.1] to derive another negative property of KW, namely that its variety $\mathcal{V}_{dg}$ is not complex, i.e. is not of the form $\mathsf{S}\,\mathsf{Cm}\,\mathcal{K}$ for any $\mathcal{K}$. These two negative properties are really two sides of the same coin, because it turns out that a variety is complex iff its associated logic is strongly complete in all cardinalities. In fact we can formulate this more strongly as the following result about *quasi-varieties*, which are classes of algebras that are closed under subalgebras (S), direct products (P), and ultraproducts (Pu). (The version for varieties was discovered independently by F. Wolter.)

Theorem 5.14 *Let $\mathcal{V}$ be a quasi-variety.*

(1) *If $\mathcal{V}$ is complex, then its associated logic is strongly κ-complete for all infinite cardinals κ.*

(2) *If the logic associated with $\mathcal{V}$ is strongly κ-complete for all infinite κ, then the homomorphic closure $\mathsf{H}\mathcal{V}$ of $\mathcal{V}$ is complex.*

Consequently, if a quasi-variety $\mathcal{V}$ is complex, then the variety $\mathsf{H}\mathcal{V}$ generated by $\mathcal{V}$ is also complex.

Proof. Recall that the logic associated with $\mathcal{V}$ is

$$\Lambda = \{\varphi \text{ in } \mathcal{L}_\omega(\tau) : \mathcal{V} \models \varphi\}.$$

Then $\mathsf{Var}\,\Lambda$ is the variety generated by $\mathcal{V}$, so as $\mathcal{V}$ is S-P-closed we do have $\mathsf{Var}\,\Lambda = \mathsf{H}\mathcal{V}$ as claimed.

(1) Assume that $\mathcal{V}$ is complex. Take $\kappa \geq \omega$, with Λ_κ the logic induced in $\mathcal{L}_\kappa(\tau)$ by closure of Λ under substitution. Let $\mathfrak{A} \in \mathcal{V}$ be the Lindenbaum-Tarski algebra for Λ_κ in the language $\mathcal{L}_\kappa(\tau)$. Then $\mathfrak{A}$ belongs to $\mathcal{V}$, because it is a free algebra in the variety $\mathsf{Var}\,\Lambda$ generated by $\mathcal{V}$ (5.2.1), and $\mathcal{V}$, being closed under S and P, contains all such free algebras.

Now if Δ is any Λ_κ-consistent set of $\mathcal{L}_\kappa(\tau)$-formula, then

$$\|\Delta\| = \{\|\varphi\| : \varphi \in \Delta\}$$

is a subset of $\mathfrak{A}$ with the finite meet property: every finite subset of $\|\Delta\|$ has non-zero meet in $\mathfrak{A}$. It follows, by a standard compactness argument, there is an algebra $\mathfrak{A}^*$ that has $\mathfrak{A}$ as a subalgebra and has a non-zero element x that is a lower bound for $\|\Delta\|$:

$0 \neq x \leq \|\varphi\|$ for all $\varphi \in \Delta$.

Indeed $\mathfrak{A}^*$ can be constructed as an ultrapower of $\mathfrak{A}$, so Pu-closure of $\mathcal{V}$ allows us to conclude that $\mathfrak{A}^* \in \mathcal{V}$.

Since $\mathcal{V}$ is complex, we can assume that $\mathfrak{A}^*$, and hence $\mathfrak{A}$, is a subalgebra of the complex algebra $\mathsf{Cm}\,\mathfrak{S}$ of some structure that belongs to $\mathbb{S}\mathsf{tr}\mathcal{V}$, and so has $\mathfrak{S} \models \Lambda_\kappa$. Let s be an element of x in $\mathfrak{S}$. Then $s \in \|\varphi\|$ for all $\|\varphi\| \in \Delta$. Putting

$$V(p_\lambda) = \|p_\lambda\| \subseteq S$$

defines a model $\mathfrak{M}$ on $\mathfrak{S}$ having $\mathfrak{M}(\varphi) = \|\varphi\|$ for all φ, and hence

$$\mathfrak{M} \models_s \Delta.$$

This establishes that every Λ_κ-consistent set of $\mathcal{L}_\kappa(\tau)$-formulae is satisfiable in a model on a structure validating Λ_κ, giving strong κ-completeness.

(2) Suppose that Λ is strongly κ-complete for $\kappa \geq \omega$. To prove $\mathsf{H}\mathcal{V}$ complex we need to show that if $\mathfrak{A} \in \mathsf{H}\mathcal{V}$, then $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{T}$ and $\mathsf{Cm}\,\mathfrak{T} \in \mathsf{H}\mathcal{V}$ for some structure $\mathfrak{T}$. We will show this first for the case that $\mathfrak{A}$ is a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$ for some structure $\mathfrak{S}$ that is generated by a point s (N.B. we do not assume $\mathsf{Cm}\,\mathfrak{S} \in \mathsf{H}\mathcal{V}$ here). Let κ be any infinite cardinal for which there is a surjection

$$V : \Phi_\kappa \twoheadrightarrow \mathfrak{A},$$

and put $\mathfrak{M} = \langle \mathfrak{S}, V \rangle$. Then each truth-set $\mathfrak{M}(\varphi)$ is in $\mathfrak{A}$, and each member of $\mathfrak{A}$ is such a truth-set, indeed is one of the form $\mathfrak{M}(p_\lambda)$. Let

$$\Delta = \{\varphi \text{ in } \mathcal{L}_\kappa(\tau) : \mathfrak{M} \models_s \varphi\}.$$

Since $\mathfrak{A} \models \Lambda_\kappa$, Δ is Λ_κ-consistent (in fact it is Λ_κ-maximal). By strong κ-completeness there exists a structure $\mathfrak{T}'$ validating Λ_κ and a model $\mathfrak{N}'$ on $\mathfrak{T}$ such that $\mathfrak{N}' \models_t \Delta$ for some t. Let $\mathfrak{T}$ be the inner substructure of $\mathfrak{T}'$ generated by the point t, and $\mathfrak{N}$ the restriction to T of the model $\mathfrak{N}'$, having

$$\mathfrak{N}(\varphi) = \mathfrak{N}'(\varphi) \cap T.$$

Then $\mathfrak{N} \models_t \Delta$, and so

$$(\dagger) \qquad \mathfrak{M} \models_s \varphi \quad \text{iff} \quad \mathfrak{N} \models_t \varphi$$

for all φ in $\mathcal{L}_\kappa(\tau)$. Moreover $\mathfrak{T}$ validates Λ_κ, so $\mathsf{Cm}\,\mathfrak{T} \in \mathsf{Var}\,\Lambda_\kappa = \mathsf{H}\mathcal{V}$.
It thus remains to show that $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{T}$. For this purpose, consider the correspondence

$$\theta : \mathfrak{M}(\varphi) \mapsto \mathfrak{N}(\varphi)$$

between $\mathfrak{A}$ and $Sb\,T$. First we need to show that θ is a well-defined injection, i.e.

$$\mathfrak{M}(\varphi) = \mathfrak{M}(\psi) \quad \text{iff} \quad \mathfrak{N}(\varphi) = \mathfrak{N}(\psi).$$

This will be explained for the case of the simplest language with a single modality $\Diamond$, so that $\mathfrak{S}$ and $\mathfrak{T}$ are type 1 frames with a single binary relation. If $\mathfrak{M}(\varphi) \neq \mathfrak{M}(\psi)$ then there is some point u in $\mathfrak{S}$ with, say, $\varphi \wedge \neg\psi$ true in $\mathfrak{M}$ at u. Since $\mathfrak{S}$ is generated by s, the analysis of Section 4.3 shows that $sR^n_{\mathfrak{S}}u$ for some n. Hence the formula

$$(\ddagger) \qquad \underbrace{\Diamond\Diamond\cdots\Diamond}_{n \text{ times}}(\varphi \wedge \neg\psi)$$

is true in $\mathfrak{M}$ at s, and so by $(\dagger)$ is true in $\mathfrak{N}$ at t. From this is follows that $\varphi \wedge \neg\psi$ is true at some point in $\mathfrak{N}$, showing that $\mathfrak{N}(\varphi) \neq \mathfrak{N}(\psi)$. The proof that $\mathfrak{N}(\varphi) \neq \mathfrak{N}(\psi)$ implies $\mathfrak{M}(\varphi) \neq \mathfrak{M}(\psi)$ is the same, using the other implication of $(\dagger)$ and the fact that t generates $\mathfrak{T}$.

The argument for structures of arbitary type follows the same pattern, using the general description of point-generated structures from Section 4.3 and some more complicated formulae in place of (‡). The properties of truth-sets ensure that θ is a homomorphism, and hence gives an embedding $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{T} \in \mathsf{H}\mathcal{V}$ as desired.

For the case of an arbitrary $\mathfrak{A} \in \mathsf{H}\mathcal{V}$, by Theorem 4.5 there is a subdirect representation

$$\mathfrak{A} \rightarrowtail \prod_{s \in S} \mathfrak{A}_s$$

of $\mathfrak{A}$ by complex algebras $\mathfrak{A}_s$ based on point-generated structures. Each $\mathfrak{A}_s$ is in $\mathsf{H}\mathcal{V}$, as $\mathfrak{A} \twoheadrightarrow \mathfrak{A}_s$, so by the above argument there is a structure $\mathfrak{T}_s \in \mathbb{S}\mathsf{tr}\mathsf{H}\mathcal{V}$ such that $\mathfrak{A}_s \rightarrowtail \mathsf{Cm}\,\mathfrak{T}_s$. Then

$$\mathfrak{A} \rightarrowtail \prod_{s \in S} \mathfrak{A}_s \rightarrowtail \prod_{s \in S} \mathsf{Cm}\,\mathfrak{T}_s \cong \mathsf{Cm}(\mathfrak{T}),$$

where

$$\mathfrak{T} = \coprod_{s \in S} \mathfrak{T}_s \in \mathbb{S}\mathsf{tr}\mathsf{H}\mathcal{V}$$

by closure of $\mathbb{S}\mathsf{tr}\mathsf{H}\mathcal{V}$ under disjoint unions. This proves that $\mathsf{H}\mathcal{V}$ is a complex variety.

The last part of the statement of Theorem 5.6.1 now follows directly by applying (1) and then (2). ■

The assumption of Pu-closure is essential in Theorem 5.6.1, as may be seen by taking $\mathcal{V}$ as the complex class $\mathsf{S}\,\mathsf{Cm}\,\mathbb{S}\mathsf{tr}$KW discussed just before 5.6.1. In this example $\mathcal{V}$ is closed under S and P, but not under Pu, and the variety it generates is the *non-complex* class $\mathcal{V}_{dg}$ of diagonalisable algebras.

The question of whether there exist complex varieties that are not canonical is now seen to be equivalent to the question of the existence of strongly complete logics that are not canonical. In fact one such is the logic $\Lambda_{\mathbb{R}}$ characterised by the type 2 frame

$$\langle \mathbb{R}, <, > \rangle,$$

where $\mathbb{R}$ is the set of real numbers. This logic is not canonical, for reasons that will be clarified below, but was shown in [13] to be strongly ω-complete: every consistent set of $\mathcal{L}_\omega(2)$-formulae is satisfiable in a model on the real-number frame itself.

It can be inferred from this that the variety $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ defined by $\Lambda_{\mathbb{R}}$ is complex but not canonical, as was first noticed by F. Wolter [61]. But instead of appealing to Theorem 5.14, the idea of Gabbay's strong completeness proof can be adapted to give an interesting direct algebraic construction showing $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ complex. We will carry this out now for a slightly simpler example: the temporal logic of Dedekind complete strict orderings. For this purpose the two diamond modalities of a type 2 language will be written $\langle\,\mathrm{F}\,\rangle$ and $\langle\,\mathrm{P}\,\rangle$, with their duals being $[\,\mathrm{F}\,]$ and $[\,\mathrm{P}\,]$. Here "F" is for "future" and "P" for "past". The additive operators of a type 2 algebra are m_{F} and m_{P}, with duals l_{F} and l_{P}. A type 2 frame will be written as $\mathfrak{S} = \langle S, R_{\mathrm{F}}, R_{\mathrm{P}} \rangle$. The operators on $\mathsf{Cm}\,\mathfrak{S}$ induced by R_{F} are

$$m_{R_{\mathrm{F}}}(T) = \{s \in S : \exists t\,(sR_{\mathrm{F}}t \text{ and } t \in T)\}$$

and its dual

$$l_{R_{\mathrm{F}}}(T) = \{s \in S : \forall t\,(sR_{\mathrm{F}}t \text{ implies } t \in T)\},$$

and similarly for $m_{R_{\mathrm{P}}}$ and $l_{R_{\mathrm{P}}}$.

A *linear temporal logic* is any normal logic containing the following schemata, which come in three pairs that are "mirror images", i.e. each member of the pair is obtained from the other by interchanging F and P.

$$\begin{aligned}
&\varphi \to [\,\mathrm{P}\,]\langle\,\mathrm{F}\,\rangle\varphi \\
&\varphi \to [\,\mathrm{F}\,]\langle\,\mathrm{P}\,\rangle\varphi \\
&\langle\,\mathrm{F}\,\rangle\langle\,\mathrm{F}\,\rangle\varphi \to \langle\,\mathrm{F}\,\rangle\varphi \\
&\langle\,\mathrm{P}\,\rangle\langle\,\mathrm{P}\,\rangle\varphi \to \langle\,\mathrm{P}\,\rangle\varphi \\
&[\,\mathrm{F}\,](\varphi \wedge [\,\mathrm{F}\,]\varphi \to \psi) \vee [\,\mathrm{F}\,](\psi \wedge [\,\mathrm{F}\,]\psi \to \varphi) \\
&[\,\mathrm{P}\,](\varphi \wedge [\,\mathrm{P}\,]\varphi \to \psi) \vee [\,\mathrm{P}\,](\psi \wedge [\,\mathrm{P}\,]\psi \to \varphi).
\end{aligned}$$

The first pair are valid in a frame precisely when $R_{\mathrm{P}} = R_{\mathrm{F}}^{-1}$ (cf. the discussion of conjugate operators in Sections 2.1 and 2.2), so that frames for this pair are uniquely determined as soon as R_{F} is specified. The second pair characterise transitivity of R_{F} and R_{P}. The last pair ensure that

$$sR_{\mathrm{F}}t \wedge sR_{\mathrm{F}}u \quad \text{implies} \quad (t = u \text{ or } tR_{\mathrm{F}}u \text{ or } uR_{\mathrm{F}}t),$$

and correspondingly for R_{P}. In a frame $\mathfrak{S}$ validating any linear temporal logic, the inner subframe $\mathfrak{S}_s$ generated by a point s is based on the set

$$\{t \in S : sR_{\mathrm{F}}t \text{ or } s = t \text{ or } tR_{\mathrm{F}}s\}.$$

$\mathfrak{S}_s$ is *connected*, i.e. satisfies

$$\forall t \forall u\,(t \neq u \quad \text{implies} \quad tR_{\mathrm{F}}u \text{ or } uR_{\mathrm{F}}t),$$

and consists of a linear sequence of *clusters* as defined in Section 4.2. These clusters can then be flattened by the bulldozer construction to show that there is a bounded epimorphism $\mathfrak{T} \twoheadrightarrow \mathfrak{S}_s$ with $\mathfrak{T}$ a *strict linear ordering* (irreflexive, transitive, connected) having $\mathsf{Cm}\,\mathfrak{S}_s \rightarrowtail \mathsf{Cm}\,\mathfrak{T}$.

Now the schemata defining a linear temporal logic Λ are preserved by canonical extensions. Thus if $\mathfrak{A} \models \Lambda$ then $\mathbb{C}\mathsf{st}\,\mathfrak{A} \models \Lambda$. Then taking the subdirect representation in terms of point-generated structures of $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ that underlies Theorem 4.5, and applying the observations of the previous paragraph, the following can be concluded.

Theorem 5.15 *Any algebra validating a linear temporal logic has a subdirect representation by complex algebras based on strict linear orderings.* □

Now if $\mathfrak{S} = \langle S, R_{\mathrm{F}}, R_{\mathrm{P}} \rangle$ is a strict linear ordering with $R_{\mathrm{P}} = R_{\mathrm{F}}^{-1}$, then a subset I of S is an *initial segment* of $\mathfrak{S}$ if

$$sR_{\mathrm{F}}t \text{ and } t \in I \quad \text{implies} \quad s \in I.$$

Then $I \subseteq l_{R_{\mathrm{P}}}(I)$, and the complement $I^c = S - I$ of I satisfies $I^c \subseteq l_{R_{\mathrm{F}}}(I^c)$ since $s \in I^c$ and $sR_{\mathrm{F}}t$ implies $t \in I^c$. I is a *proper* initial segment if its complement is non-empty. All members of this complement are upper bounds of I.

A *gap* is a non-empty proper initial segment that has no least upper bound. If I is a gap then I has no greatest member, so if s is in I then s is in $m_{R_{\mathrm{F}}}(I)$, and therefore not in $l_{R_{\mathrm{F}}}(I^c)$. Thus $I^c = l_{R_{\mathrm{F}}}(I^c)$. Also I^c has no least member, so each member of I^c is in $m_{R_{\mathrm{P}}}(I^c)$. It follows that

$$l_{R_{\mathrm{F}}}(I^c) - m_{R_{\mathrm{P}}} l_{R_{\mathrm{F}}}(I^c) = I^c - m_{R_{\mathrm{P}}}(I^c) = \emptyset.$$

A strict linear order is *Dedekind complete* if it has no gaps. Both $\langle \omega, < \rangle$ and $\langle \mathbb{R}, < \rangle$ are Dedekind complete. Any strict linear order $\mathfrak{T}$ has a *Dedekind completion*, an extension to a Dedekind complete order $\mathfrak{S}$ obtained by "filling in the gaps in $\mathfrak{T}$". Formally this can be achieved by taking $\mathfrak{S}$ as the set of proper initial segments of $\mathfrak{T}$ ordered by proper inclusion $\subset$. $\mathfrak{T}$ can be regarded as a subordering of $\mathfrak{S}$ by identifying each s in $\mathfrak{T}$ with the initial segment $\{t \in T : tR_{\mathrm{F}}s\}$. When $s \in S - T$, this initial segment is a gap in $\mathfrak{T}$.

It was discovered by Arthur Prior that there is a type 2 formula that characterises Dedekind completeness (there is no such type 1 formula). *Prior's axiom* is the schema

$$\langle \mathrm{F} \rangle \neg\varphi \wedge \langle \mathrm{F} \rangle [\mathrm{F}]\varphi \to \langle \mathrm{F} \rangle ([\mathrm{F}]\varphi \wedge \neg \langle \mathrm{P} \rangle [\mathrm{F}]\varphi),$$

which is valid in any Dedekind complete strict linear ordering.

Lemma 5.16 (**Gap Lemma**) *Let $\mathfrak{S}$ be a strict linear ordering and $\mathfrak{A}$ a subalgebra of $\mathsf{Cm}\,\mathfrak{S}$. If $\mathfrak{A}$ validates Prior's axiom, then no gap of $\mathfrak{S}$ can belong to $\mathfrak{A}$.*

PROOF. Suppose there is a gap $I \in \mathfrak{A}$, with complement $I^c \in \mathfrak{A}$, and take $s \in I$. As I has no greatest element, there exists $t \in I$ with $sR_{\mathrm{F}}t$, and so $s \in m_{R_{\mathrm{F}}}(I)$. But I has as upper bound any $u \in I^c = l_{R_{\mathrm{F}}}(I^c)$, with $sR_{\mathrm{F}}u$, so $s \in m_{R_{\mathrm{F}}} l_{R_{\mathrm{F}}}(I^c)$.

Now we saw above that $l_{R_{\mathrm{F}}}(I^c) - m_{R_{\mathrm{P}}} l_{R_{\mathrm{F}}}(I^c) = \emptyset$, and so

$$s \notin m_{R_{\mathrm{F}}}(l_{R_{\mathrm{F}}}(I^c) - m_{R_{\mathrm{P}}} l_{R_{\mathrm{F}}}(I^c)) = m_{R_{\mathrm{F}}}\emptyset = \emptyset.$$

This shows that

$$m_{R_{\mathrm{F}}}(I) \cap m_{R_{\mathrm{F}}} l_{R_{\mathrm{F}}}(I^c) \nsubseteq m_{R_{\mathrm{F}}}(l_{R_{\mathrm{F}}}(I^c) - m_{R_{\mathrm{P}}} l_{R_{\mathrm{F}}}(I^c)),$$

in violation of Prior's axiom. ∎

Theorem 5.17 *If Λ_{D} is the smallest linear temporal logic that includes Prior's axiom, then the variety $\mathsf{Var}\,\Lambda_{\mathrm{D}}$ of all type 2 algebras that validate Λ_{D} is complex but not canonical.*

PROOF. We deal with non-canonicity first. The type 2 frame $\langle \omega, <, > \rangle$ is a Dedekind complete strict linear ordering, and the set of finite or cofinite subsets of ω forms a subalgebra $\mathfrak{A}$ of $\mathsf{Cm}\,\mathfrak{S}$ which validates Λ_{D}. The canonical structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ consists of the principal ultrafilters

$$\{X \in \mathfrak{A} : n \in X\}$$

for each $n < \omega$, together with the set s of all cofinite sets, which satisfies sRs in $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. Thus $\mathbb{C}\mathsf{st}\,\mathfrak{A}$ looks like a copy of $\langle \omega, < \rangle$ with a single reflexive point added at the right end, so that the copy of ω functions like a gap (although the linear ordering is no longer strict). Precisely, in $\mathsf{Em}\,\mathfrak{A}$ we have

$$l_{\mathrm{F}}(\{s\}) = m_{\mathrm{P}} l_{\mathrm{F}}(\{s\}) = (\{s\}),$$

and so

$$m_{\mathrm{F}}(l_{\mathrm{F}}(\{s\}) - m_{\mathrm{P}} l_{\mathrm{F}}(\{s\})) = \emptyset,$$

while each of the principal ultrafilters belongs to

$$m_{\mathrm{F}}(-\{s\}) \cap m_{\mathrm{F}} l_{\mathrm{F}}(\{s\}).$$

Thus Prior's axiom fails in $\mathsf{Em}\,\mathfrak{A}$.

To show that $\mathsf{Var}\,\Lambda_{\mathrm{D}}$ is complex, we have to show that if $\mathfrak{A}$ is any member of $\mathsf{Var}\,\Lambda_{\mathrm{D}}$ then there is a structure $\mathfrak{S}$ with $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{S}$ and $\mathsf{Cm}\,\mathfrak{S} \in \mathsf{Var}\,\Lambda_{\mathrm{D}}$. As explained in the latter part of the proof of Theorem 5.14, it suffices to prove this for a class of algebras that provide subdirect representations of all other members of $\mathsf{Var}\,\Lambda_{\mathrm{D}}$. Therefore by Theorem 5.15 we can assume that $\mathfrak{A}$ is a subalgebra of $\mathsf{Cm}\,\mathfrak{T}$ for some strict linear ordering $\mathfrak{T} = \langle T, R_{\mathrm{F}} \rangle$. Now let $\mathfrak{S} = \langle S, R_{\mathrm{F}} \rangle$ be the Dedekind completion of $\mathfrak{T}$. Then $\mathsf{Cm}\,\mathfrak{S}$ validates Prior's axiom and so belongs to $\mathsf{Var}\,\Lambda_{\mathrm{D}}$. Thus it is enough to show that $\mathfrak{A} \rightarrowtail \mathsf{Cm}\,\mathfrak{S}$ to complete the argument. By Theorem 4.3, this in turn reduces to the problem of showing that there is an ultrafilter map from $\mathfrak{S}$ to $\mathfrak{A}$ that covers $\mathfrak{A}$. This map is to be a function $f : \mathfrak{S} \to \mathbb{C}\mathsf{st}\,\mathfrak{A}$ satisfying, for all $s \in S$ and $X \in \mathfrak{A}$,

(i) $m_{\mathrm{F}}(X) \in f(s)$ iff for some $t \in S$, $sR_{\mathrm{F}}t$ and $X \in f(t)$;
(ii) $m_{\mathrm{P}}(X) \in f(s)$ iff for some $t \in S$, $tR_{\mathrm{F}}s$ and $X \in f(t)$.

Note that m_{F} here means the operation on $\mathsf{Cm}\,\mathfrak{T}$, and hence on $\mathfrak{A}$, induced by R_{F} in $\mathfrak{T}$, rather than the operation $m_{R_{\mathrm{F}}}$ induced on $\mathsf{Cm}\,\mathfrak{S}$. Thus for $X \subseteq T$,

$$m_{\mathrm{F}}(X) = \{u \in T : \exists t \in T\,(uR_{\mathrm{F}}t)\}.$$

Since $\mathfrak{T}$ will not in general be an inner substructure of $\mathfrak{S}$, we may well have $m_{\mathrm{F}}(X) \neq m_{R_{\mathrm{F}}}(X)$ for $X \in \mathfrak{A}$. Similarly,

$$m_{\mathrm{P}}(X) = \{u \in T : \exists t \in T\,(tR_{\mathrm{F}}u)\}.$$

Now for s in $\mathfrak{T}$, put

$$f(s) = \{X \in \mathfrak{A} : s \in X\}.$$

This already ensures that f covers $\mathfrak{A}$, for if $\emptyset \neq X \in \mathfrak{A}$ then any $s \in X$ has $X \in f(s)$. For $s \in S - T$, let

$$\begin{aligned} U_s &= \{m_{\mathrm{F}}(X) : X \in \mathfrak{A} \text{ and } \exists t \in T\,(sR_{\mathrm{F}}t \text{ and } t \in X)\}, \\ L_s &= \{m_{\mathrm{P}}(X) : X \in \mathfrak{A} \text{ and } \exists t \in T\,(tR_{\mathrm{F}}s \text{ and } t \in X)\}. \end{aligned}$$

Then $U_s \cup L_s$ has the finite intersection property. To see this, suppose that for some $n < \omega$ there are elements $X_i \in \mathfrak{A}$ and $t_i \in T$ such that $sR_{\mathrm{F}}t_i \in X_i$, and hence $m_{\mathrm{F}}(X_i) \in U_s$, for all $i < n$. Now since s is not in T, it represents a gap in $\mathfrak{T}$, and so the set $\{t \in T : sR_{\mathrm{F}}t\}$ has no least element. Thus there exists some $t \in T$ such that $sR_{\mathrm{F}}tR_{\mathrm{F}}t_i$ and hence $t \in m_{\mathrm{F}}(X_i)$ for all $i < n$. Moreover, because $sR_{\mathrm{F}}t$, t is in *every* set $m_{\mathrm{P}}(X)$ from L_s.

Since $U_s \cup L_s$ has the finite intersection property, it is contained in an ultrafilter of $\mathfrak{A}$, which we take to be $f(s)$. This completes the definition of f.

To derive (i) and (ii), we need two preliminary facts about this definition.

(iii) *Let $s \in S - T$, $X \in f(s)$, and $t \in T$. Then $tR_{\mathrm{F}}s$ implies $t \in m_{\mathrm{F}}(X)$, and $sR_{\mathrm{F}}t$ implies $t \in m_{\mathrm{P}}(X)$.*

To prove this, observe that validity of schema $(\varphi \to [\mathrm{P}]\langle \mathrm{F}\rangle\varphi)$ in $\mathfrak{A}$ entails that $X \subseteq l_{\mathrm{P}}m_{\mathrm{F}}(X)$, so $l_{\mathrm{P}}m_{\mathrm{F}}(X) \in f(s)$, and hence $m_{\mathrm{P}}l_{\mathrm{F}}(-X) \notin f(s)$. Thus $m_{\mathrm{P}}l_{\mathrm{F}}(-X) \notin L_s$. But then if $tR_{\mathrm{F}}s$, the definition of L_s implies that $t \notin l_{\mathrm{F}}(-X)$, giving $t \in m_{\mathrm{F}}(X)$ as desired. The other part of (iii) follows by the "mirror image" of this argument.

As a corollary to (iii) we obtain

(iv) *Let $s \in S - T$, $X \in f(s)$, and $t \in T$. Then if $tR_{\mathrm{F}}s$, $m_{\mathrm{F}}(X) \in f(s)$ implies $t \in m_{\mathrm{F}}(X)$, and if $sR_{\mathrm{F}}t$, $m_{\mathrm{P}}(X) \in f(s)$ implies $t \in m_{\mathrm{P}}(X)$.*

For the proof, supposing $tR_{\mathrm{F}}s$ and $m_{\mathrm{F}}(X) \in f(s)$, applying (iii) with X replaced by $m_{\mathrm{F}}(X)$ gives $t \in m_{\mathrm{F}}m_{\mathrm{F}}(X)$. Validity of $\langle \mathrm{F}\rangle\langle \mathrm{F}\rangle\varphi \to \langle \mathrm{F}\rangle\varphi$ in $\mathfrak{A}$ then implies $t \in m_{\mathrm{F}}(X)$. Again the other part of the proof is a mirror image argument.

Now for the proof of (i), first from right to left. Suppose that $sR_{\mathrm{F}}t$ and $X \in f(t)$. We want $m_{\mathrm{F}}(X) \in f(s)$. There are two main cases. Firstly, if $t \in T$ then $t \in X$ and either $s \in T$, giving then $s \in m_{\mathrm{F}}(X)$ and so $m_{\mathrm{F}}(X) \in f(s)$, or else $s \in S - T$, so that $m_{\mathrm{F}}(X) \in U_s$ by definition of U_s, and again $m_{\mathrm{F}}(X) \in f(s)$. For the second case, suppose $t \notin T$. Again there are two subcases. If $s \in T$ then applying result (iii) with s and t interchanged gives $s \in m_{\mathrm{F}}(X)$, so $m_{\mathrm{F}}(X) \in f(s)$. If however $s \notin T$, then since $sR_{\mathrm{F}}t$ and s and t both define gaps in $\mathfrak{T}$ there must be some $u \in T$ with $sR_{\mathrm{F}}uR_{\mathrm{F}}t$. But then by (iii) with t in place of s and u in place of t gives $u \in m_{\mathrm{F}}(X)$, so there exists $w \in T$ such that $uR_{\mathrm{F}}w \in X$. Then $sR_{\mathrm{F}}w$, so $m_{\mathrm{F}}(X) \in U_s \subseteq f(s)$.

To prove (i) from left to right we invoke at last the validity of Prior's axiom in $\mathfrak{A}$, in the form of the Gap Lemma 5.16. Suppose $m_{\mathrm{F}}(X) \in f(s)$. If $s \in T$, then $s \in m_{\mathrm{F}}(X)$ so there is a $t \in T$ with $sR_{\mathrm{F}}t$ and $t \in X$, whence $X \in f(t)$ as desired. If however $s \notin T$, then s defines the gap $\{t \in T : tR_{\mathrm{F}}s\}$ in $\mathfrak{T}$, and by (iv) every member of this gap belongs to $m_{\mathrm{F}}(X)$. But now if every $t \in T$ such that $sR_{\mathrm{F}}t$ had $t \notin m_{\mathrm{F}}(X)$ we would have

$$\{t \in T : tR_{\mathrm{F}}s\} = m_{\mathrm{F}}(X) \in \mathfrak{A},$$

contradicting the Gap Lemma. Therefore there must be some $t \in T$ with $sR_{\mathrm{F}}t$ and $t \in m_{\mathrm{F}}(X)$, so that $tR_{\mathrm{F}}u$ and $u \in X$ for some $u \in T$. Then $sR_{\mathrm{F}}u$ and $X \in f(u)$, and the proof of (i) is finished.

The proof of (ii) would be the mirror image of that of (i) if we assumed that the mirror image of Prior's axiom was valid in $\mathfrak{A}$. But in fact we can directly use the axiom itself again. The only essentially new situation arises when $m_{\mathrm{P}}(X) \in f(s)$ and $s \notin T$. Then similarly to the case of (i) we find that if there was no $t \in T$ such that $tR_{\mathrm{F}}s$ and $t \in m_{\mathrm{P}}(X)$ we would have

$$\{t \in T : sR_{\mathrm{F}}t\} = m_{\mathrm{P}}(X).$$

But in that case

$$\{t \in T : tR_{\mathrm{F}}s\} = T - m_{\mathrm{P}}(X) \in \mathfrak{A},$$

again contradicting the Gap Lemma. Therefore there must be some $t \in T$ with $tR_{\mathrm{F}}s$ and $t \in m_{\mathrm{P}}(X)$, leading to a $u \in T$ with $uR_{\mathrm{F}}s$ and $X \in f(u)$.

This completes the proof of the Theorem. ∎

In conclusion, let $\Lambda_{\mathbb{R}}$ be the smallest linear temporal logic containing Prior's axiom and the schema

$$\langle \mathrm{F} \rangle \varphi \to \langle \mathrm{F} \rangle \langle \mathrm{F} \rangle \varphi$$

which corresponds to the *density* condition that

$$sR_{\mathrm{F}}t \quad \text{implies} \quad \exists u\,(sR_{\mathrm{F}}uR_{\mathrm{F}}t).$$

The following can be shown about the variety $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ defined by this logic.

- $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ is generated by the complex algebra of the real-number frame $\langle \mathbb{R}, <, > \rangle$. This follows from the fact, due to Bull [9], that $\Lambda_{\mathbb{R}}$ is characterised by this frame.
- $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ is a complex variety. This is proved by an adaptation of the above argument, establishing that any algebra in the variety can be embedded into the complex algebra of a disjoint union of dense Dedekind complete orderings.
- $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ is not canonical. The canonical extension of the real-number frame has gaps and violates Prior's axiom. This extension looks similar to the nonstandard hyperreal number system, except that the "positive infinite" elements form a single cluster "at infinity", and likewise for the negative infinite elements. In fact one can take the countable subalgebra $\mathfrak{A}$ of $\mathsf{Cm}\,\mathbb{R}$ generated by the semi-infinite intervals $(-\infty, q)$, (q, ∞) with q rational and show that $\mathfrak{A}$ is in $\mathsf{Var}\,\Lambda_{\mathbb{R}}$ but $\mathsf{Em}\,\mathfrak{A}$ is not.

6 The Finite Model Property

We will now briefly review a concept that has been important in the development of general theory about modal logic, as well as in determining the properties of particular logics. Essentially, a logic Λ has the finite model property if it is characterised by its finite models. Precisely what this means depends on the notion of "model" involved, and there are three natural candidates: if Λ is a normal logic, then

(1) Λ has the *finite algebra property* if $\vdash_\Lambda \varphi$ whenever φ is valid in all finite algebras $\mathfrak{A}$ such that $\mathfrak{A} \models \Lambda$;

(2) Λ has the *finite frame property* if $\vdash_\Lambda \varphi$ whenever φ is valid in all finite structures $\mathfrak{S}$ such that $\mathfrak{S} \models \Lambda$;

(3) Λ has the *finite model property* if $\vdash_\Lambda \varphi$ whenever φ is true in all finite models $\mathfrak{M}$ such that $\mathfrak{M} \models \Lambda$.

It is readily seen that (1) and (2) are equivalent. This is because a structure $\mathfrak{S}$ validates the same formulae that the algebra $\mathsf{Cm}\,\mathfrak{S}$ does, while a *finite* normal BAO $\mathfrak{A}$ is isomorphic to $\mathsf{Cm}\,\mathbb{C}\mathsf{st}\,\mathfrak{A}$ (Theorem 3.2.2) and so validates the same formulae as the finite structure $\mathbb{C}\mathsf{st}\,\mathfrak{A}$. Hence a formula is valid in all finite Λ-algebras iff it is valid in all finite Λ-structures.

It is immediate from the definitions that (2) implies (3), since a formula true in all Λ-models will be valid in all Λ-structures. But it turns out that for *uniform* logics, (3) implies (2) as well (for a proof, see [52, Corollary 3.8] or [17, Exercise 4.9]). Thus for normal uniform logics, all three notions coincide, and are generally referred to as the "finite model property".

An example of a type 1 logic lacking the finite model property was provided by Makinson [38]: this is the smallest normal logic containing the schemata

(T) $\varphi \to \Diamond\varphi$, and
(Mk) $\Box\varphi \wedge \neg\Box\Box\varphi \to \Diamond(\Box\Box\varphi \wedge \neg\Box\Box\Box\varphi)$.

The schema $\Diamond\Diamond\varphi \to \Diamond\varphi$ (4) is valid in all *finite* algebras for this logic, but is not a theorem of the logic since there are infinite algebras validating (T) and (Mk) but not (4). An example is the algebra of finite and cofinite subsets of $\langle\omega, R\rangle$, where mRn iff $m \leq n+1$. This structure has become known as the *recession frame* and has significant application beyond this example for which it was originally constructed by Makinson. In particular, it underlies the major work of Blok [5] on incompleteness.

The finite model property provides a powerful method for demonstrating the decidability of various logics, in view of the fact that

- *if a logic Λ is finitely axiomatisable and has the finite model property, then it is decidable, i.e. there is an algorithm for deciding of an arbitrary formula φ whether or not $\vdash_\Lambda \varphi$.*

Here "finitely axiomatisable" means that Λ is the smallest logic containing some finite number of prescribed schemata. To sketch briefly why this result holds, observe that it can be algorithmically determined whether a given finite algebra $\mathfrak{A}$ satisfies some finite number of given equations, and hence whether $\mathfrak{A}$ validates a given finitely axiomatisable logic Λ. Therefore by systematically enumerating the finite algebras and testing formulae for validity in them, as well as testing whether they are Λ-algebras, we can generate a list of formulae that are invalidated by at least one Λ-algebra. But the finite model property implies that if $\not\vdash_\Lambda \varphi$ then there is a finite Λ-algebra that will invalidate φ, a fact that will then be discovered by the systematic testing procedure. Thus every non-Λ-theorem will appear in the list, and so the procedure provides an effective enumeration of the set $\Phi - \Lambda$ of formulae not in Λ. But Λ itself is effectively enumerable, since it is a finitely axiomatisable logic. Since now both Λ and $\Phi - \Lambda$ are effectively enumerable, it follows that Λ is decidable.

The restriction to finitely axiomatisable logics in this analysis is essential. Logics with the finite model property need not be decidable if they are not finitely axiomatisable. Indeed it has been shown in [58] that for each set X of natural numbers there is a modal logic Λ_X that has the finite frame property but whose degree of unsolvability is the same as that of X.

The first application of algebraic methods to prove decidability of modal logics in this way was made by J. C. C. McKinsey in [40]. If $\mathfrak{A} = \langle\mathfrak{B}, m\rangle$ is the Lindenbaum-Tarski algebra of a type 1 logic Λ and $\not\vdash_\Lambda \varphi$, then, as we saw in Section 5.2, there is an interpretation of the variables of φ in $\mathfrak{A}$ that invalidates φ. If C is the finite set of elements of $\mathfrak{B}$ "named" by subformulae of φ under this interpretation, and $\mathfrak{B}'$ is the sub-Boolean algebra of $\mathfrak{B}$ generated by C, then $\mathfrak{B}'$ is finite and can be made into a modal algebra under the new operator $m' : B' \to B'$ defined by

$$m'x = \prod\{my : x \leq y \in B' \text{ and } my \in B'\}.$$

The resulting finite modal algebra still invalidates φ. McKinsey showed further that it also validates Λ in the case that Λ is either of the well-known logics S2 and S4, thereby establishing the finite model property and decidability for them.

Pioneering studies of the finite model property were made in a series of papers by R. A. Bull (cf. [8, 9] and references cited therein). This involved a sophisticated analysis and modification of the finite algebras produced by McKinsey's method, and led to a demonstration that every normal uniform extension of the type 1 logic S4.3 (characterised by linearly ordered K-frames) has the finite model property. The method subsequently yielded completeness proofs for the linear temporal logics characterised by the frames $\langle \mathbb{Z}, <, > \rangle$, $\langle \mathbb{Q}, <, > \rangle$, and $\langle \mathbb{R}, <, > \rangle$. Bull's work also contained the first application to logical systems of Birkhoff's theory of subdirect representation of algebras in terms of subdirectly irreducibles.

The method of McKinsey was adapted to the complex algebra setting by Lemmon [34, Part IV]. A model-theoretic version of his approach appeared in [35], and was further developed by Segerberg [51, 52] under the name of *filtration*. In essence, filtration of a model $\mathfrak{M}$ involves collapsing $\mathfrak{M}$ to a finite model by identifying points that assign the same truth-values to the members of some fixed set Γ of formulae. Typically Γ will be (based on) the set of subformulae of a particular non-theorem φ that is to be falsified in the resulting finite model.

Now if a logic has the finite algebra/frame property then its associated variety will be generated by its finite members, and hence generated by the finite structures in the variety. In other words, such a logic must be complete (Section 5.4). Construction of finite models has in fact been an important procedure for proving completeness or axiomatisation results for many logics. This is inevitable if the logic is *defined* by reference to finite structures (e.g. the logic characterised by finite linear orderings), but the procedure has also proved vital when the canonical frame method breaks down because the logic in question is not canonical. This applies for instance to the temporal logic $\Lambda_{\mathbb{R}}$ of real time. Another particularly notable case is propositional dynamic logic [53, 25], where the only known method for proving completeness involves some variant of the filtration approach.

7 Other Topics

This article has sought to indicate how the basic theory of Boolean algebras with operators can be used to investigate properties of modal logics and similar logical systems. There are other topics in this and related areas that could be considered, including

- the connection between interpolation properties of logics and amalgamation properties of algebras;
- the relationship between the Beth definability property of logics and the question of surjectivity of epimorphisms between algebras;
- the study of non-normal operators and associated non-normal logics;
- the investigation of logics whose algebraic semantics is based on something other than Boolean algebras, such as distributive lattices, Heyting algebras, "semilattice-ordered residuated semigroups", and many others.

Those who wish to pursue such topics may find it profitable to explore such sources as the papers [46, 50, 2, 37], the books [1, 48], the dissertations [60, 39, 41], and the references they contain.

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