An Alternative Semantics for Quantified Relevant Logic*

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Abstract

The quantified relevant logic RQ is given a new semantics in which a formula $\forall xA$ is true when there is some true proposition that implies all x-instantiations of A. Formulae are modelled as functions from variable-assignments to propositions, where a proposition is a set of worlds in a relevant model structure. A completeness proof is given for a basic quantificational system QR from which RQ is obtained by adding the axiom EC of 'extensional confinement': $\forall x(A \vee B) \to (A \vee \forall xB)$, with x not free in A. Validity of EC requires an additional model condition involving the boolean difference of propositions. A QR-model falsifying EC is constructed by forming the disjoint union of two natural arithmetical structures in which negation is interpreted by the minus operation.

1 Introduction

> You can never arrive at a general proposition by inference from particular propositions alone. You will always have to have at least one general proposition in your premisses.
>
> Bertrand Russell [17]

Kit Fine [10] showed that the quantified relevant logic RQ is incomplete over Routley and Meyer's relational semantics with constant domains. In its place he

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developed [9] an increasing domain semantics over which RQ is complete, and for which a model includes a set of frames which are related to one another by a number of relations and operators. This semantics is powerful and ingenious.

But Fine's semantics is very complicated. Since it was produced it in the mid-1980s relevant logicians have wanted to simplify it. J. Michael Dunn and Greg Restall say [7, p 83]:

> [I]t must be said that while the semantic structure pins down the behaviour of RQ and related systems exactly, it is not altogether clear whether the rich and complex structure of Fine's semantics is necessary to give a semantics for quantified relevance logics.[1]

To those of us who have seriously attempted to simplify Fine's semantics, it is becoming clear that the "rich and complex structure" is in a certain sense necessary. The elements of the theory seem to work in concert with one another and eliminating even one operator or relation seems to make the entire structure collapse. Thus, in order to create a simpler semantics for quantified relevant logic we need to take an alternative approach.

In this paper, we take an alternative approach.[2] The idea comes from the proof theory for quantified relevant logic. We start with the schema,

$$\text{(UI) } \forall x A \rightarrow A[\tau/x],$$

and the rule,

$$\text{(RIC) } \frac{\vdash A \rightarrow B}{\vdash A \rightarrow \forall x B}$$

(restriction on the rule: x does not occur free in A). The truth condition of the present theory draws on both UI and RIC. Here is an informal paraphrase. For the sake of simplicity we will only look at the case in which A has at most the one variable x free. Our truth condition says:

> '$\forall x A$' is true at a world a if and only if there is some proposition X such that (i) X entails that A holds of every individual and (ii) X obtains at a.

Thus, in order to incorporate RIC into the semantics, we add propositions to the model theory. A proposition is a set of worlds. As we shall see soon, not every set

[1] Restall also says [14, p 5]: "The groundbreaking work of [9] is formally astounding but philosophically opaque".

[2] After seventeen years of trying to modify Fine's semantics, Mares in particular is glad that an alternative has suggested itself.

of worlds is a proposition. A proposition X entails a proposition Y if X is a subset of Y and a proposition X obtains at a world a if a is in X. Thus, our truth condition says that '$\forall xA$' is true at a if and only if there is some proposition X that is a subset of any proposition expressed by A on any assignment of x and $a \in X$. Clause (i) of the truth condition reflects the rule RIC – i.e. that a proposition that entails an arbitrary instantiation of a formula also entails its universal generalisation. Clause (ii) ensures that each instantiation of A is true at a if $\forall xA$ is true at a – hence it ensures that UI obtains.

Philosophers call propositions like X "general propositions" or, more commonly, "general facts". The existence of general facts has been supported by Bertrand Russell [17, pp 101-103] and David Armstrong [2, ch 13]. We will not present Russell's or Armstrong's arguments for general facts here, nor will we give an argument for them that is independent of their use in the semantics. Rather, we depend on their utility in the semantics to justify the claim that there are such entities.[3]

This truth condition for the universal quantifier also has its origins, not only in the writings of Russell and Armstrong, but in the Brouwer-Heyting-Komologorov (BHK) interpretation of intuitionist logic. On the BHK interpretation, a universally quantified formula $\forall xA(x)$ is proved if and only if there is a proof that takes any object i and returns a proof of $A(i)$. The propositions in our frames are analogous to proofs in the BHK interpretation. It is interesting that both Fine's semantics and the present theory borrow ideas from intuitionist logic. Fine's truth condition for the universal quantifier is essentially the condition from Kripke's semantics for intuitionist logic.

The most straightforward version of the semantics does not make valid the "extensional confinement axiom", viz.,

$$\text{(EC)} \quad \forall x(A \vee B) \rightarrow (A \vee \forall xB),$$

where x is not free in A. In order to make EC valid, we need an additional model condition involving the boolean difference of propositions. In this paper we prove soundness and completeness both for the logic QR which adds UI and RIC to an axiomatization of the Anderson-Belnap system R, and for RQ which results by adding EC to QR.[4] We also construct a QR-model that falsifies EC, and indeed

[3]Moreover, it is not clear that our semantical theory is committed to a full metaphysics of general facts in the sense of Russell and Armstrong. For, what propositions count as general propositions is determined by the frame itself. Russell and especially Armstrong think that the generality of general facts precedes any considerations of how they fit into a scheme that determines notions of possibility, necessity, or logical entailment.

[4]Before going on, we should note that Ross Brady has also constructed two semantics for quantified relevant logics. The first is based on his own content semantics, rather than on the Routley-Meyer

falsifies the weaker principle

$$(!) \quad \neg\forall x(A \vee B) \vee A \vee \forall xB,$$

where x is not free in A. This construction involves certain arithmetical model structures, which we call "Sugihara frames", in which negation is interpreted by the minus operation.

2 R Frames

We begin by introducing the notion of an R-frame, due to Richard Routley and Robert Meyer [15].

An R-frame is a structure $\langle K, 0, R, *\rangle$, such that K is a non-empty set (of "worlds"), 0 is a non-empty subset of K (of "base worlds"), R is a ternary relation on K, and $*$ is a unary operator on K, which obey the conditions set out below, where

$$a \leq b \ =_{df} \exists x(x \in 0 \ \& \ Rxab).$$

F1 if $a \in 0$ and $a \leq b$, then $b \in 0$;

F2 $\leq$ is transitive and reflexive;

F3 if $Rabc$, then $Rbac$;

F4 if $\exists x(Rabx \ \& \ Rxcd)$, then $\exists x(Racx \ \& \ Rxbd)$;

F5 $Raaa$;

F6 if $Rabc$, then Rac^*b^*;

F7 if $Rbcd$ and $a \leq b$, then $Racd$;

F8 $a^{**} = a$.

It follows from F6 that $*$ is *order-reversing* in the following sense:

F6′ if $b \leq c$, then $c* \leq b*$.

frame theory (see [3, §13.2]) and the other uses relevant logic as an "interpretational metalanguage" (ibid. §13.4). The latter was used mainly to show that RQ is conservatively extended by the addition of Boolean negation.

Let $\wp K$ be the powerset of K. Operations $-$ and $\Rightarrow$ on $\wp K$ are defined by

$$-X = \{a \in K : a^* \notin X\},$$
$$X \Rightarrow Y = \{a \in K : \forall b \in K \forall c \in K((Rabc \ \&\ b \in X) \supset c \in Y)\},$$

for all $X, Y \subseteq K$. The operation $-$ satisfies the De Morgan laws $-(X \cap Y) = -X \cup -Y$ and $-(X \cup Y) = -X \cap -Y$, and is inclusion-reversing, i.e. $X \subseteq Y$ implies $-Y \subseteq -X$. Those fact require no special properties of $*$. When $*$ is involutary (F8), then so is $-$, i.e. $- - X = X$.

A subset X of K is *hereditary* if it is closed upwards under $\leq$, which means that if $a \in X$ and $a \leq b$, then $b \in X$. The set 0 will provide the interpretation of a propositional constant t (see Section 4), so condition F1 states that t is interpreted as an hereditary set. This requirement is not strictly necessary, in that t could be interpreted as the upward closure of 0, but it makes some definitions and proofs easier. The other conditions are the standard conditions for R-frames (see, e.g., [16, 1]).

If X and Y are hereditary, then so too are $-X$, $X \Rightarrow Y$, $X \cap Y$ and indeed $X \cup Y$.

3 Propositions and Propositional Functions

Our semantics requires two other notions in addition to that of an R-frame – those of a propositional function and a proposition. All the complexity in our theory, over and above what is already present in the Routley-Meyer semantics, has to do with the definitions of the sets of propositional functions and propositions. The "truth set" of a formula is the set of worlds at which it is true. For our soundness proof we need to show that every formula's truth set is a proposition. An easy way to do so would be to assume that every set of worlds is a proposition (or better, that every hereditary set is a proposition), but then our theory would collapse into the constant domain semantics, over which RQ is incomplete. Thus, to prove completeness we define a class of general frames. In aid of this we close our set of propositions under conditions that correspond to the truth conditions for the various connectives. We need to include a set of propositional functions as well in order to state a closure condition that corresponds to the truth condition for the universal quantifier.

The resulting theory is reminiscent of that of the functional polyadic algebras of Paul Halmos [11]. Halmos's propositional functions have the form $I^{\mathcal{V}} \to B$, where I is a domain of individuals, $\mathcal{V}$ is a set of "variables", and B is a Boolean algebra whose members are thought of as propositions. In place of B we take here an algebra of hereditary subsets of an R-frame.[5] For $\mathcal{V}$ we take the set ω of natural

[5]This suggests that our theory can be generalised, or further "algebraised", by taking the set of propositions to be an arbitrary de Morgan monoid.

numbers. A member f of the set I^ω of all functions from ω to I may be viewed as an infinite sequence $\langle f0, \ldots, fn, \ldots\rangle$ of elements of I. Later we will view it as a value-assignment to variables, assigning value fn to variable x_n.

Now fix an R-frame $\langle K, 0, R, *\rangle$. The operations $\cap$, $\Rightarrow$ and $-$ on $\wp K$ can be lifted pointwise to operations on functions of the form $I^\omega \to \wp K$. If φ and ψ are two such functions, we define functions $\varphi \cap \psi$, $\varphi \Rightarrow \psi$ and $-\varphi$ of the same form by putting, for each $f \in I^\omega$,

$$(\varphi \cap \psi)f = \varphi f \cap \psi f$$
$$(\varphi \Rightarrow \psi)f = \varphi f \Rightarrow \psi f$$
$$(-\varphi)f = -(\varphi f).$$

Next, fix a set *Prop* of hereditary subsets of K. Then *Prop* determines an operation $\sqcap : \wp\wp K \to \wp K$ defined, for each $S \subseteq \wp K$, by putting

$$\sqcap S = \bigcup\{X \in \mathit{Prop} : X \subseteq \bigcap S\}.$$

In general $\sqcap S \subseteq \bigcap S$, so $\sqcap S$ is a lower bound of S in the partially-ordered set $(\wp K, \subseteq)$. If $S \subseteq \mathit{Prop}$ and $\sqcap S \in \mathit{Prop}$, then $\sqcap S$ is the *greatest lower bound* of S in $(\mathit{Prop}, \subseteq)$. But it may be that $\sqcap S \notin \mathit{Prop}$ even when $S \subseteq \mathit{Prop}$. Note that if $\bigcap S \in \mathit{Prop}$, then $\sqcap S = \bigcap S$. But it is also possible to have $\sqcap S \in \mathit{Prop}$ while $\bigcap S \notin \mathit{Prop}$, as an example in Section 5 will show.

For each function $\varphi : I^\omega \to \wp(K)$ we use $\sqcap$ to define functions $\forall_n\varphi : I^\omega \to \wp(K)$ for each $n \in \omega$. First, we introduce the notation $f[j/n]$ for the function that "updates" a function $f \in I^\omega$ by assigning the value $j \in I$ to n and otherwise acting as f:

$$f[j/n] = \langle f0, \ldots, f(n-1), j, f(n+1), \ldots\rangle.$$

Then we put

$$(\forall_n\varphi)f = \sqcap_{j\in I}\varphi(f[j/n]). \tag{3.1}$$

Thus the definition of the operations $\forall_n$ depends on $\sqcap$, and hence on the particular choice of the set *Prop*.

Now we define a *QR-frame* to be a structure $\mathcal{K} = \langle K, 0, R, *, I, \mathit{Prop}, \mathit{PropFun}\rangle$ such that $\langle K, 0, R, *\rangle$ is an R-frame, I is a non-empty set (of individuals), *Prop* is a set of hereditary subsets of K, and *PropFun* is a set of functions from I^ω to *Prop*, such that the following conditions hold:

CProp $0 \in \mathit{Prop}$ and if X and Y are in *Prop*, then $X \cap Y \in \mathit{Prop}$, $X \Rightarrow Y \in \mathit{Prop}$ and $-X \in \mathit{Prop}$.

CTee: The constant function φ_0, with $\varphi_0(f) = 0$ for all f, is in *PropFun*.

CImp: If $\varphi, \psi \in PropFun$, then $\varphi \Rightarrow \psi \in PropFun$.

CConj: If $\varphi, \psi \in PropFun$, then $\varphi \cap \psi \in PropFun$.

CNeg: If $\varphi \in PropFun$, then $-\varphi \in PropFun$.

CAll: If $\varphi \in PropFun$, then $\forall_n \varphi \in PropFun$ for all $n \in \omega$.

A QR-frame will be called *full* if *Prop* is the set $Prop_{\mathcal{K}}$ of *all* hereditary subsets of K, and *PropFun* is the set $PropFun_{\mathcal{K}}$ of all functions from I^{ω} to $Prop_{\mathcal{K}}$.

4 Models

Before we can define the class of QR-models, we need to set out our language. Its alphabet consists of a countable sequence $x_0, x_1, x_2, \ldots$ of distinct individual variables; a set *Con* of individual constants; a set of predicate letters, each with a specified arity; the quantifier $\forall$; the unary connective $\neg$; binary connectives $\rightarrow$ and $\wedge$; parentheses; and a propositional constant t. The letters 'x' and 'y' will used to designate arbitrary variables, and the letter 'τ' (often with subscripts) will be used for individual terms (variables or constants). The usual formation rules apply. We use 'closed formula' to denote formulae with no free variables and 'formula' to denote formulae which *may* have free variables (we also use 'open formula' sometimes to emphasise the possibility that a formula has free variables).

We may also make use of the following defined connectives:

$$A \vee B =_{df} \neg(\neg A \wedge \neg B)$$

$$A \circ B =_{df} \neg(A \rightarrow \neg B)$$

$$A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$$

$$\exists x A =_{df} \neg \forall x \neg A$$

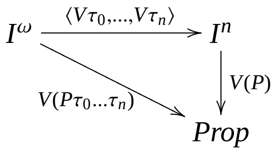
The connective $\circ$ is known as "fusion". The notation $[\tau_0/x_0, \ldots, \tau_n/x_n, \ldots]$ will be used for the substitution operator that uniformly substitutes the term τ_n for all free occurrences of the variable x_n, for all $n \in \omega$. This operator can be applied to any formula A to give a formula $A[\tau_0/x_0, \ldots, \tau_n/x_n, \ldots]$. It can also be applied to terms. The notation may be abbreviated to $[\tau_{n_0}/x_{n_0}, \ldots, \tau_{n_p}/x_{n_p}]$ to indicate that the substitution alters only $x_{n_0}, \ldots, x_{n_p}$, i.e. $\tau_n = x_n$ for all $n \notin \{n_0, \ldots, n_p\}$. We will make particular use of operators of the type $[c/x]$ that replace all free occurrences of the variable x by the constant c, leaving all other variables unchanged.

A *valuation* on a QR-frame $\mathcal{K}$ is a function V that assigns to each constant $c \in Con$ an element $V(c)$ of I, and to each n-ary predicate letter P a function

$V(P) : I^n \to \mathit{Prop}$. Given such a valuation V, each $f \in I^\omega$ determines a value $Vf\tau$ in I for each term τ, defined by putting $Vfc = V(c)$ for each constant c and $Vfx_n = fn$ for each variable x_n (this is the sense in which f is a variable-assignment). Then each term τ gets the value $V\tau : I^\omega \to I$ where $V\tau(f) = Vf\tau$; and each *atomic* formula $P\tau_0 \dots \tau_n$ gets the value $V(P\tau_0 \dots \tau_n) : I^\omega \to \mathit{Prop}$ defined by

$$V(P\tau_0 \dots \tau_n)(f) = V(P)(Vf\tau_0, \dots, Vf\tau_n).$$

Thus the diagram



commutes, where $\langle V\tau_0, \dots, V\tau_n\rangle$ is the product map $f \mapsto \langle Vf\tau_0, \dots, Vf\tau_n\rangle$.

The pair $\langle \mathcal{K}, V\rangle$ is called a *QR-model* on the QR-frame $\mathcal{K}$ if:

$V(A)$ belongs to *PropFun* for all *atomic* formulae A.

Each model has a truth/satisfaction relation $a \models_{Vf} A$ between worlds $a \in K$, variable-assignments $f \in I^\omega$ and formulae A. This is defined for each a and f by induction on the complexity of A. For this, let $|A|_{Vf} =_{df} \{b \in K : b \models_{Vf} A\}$, and write '$xvf$' for the set of x-*variants* of a variable-assignment $f \in I^\omega$. This is the set of all those $g \in I^\omega$ that differ from f only in their assignment to x. Hence

$$x_n v f = \{f[j/n] : j \in I\}. \tag{4.1}$$

The inductive definition of $a \models_{Vf} A$ is as follows:

- $a \models_{Vf} P\tau_0 \dots \tau_{n-1}$ if and only if $a \in V(P\tau_0 \dots \tau_n)(f)$;
- $a \models_{Vf} t$ if and only if $a \in 0$;
- $a \models_{Vf} A \wedge B$ if and only if $a \models_{Vf} A$ and $a \models_{Vf} B$;
- $a \models_{Vf} \neg A$ if and only if $a^* \not\models_{Vf} A$;
- $a \models_{Vf} A \to B$ if and only if $\forall b \forall c((Rabc \mathbin{\&} b \models_{Vf} A) \supset c \models_{Vf} B)$;
- $a \models_{Vf} \forall x A$ if and only if there is a proposition X such that $X \subseteq \bigcap_{g \in xvf} |A|_{Vg}$ and $a \in X$.

These truth conditions can alternatively be given in a more functional form as the following inductive definition of the "truth sets" $|A|_{Vf}$:

- $|P\tau_0 \dots \tau_{n-1}|_{Vf} = V(P\tau_0 \dots \tau_{n-1})(f)$;
- $|t|_{Vf} = 0$;
- $|A \wedge B|_{Vf} = |A|_{Vf} \cap |B|_{Vf}$;
- $|\neg A|_{Vf} = -|A|_{Vf}$;
- $|A \rightarrow B|_{Vf} = |A|_{Vf} \Rightarrow |B|_{Vf}$;
- $|\forall x A|_{Vf} = \bigsqcap_{g \in xvf} |A|_{Vg}$.

Hence $|A \vee B|_{Vf} = |A|_{Vf} \cup |B|_{Vf}$.

Each formula A determines the function $|A|_V : I^\omega \to \wp K$ such that $|A|_V(f) = |A|_{Vf}$. The above equations for $|A|_{Vf}$ then give

- $|P\tau_0 \dots \tau_{n-1}|_V = V(P\tau_0 \dots \tau_{n-1})$;
- $|t|_V = \varphi_0$, the constantly 0-valued function;
- $|A \wedge B|_V = |A|_V \cap |B|_V$;
- $|\neg A|_V = -|A|_V$;
- $|A \rightarrow B|_V = |A|_V \Rightarrow |B|_V$;
- $|\forall x_n A|_V = \forall_n |A|_V$.

The last equation follows with the help of (4.1) and (3.1), showing that for all $f \in I^\omega$,

$$(|\forall x_n A|_V) f = \bigsqcap_{g \in x_n v f} |A|_V(g) = \bigsqcap_{j \in I} |A|_V(f[j/n]) = (\forall_n |A|_V) f.$$

Corollary 4.1 *Let A be an arbitrary formula. Then in any model, $|A|_V$ is a propositional function, i.e. belongs to PropFun. Hence $|A|_{Vf}$ is a proposition, i.e. a member of Prop, for all $f \in I^\omega$.*

Proof. The second conclusion follows from the first because any member of *PropFun* takes its values in *Prop*. So it suffices to prove the first conclusion, by induction on the complexity of A. For A atomic, the result follows by the definition of ‘model’. For $A = t$ it follows by the condition CTee on the frame $\mathcal{K}$. The inductive cases for the connectives $\wedge$, $\neg$, $\rightarrow$, and the quantifiers $\forall x_n$, follow by CConj, CNeg, CImp, and CAll, and the above equations. □

A formula A is defined to be *satisfied by the assignment f in the model $\langle \mathcal{K}, V \rangle$* if $a \models_{Vf} A$ for all base worlds $a \in 0$. A is *valid in the model* if it is satisfied by every assignment $f \in I^{\omega}$. A is *valid in the frame* $\mathcal{K}$ if it is valid in every model based on $\mathcal{K}$.

By unravelling the semantics of $\neg$ and $\forall$ we obtain the following truth condition for the existential quantifier:

> $a \models_{Vf} \exists xA$ if and only if for all $X \in Prop$ such that $a^* \in X$ there exists $b \in X$ and $g \in xvf$ such that $b^* \in |A|_{Vg}$.

More informally, this criterion for $a \models_{Vf} \exists xA$ is that for every proposition X containing a^* there is some world b in X with b^* in some instantiation of A. This truth condition might seem rather counter-intuitive, but there is a way of making philosophical sense of it. To do so, we take a detour into Dunn's "compatibility" interpretation of the star operator [6]. Let us say that two worlds are compatible if one does not contain any information that is incompatible with the other. Thus, for example, two worlds are incompatible if one says that a given object is red all over at a particular time and the other says that it is green all over at that time. Then a^* can be understood as the $\leq$-maximal world that is compatible with a. On this interpretation, '$\neg A$' holds at a world if and only if no world compatible with a makes A true.

Now let us say that proposition X is compatible with world a, and that a is compatible with X, when X is true in some world that is compatible with a, or equivalently, when X is true at a^*. Then we may say that X is compatible with some other proposition Y if X is true in some world that is compatible with Y. This means that there is some b in X such that b^* is in Y. (From $a^{**} = a$ we get that if X is compatible with Y, then Y is compatible with X.) In these terms the rule for $\exists xA$ to be true at a can be expressed as follows:

every proposition compatible with A is compatible with some x-instantiation of A.

From a more algebraic point of view, we can define an operation $\bigsqcup$ dual to $\bigsqcap$ in order to treat the existential quantifier:

$$\bigsqcup S =_{df} \bigcap\{X \in Prop : \bigcup S \subseteq X\}.$$

When $S \subseteq Prop$ and $\bigsqcup S \in Prop$, then $\bigsqcup S$ is the *least upper bound* of S in $(Prop, \subseteq)$.

Fact 4.2 *$a \models_{Vf} \exists xA$ if and only if $a \in \bigsqcup_{g \in xvf} |A|_{Vg}$.*

Proof. It is readily seen that the operation $-$ on $\wp K$ obeys the De Morgan law

$$-\bigcup S = \bigcap\{-X : X \in S\}.$$

From this it can be shown that

$$-\bigsqcap S = \bigsqcup\{-X : X \in S\}. \tag{4.2}$$

For, the left side of this last equation is $-\bigcup\{X \in Prop : X \subseteq \bigcap S\}$, which by the previous equation is

$$\bigcap\{-X : X \in Prop \,\&\, X \subseteq \bigcap S\}, \tag{4.3}$$

while the right side of (4.2) is

$$\bigcap\{Y \in Prop : \bigcup\{-X : X \in S\} \subseteq Y\}. \tag{4.4}$$

Now in general $-X \subseteq Y$ iff $-Y \subseteq X$, so $\bigcup\{-X : X \in S\} \subseteq Y$ iff $-Y \subseteq \bigcap S$. Moreover, by the double negation law $--Y = Y$ and the closure of *Prop* under $-$, every $Y \in Prop$ is of the form $-X$ with $X \in Prop$, so (4.3) and (4.4) are identical, proving (4.2).

Now by the definition of ∃, the semantics of ∀ and ¬, the double negation law and (4.2), we get

$$|\exists x A|_{Vf} = -\bigsqcap_{g \in x_v f} -|A|_{Vg} = \bigsqcup_{g \in x_v f} |A|_{Vg},$$

which is what we set out to prove. □

As in [15, Lemmas 2 and 3], we can show:

Lemma 4.3 (Semantic Entailment) *In any model, a formula $A \to B$ is satisfied by f if and only if, for all worlds a in K, if $a \models_{Vf} A$, then $a \models_{Vf} B$, i.e. iff $|A|_{Vf} \subseteq |B|_{Vf}$.* □

Next we show that the satisfaction relation depends only on the value-assignment to *free* variables. Two functions $f, g \in I^\omega$ will be said to *agree on the free variables of A* if $fn = gn$ whenever x_n has a free occurrence in formula A. This holds vacuously if A has no free variables.

Lemma 4.4 *For any formula A, if f and g agree on the free variables of A, then $|A|_{Vf} = |A|_{Vg}$.*

Proof. By induction on the complexity of A.

If $A = t$, then $|A|_{Vf} = |A|_{Vg} = 0$. If A is atomic, then the hypothesis is that f and g agree on every variable of A, so $Vf\tau = Vg\tau$ for all terms τ occurring in A, hence $V(A)(f) = V(A)(g)$. The inductive cases for the propositional connectives are straightforward.

If $A = \forall x_n B$, then a variable is free in B only if it is free in A or equal to x_n. Hence for all $j \in I$, $f[j/n]$ and $g[j/n]$ agree on all free variables of B, so $|B|_{Vf[j/n]} = |B|_{Vg[j/n]}$ by induction hypothesis on B. Thus

$$|A|_{Vf} = \bigsqcap_{j \in I} |B|_{Vf[j/n]} = \bigsqcap_{j \in I} |B|_{Vg[j/n]} = |A|_{Vg}$$

as required. □

Now we consider conditions under which the present semantics collapses into the Tarskian semantics for the universal quantifier:

Fact 4.5 *If Prop contains all hereditary subsets of K, or if Prop or I is finite, then*

$$a \models_{Vf} \forall x A \text{ if, and only if, for all } x\text{-variants } g \text{ of } f,\ a \models_{Vg} A. \quad (4.5)$$

Proof. The desired conclusion is that

$$|\forall x A|_{Vf} = \bigcap_{g \in xvf} |A|_{Vg}.$$

For this it is enough to show that $\bigcap_{g \in xvf} |A|_{Vg}$ belongs to *Prop*, since it is then equal to $\bigsqcap_{g \in xvf} |A|_{Vg}$. Since $\{|A|_{Vg} : g \in xvf\} \subseteq Prop$ (Corollary 4.1), this will certainly hold when *Prop* contains all hereditary sets, since the class of hereditary sets is closed under arbitrary intersections. It will also hold when *Prop* is finite, since *Prop* is closed under finite intersections by CProp.

Finally, when I is finite, then so is $xvf = \{f[j/n] : j \in I\}$, hence so is $\{|A|_{Vg} : g \in xvf\}$, so again the results follows by closure of *Prop* under finite intersections. □

Note that when a model satisfies (4.5) it also satisfies the Tarskian semantics for the existential quantifier:

$$a \models_{Vf} \exists x A \text{ if, and only if, for some } x\text{-variant } g \text{ of } f,\ a \models_{Vg} A. \quad (4.6)$$

Routley and Meyer in [15] defined a *relevant quantificational model structure* to be a pair $\langle \mathcal{K}, I \rangle$ with $\mathcal{K}$ an R-frame and I a non-empty set of individuals. A valuation ν on such a structure assigns an element of I to each variable and constant, and assigns to each n-ary predicate letter P and each element a of $\mathcal{K}$ an n-ary relation $\nu(P, a) \subseteq I^n$, subject to the requirement that

$$a \leq b \text{ implies } \nu(P, a) \subseteq \nu(P, b). \quad (4.7)$$

This yields an inductively defined satisfaction relation $a \models_\nu A$, with the atomic case given by

$$a \models_\nu P\tau_0 \ldots \tau_{n-1} \quad \text{iff} \quad \langle \nu\tau_0, \ldots, \nu\tau_{n-1} \rangle \in \nu(P, a),$$

and the quantifier cases given by (4.5) and (4.6). A formula is *valid* in the model structure $\langle \mathcal{K}, I \rangle$ if it is satisfied at every base world in every such valuation on the structure.

The formulas valid in $\langle \mathcal{K}, I \rangle$ in this sense are precisely those that are valid in our sense in the *full* QR-frame $\langle \mathcal{K}, I, Prop_{\mathcal{K}}, PropFun_{\mathcal{K}} \rangle$, where $PropFun_{\mathcal{K}}$ is the set of all functions from I^ω to the set $Prop_{\mathcal{K}}$ of all hereditary subsets of $\mathcal{K}$. To indicate why: given a valuation ν as above, for each n-ary P we can define $V_\nu(P) : I^n \to \wp K$ by putting $V_\nu(P)(i_0, \dots, i_{n-1}) = \{a \in K : \langle i_0, \dots, i_{n-1} \rangle \in \nu(P, a)\}$. Then (4.7) ensures that $V_\nu(P)(a_0, \dots, a_{n-1})$ belongs to $Prop_{\mathcal{K}}$. Putting also $V_\nu(c) = \nu(c)$ for constants c makes V_ν a valuation on the QR-frame. Then defining $f_\nu \in I^\omega$ by $f_\nu n = \nu(x_n)$, we find that in general, $|A|_{V_\nu f_\nu} = \{a \in K : a \models_\nu A\}$. Moreover, any member of $Prop_{\mathcal{K}}$ of the form $|A|_{Vf}$ is equal to $|A|_{V_\nu f_\nu}$ for a unique valuation ν on the model structure $\langle \mathcal{K}, I \rangle$.

In [10] an example is given of a formula that is not derivable in the system RQ, but is valid in all model structures $\langle \mathcal{K}, I \rangle$. Hence this formula is valid in all full QR-frames. It is only by allowing *Prop* to be a proper subset of $Prop_{\mathcal{K}}$ that a complete semantics for RQ itself can be obtained. However the schema EC is RQ-derivable but not valid in some QR-frames (Section 11), so the class of QR-frames has to be further restricted to obtain this complete RQ-semantics.

5 An Example: Sugihara Frames

We now give a construction of QR-models in which the semantics of $\forall$ is non-Tarskian. According to Fact 4.5, this will require us to use infinite sets for I and *Prop*, and hence for K, and to ensure that *Prop* does not contain all hereditary sets. The following is an outline of the construction that leaves a lot of the detailed verification to the reader. Later in Section 11 we will adapt this to construct a QR-model falsifying the axiom EC.

By a *Sugihara frame* we mean a structure $\langle K, 0, \leq, * \rangle$ such that 0 is a *member* of K (not a subset), $\leq$ is a *linear* ordering of K, and $*$ is a unary operation on K that is order-reversing (F6′) and involutary (F8). Natural examples include $\langle \mathbb{Z}, 0, \leq, * \rangle$, where $\leq$ is the usual ordering of the set $\mathbb{Z}$ of all integers, 0 is the number zero, and $x^* = -x$; as well as any subset of $\mathbb{Z}$ that contains 0 and is closed under the minus operation. In Section 11 we will make significant use of the 3-element frame $\{-1, 0, 1\}$.

In any Sugihara frame the $*$-operation defines an inclusion-reversing involution $-X = \{a \in K : a^* \notin X\}$ on $\wp K$ as for R-frames. If X is $\leq$-hereditary, then so is $-X$.

For each $a \in K$, let $[a) = \{x \in K : a \leq x\}$ be the right-open interval starting at a. Then $-[a) = \{x \in K : a^* < x\}$, while $[a) \cap [b) = [\max(a, b))$ and $[a) \cup [b) =$

$[\min(a,b))$. If X is hereditary, then $[0) \subseteq X$ iff $-X \subseteq X$. We will turn the Sugihara frame into an R-frame in which the set $[0)$ of *non-negative* elements is the set of base worlds.[6] The set $\mathcal{H}$ of all hereditary subsets of K forms a chain under inclusion: the linearity of $\leq$ ensures that $X \subseteq Y$ or $Y \subseteq X$ for all $X, Y \in \mathcal{H}$. In this situation there is a natural operation $\Rightarrow$ on $\mathcal{H}$ introduced by Takeo Sugihara in the 1950's: $X \Rightarrow Y$ is $-X \cup Y$ if $X \subseteq Y$, and $-X \cap Y$ otherwise. Then $[0) \subseteq X \Rightarrow Y$ iff $X \subseteq Y$. The structure

$$\langle \mathcal{H}, D, -, \Rightarrow, \cap, [0) \rangle$$

is a logical matrix for propositional logic in which the set D of designated elements is $\{X \in \mathcal{H} : -X \subseteq X\} = \{X : [0) \subseteq X\}$; the operations $-, \Rightarrow, \cap$ interpret the connectives $\neg, \rightarrow, \wedge$; and $[0)$ is the value of the constant t. This matrix validates all theorems of the propositional logic RM, obtained by adding the "mingle" axiom $A \rightarrow (A \rightarrow A)$ to Anderson and Belnap's propositional logic R (see [5]).

To define a ternary relation R on K we use the fusion operator $X \circ Y = -(X \Rightarrow -Y)$ associated with $\Rightarrow$. This has $X \circ Y = X \cap Y$ when $X \subseteq -Y$, and $X \circ Y = X \cup Y$ otherwise. It follows readily that fusion is commutative, and monotonic in each argument. Using the validity of RM-theorems in $\mathcal{H}$ we can establish other properties of $\circ$, including that it is associative.

Define a binary operation $\bullet$ on K by putting $a \bullet b = \max(a,b)$ if $b^* < a$, and $a \bullet b = \min(a,b)$ otherwise. Then in fact $[a) \circ [b) = [a \bullet b)$. Put

$$Rabx \quad \text{iff} \quad x \in [a) \circ [b) \quad \text{iff} \quad a \bullet b \leq x.$$

The structure $\langle K, [0), R, * \rangle$ proves to be an R-frame whose ordering, defined by the condition "$\exists x(x \in [0) \;\&\; Rxab)$", turns out to be the original linear ordering $\leq$ of the Sugihara frame. The demonstration of all this involves some lengthy but fairly routine reasoning. We illustrate by sketching the proof of "Pasch's law" F4. If $Rabx$ and $Rxcd$, then $[x) \subseteq [a) \circ [b)$, and

$$d \in [x) \circ [c) \subseteq ([a) \circ [b)) \circ [c) = ([a) \circ [c)) \circ [b) = [a \bullet c) \circ [b).$$

Putting $z = a \bullet c$ then gives $Racz$ and $Rzbd$.

A *discrete* Sugihara frame is one meeting the following description. Its 0-element is the number zero and K includes the set $\mathbb{Z}$ of integers; $m^* = -m$ for all $m \in \mathbb{Z}$; $\leq$ is the usual ordering when restricted to $\mathbb{Z}$; and for each $m \in \mathbb{Z}$ there is no $x \in K$ with $m < x < m + 1$. Then $-[m) = [-m + 1)$ for all $m \in \mathbb{Z}$. Moreover, each $x \in K - \mathbb{Z}$ is either positively infinite: $x > m$ for all $m \in \mathbb{Z}$, or negatively infinite:

[6] It is also possible to study frames having $x^* \neq x$ in general, so there is no "zero" element. Then the base worlds would be given by the the set $P = \{x : x^* < x\}$ of *positive* elements, which would have $-P = P$, and $P \subseteq X$ iff $-X \subseteq X$ for hereditary X.

$x < m$ for all $m \in \mathbb{Z}$, and indeed x has one of these properties iff x^* has the other. We write K_∞ for the set of all positively infinite elements of K. Note that $K_\infty \subseteq [m)$ for all $m \in \mathbb{Z}$. The subcollection

$$Prop_{\mathbb{Z}} = \{K, \emptyset\} \cup \{[m) : m \in \mathbb{Z}\}$$

of $\mathcal{H}$ is closed under $-$, $\cap$, and $\cup$, and hence under the Sugihara operation $\Rightarrow$. If $K \neq \mathbb{Z}$, then this collection will not include all hereditary sets, since $[x) \notin Prop_{\mathbb{Z}}$ when $x \in K - \mathbb{Z}$. Now define

$$\mathcal{K} = \langle K, [0), R, *, I, Prop_{\mathbb{Z}}, PropFun_{\mathbb{Z}} \rangle,$$

where $\langle K, [0), R, * \rangle$ is the R-frame defined from the Sugihara frame $\langle K, 0, \leq, * \rangle$ as above; $I = \omega$; and $PropFun_{\mathbb{Z}}$ is the set of *all* functions from I^ω to $Prop_{\mathbb{Z}}$.

It is immediate that $\mathcal{K}$ satisfies all of CProp, CTee, CNeg, CImp and CConj. For CAll it suffices, by (3.1), to show that $Prop_{\mathbb{Z}}$ is $\sqcap$-complete, i.e. $\sqcap S \in Prop_{\mathbb{Z}}$ for every $S \subseteq Prop_{\mathbb{Z}}$, where $\sqcap$ is the operation that $Prop_{\mathbb{Z}}$ determines. This is a matter of cases: if $\emptyset \in S$ then $\sqcap S = \bigcap S = \emptyset$, while if $S = \{K\}$ or $S = \emptyset$ then $\sqcap S = \bigcap S = K$. Otherwise, if there is a largest integer m having $[m) \in S$, then $\sqcap S = \bigcap S = [m)$; and finally, if there are arbitrarily large integers m having $[m) \in S$, then $\bigcap S = K_\infty$, and so $\sqcap S = \emptyset$ since $\emptyset$ is the only member of $Prop_{\mathbb{Z}}$ included in K_∞.

This establishes that $\mathcal{K}$ is a QR-frame. Now take a language having a single unary predicate letter P, and define $V(P) : I \to Prop_{\mathbb{Z}}$ by putting $V(P)n = [n)$ for all $n \in \omega$. This gives a QR-model on $\mathcal{K}$. If A is the formula Px_0, then for each $f \in I^\omega$,

$$|A|_{Vf} = V(Px_0)f = V(P)(Vfx_0) = [f0),$$

so that $a \models_{Vf} A$ iff $f0 \leq a$, for all $a \in K$.

Now for each $g \in x_0 v f$, with $g = f[j/0]$ for some $j \in \omega$, we have $g0 = j$, and as above, $|A|_{Vg} = [g0) = [j)$. Thus

$$\bigcap_{g \in x_0 v f} |A|_{Vg} = \bigcap_{j \in \omega} [j) = K_\infty,$$

and so $|\forall x_0 A|_{Vf} = \bigsqcap_{g \in x_0 v f} |A|_{Vg} = \emptyset$. This means that if $a \in K_\infty$, then for any $f \in I^\omega$, we have $a \models_{Vg} A$ for all $g \in x_0 v f$ but $a \not\models_{Vf} \forall x_0 A$. So if $K \neq \mathbb{Z}$, and hence $K_\infty \neq \emptyset$, then the QR-model $\mathcal{K}$ has non-Tarskian semantics for the quantifier $\forall$. Such a $\mathcal{K}$ can be constructed from any nonstandard model of the first-order theory of the discrete Sugihara frame $\langle \mathbb{Z}, 0, \leq, - \rangle$. More simply we could just take $K = \mathbb{Z} \cup \{-\infty, \infty\}$, where $-\infty$ and ∞ are two new objects having $-\infty < m < \infty$ for all $m \in \mathbb{Z}$.

6 The Logic QR

The base logic of our semantics is called "QR" because it bears a strong resemblance to Meyer's logic NR, which adds necessity to R.

Axiom Schemes

1. $A \to A$
2. $(B \to C) \to ((A \to B) \to (A \to C))$
3. $(A \to (B \to C)) \to (B \to (A \to C))$
4. $(A \to (A \to B)) \to (A \to B)$
5. $(A \wedge B) \to A$
6. $(A \wedge B) \to B$
7. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
8. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
9. $A \to (A \vee B)$
10. $A \to (B \vee A)$
11. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (B \wedge C))$
12. $(A \to \neg A) \to \neg A$
13. $A \leftrightarrow \neg\neg A$
14. $(A \to \neg B) \to (B \to \neg A)$
15. t
16. $A \leftrightarrow (t \to A)$
17. $(A \to (B \to C)) \leftrightarrow ((A \circ B) \to C)$
18. $\forall x A \to A[\tau/x]$, where x is free for τ in A (i.e. τ is a constant, or is a variable that does not become bound in $A[\tau/x]$.)

Rules

$$\text{(MP)}\ \frac{\vdash A \to B \quad \vdash A}{\vdash B}$$

$$\text{(Adjunction)}\ \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}$$

$$\text{(RIC)}\ \frac{\vdash A \to B}{\vdash A \to \forall x B},\ \text{where } x \text{ is not free in } A.$$

The next few lemmas show that QR contains many of the standard theorems of first order logic.

Lemma 6.1 $\forall x(A \wedge B) \leftrightarrow (\forall xA \wedge \forall xB)$ *is a theorem of QR.*

Proof. First we show $\vdash \forall x(A \wedge B) \to (\forall xA \wedge \forall xB)$:

1.	$\vdash \forall x(A \wedge B) \to A$	Axioms 18, 5
2.	$\vdash \forall x(A \wedge B) \to \forall xA$	1, RIC
3.	$\vdash \forall x(A \wedge B) \to B$	Axioms 18, 6
4.	$\vdash \forall x(A \wedge B) \to \forall xB$	3, RIC
5.	$\vdash \forall x(A \wedge B) \to (\forall xA \wedge \forall xB)$	2, 4, Axiom 7

Now we show the converse:

1.	$\vdash (\forall xA \wedge \forall xB) \to A$	Axioms 5, 18
2.	$\vdash (\forall xA \wedge \forall xB) \to B$	Axioms 6, 18
3.	$\vdash (\forall xA \wedge \forall xB) \to (A \wedge B)$	1, 2, Adjunction, Axiom 7
4.	$\vdash (\forall xA \wedge \forall xB) \to \forall x(A \wedge B)$	3, RIC

Putting these two theorems together using the rule of adjunction, we obtain
$\vdash \forall x(A \wedge B) \leftrightarrow (\forall xA \wedge \forall xB)$. □

Lemma 6.2 *The rule*

$$\frac{\vdash A}{\vdash \forall xA}\ \text{(UG)}$$

is derivable in QR.

Proof.

1.	$\vdash A$	hypothesis
2.	$\vdash A \to (t \to A)$	Axiom 16
3.	$\vdash t \to A$	1, 2, MP
4.	$\vdash t \to \forall xA$	3, RIC
5.	$\vdash t$	Axiom 15
6.	$\vdash \forall xA$	4, 5, MP

□

Lemma 6.3 *The following are theorems of QR:*

(a) $\forall x(A \to B) \to (A \to \forall xB)$*, where x is not free in A;*

(b) $\forall x(A \to B) \to (\forall xA \to \forall xB)$;

(c) $A \leftrightarrow \forall xA$*, where x is not free in A;*

(d) $\forall x\forall yA \to \forall y\forall xA$;

(e) $\exists x\forall yA \to \forall y\exists xA$.

Proof.
(a) Suppose that x is not free in A.

1.	$\forall x(A \to B) \to (A \to B)$	Axiom 18
2.	$(\forall x(A \to B) \circ A) \to B$	1, Axiom 17
3.	$(\forall x(A \to B) \circ A) \to \forall xB$	2, RIC
4.	$\forall x(A \to B) \to (A \to \forall xB)$	3, Axiom 17

(b)

1.	$\forall x(A \to B) \to (A \to B)$	Axiom 18
2.	$\forall xA \to A$	Axiom 18
3.	$\forall x(A \to B) \to (\forall xA \to B)$	1, 2, transitivity of $\to$
4.	$(\forall x(A \to B) \circ \forall xA) \to B$	3, Axiom 17
5.	$(\forall x(A \to B) \circ \forall xA) \to \forall xB$	4, RIC
6.	$\forall x(A \to B) \to (\forall xA \to \forall xB)$	6, Axiom 17

(c) Suppose that x does not occur free in A.

1.	$\forall xA \to A$	Axiom 18
2.	$A \to A$	Axiom 1
3.	$A \to \forall xA$	2, RIC
4.	$A \leftrightarrow \forall xA$	1, 4, Adjunction

(d)

1.	$\forall x\forall yA \to A$	Axiom 18
2.	$\forall x\forall yA \to \forall xA$	1, RIC
3.	$\forall x\forall yA \to \forall y\forall xA$	2, RIC

(e)

1.	$\forall x \neg A \to \neg A$	Axiom 18
2.	$A \to \exists x A$	1, Axiom 14
3.	$\forall y A \to \exists x A$	Axiom 18, 2
4.	$\forall y A \to \forall y \exists x A$	3, RIC
5.	$\exists x \forall y A \to \forall y \exists x A$	4, RIC, fiddling

□

We also will need the following fact:

Fact 6.4 *The following schemata are theorems of R (and hence of QR):*

(a) $(A \to B) \to (\neg A \vee B)$;

(b) $A \to ((A \to B) \to B)$. □

In our completeness proof we use the following variation on the rule RIC:

$$\frac{\vdash A \to B[c/x]}{\vdash A \to \forall x B} \text{ (RIC(Con))},$$

where c (an individual constant) is in neither A nor B and x is not free in A. To prove that RIC(Con) is derivable in QR, we first show the following lemma:

Lemma 6.5 $\forall y(A[y/x]) \to \forall x A$ *is a theorem of QR if y does not occur free in A.*

Proof.

1.	$\forall y(A[y/x]) \to (A[y/x])[x/y]$	axiom 18
2.	$(A[y/x])[x/y] = A$	as y not free in A
3.	$\forall y(A[y/x]) \to A$	1, 2
4.	$\forall y(A[y/x]) \to \forall x A$	3, RIC

□

Now we prove

Lemma 6.6 *RIC(Con) is derivable in QR.*

Proof. Given a proof sequence ending in $A \to B[c/x]$, where c does not occur in either A or B, let y be a new variable not occurring anywhere in this sequence. Replace c throughout the sequence by y. The result is a new proof sequence whose last member is $A \to B[y/x]$, since c does not occur in A or B. Thus y does not occur in A, so by RIC,

$$\vdash A \to \forall y(B[y/x]).$$

Therefore, by lemma 6.5 and the transitivity of $\to$, we obtain $\vdash A \to \forall x B$. □

Corollary 6.7 *The rule*

$$\frac{\vdash A[c/x]}{\vdash \forall x A}\ \text{(UG(Con))},$$

where c is not in A, is derivable in QR.

Proof. This follows using RIC(Con), in an exactly parallel way to the derivation of UG from RIC in Lemma 6.2. □

7 Soundness for QR

The proof that every instance of axioms 1-17 is valid over our semantics follows the usual pattern [1, 16]. Thus we need to prove here that axiom 18 and RIC are valid in QR-models. We make use of the result of Lemma 4.4 that $|A|_{Vf} = |A|_{Vg}$ whenever f and g agree on the free variables of A.

Lemma 7.1 *Let A be any formula and let x be free for τ in A. In any QR-model, if g is an x-variant of f such that* $Vgx = Vf\tau$*, then* $|A[\tau/x]|_{Vf} = |A|_{Vg}$.

Proof. By induction on the complexity of A. The case of atomic formulae is exemplified by $A := Px$ with P a unary predicate letter. Then $|A[\tau/x]|_{Vf} = V(P\tau) = V(P)(Vf\tau)$, while $|A|_{Vg} = V(P)(Vgx)$. The inductive cases for the propositional connectives are straightforward.

Now let A be $\forall yB$ and assume that the result holds for B. First, if x is not free in A, then $A[\tau/x] = A$, and f and g agree on the free variables of A since $g \in xvf$, so $|A[\tau/x]|_{Vf} = |A|_{Vf} = |A|_{Vg}$ by Lemma 4.4.

The most intricate case is when x occurs free in $A = \forall yB$. Then $x \neq y$ and $A[\tau/x] = \forall y(B[\tau/x])$. Then since x is free for τ in A, we must have $y \neq \tau$ and x free for τ in B. Thus if y is the variable x_n,

$$|A[\tau/x]|_{Vf} = \prod_{j\in I} |B[\tau/x]|_{Vf[j/n]}, \quad \text{and}$$

$$|A|_{Vg} = \prod_{j\in I} |B|_{Vg[j/n]}.$$

But for any $j \in I$, $f[j/n]$ is an x-variant of $g[j/n]$ since $g \in xvf$ and $x_n = y \neq x$. Moreover, since $x_n \neq \tau$ we have $(Vf[j/n])\tau = Vf\tau = Vgx = (Vg[j/n])x$ (as $x \neq x_n$). Hence by the induction hypothesis on B,

$$|B[\tau/x]|_{Vf[j/n]} = |B|_{Vg[j/n]}$$

for all $j \in I$. From the above, this implies $|A[\tau/x]|_{Vf} = |A|_{Vg}$ as required. □

Lemma 7.2 *Axiom 18 is valid on the class of QR-frames.*

Proof. Suppose that $a \models_{Vf} \forall x A$, and x is free for τ in A. Then, by the truth condition for the universal quantifier, $a \models_{Vg} A$, for all g an x-variant of f. In particular $a \in |A|_{Vg}$ where g is the x-variant of f defined by $Vgx = Vf\tau$. But Lemma 7.1 gives $|A|_{Vg} = |A[\tau/x]|_{Vf}$, so $a \models_{Vf} A[\tau/x]$.

Therefore, by the Semantic Entailment Lemma 4.3, axiom 18 is valid on the class of QR-frames. □

Lemma 7.3 *The rule RIC preserves validity on the class of QR-frames.*

Proof. Suppose that x does not occur free in A, and that $A \to B$ is valid in all models on QR-frames. We show that $A \to \forall x B$ is also valid. Consider an arbitrary QR-model $\langle \mathcal{K}, V \rangle$. Then for every $g \in I^\omega$, by hypothesis and Lemma 4.3, we have $|A|_{Vg} \subseteq |B|_{Vg}$. Now given an $f \in I^\omega$, if g is any x-variant of f, then f and g agree on the free variables of A, since x is not free in A, and hence by Lemma 4.4, $|A|_{Vf} = |A|_{Vg} \subseteq |B|_{Vg}$. Thus

$$|A|_{Vf} \subseteq \bigsqcap_{g \in xvf} |B|_{Vg} = |\forall x B|_{Vf}.$$

By Lemma 4.3 again, f satisfies $A \to \forall x B$. Since f was arbitrary, the result follows. □

8 Theories

We now turn to the proof that QR is complete over the class of QR-models. The construction we use is a fairly standard Henkin-Lemmon-Scott-Routley-Meyer-style canonical model construction.

We need a few definitions before we can state the relevant version of the Lindenbaum extension lemma. We use the notation $\Gamma \Rrightarrow \Delta$, where Γ and Δ are sets of formulae, to mean that there is some $A_1, ..., A_n \in \Gamma$ and $B_1, ..., B_m \in \Delta$ such that $\vdash_{QR} (A_1 \wedge ... \wedge A_n) \to (B_1 \vee ... \vee B_m)$ (cf. [8]). The pair (Γ, Δ) is said to be *QR-independent*, or just *independent*, if and only if $\Gamma \not\Rrightarrow \Delta$. For example, if $(\Gamma, \{B\})$ is independent, then by axiom 1 we must have $B \notin \Gamma$.

A *QR-theory* Γ is a set of formulae such that, for every formula B, if $\Gamma \Rrightarrow \{B\}$, then $B \in \Gamma$. Note that if $A \in \Gamma$ and $\vdash A \to B$, then $\Gamma \Rrightarrow \{B\}$. Thus a theory satisfies the following form of detachment:

$$\text{if } A \in \Gamma \text{ and } \vdash A \to B, \text{ then } B \in \Gamma. \quad (8.1)$$

A theory Γ is *prime* if and only if for every disjunction $A \vee B \in \Gamma$, either $A \in \Gamma$ or $B \in \Gamma$. Γ is *regular* if and only if it contains every theorem. We can now state the extension lemma (the proof of which is due to Belnap – see [7]).

Lemma 8.1 (Extension) *If* (Γ, Δ) *is independent, then there is some prime theory* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* (Γ', Δ) *is independent.*

Note that this a fairly simple version of the extension lemma. We do not need to show a fact that is often required in completeness proofs for first order logics. That is, we do not need to show that, given an independent pair, we can extend to an independent pair in which the first member is a prime theory that is $\forall$-complete (i.e. if it contains $A[c/x]$ for every constant c, then it contains $\forall x A$).

Corollary 8.2 *If A is a non-theorem of QR, then there is a regular prime theory* Γ *such that* $A \notin \Gamma$.

Proof. Suppose that A is not a theorem of QR. Then $(T, \{A\})$ is independent, where T is the set of theorems of QR. Thus, by the extension lemma, T extends to a prime theory Γ such that $(\Gamma, \{A\})$ is independent, hence $A \notin \Gamma$. □

9 The Canonical Model

We now turn to the task of proving completeness. Our definition of the canonical model extends that of [15] and [8]. The worlds of the model are prime QR-theories and the set of individuals is the set of individual constants. For this purpose we make the assumption from now that

the set *Con* of individual constants is *infinite*.

If this were not so, then we could conservatively add infinitely many constants to our alphabet. Then a formula from the old language would be QR-derivable in the new language if and only if it was already derivable in the old. This follows by a standard argument, using the fact that in any proof sequence there are only finitely many formulae, and any new constants can be replaced by fresh variables from the infinitely many variables available in the old language (see the proof of Lemma 6.6).

The full description of the canonical model is as follows:

- K is the set of all prime theories of QR.
- 0 is the set of all regular prime theories of QR.

- for all a, b, c in K, $Rabc$ if and only if $\{A \circ B : A \in a \ \& \ B \in b\} \subseteq c$.
- a^* is the set of all formulae A such that $\neg A \notin a$.
- For each *closed* formula A, $\|A\| =_{df} \{a \in K : A \in a\}$.
- $Prop =_{df} \{\|A\| : A$ is a closed formula$\}$.
- $I = Con$, the set of individual constants.
- For any $f \in I^\omega$, each fn is a constant, so can be substituted for free occurrences of the variable x_n. The result of applying this substitution to a formula A is a *closed* formula we denote A^f. Thus

$$A^f = A[f0/x_0, \ldots, fn/x_n, \ldots].$$

- For each formula A, a function $\varphi_A : I^\omega \to K$ is defined by $\varphi_A f = \|A^f\|$.
- *PropFun* is the set of functions φ_A for all formulae A.
- $V(c) = c$ for all individual constants c.
- Where P is an n-ary predicate letter, $V(P) : I^n \to Prop$ is defined by

$$V(P)(c_0, \ldots, c_{n-1}) = \|Pc_0 \ldots c_{n-1}\|.$$

We now have the task of show that this structure is a QR-model. First, by Routley and Meyer's arguments [15, 16, 1], it can be shown that the conditions F1–F8 hold, so we have an R-frame, and moreover that this frame satisfies

$$a \leq b \quad \text{iff} \quad a \subseteq b.$$

To show that it is a QR-frame, we need to show that *Prop* and *PropFun* satisfy the closure conditions CProp-CAll.

Lemma 9.1 *Prop satisfies CProp and CTee.*

Proof. To show that 0 is in *Prop* we only need to note that t is in a prime theory a if and only if a is regular. Thus $0 = \|t\| \in Prop$.

Closure of *Prop* under $\cap$, $\Rightarrow$ and $-$ follows because the equations

$$\|A\| \cap \|B\| = \|A \wedge B\|$$
$$\|A\| \Rightarrow \|B\| = \|A \to B\|$$
$$-\|A\| = \|\neg A\|$$

hold for all formulas A, B. The proof of these are as given by Routley and Meyer.

For CTee, just note that the constant function φ_0 has $\varphi_0 f = \|t\| = \|t^f\|$ for all f, so φ_0 is $\varphi_t \in PropFun$. □

Lemma 9.2 *PropFun satisfies CConj, CImp and CNeg.*

Proof. For any $f \in Con^\omega$, the substitution operator induced by f commutes with the propositional connectives. Thus $A^f \wedge B^f = (A \wedge B)^f$, implying that $\|A^f\| \cap \|B^f\| = \|(A \wedge B)^f\|$. But this says that $\varphi_A f \cap \varphi_B f = (\varphi_{A \wedge B}) f$ for any f, hence $\varphi_A \cap \varphi_B = \varphi_{A\wedge B} \in PropFun$, establishing CConj.

Likewise we can show $\varphi_A \Rightarrow \varphi_B = \varphi_{A \to B}$ and $-\varphi_A = \varphi_{\neg A}$ to establish CImp and CNeg. □

Lemma 9.3 *If the formula $\forall xA$ is* ***closed****, then for all prime theories a, $\forall xA \in a$ if and only if there is some proposition $X \in Prop$ such that $a \in X$ and $X \subseteq \|A[c/x]\|$ for all constants c. In other words,*

$$\|\forall xA\| = \bigsqcap_{c \in Con} \|A[c/x]\|.$$

Proof. Suppose that $\forall xA \in a$. Set $X = \|\forall xA\|$. Then $a \in X \in Prop$ by definition, so by axiom 18 and the detachment result (8.1), $X \subseteq \|A[c/x]\|$ for all constants c.

Conversely, suppose that there is some $X \in Prop$ such that $X \subseteq \|A[c/x]\|$ for all constants c, and $a \in X$. By the definition of $Prop$ there is some closed formula B such that $\|B\| = X$, so $B \in a$.

Choose a constant c that does not occur in A or B (remember Con is infinite). Now if $\nvdash B \to \forall xA$, then $(\{B\}, \{A[c/x]\})$ is an independent pair, so by the Extension Lemma 8.1 there is a prime theory Γ extending B such that $(\Gamma, \{A[c/x]\})$ is independent, hence $A[c/x] \notin \Gamma$, giving $\Gamma \in \|B\| \backslash \|A[c/x]\|$ contrary to hypothesis.

Therefore $\vdash B \to A[c/x]$, so by Lemma 6.6, $\vdash B \to \forall xA$. Since a is a theory and $B \in a$, this implies $\forall xA \in a$ as desired. □

This last result enables us to prove:

Lemma 9.4 *For any formula A and any $n \in \omega$, $\forall_n \varphi_A = \varphi_{\forall x_n A}$. Hence PropFun satisfies CAll.*

Proof. Given $f \in I^\omega$, we define $A^{f\backslash n}$ to be the formula

$$A[f0/x_0, \ldots, f(n-1)/x_{n-1}, x_n/x_n, f(n+1)/x_{n+1}, \ldots],$$

obtained from A by leaving x_n alone and otherwise applying the substitution f. Then

$$A^{f\backslash n}[c/x_n] = A^{f[c/n]}, \tag{9.1}$$

since each formula in this equation can be seen as arising from A by substitution of c for x_n and fm for x_m when $m \neq n$. Also

$$\forall x_n(A^{f\backslash n}) = (\forall x_n A)^f, \tag{9.2}$$

since both formulas have x_n bound and can be obtained from A by binding x_n by $\forall$ and applying f to the other variables. Now

$$\begin{aligned}
(\forall_n \varphi_A)f &= \prod_{c \in Con} \varphi_A(f[c/n]) && \text{by definition of } \forall_n \text{ and (4.1)},\\
&= \prod_{c \in Con} \|A^{f[c/n]}\| && \text{by definition of } \varphi_A,\\
&= \prod_{c \in Con} \|A^{f\backslash n}[c/x_n]\| && \text{by (9.1)},\\
&= \|\forall x_n(A^{f\backslash n})\| && \text{by Lemma 9.3},\\
&= \|(\forall x_n A)^f\| && \text{by (9.2)},\\
&= \varphi_{\forall x_n A} f && \text{by definition of } \varphi_{\forall x_n A}.
\end{aligned}$$

Hence $\forall_n \varphi_A = \varphi_{\forall x_n A} \in PropFun$. □

This completes the proof that we have a QR-frame. Now to analyse the treatment of atomic formulae:

Lemma 9.5 *For all n-ary predicate letters P, terms $\tau_0, \ldots, \tau_{n-1}$, and variable-assignments $f \in I^\omega$,*

(1) $(P\tau_0 \ldots \tau_{n-1})^f = P(Vf\tau_0) \ldots (Vf\tau_{n-1})$.

(2) $V(P\tau_0 \ldots \tau_{n-1}) = \varphi_{P\tau_0 \ldots \tau_{n-1}}$.

Proof.

(1) The left side is $P(\tau_0^f) \ldots (\tau_{n-1}^f)$. But for $i < n$, if $\tau_i = x_m$, then $(\tau_i)^f = fm = Vf\tau_i$; and if $\tau_i \in Con$, then $(\tau_i)^f = \tau_i = Vf\tau_i$.

(2)
$$\begin{aligned}
V(P\tau_0 \ldots \tau_{n-1})f &= V(P)(Vf\tau_0, \ldots, Vf\tau_{n-1}) && \text{by definition of } V,\\
&= \|P(Vf\tau_0) \ldots (Vf\tau_{n-1})\| && \text{by definition of } V(P),\\
&= \|(P\tau_0 \ldots \tau_{n-1})^f\| && \text{by part (1)},\\
&= \varphi_{P\tau_0 \ldots \tau_{n-1}} f && \text{by definition of } \varphi.
\end{aligned}$$

□

Part (2) of this last Lemma shows that $V(A)$ belongs to *PropFun* for any atomic A, completing the proof that the canonical model is a QR-model.

Lemma 9.6 (Truth Lemma) *For any formula A, $|A|_V = \varphi_A$. Thus for all $f \in I^\omega$, $|A|_{Vf} = \|A^f\|$, and so for all $a \in K$, $a \models_{Vf} A$ iff $A^f \in a$.*

Proof. By induction on the complexity of A.

If A is an atomic formula $P\tau_0 \dots \tau_{n-1}$, then $|A|_V = V(A)$, which is φ_A by Lemma 9.5(2).

If $A = t$, then $|A|_V = \varphi_0$ which is φ_A in this case.

If $A = \neg B$ and inductively $|B|_V = \varphi_B$, then $|A|_V = -|B| = -\varphi_B = \varphi_A$.

The inductive cases of the connectives $\wedge$ and $\rightarrow$ are similar.

If $A = \forall x_n B$ and $|B|_V = \varphi_B$, then $|A|_V = \forall_n |B|_V = \forall_n \varphi_B$, which is φ_A by Lemma 9.4.

□

Theorem 9.7 (Completeness for QR) *For any formula A, the following are equivalent:*

(1) *A is a theorem of QR.*

(2) *A is valid in all QR-models.*

(3) *A is valid in all QR-frames.*

Proof. Since every QR-model is based on a QR-frame by definition, it is evident that (2) and (3) are equivalent. That (1) implies (2) follows by the soundness results of Section 7.

If (2) holds, then in particular A is valid in the canonical model, so in that model, for every regular prime theory $a \in 0$ and every $f \in Con^\omega$, $A^f \in a$ by the Truth Lemma 9.6. If the free variables of A are among $x_0, \dots, x_n$ for some n, choose constants $c_0, \dots, c_n$ that do not occur in A (remember *Con* is infinite). Then $A[c_0/x_0, \dots, c_n/x_n]$ belongs to every regular prime theory, so is a QR-theorem by Corollary 8.2. Hence $\vdash \forall x_0 \cdots \forall x_n A$ by (repeated application of) the rule UG(Con) (Corollary 6.7). Therefore $\vdash A$ by axiom 18. □

Note that this proof shows that A is a QR-theorem iff it is valid in the canonical QR-model iff it is valid in the canonical QR-frame.

10 Extensional Confinement

QR is an elegant logic with an elegant semantics, but it has shortcomings. Most relevant logicians would accept as valid any classically valid formula that is implication-free. But in the next Section we will show that the thesis

$$(!) \quad \neg\forall x(A \vee B) \vee A \vee \forall xB,$$

where x is not free in A, is not derivable in QR,[7] even when A and B are atomic formulae.

One way to ensure that we can prove (!) in our logic is to add the principle of extensional confinement, that is,

$$(\mathrm{EC}) \quad \forall x(A \vee B) \rightarrow (A \vee \forall xB)$$

where x is not free in A. Adding EC to QR yields Anderson and Belnap's logic RQ. We could of course add (!) directly, but it is more natural to add EC. Notice that every axiom of QR has implication as its main connective. Relevant logicians see a logic as codifying inference, not merely the laws that are true at every base world. By the deduction theorem, each implicational theorem is equivalent to a general rule of inference, that is, to a rule of inference that governs the behaviour of formulae at every world in the frame. A disjunctive theorem, like (!), does not codify a rule of inference in this sense.

We can make EC valid by appealing to Ross Brady's result that RQ is conservatively extended by the addition of boolean negation [3, §13.4]. Using this result, we can show that RQ is sound and complete over the class of QR-frames that satisfy the condition that if $a \leq b$, then $a = b$ and in which the set of propositions is closed under boolean complement, that is, if X is a proposition then so is $K \backslash X = \{a \in K : a \notin X\}$.

For suppose that the class of propositions is closed under boolean complement. Also suppose that

$$a \models_{Vf} \forall x(A \vee B),$$

where x does not occur free in A. Then there is some proposition X such that $a \in X$ and

$$X \subseteq |A \vee B|_{Vg},$$

for all x-variants g of f, i.e.

$$X \subseteq |A|_{Vg} \cup |B|_{Vg}$$

[7] This verifies a conjecture of the first author, made because the analogous formula in the modal logic NR (i.e. $\neg\Box(A \vee B) \vee \Diamond A \vee \Box B$) is not a theorem, and the two logics are very similar.

and so

$$X\backslash|A|_{Vg} \subseteq |B|_{Vg}.$$

But, if the class of propositions is closed under boolean complement, $X\backslash|A|_{Vg}$ is itself a proposition, that is, $X \cap (K\backslash|A|_{Vg})$. But for any g an x-variant of f, $|A|_{Vg} = |A|_{Vf}$ by Lemma 4.4 since x is not free in A. Thus, we have

$$X\backslash|A|_{Vf} \subseteq |B|_{Vg},$$

for all g, x-variants of f. By assumption, $a \in X$. If a is also in $|A|_{Vf}$, then, by the truth condition for disjunction, $a \models_{Vf} A \vee \forall xB$. If a is not in $|A|_{Vf}$, then $a \in X\backslash|A|_{Vf}$ and so, by the truth condition for the universal quantifier, $a \models_{Vf} \forall xB$, hence $a \models_{Vf} A \vee \forall xB$.

This method is somewhat unsatisfactory, since it requires that boolean negation be added to the semantics if not to the logic itself. If we do require that our semantics be boolean, then we are admitting that relevant logic is somehow parasitic on classical logic. However, using the above argument we can modify our semantics to make it more like the boolean semantics and in so doing make EC valid without going all the way to closing the class of propositions under boolean negation.

Define an *RQ-model* to be any QR-model in which

$$X\backslash Y \subseteq \bigcap_{g \in xvf} |B|_{Vg} \quad \text{implies} \quad X\backslash Y \subseteq |\forall xB|_{Vf} \tag{10.1}$$

for all formulae $\forall xB$, all $X, Y \in Prop$, and all $f \in I^{\omega}$. Note that (10.1) holds if $X\backslash Y \in Prop$, by the semantics of $\forall$, but in general it may be that $X\backslash Y \notin Prop$. Also, (10.1) is trivially satisfied when the model gives Tarskian semantics to $\forall$, i.e. when $|\forall xB|_{Vf} = \bigcap_{g \in xvf} |B|_{Vg}$. In particular it holds under any of the hypotheses of Lemma 4.5.

Lemma 10.1 *In any RQ-model, for any world a and any formulae A and B with x not free in A, if* $a \models_{Vf} \forall x(A \vee B)$*, then* $a \models_{Vf} A \vee \forall xB$*.*

Proof. Suppose that x is not free in A and that $a \models_{Vf} \forall x(A \vee B)$. Repeating the above argument regarding the boolean semantics, there is an $X \in Prop$ such that $a \in X$ and

$$X\backslash|A|_{Vf} \subseteq \bigcap_{g \in xvf} |B|_{Vg}.$$

Since $|A|_{Vf} \in Prop$ it follows from (10.1) that $X\backslash|A|_{Vf} \subseteq |\forall xB|_{Vf}$.

If $a \in |A|_{Vf}$, then $a \models_{Vf} A \vee \forall xB$. But, if $a \notin |A|_{Vf}$, then $a \in X\backslash|A|_{Vf} \subseteq |\forall xB|_{Vf}$, so again $a \models_{Vf} A \vee \forall xB$. □

Corollary 10.2 (Soundness for RQ) *Every theorem of RQ is valid in all RQ models.*

Proof. By the preceding lemma and the semantical entailment Lemma 4.3, EC is valid in all RQ-models. □

Similarly to the notion of RQ-model, we define an *RQ-frame* to be any QR-frame in which

$$X\backslash Y \subseteq \bigcap_{j\in I}\varphi(f[j/n]) \quad \text{implies} \quad X\backslash Y \subseteq (\forall_n\varphi)f \tag{10.2}$$

for all $\varphi \in PropFun$, all $n \in \omega$, all $X, Y \in Prop$, and all $f \in I^\omega$.

Theorem 10.3 (Completeness for RQ) *For any formula A, the following are equivalent:*

(1) *A is a theorem of RQ.*

(2) *A is valid in all RQ-models.*

(3) *A is valid in all RQ-frames.*

Proof. (1) implies (2) by Corollary 10.2, and (2) implies (3) because every model on an RQ-frame is an RQ-model.

To show that (3) implies (1), we construct a canonical model for RQ in the same way as for QR, taking the elements of K to be the prime RQ-theories. The resulting model has the property that any formula valid in it is an RQ-theorem. So it suffices to show that the underlying frame of this model is an RQ-frame.

To show this, suppose that the antecedent of (10.2) holds in this frame for certain suitable φ, X, Y, n, f. By the canonical construction, $Y = \|A\|$ for some *closed* formula A, and $\varphi = \varphi_B$ for some formula B (possibly open). Recalling that $I = Con$, we are thus supposing that for all $c \in Con$, $X\backslash\|A\| \subseteq \varphi_B(f[c/n])$, and so

$$X \subseteq \|A\| \cup \|B^{f[c/n]}\| = \|A \vee (B^{f[c/n]})\|$$

But by (9.1) and the fact that A is closed and hence unchanged by substitution,

$$A \vee (B^{f[c/n]}) = A \vee (B^{f\backslash n}[c/x_n]) = (A \vee B)^{f\backslash n}[c/x_n].$$

Now suppose $a \in X\backslash Y$. Then $a \in X \subseteq \|(A \vee B)^{f\backslash n}[c/x_n]\|$ for all $c \in Con$. But $(A \vee B)^{f\backslash n}[c/x_n]$ is a closed formula, so by Lemma 9.3, $\forall x_n((A \vee B)^{f\backslash n}) \in a$, i.e. $\forall x_n(A \vee (B^{f\backslash n})) \in a$. Hence by EC and detachment (8.1), $A \vee \forall x_n(B^{f\backslash n}) \in a$.

But $a \notin Y = \|A\|$, so $A \notin a$, implying that $\forall x_n(B^{f\backslash n}) \in a$. Using (9.2) we then get

$$a \in \|(\forall x_n B)^f\| = (\varphi_{\forall x_n B})f = (\forall_n\varphi_B)f$$

by Lemma 9.4. This proves that $X\backslash Y \subseteq (\forall_n\varphi)f$, as required for (10.2). □

11 Independence of EC

In Section 5 we constructed a QR-frame $\mathcal{K} = \langle K, [0), R, *, I, Prop_{\mathbb{Z}}, PropFun_{\mathbb{Z}} \rangle$ from any discrete Sugihara frame $\langle K, 0, \leq, * \rangle$. This $\mathcal{K}$ always satisfies (10.2) and so is an RQ-frame. For, the only way it can have $\bigcap_I \varphi(f[j/n]) \neq (\forall_n \varphi) f$ is to have $\bigcap_I \varphi(f[j/n]) = K_\infty \neq \emptyset$ while $(\forall_n \varphi) f = \emptyset$. But for any $X, Y \in Prop_{\mathbb{Z}}$, if $X \backslash Y$ is non-empty it must contain some integer, and hence is not a subset of K_∞.

We can modify any such $\mathcal{K}$ to a frame falsifying EC by taking an object $\theta \notin K$ and adding it as a new point to K with $\theta^* = \theta$, but without adding $\{\theta\}$ to *Prop*. New members of *Prop* are created by adding θ to each $[m)$ while retaining $[m)$ itself. Then if, say, $X = [0) \cup \{\theta\}$ and $Y = [0)$, we get $X \backslash Y = \{\theta\} \neq \emptyset$. There will be cases where $\bigcap_I \varphi(f[j/n]) = K_\infty \cup \{\theta\}$ while still having $(\forall_n \varphi) f = \emptyset$, so we get $X \backslash Y \subseteq \bigcap_I \varphi(f[j/n])$ while $X \backslash Y \nsubseteq (\forall_n \varphi) f$. Actually we do not need any elements of K_∞ for this, so the new frame could just be a one-point extension of $\mathbb{Z}$ itself.

However, although this approach provides a falsifying QR-model for EC, it does not yield the stronger result that the weaker principle (!) is falsifiable. To achieve that seems to require the addition of more points. One solution is to form the disjoint union of $\mathbb{Z}$ with a copy of the 3-element Sugihara frame

$$\langle \{-1, 0, 1\}, \{0, 1\}, \leq, - \rangle.$$

We take this copy on a set $T = \{\varepsilon^*, \theta, \varepsilon\}$ with $T \cap \mathbb{Z} = \emptyset$, $\theta^* = \theta$, and the ordering of T given by $\varepsilon^* < \theta < \varepsilon$. The base worlds of T are θ and ε. Let R^T be the ternary relation making T into an R-frame. Recall from Section 5 that this is characterized by the binary operation $\bullet$ having $a \bullet b = \max(a, b)$ if $b^* < a$, and $a \bullet b = \min(a, b)$ otherwise. This operation on T is given by the table

$\bullet$	ε^*	θ	ε
ε^*	ε^*	ε^*	ε^*
θ	ε^*	θ	ε
ε	ε^*	ε	ε

from which R^T can be directly read off: $R^T abc$ iff $a \bullet b \leq c$.

For $m \in \mathbb{Z}$ we continue to write $[m)$ for the interval $\{x \in \mathbb{Z} : m \leq x\}$ in $\mathbb{Z}$ starting at m. Let $\mathbb{Z}^T = \mathbb{Z} \cup T$, and for each $X \subseteq \mathbb{Z}$, let $X^\varepsilon = X \cup \{\varepsilon\}$ and $X^{\theta\varepsilon} = X \cup \{\theta, \varepsilon\}$. Define

$$Prop = \{\mathbb{Z}^T, \emptyset\} \cup \{[m)^\varepsilon, [m)^{\theta\varepsilon} : m \in \mathbb{Z}\}.$$

For every $S \subseteq Prop$ we have $\bigsqcap S \in Prop$, where $\bigsqcap$ is the operation that *Prop* determines. There are two significant cases in addition to those of $\emptyset \in S$, $S = \{\mathbb{Z}^T\}$

or $S = \emptyset$. If there is a largest integer m with one of $[m)^{\varepsilon}, [m)^{\theta\varepsilon}$ in S, then

$$\bigsqcap S = \bigcap S = \begin{cases} [m)^{\theta\varepsilon} & \text{if } \theta \in \bigcap S; \\ [m)^{\varepsilon} & \text{if } \theta \notin \bigcap S. \end{cases}$$

Alternatively, if there are arbitrarily large integers m with one of $[m)^{\varepsilon}, [m)^{\theta\varepsilon}$ in S, then $\bigcap S \subseteq \{\theta, \varepsilon\}$, and so $\bigsqcap S = \emptyset$ since $\emptyset$ is the only member of *Prop* included in $\{\theta, \varepsilon\}$.

If R is the ternary relation making $\mathbb{Z}$ into an R-frame, put

$$\mathcal{K} = \langle \mathbb{Z}^T, [0)^{\theta\varepsilon}, R^+, *, I, Prop, PropFun \rangle,$$

where $R^+ = R \cup R^T$, $*$ is the union of the corresponding operations on $\mathbb{Z}$ and T, $I = \omega$, and *PropFun* is the set of *all* functions from I^{ω} to *Prop*.

The structure $\langle \mathbb{Z}^T, [0)^{\theta\varepsilon}, R^+, * \rangle$ is an R-frame, since the disjoint union of any two R-frames satisfies F1–F8. Hence the ordering $\leq$ of $\mathbb{Z}^T$ induced by R^+ is just the disjoint union of the orderings of $\mathbb{Z}$ and T. In $\mathcal{K}$, we have $-[m)^{\varepsilon} = [-m+1)^{\theta\varepsilon}$; $-[m)^{\theta\varepsilon} = [-m+1)^{\varepsilon}$; $[m)^{\varepsilon} \cap [n)^{\theta\varepsilon} = [m)^{\varepsilon} \cap [n)^{\varepsilon} = [\max(m,n))^{\varepsilon}$; and $[m)^{\theta\varepsilon} \cap [n)^{\theta\varepsilon} = [\max(m,n))^{\theta\varepsilon}$ for all $m, n \in \mathbb{Z}$. Together with $-\mathbb{Z}^T = \emptyset$, this shows that *Prop* is closed under $-$ and $\cap$.

Then to show that *Prop* is closed under the operation $\Rightarrow$ defined by R^+, it suffices to show that *Prop* is closed under the fusion operation $X \circ^+ Y$ that R^+ defines, since $X \Rightarrow Y = -(X \circ^+ -Y)$. Fusion satisfies

$$X \circ^+ Y = \{z : (\exists x \in X)(\exists y \in Y)R^+xyz\},$$

is commutative, and has $\emptyset \circ^+ Y = \emptyset$. In $\mathcal{K}$ we can show that

$$X^{\varepsilon} \circ^+ \mathbb{Z}^T = X^{\theta\varepsilon} \circ^+ \mathbb{Z}^T = \mathbb{Z}^T \circ^+ \mathbb{Z}^T = \mathbb{Z}^T \text{ for all nonempty hereditary } X \subseteq \mathbb{Z};$$

$$[m)^{\varepsilon} \circ^+ [n)^{\varepsilon} = [m)^{\varepsilon} \circ^+ [n)^{\theta\varepsilon} = [m \bullet n)^{\varepsilon} \text{ and } [m)^{\theta\varepsilon} \circ^+ [n)^{\theta\varepsilon} = [m \bullet n)^{\theta\varepsilon},$$

which gives the closure of *Prop* under $\circ^+$. Thus $\mathcal{K}$ is a QR-frame.

Now take a language having two unary predicate letter P and Q, and define a model on $\mathcal{K}$ by putting $V(P)n = [0)^{\varepsilon}$ and $V(Q)n = [n)^{\theta\varepsilon}$ for all $n \in \omega$. Let A be the formula Px_1, and B be Qx_0. Then for each $f \in I^{\omega}$,

$$|A \vee B|_{Vf} = V(P)(Vfx_1) \cup V(Q)(Vfx_0) = [0)^{\varepsilon} \cup [f0)^{\theta\varepsilon} = [0)^{\theta\varepsilon},$$

since $f0 \geq 0$. Thus

$$|\forall x_0(A \vee B)|_{Vf} = \bigsqcap_{g \in x_0 vf} |A \vee B|_{Vg} = \bigsqcap_{g \in x_0 vf} [0)^{\theta\varepsilon} = [0)^{\theta\varepsilon}.$$

In particular, $\forall x_0(A \vee B)$ is satisfied at $\theta^* = \theta$, so $\theta \not\models_{Vf} \neg\forall x_0(A \vee B)$. But

$$|\forall x_0(B)|_{Vf} = \prod_{j\in\omega} |B|_{Vf[j/0]} = \prod_{j\in\omega} [j)^{\varepsilon} = \emptyset,$$

and so $|A \vee \forall x_0 B|_{Vf} = |A|_{Vf} = [0)^{\varepsilon}$. Hence $\theta \not\models_{Vf} A \vee \forall x_0 B$, and altogether

$$\theta \not\models_{Vf} \neg\forall x_0(A \vee B) \vee A \vee \forall x_0 B.$$

Since θ belongs to the set $[0)^{\theta\varepsilon}$ of base worlds of $\mathcal{K}$, this shows that (!) is not valid in the QR-model $\langle \mathcal{K}, V \rangle$. By the soundness of QR in QR-frames, it follows that (!) is not derivable in QR, and hence nor is EC by Fact 6.4(a).

12 Further Modifications to the Semantics

RQ is known to have some very nice properties. We can use these properties to make its semantics even more intuitive.

Meyer, Dunn and Leblanc [13] showed that RQ is characterised by the class of its regular prime theories Γ that are $\forall$-complete. Using this fact, we can modify the definition of a frame as follows. We remove 0 as a primitive element of frames and replace it with a set of worlds PW (for 'possible worlds'). We can then define 0 as the set of worlds a such that there is some world $b \leq a$ that is in PW. We then take a truth condition for the universal quantifier stating that $a \models_{Vf} \forall xA$ if and only if either the previous condition for RQ holds or there is a world $b \in PW$, such that $b \leq a$ and $b \models_{Vg} A$ for all g, x-variants of f. We can then show that for any possible world b, $b \models_{Vf} \forall xA$ if and only if $b \models_{Vg} A$ for all g, x-variants of f.

We can go even further. Meyer and Dunn [12] have shown that RQ is characterised by the class of its regular, prime theories Γ that are $\forall$-complete and that are consistent. Thus, we can even require that for all worlds $a \in PW$, $a = a^*$. For any world $a = a^*$, $a \models_{Vf} \neg A$ if and only if $a \not\models_{Vf} A$. Thus, on this semantics the set of possible worlds, considered in terms of the formulae that include only predicates, individual variables and constants, parentheses, $\wedge$, $\neg$, and $\forall$, are very much like possible worlds in the classical sense.

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