AN EFFECTIVE CLASSIFICATION OF BOREL WADGE CLASSES

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Abstract. We give a new and effective classification of all Borel Wadge classes of subsets of Baire space $\mathcal{N}$. This relies on the true stage machinery originally developed by Montalbán. We use this machinery to give a new proof of Louveau and Saint-Raymond's separation theorem for Borel Wadge classes. This gives a proof of Borel Wadge determinacy in the subsystem $\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$ of second-order arithmetic.

Contents

1. Introduction 1
2. Preliminaries 3
3. Described classes 9
4. Classification of acceptable classes 20
5. The separation theorem(s) 29
6. Borel Wadge determinacy 37
7. Developing the true stage machinery 41
References 50

1. Introduction

Let $\mathcal{N}$ denote Baire space, ω^ω. A set $A \subseteq \mathcal{N}$ is *Wadge reducible* to a set $B \subseteq \mathcal{N}$ if A is a continuous pre-image of B; we write $A \leqslant_W B$. The *Wadge class* of B is the collection of all sets which are Wadge reducible to B. What are all Wadge classes of Borel sets? This question was essentially answered by Wadge ([Wad84]), with a full description appearing in [Lou83] and another one in [LSR88b]. This analysis was then extended by Duparc [Dup01] to spaces κ^ω, Selivanov [Sel07, Sel17] to k-partitions rather than sets, and Kihara and Montalbán [KM19] to BQO-valued functions.

Once a system of descriptions for Borel Wadge classes is suggested, how does one prove that indeed every such class is described? The standard route, as suggested by Wadge, relies on *Borel Wadge determinacy*. For any two sets $A, B \subseteq \mathcal{N}$, players in a game $G_L(A, B)$ alternate choosing natural numbers, resulting in a real $x \in \mathcal{N}$ built by player I and another $y \in \mathcal{N}$ built by player II; player II wins if and only if $x \in A \iff y \in B$. A winning strategy for player II gives $A \leqslant_W B$; for player I, it gives $B \leqslant_W A^{\complement}$. Borel Wadge determinacy is the statement that every such game, for Borel sets A and B, is determined. Borel Wadge determinacy, thus, implies

Wadge's *semi-linear ordering principle* for Borel Wadge degrees: for all Borel A and B, either $A \leqslant_W B$ or $B \leqslant_W A^{\complement}$.

Further, using Borel Wadge determinacy, Martin showed (see [KM78]) that the Wadge ordering of Borel sets is well-founded. Thus, theorems about Borel Wadge classes can be proved inductively. One such theorem is that every Borel Wadge class has a description.

In [LSR87, LSR88b], Louveau and Saint-Raymond follow a different route. They give two systems of descriptions (one is from [Lou83], based on work of Wadge's). They then give a new proof of Borel Wadge determinacy, but restricted to described classes. Using this, and a detailed analysis from [Lou83] of the ambiguous classes of described classes, they then show that all Borel Wadge degrees are described, and so their determinacy result gives full Borel Wadge determinacy.

One motivation for taking this different route is that their argument gives a proof of Borel Wadge determinacy within second-order arithmetic. This was quite surprising, since the most straightforward proof relies on Borel determinacy, which is known to require strong axioms [Fri71]; and since $\mathbf{\Pi}^1_1$ Wadge determinacy was known to be equivalent to full $\mathbf{\Pi}^1_1$ determinacy (Harrington [Har78]). The argument also gives an extension of Louveau's separation theorem [Lou80] to all non-self-dual Borel Wadge classes.

It is then natural to ask for the weakest subsystem of second-order arithmetic in which Borel Wadge determinacy, or the semi-linear ordering principle, are provable. Cordón-Franco, Lara-Martín and Loureiro [Lou16, CFLML] established some lower and upper bounds for lower levels of the Borel hierarchy.

In the current paper we give a new analysis of all Borel Wadge classes. The main feature of our descriptions of Wadge classes is their *dynamic* nature. Rather than a "static" description of sets in these classes, as constructed from simpler sets by using a variety of Boolean operations on sets, we describe approximation procedures that determine whether a given real is in the set described, based on its (finite) initial segments.

To do this, we rely on the *true stage machinery* developed by Montalbán [Mon14] to organise iterated priority arguments in computable structure theory. Applications of this technique to descriptive set theory were discovered by Day, Downey and Westrick [DDW] and Day and Marks [DM]. Our previous paper [DGHTT] is an expository paper, in which we give a description of the true stage machinery (as re-developed in [GTa]), and showed how to use it to prove effective versions of the Hausdorff-Kuratowski theorem, Wadge's theorem on the structure of $\mathbf{\Delta}^0_\lambda$ for limit λ, and Louveau's separation theorem [Lou80]. In the current paper, we use our descriptions of Borel Wadge classes to prove Louveau and Saint-Raymond's separation theorem mentioned above, extending our argument from [DGHTT], where it was restricted to the classes $\mathbf{\Sigma}^0_\xi$. This argument is more direct compared to the original one from [LSR87, LSR88b], which relied on the ramification method of unravelling games. We also provide the full details of the development of the true stage machinery, which was delayed from [DGHTT].

Our reliance on the true stage method means that our analysis of the ambiguous classes described classes is inherently effective: computability of class descriptions and of names for sets in these classes are a necessary feature in this analysis, rather than an afterthought.

Finally, we will observe that our methods are sufficiently effective so that they can be proved within the system ATR_0 of second-order arithmetic, which is, in a sense, the natural base system to use when analysing Borel sets. We can then complete the argument sketched in [LSR88b] to prove Borel Wadge determinacy within that system (with the added requirement of some induction).

2. Preliminaries

2.1. **An informal description of class descriptions.** In some sense, our descriptions of Borel Wadge classes are a common generalisation of the two systems of descriptions presented in [LSR88b]. A *class description* Γ will include a well-founded tree T_Γ, whose leaves will be labelled by 0's and 1's. For such a class description Γ, a Γ-*name* N for a subset of $\mathcal{N}$ will determine functions f_s for *non-leaf* $s \in T_\Gamma$. Such a function f_s will choose, for each real $x \in \mathcal{N}$, a child t of s on T_Γ. The subset of $\mathcal{N}$ named by such a name N is then defined as follows: for a real x, we determine a path $s_0, s_1, \dots, s_k$ through the tree T_Γ, starting at the root $s_0 = \langle\rangle$, and recursively choosing the child $s_{i+1} = f_{s_i}(x)$. When we get to a leaf s_k, we decide that x is in the set or not, depending on the label (0 or 1) of s_k given by Γ. The class description Γ will provide information stating which functions f_s can be used by Γ-names. The class $\mathbf{\Gamma}$ will consist of all sets that are named by a Γ-name.

More specifically, for a *non-leaf* $s \in T_\Gamma$, the class description Γ will specify two ordinals ξ_s and η_s. A function f_s used at the node s by a Γ-name will be the limit of an approximation at level ξ_s, with bound $\eta_s + 1$, and with a "default outcome" being the leftmost child of s on T_Γ. Set-theoretically, this means that for each non-default child t of s, the collection of x mapped by f_s to t form a $D_{\eta_s}(\mathbf{\Sigma}^0_{1+\xi_s})$ set.

By induction on the rank of T_Γ, we can view our class descriptions as being built recursively, starting with the very simplest class descriptions: those which consists only of the root, which name the classes $\{\varnothing\}$ and $\{\mathcal{N}\}$ (depending on the label 0 or 1 of the root). In the typical application, given class descriptions $\Gamma_0, \Gamma_1, \dots$, we construct a more complicated class description Γ by combining the trees T_{Γ_i} to a single tree T_Γ by adding a root below them all and specifying the ordinals ξ and η for the new root. A Γ-name N will consist of a list $N_0, N_1, \dots$ with N_i a Γ_i-name, and a function $f = f_{\langle\rangle}$ which for each real x tells us which name N_i we should apply to x, in order to determine whether x is in the set named by N. That is, if A_i is the set named by N_i, then the set named by N is $\{x \in \mathcal{N} : x \in A_{f(x)}\}$.

There are a couple of comments we should make now. The first is that even though Γ-names can be described "statically", using $D_{\eta_s}(\mathbf{\Sigma}^0_{1+\xi_s})$ sets rather than functions f_s, the way we will reason about these classes will be *effective*. Thus, it will be important to view the functions f_s as the results of approximation procedures using true ξ_s-initial segments of a real x, as in our analysis of the classes $D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ in [DGHTT]. Combining these approximation procedures, a Γ-name provides a "dynamic" approximation procedure for membership in the set it names.

Further, the effective nature of our analysis, and our reliance on the true stage machinery, mean that we need to say what it means for an oracle y to compute a description of a class, and we need to choose specific computable copies of computable ordinals. For this reason, by "ordinals" we will mean well-orderings of subsets of $\mathbb{N}$, and we will specify what it means to compute such orderings.

The second point is that the nature of our approximation procedures indicates that the objects being approximated are *functions* $F\colon \mathcal{N} \to \{0,1\}$, the characteristic functions of Borel sets. Our class descriptions can therefore be generalised to discuss Wadge classes of functions with ranges other than $\{0,1\}$, as was done by Selivanov and Kihara-Montalbán. Our classification of these classes, though, is restricted to $\{0,1\}$-valued functions.

Throughout our classification, we will be decomposing functions into simpler pieces. Proposition 4.13 is illustrative – here we begin with an F belonging to a described class $\Delta(\mathbf{\Gamma})$, and we find $Y \subseteq \mathcal{N}$ such that $F \restriction Y$ and $F \restriction \mathcal{N}\setminus Y$ are from the simpler classes $\Delta(\mathbf{\Gamma}_0)$ and $\mathbf{\Gamma}^*$, respectively (the particulars of what this means are not important now; the reader should just accept that these are simpler classes). In the past, authors usually extended partial functions such as $F \restriction Y$ to total functions by using retraction maps. While these are not effective, there are other, effective methods of extending partial functions to total functions. Thus, Proposition 4.13 could be stated as $F \restriction \mathcal{N}\setminus Y$ being the restriction of some function from $\mathbf{\Gamma}^*$. This, however, masks the fact that we are really only interested in the partial functions $F \restriction Y$ and $F \restriction \mathcal{N}\setminus Y$. Our arguments indicate that the more natural approach is to say that each description describes a class of *partial* functions from $\mathcal{N}$ to $\{0,1\}$. Then we can simply write $F \restriction \mathcal{N}\setminus Y \in \mathbf{\Gamma}^*$.

In particular, at each level ξ, we will need to consider the restrictions of functions to both $\mathbf{\Sigma}^0_{1+\xi}$ and $\mathbf{\Pi}^0_{1+\xi}$ subsets of their domains. To do this, we will work with functions defined on subsets of $\mathcal{N}$ in the second level of the Hausdorff hierarchy starting with each $\mathbf{\Sigma}^0_{1+\xi}$: the intersections of $\mathbf{\Sigma}^0_{1+\xi}$ and $\mathbf{\Pi}^0_{1+\xi}$ sets. To be able to properly discuss (and compute) names for such functions, we will represent their domains using the true stage relations, via the notion of (z,α)-forests.

In this section we introduce the tools required for our definitions. We discuss computable ordinals. We detail the properties of the true stage machinery that we will use; but we relegate the development and verification of these properties to an appendix-like Section 7. We then introduce (z,α)-forests and approximations of functions on such forests, and state a corresponding "limit lemma". We define our class descriptions in Section 3.

2.2. **Computable ordinals.** Henceforth, by an *ordinal* we mean a well-ordering of a subset of $\mathbb{N}$. We will nonetheless use set-theoretic terminology, and for example, talk about limit ordinals or successor ordinals; we will write 0 to denote the empty ordinal. When we want to refer to the von-Neumann (set-theoretic) ordinal isomorphic to a well-ordering α, we write $\mathrm{otp}(\alpha)$.

We say that an oracle z computes an ordinal α if z computes the field of α, the ordering relation, the successor function on α, and the set of limit points of α.

We also require z to convey the information of whether α is 0, a successor, or a limit; if a successor, what its greatest element is; if nonzero, what its initial element is. This means that when we say that z uniformly computes a sequence $\langle \alpha_n \rangle$ of ordinals, that $\{n \in \mathbb{N} : \alpha_n \text{ is limit}\}$ is z-computable, and similarly for the other properties. Note that this information allows us to tell whether α is finite or not, and if so, what its size is. If α is a limit ordinal, then it allows us to (uniformly) compute an increasing and cofinal ω-sequence in α.

For two ordinals α and β, we write $\alpha < \beta$ if α is a proper initial segment of β (this is a much stricter requirement than $\mathrm{otp}(\alpha) < \mathrm{otp}(\beta)$). Continuing set-theoretic notation, when $\alpha < \beta$, we often use α to denote n_α, the element of the

field of β which bounds α. For example, If $f\colon \mathbb{N} \to \beta$ is a function, we will often write $f(k) = \alpha$ instead of $f(k) = n_\alpha$.

If an oracle z computes β then it computes all $\alpha < \beta$, and uniformly so, in the sense that given n_α in the field of β, z computes all the required information about α.

2.3. **True stages.** For each computable ordinal α, the true stage machinery provides a binary relation $\preccurlyeq_\alpha$ on $\omega^{\leqslant\omega}$, the collection of all finite and infinite sequences of natural numbers (which we call *strings* or *sequences*). The relation $\sigma \preccurlyeq_\alpha \tau$ reads "σ appears α-true to τ." Intuitively, each finite σ makes a guess about $\Sigma^0_{1+\alpha}$ facts about infinite strings extending σ. For $x \in \mathcal{N}$, $\sigma \prec_\alpha x$ will mean that σ's guesses are correct about x. If τ is finite, then $\sigma \preccurlyeq_\alpha \tau$ will mean that τ agrees with all of σ's guesses. This will not be precisely true, but it serves as a useful intuition for this machinery. As mentioned, the details are given in Section 7.

We list some properties of the true stage relations $\preccurlyeq_\alpha$ that we will use.[1] Fix a computable ordinal α.

TSP(1): The relations are nested: If $\alpha \leqslant \beta$ then $\sigma \preccurlyeq_\beta \tau$ implies $\sigma \preccurlyeq_\alpha \tau$. We have $\sigma \preccurlyeq_0 \tau \iff \sigma \preccurlyeq \tau$. Hence, for all α, $\sigma \preccurlyeq_\alpha \tau$ implies $\sigma \preccurlyeq \tau$.

TSP(2): The relation $\preccurlyeq_\alpha$ is a partial ordering of $\omega^{\leqslant\omega}$ which is a tree (for all $\tau \in \omega^{\leqslant\omega}$, $\{\sigma : \sigma \preccurlyeq_\alpha \tau\}$ is linearly ordered). The root of the tree is $\langle\rangle$ (the empty string).

TSP(3): For every $x \in \mathcal{N}$,

$$\{\sigma \in \omega^{<\omega} : \sigma \prec_\alpha x\}$$

is the unique infinite path of the restriction of $\preccurlyeq_\alpha$ to $\{\sigma \in \omega^{<\omega} : \sigma \prec x\}$.

TSP(4): The restriction of the relation $\preccurlyeq_\alpha$ to pairs of finite strings is computable.

TSP(5): If λ is a limit, then for all $\sigma, \tau \in \omega^{\leqslant\omega}$,

$$\sigma \preccurlyeq_\lambda \tau \iff (\forall \alpha < \lambda)\, \sigma \preccurlyeq_\alpha \tau.$$

Furthermore, there is a computable and increasing sequence $\langle \lambda_k \rangle$, cofinal in λ, such that for all $\sigma \in \omega^{<\omega}$, if $k = |\sigma|_\lambda$ is the height of σ in the tree $(\omega^{<\omega}, \preccurlyeq_\lambda)$, then $|\sigma|_{\lambda_k} = k$, and for all $\tau \in \omega^{\leqslant\omega}$,

$$\sigma \preccurlyeq_\lambda \tau \iff \sigma \preccurlyeq_{\lambda_k} \tau.$$

For TSP(5), note that in light of TSP(1), the condition $|\sigma|_\lambda = |\sigma|_{\lambda_k}$ says that for all $\rho \preccurlyeq \sigma$, $\rho \preccurlyeq_\lambda \sigma \iff \rho \preccurlyeq_{\lambda_k} \sigma$.

The next property deals with successor ordinals. For an ordinal α, we write $\alpha+1$ to denote any ordinal β such that $\alpha < \beta$ and $\mathrm{otp}(\beta) = \mathrm{otp}(\alpha) + 1$.

TSP(6): There is a computable function $p_{\alpha+1}\colon \omega^{<\omega} \to \mathbb{N} \cup \{-1\}$ such that for all $\sigma, \tau \in \omega^{\leqslant\omega}$, $\sigma \preccurlyeq_{\alpha+1} \tau$ if and only if $\sigma \preccurlyeq_\alpha \tau$, and for all finite ρ with $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$, $p_{\alpha+1}(\rho) \geqslant p_{\alpha+1}(\sigma)$.

We note that TSP(6) implies the following:

(♣) For all $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau \in \omega^{\leqslant\omega}$, if $\sigma \preccurlyeq_{\alpha+1} \tau$ then $\sigma \preccurlyeq_{\alpha+1} \rho$.

The connection with the (effective) Borel hierarchy is the following:

TSP(7): A set $A \subseteq \mathcal{N}$ is $\Sigma^0_{1+\alpha}$ if and only if there is a c.e. set $U \subseteq \omega^{<\omega}$ such that

$$A = \{x \in \mathcal{N} : (\exists \sigma \prec_\alpha x)\ \sigma \in U\}.$$

[1] The numbering of the properties differs from that in [DGHTT].

Remark 2.1. We will not repeatedly state this, but every computable procedure in this paper is uniform, in every conceivable way. This means that in each instance, there is a single computable operator which takes the input and produces the desired output. In particular, for example, if $\langle \alpha_n \rangle$ is a uniformly computable sequence of ordinals, then the relations $\preccurlyeq_{\alpha_n}$ are uniformly computable. Similarly, for each α (uniformly in α) there are computable functions translating between c.e. indices of sets $U \subseteq \omega^{<\omega}$ and $\Sigma^0_{1+\alpha}$-indices of the related subsets of $\mathcal{N}$ as in TSP(7).

Relativised true stages. The machinery above was stated for computable ordinals α, to keep notation simple. In fact, we can relativise this machinery to any oracle. For any ordinal α, and for any oracle z that computes α, we obtain a relation $\preccurlyeq^z_\alpha$, which satisfies all of the properties above, with the following changes: "computable" (in TSP(4), TSP(5) and TSP(6)) is replaced by "z-computable"; and in TSP(7), "c.e." is replaced by "z-c.e." and $\Sigma^0_{1+\alpha}$ is replaced by $\Sigma^0_{1+\alpha}(z)$. Note that even when α is computable, the relation $\preccurlyeq^z_\alpha$ is not the same as $\preccurlyeq_\alpha$, though we can think of $\preccurlyeq_\alpha$ as $\preccurlyeq^\varnothing_\alpha$.

As expected, this relativisation process is uniform. Suppose that α is a computable ordinal. Then the relations $\preccurlyeq^z_\alpha$ on pairs of finite strings are z-computable, uniformly in z. In fact, they naturally turn out to be Lipschitz-computable (computable with constant use): there is a ternary computable relation $\sigma \preccurlyeq^\rho_\alpha \tau$, defined when $|\rho| \geqslant |\tau|$, such that for all z, $\sigma \preccurlyeq^z_\alpha \tau \iff \sigma \preccurlyeq^{z\restriction|\tau|}_\alpha \tau$.

2.4. **α-forests.** Fix a computable ordinal α.

Definition 2.2. Let $S \subseteq \omega^{<\omega}$.

(a) A set $R \subseteq S$ is *α-open in S* if for all $\sigma \in R$ and $\tau \in S$, if $\sigma \preccurlyeq_\alpha \tau$ then $\tau \in R$.
(b) A set $R \subseteq S$ is *α-closed in S* if for all $\sigma \in R$ and $\tau \in S$, if $\tau \preccurlyeq_\alpha \sigma$ then $\tau \in R$.

TSP(7) can be modified as follows.

Lemma 2.3. Let $A \subseteq \mathcal{N}$.

(a) A is $\Sigma^0_{1+\alpha}$ if and only if there is a computable $W \subseteq \omega^{<\omega}$, α-open in $\omega^{<\omega}$, such that
$$A = \{x \in \mathcal{N} : (\exists \sigma \prec_\alpha x)\ \sigma \in W\} = \{x \in \mathcal{N} : (\forall^\infty \sigma \prec_\alpha x)\ \sigma \in W\}.$$
(b) A is $\Pi^0_{1+\alpha}$ if and only if there is a computable $T \subseteq \omega^{<\omega}$, α-closed in $\omega^{<\omega}$, such that
$$A = \{x \in \mathcal{N} : (\forall \sigma \prec_\alpha x)\ \sigma \in T\}.$$

Proof. (a): this is a standard trick. If $\langle U_s \rangle$ is a computable enumeration of the set U given by TSP(7), then we let $\sigma \in W$ if there is some $\rho \preccurlyeq_\alpha \sigma$ in $U_{|\sigma|_\alpha}$, where recall that $|\sigma|_\alpha$ is the height of σ in the tree $(\omega^{<\omega}, \preccurlyeq_\alpha)$. (b) is obtained from (a) by taking the complement. □

As mentioned above, we need to generalise these two notions.

Definition 2.4. An *α-forest* is a subset S of $\omega^{<\omega}$ which is $\preccurlyeq_\alpha$-convex: if $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$ and $\sigma, \tau \in S$ then $\rho \in S$.

For an α-forest S we let
$$[S]_\alpha = \{x \in \mathcal{N} : (\exists^\infty \sigma \prec_\alpha x)\ \sigma \in S\} = \{x \in \mathcal{N} : (\forall^\infty \sigma \prec_\alpha x)\ \sigma \in S\}.$$

We can visualise an α-forest as the disjoint union of trees: the collection of $\preccurlyeq_\alpha$-minimal elements of S forms an antichain in $(\omega^{<\omega}, \preccurlyeq_\alpha)$, and for each σ in that antichain, the restriction of $(S, \preccurlyeq_\alpha)$ to strings $\tau \succcurlyeq_\alpha \sigma$ is an α-tree with root σ.

Every α-open subset W of $\omega^{<\omega}$ is an α-forest, as is every α-closed set T; and $[W]_\alpha$ and $[T]_\alpha$ coincide with the sets described in Lemma 2.3. Further:

Lemma 2.5. Let S be an α-forest. If $R \subseteq S$ is either α-open in S or α-closed in S, then R is an α-forest.

For the following lemma, let $\Sigma^0_\beta \wedge \Pi^0_\beta$ denote the class of sets which are intersections $A \cap B$ where A is Σ^0_β and B is Π^0_β (this is the class $D_2(\Sigma^0_\beta)$ in the effective Hausdorff hierarchy).

Lemma 2.6. A set $A \subseteq \mathcal{N}$ is $\Sigma^0_{1+\alpha} \wedge \Pi^0_{1+\alpha}$ if and only if there is a computable α-forest S such that $A = [S]_\alpha$.

Proof. Suppose that A is $\Sigma^0_{1+\alpha} \wedge \Pi^0_{1+\alpha}$. By Lemma 2.3, let $U, T \subseteq \omega^{<\omega}$ be computable with U α-open and T α-closed such that $A = [U]_\alpha \cap [T]_\alpha$. Let $S = U \cap T$; then S is an α-forest and $A = [S]_\alpha$ (note that the intersection of any number of α-forests is an α-forest, though the class $\Sigma^0_{1+\alpha} \wedge \Pi^0_{1+\alpha}$ is not closed under taking infinite intersections, even if effective).

In the other direction, suppose that S is a computable α-forest. Let W be the upwards closure of S in $\preccurlyeq_\alpha$; let $V = W \backslash S$. Then both W and V are computable and α-open, and $A = [U]_\alpha \backslash [V]_\alpha$, so is $\Sigma^0_{1+\alpha} \wedge \Pi^0_{1+\alpha}$. □

Suppose that X is $\Sigma^0_\beta \wedge \Pi^0_\beta$. A set $Y \subseteq X$ is Σ^0_β *within* X if $Y = X \cap A$ for some Σ^0_β set; similarly define Π^0_β within X. Both sets which are Σ^0_β within X or Π^0_β within X are $\Sigma^0_\beta \wedge \Pi^0_\beta$. A set $Y \subseteq X$ is $\Sigma^0_\beta \wedge \Pi^0_\beta$ within X if and only if it is $\Sigma^0_\beta \wedge \Pi^0_\beta$.

Lemma 2.7. Let S be a computable α-forest.

(a) A set $Y \subseteq [S]_\alpha$ is $\Sigma^0_{1+\alpha}$ within $[S]_\alpha$ if and only if $Y = [W]_\alpha$ for some computable $W \subseteq S$ which is α-open in S.
(b) A set $Y \subseteq [S]_\alpha$ is $\Pi^0_{1+\alpha}$ within $[S]_\alpha$ if and only if $Y = [T]_\alpha$ for some computable $T \subseteq S$ which is α-closed in S.
(c) A set $Y \subseteq [S]_\alpha$ is $\Sigma^0_{1+\alpha} \wedge \Pi^0_{1+\alpha}$ if and only if $Y = [R]_\alpha$ for some computable α-forest $R \subseteq S$.

Remark 2.8. TSP(1) implies that if $\alpha < \beta$ and S is an α-forest, then S is also a β-forest, and $[S]_\alpha = [S]_\beta$. Thus, when α is clear, we will just write $[S]$.

Remark 2.9. As above, all notions can be relativised to an oracle z, giving the notion of a (z, α)-forest. We write $[S]^z_\alpha$ for the set of "infinite paths" through S.

Orthogonal forests.

Definition 2.10. Two sets $A, B \subseteq \omega^{<\omega}$ are *α-orthogonal* if every $\sigma \in A$ and $\tau \in B$ are $\preccurlyeq_\alpha$-incomparable.

If A and B are α-orthogonal α-forests then $[A]_\alpha$ and $[B]_\alpha$ are disjoint, indeed they are contained in disjoint $\Sigma^0_{1+\alpha}$ sets (take the $\preccurlyeq_\alpha$-upward closures of A and B). On the other hand it is possible that U and W are α-open and not orthogonal, and so not disjoint, while $[U]_\alpha$ and $[W]_\alpha$ are disjoint. The reason is that when $\alpha > 0$, there are $\sigma \in \omega^{<\omega}$ such that $[\sigma]_\alpha$ is empty, that is, the tree $(\omega^{<\omega}, \preccurlyeq_\alpha)$ is

well-founded above σ. The set of such σ is Π^1_1-complete, in particular, we cannot identify such σ computably and exclude them from α-forests. However, if we know that $[U]_\alpha \cap [W]_\alpha = \varnothing$ but $\sigma \in U \cap W$, this gives us a proof that $[\sigma]_\alpha = \varnothing$, and so we can ignore such σ.

Lemma 2.11. Let S be an α-forest. Suppose that $Z_0, Z_1, \dots$ are pairwise disjoint sets which are uniformly $\Sigma^0_{1+\alpha}$ in $[S]_\alpha$. Then there are pairwise orthogonal, uniformly computable α-forests $W_0, W_1, \dots$, each α-open in S, such that $Z_i = [W_i]_\alpha$ for all i, and such that $\bigcup_i W_i$ also is computable.

Proof. There are uniformly computable U_i, α-open in S, with $Z_i = [U_i]_\alpha$. To make them pairwise orthogonal, we may first assume that $|\sigma|_\alpha > i$ for all $\sigma \in U_i$. Then define $W_i \subseteq U_i$ as follows. For each σ and i, we determine if $\sigma \in W_i$ by induction on $|\sigma|_\alpha$; we declare that $\sigma \in W_i$ if $\sigma \in U_i$, and there is no $\rho \precsim_\alpha \sigma$ in W_j for any $j \neq i$ with $j < |\sigma|_\alpha$. □

2.5. **Approximations and translations.**

α-approximations. Some of the following was discussed in [DGHTT]. We include it in this paper for completeness. The main difference is that we work with α-forests rather than all of $\omega^{<\omega}$.

Definition 2.12. Let S be an α-forest. An *α-approximation* of a function $F\colon [S] \to \mathbb{N}$ is a function $f\colon S \to \mathbb{N}$ such that for all $x \in [S]$, for all but finitely many $\sigma \prec_\alpha x$, $f(\sigma) = F(x)$.

Remark 2.13. Continuing Remark 2.8, if $\alpha < \beta$, S is an α-forest and $f\colon S \to \mathbb{N}$ is an α-approximation of some function on $[S]$, then f is also a β-approximation of the same function on $[S]$.

Let X be $\Sigma^0_\beta \wedge \Pi^0_\beta$. A function $F\colon X \to \mathbb{N}$ is *Σ^0_β-measurable* if the sets $F^{-1}\{n\}$ for $n \in \mathbb{N}$ are uniformly Σ^0_β within X.

Proposition 2.14. *Let S be an α-forest. A function $F\colon [S] \to \mathbb{N}$ is $\Sigma^0_{1+\alpha+1}$-measurable if and only if it has a computable α-approximation $f\colon S \to \mathbb{N}$.*

We prove this proposition, and the rest of the results of this section, in Section 7.

Definition 2.15. Let S be an α-forest and let $f\colon S \to \mathbb{N}$ be an α-approximation of a function F. An *ordinal witness for the convergence of f* is a function $\sigma \mapsto \beta(\sigma)$ from S to some ordinal γ satisfying:

(i) For all $\sigma \preccurlyeq_\alpha \tau$ from S, $\beta(\tau) \leqslant \beta(\sigma)$;
(ii) If $\sigma \preccurlyeq_\alpha \tau$ are from S and $f(\sigma) \neq f(\tau)$ then $\beta(\tau) < \beta(\sigma)$.

Thus, the well-foundedness of γ "proves" that for all $x \in [S]$, the sequence $\langle f(\sigma)\rangle_{\sigma \prec_\alpha x}$ stabilises to the value $F(x)$.

Lemma 2.16. For every computable α-approximation $f\colon S \to \mathbb{N}$ there is a computable ordinal γ and a computable witness $\beta\colon S \to \gamma$ for the convergence of f.

The proof of Lemma 2.16 will use the following, which we will also need on its own.

Lemma 2.17. If T is a well-founded tree, then there is a T-computable ordinal η and a T-computable rank function $r\colon T \to \eta$.

Translating between oracles, ordinals, and spaces. We will use four "translation propositions". For the first, let α be an ordinal and suppose that an oracle z computes α. Every $w \geqslant_{\mathrm{T}} z$ also computes α, and every $\Sigma^0_{1+\alpha}(z)$ set is also $\Sigma^0_{1+\alpha}(w)$, uniformly in the indices by a w-computable function (as usual for functions on indices, it can be made computable rather than just w-computable, but that is not important). This can be translated to true stage relations.

Proposition 2.18. *Suppose that $w \geqslant_{\mathrm{T}} z$ and that α is a z-computable ordinal. Then there is a w-computable function $h\colon \omega^{<\omega} \to \omega^{<\omega}$ such that:*

(i) if $\sigma \preccurlyeq^w_\alpha \tau$ then $h(\sigma) \preccurlyeq^z_\alpha h(\tau)$;
(ii) for all $x \in \mathcal{N}$, $\{h(\sigma) : \sigma \prec^w_\alpha x\} = \{\rho : \rho \prec^z_\alpha x\}$.

We can also pull back true stage relations by computable functions.

Proposition 2.19. *Let α be a computable ordinal; let $\Phi\colon \mathcal{N} \to \mathcal{N}$ be a computable function. Then there is a computable function $h\colon \omega^{<\omega} \to \omega^{<\omega}$ such that:*

(i) if $\sigma \preccurlyeq_\alpha \tau$ then $h(\sigma) \preccurlyeq_\alpha h(\tau)$;
(ii) for all $x \in \mathcal{N}$, $\{h(\sigma) : \sigma \prec_\alpha x\} = \{\rho : \rho \prec_\alpha \Phi(x)\}$.

Computably isomorphic ordinals give rise to equivalent true stage relations.

Proposition 2.20. *Suppose that α and β are two computably isomorphic ordinals. Then there is a computable function $h\colon \omega^{<\omega} \to \omega^{<\omega}$ such that:*

(i) if $\sigma \preccurlyeq_\beta \tau$ then $h(\sigma) \preccurlyeq_\alpha h(\tau)$;
(ii) for all $x \in \mathcal{N}$, $\{h(\sigma) : \sigma \prec_\beta x\} = \{\rho : \rho \prec_\alpha x\}$.

We can "unpair" true stage relations. Suppose that $(z, x) \mapsto \langle z, x\rangle$ is a computable bijection between $\mathcal{N}^2$ and $\mathcal{N}$ (a computable "pairing function" for Baire space). Let $\sigma \mapsto ((\sigma)_0, (\sigma)_1)$ be a corresponding computable "unpairing" of finite strings (so for all $(x_0, x_1) \in \mathcal{N}^2$, $x_i = \bigcup_{\sigma \prec \langle x_0, x_1\rangle} (\sigma)_i$). Recall that for a computable ordinal α, the relation $\sigma \preccurlyeq^\rho_\alpha \tau$ is defined when $|\rho| \geqslant |\tau|$.

Proposition 2.21. *For any computable ordinal α, there is a computable function $h\colon \omega^{<\omega} \to \omega^{<\omega}$ such that:*

(i) For all σ, $|h(\sigma)| \leqslant |(\sigma)_0|$;
(ii) if $\sigma \preccurlyeq_\alpha \tau$ then $h(\sigma) \preccurlyeq^{(\tau)_0}_\alpha h(\tau) \preccurlyeq_0 (\tau)_1$;
(iii) for all $(z, x) \in \mathcal{N}^2$, $\{h(\sigma) : \sigma \prec_\alpha \langle z, x\rangle\} = \{\rho : \rho \prec^z_\alpha x\}$.

Propositions 2.19 to 2.21 relativise to any oracle.

3. Described classes

We introduce our class descriptions. We will start with very general descriptions. These are sufficiently general to include all classes described by Wadge [Wad84], Louveau [Lou83] and by Louveau and Saint-Raymond [LSR88b]. We will then consider a restricted collection of descriptions, the *acceptable* ones. In Section 4 we will be able to classify all acceptable classes in terms of an analysis of their ambiguous class.

3.1. **General class descriptions.** As in [Lou83] and [LSR88b], we think of classes as defined recursively. The simplest classes contain the constant partial functions. Suppose that $\{\Upsilon_i : i \in I\}$ is a finite or countable collection of class descriptions. Among these, we choose a special "initial" class Υ_{i*}. We then combine these classes to a more complicated class Γ by specifying two ordinals ξ and η. The ordinal ξ (which we will also denote by $o(\Gamma)$) denotes the *level* of Γ: Γ-functions will be defined on $\mathbf{\Sigma}^0_{1+\xi} \wedge \mathbf{\Pi}^0_{1+\xi}$ sets. A Γ-function F on such a set $X = [S]_\xi$ will be defined by choosing a function $f : X \to I$ and a function F_i on X in the class Υ_i for each $i \in I$. We will then set $F(x) = F_{f(x)}(x)$. Thus f's role is to decide, for each $x \in X$, which F_i to apply to x. The function f will be approximated on S in the sense of Definition 2.12. The ordinal $\eta + 1$ bounds the "mind-changes" permitted along this approximation, in the sense of Definition 2.15. An approximation of the function f is supplied with a witness $\beta\colon S \to \eta + 1$. We require that the chosen class Υ_{i*} is the initial value, meaning: if $\beta(\sigma) = \eta$ then $f(\sigma) = i^*$. This requirement to have an initial choice is what makes the class Γ non-self-dual.

Definition 3.1. A *class description* consists of an oracle y, and a labelled tree Γ satisfying:

(i) The domain T_Γ of Γ is a well-founded subtree of $\omega^{<\omega}$;
(ii) If $s \in T_\Gamma$ is not a leaf, then $\Gamma(s)$ is a pair of ordinals (ξ, η), with η nonzero, which we denote by $(\xi_s, \eta_s) = (\xi^\Gamma_s, \eta^\Gamma_s)$;
(iii) If s is a leaf of T_Γ then $\Gamma(s) \in \{0, 1\}$;
(iv) If $s, t \in T_\Gamma$ are not leaves and $s \preccurlyeq t$ then $\xi_s \leqslant \xi_t$;
(v) y computes Γ.

We use Γ to denote the pair (Γ, y). If T_Γ consists only of the root $\langle\rangle$, we declare $o(\Gamma) = \omega_1$. Otherwise, we set $o(\Gamma) = \xi_{\langle\rangle}$. We write η^Γ for $\eta^\Gamma_{\langle\rangle}$. We write y^Γ for y.

Here computing Γ means computing T_Γ, the set of leaves of T_Γ, the function $s \mapsto \Gamma(s)$ for leaves s of T_Γ, and the ordinals ξ_s and η_s, uniformly in non-leaf $s \in T_\Gamma$. There is a Turing-least oracle computing Γ, but at times we will want to choose y^Γ which is strictly Turing above that real. This will restrict the class of oracles that are allowed to compute a name of a Γ-function.

As our ordinals are well-orderings of $\mathbb{N}$ (and thus countable), the definition $o(\Gamma) = \omega_1$ when the tree is trivial requires some discussion. We consider ω_1 as a symbol and declare that $\alpha < \omega_1$ for every ordinal α. Since ω_1 is a limit, if, hypothetically, TSP(5) were to extend to ω_1, it would imply (via a regularity argument) that any ω_1-forest is an α-forest for some countable α (and the converse would hold by TSP(1)). So we will take this as our definition: for any oracle z, S is a (z, ω_1)-forest if it is a (z, α)-forest for some z-computable ordinal α.

Definition 3.2. Suppose that Γ is a class description. A Γ-*name* consists of an oracle $z \geqslant_{\mathrm{T}} y^\Gamma$, a $(z, o(\Gamma))$-forest S, and a z-computable labelled tree N, whose domain T_N is the collection of $s \in T_\Gamma$ which are not leaves of T_Γ, and such that for any $s \in T_N$, $N(s)$ is a pair $(f_s, \beta_s) = (f^N_s, \beta^N_s)$, satisfying:

(i) f_s is a (z, ξ_s)-approximation on S (where $\xi_s = \xi^\Gamma_s$) of a function on $[S]$, which we also denote by f_s;
(ii) $\beta_s\colon S \to \eta_s + 1$ is a witness for the convergence of f_s (where $\eta_s = \eta^\Gamma_s$);
(iii) For all $\sigma \in S$, $f_s(\sigma)$ is a child of s on T_Γ;
(iv) For all $\sigma \in S$, if $\beta_s(\sigma) = \eta_s$ then $f_s(\sigma)$ is the leftmost child of s on T_Γ.

We use N to denote the name. We write z^N for z and S^N for S; we write f^N for $f^N_{\langle\rangle}$ and β^N for $\beta^N_{\langle\rangle}$.

As with class description, N being z-computable requires that S be z-computable, and that $f_s \restriction S$ and β_s are z-computable functions, uniformly in $s \in T_N$. Note that since $z \geqslant_{\mathrm{T}} y^\Gamma$, the ordinals ξ_s for $s \in T_N$ are uniformly z-computable, so TSP(4) implies that the relations $\preccurlyeq^z_{\xi_s}$ are uniformly z-computable. The fact that $\xi_s \geqslant o(\Gamma)$ for all $s \in T_N$ implies that $S = S^N$ is a (z, ξ_s)-forest (Remark 2.8), so the notion of a (z, ξ_s)-approximation on S makes sense.

As we don't use the ordinals η_s for true stage relations, only as bounds, we do not require Γ to choose a particular version of $\eta_s + 1$. Rather, we can think of a function into $\eta_s + 1$ as a function into η_s which also takes an extra special value "∞", which we also denote by η_s.

Definition 3.3. Suppose that Γ is a class description, N is a Γ-name, and $w \in S^N \cup [S^N]$. We define a leaf $\ell(w) = \ell^N(w)$ of T_Γ by recursion as follows: $\langle\rangle$ is a predecessor of $\ell(w)$; if $s \in T_N$ has been decided to be a predecessor of $\ell(w)$, then we declare that $f_s(w)$ is also a predecessor of $\ell(w)$. (Recall that for $w \in S^N$, $f_s(w)$ refers to the approximation function, and for $w \in [S^N]$, $f_s(w)$ refers to the function being approximated.)

We define $F^N(w) = \Gamma(\ell(w))$. (Recall that for a leaf s, $\Gamma(s) \in \{0, 1\}$).

When we refer to the function F^N we mean $F^N \restriction [S^N]$ (that is, the function on infinite sequences, excluding the finite ones), unless we mention otherwise.

Note that if T_Γ consists of only the root ($o(\Gamma) = \omega_1$), then T_N is empty. So a Γ-name is determined by only a z and an S; in this case F^N is the constant $\Gamma(\langle\rangle)$-value function on $[S]$.

Definition 3.4. Let Γ be a class description. For any $z \geqslant_{\mathrm{T}} y^\Gamma$, we let

$$\Gamma(z) = \left\{F^N \,:\, N \text{ is a } \Gamma\text{-name } \;\&\; z^N = z\right\}.$$

We let

$$\mathbf{\Gamma} = \bigcup \{\Gamma(z) \,:\, z \geqslant_{\mathrm{T}} y^\Gamma\}.$$

Informally, a $\Gamma(z)$-*name* is a Γ-name N with $z^N = z$. Proposition 2.18 implies:

Lemma 3.5. Let Γ be a class description. If $w \geqslant_{\mathrm{T}} z \geqslant_{\mathrm{T}} y^\Gamma$ then $\Gamma(z) \subseteq \Gamma(w)$. Indeed, we can w-effectively translate a $\Gamma(z)$-name N to a $\Gamma(w)$-name N' such that $F^N = F^{N'}$, uniformly in w and z.

Remark 3.6. The class $\mathbf{\Gamma}$ depends only on the order-types of the ordinals mentioned by Γ, not their realisation as well-orderings of a subset of $\mathbb{N}$. This follows from Proposition 2.20

Sub-descriptions. Suppose that P is a labelled tree and $s \in P$. The subtree P_s issuing from s is defined to be the labelled tree defined on $T_{P,s} = \{t \,:\, s\hat{\ }t \in T_P\}$ by setting $P_s(t) = P(s\hat{\ }t)$.

Lemma 3.7. Suppose that Γ is a class description and $s \in T_\Gamma$. Then:

(a) Γ_s (equipped with $y^{\Gamma_s} = y^\Gamma$) is a class description;
(b) if N is a Γ-name, then N_s is a Γ_s-name (we set $S^{N_s} = S^N$ and $z^{N_s} = z^N$).

The classes Γ_s are those that are used in the inductive construction of Γ from simpler classes. Note that consideration of sub-descriptions is why in Definition 3.1 we asked for $\xi_s \leqslant \xi_t$ whenever $s \preccurlyeq t$, rather than just $\xi_\diamondsuit \leqslant \xi_t$, which would have sufficed up until now.

The subclasses $\Gamma_{(n)}$ (for n such that $(n) \in T_\Gamma$) are the classes that were used in the last step of the construction of Γ; so $I = \{n : (n) \in T_\Gamma\}$, and $i^* = \min I$, to use the notation of the discussion at the beginning of this section. If N is a Γ-name, then for all $x \in [S] = [S^N]$, $F^N(x) = F^{N_{(n)}}(x)$, where $(n) = f^N(x)$. That is, the function $f^N = f^N_\diamondsuit$ is the function that chooses, for each $x \in [S]$, which of the functions $F^{N_{(n)}}$ to apply to x in order to calculate $F^N(x)$. For finite $\sigma \prec_{o(\Gamma)} x$ in S, $f^N(\sigma)$ is the "stage $|\sigma|_{o(\Gamma)}$-guess" of this choice $f^N(x)$.

Duals.

Definition 3.8. Let Γ be a class description. We let $\check{\Gamma}$ be the result of switching all the labels of the leaves of Γ (i.e., 0s become 1s and 1s become 0s).

If N is a Γ-name, then in the technical sense, it is also a $\check{\Gamma}$-name, but when we interpret it as such, we write $\check{N}$. Then $\operatorname{dom} F^N = \operatorname{dom} F^{\check{N}} = [S^N]$, and

$$F^{\check{N}} = 1 - F^N.$$

3.2. **Some examples.** We show how our class descriptions include a variety of classes from the literature. We use the following terminology.

Definition 3.9. Let Γ be a class description. A Γ-name N is *total* if $S^N = \omega^{<\omega}$. A function $F \in \mathbf{\Gamma}$ is total if $F = F^N$ for some total Γ-name N.

For the following examples, if $\mathbf{\Lambda}$ is a collection of subsets of $\mathcal{N}$, we say that $\mathbf{\Gamma} = \mathbf{\Lambda}$ if the total functions in $\mathbf{\Gamma}$ are the characteristic functions of the sets in $\mathbf{\Lambda}$.

a. Constants: Let T_Γ consist only of a root. Let $i \in \{0, 1\}$ be the label of the root. Then $\mathbf{\Gamma}$ is the collection of all constant functions with value i, defined on any Borel set. Using the notation just introduced, $\mathbf{\Gamma} = \{\varnothing\}$ when $i = 0$ and $\mathbf{\Gamma} = \{\mathcal{N}\}$ when $i = 1$.

b. $\mathbf{\Sigma}^0_\alpha$: let T_Γ consist of the root, and two children (0) and (1). Let the labels of the leaves be 0 and 1, respectively. Set $o(\Gamma)$ to be some ordinal ξ, and let $\eta^\Gamma = 1$. Then TSP(7) says that $\mathbf{\Gamma}$ consists of all functions $1_A \colon X \to \{0, 1\}$ where X is $\mathbf{\Sigma}^0_{1+\xi} \wedge \mathbf{\Pi}^0_{1+\xi}$, and $A \subseteq X$ is $\mathbf{\Sigma}^0_{1+\xi}$ within X. In terms of sets, $\mathbf{\Gamma} = \mathbf{\Sigma}^0_{1+\xi}$.

c. Hausdorff difference hierarchy: More generally, let T_Γ be as in the previous example, but allow $\eta = \eta^\Gamma$ to be larger than 1. Then $\mathbf{\Gamma} = D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ of the Hausdorff difference hierarchy. This is proved in [DGHTT, Prop.3.8]. The main takeaway is the dynamic characterisation of a set C in the class $D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ as one for which membership $x \in C$ is the result of a guessing process which starts at level η with "no", and then allows mind-changes as the ordinal decreases.

d. One-sided separated unions: Generalising the previous example, let Υ be a class description. Let Γ be the class description consisting of a root, and two children (0) and (1), such that the immediate sub-descriptions are $\Gamma_{(0)} = \check{\Upsilon}$ and $\Gamma_{(1)} = \Upsilon$. Choose $\xi = o(\Gamma) \leqslant o(\Upsilon)$ and any $\eta = \eta^\Gamma$. Then $\mathbf{\Gamma} = \operatorname{Sep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Upsilon})$ is the collection of all sets of the form $(C \cap A_0) \cup (C^\complement \cap A_1)$, where $C \in D_\eta(\mathbf{\Sigma}^0_{1+\xi})$, $A_0 \in \mathbf{\Upsilon}$ and $A_1 \in \check{\mathbf{\Upsilon}}$.

e. Two-sided separated unions: Let Υ and Λ be two class descriptions. Let Γ be the class description consisting of a root, and three children (0), (1), and (2), such that the immediate sub-descriptions are $\Gamma_{(0)} = \Lambda$, $\Gamma_{(1)} = \Upsilon$ and $\Gamma_{(2)} = \check{\Upsilon}$. Choose $\xi = o(\Gamma) \leqslant o(\Lambda), o(\Upsilon)$, and $\eta^\Gamma = 1$. Then $\mathbf{\Gamma} = \mathrm{BiSep}(\mathbf{\Sigma}^0_{1+\xi}, \mathbf{\Upsilon}, \mathbf{\Lambda})$ is the collection of all sets of the form $(C_1 \cap A_1) \cup (C_2 \cap A_2) \cup ((C_1 \cup C_2)^{\complement} \cap B)$, where C_1 and C_2 are disjoint $\mathbf{\Sigma}^0_{1+\xi}$ sets, $A_1 \in \mathbf{\Upsilon}$, $A_2 \in \check{\mathbf{\Upsilon}}$, and $B \in \mathbf{\Lambda}$. For a Γ-name N, the corresponding sets are $C_i = \{x : f^N(x) = i\}$ for $i = 1, 2$, and the characteristic functions of the sets A_i are $F^{N_{(i)}}$.

Below we extend this example to separated unions. More general two-sided separated unions, of the form $\mathrm{BiSep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Upsilon}, \mathbf{\Lambda})$, as well as more complicated separated unions and separated differences, can also be described, but their descriptions are a little more complicated; a direct construction is given in [GTb].

3.3. **Pointclasses.** Each described class is a pointclass.

Proposition 3.10. *Let Γ be a class description. Then $\mathbf{\Gamma}$ is closed under taking continuous pre-images.*

Proof. Follows from Proposition 2.19 (and Proposition 2.18). Let N be a total Γ-name; let $\Phi\colon \mathcal{N} \to \mathcal{N}$ be continuous. By Lemma 3.5, we may assume that Φ is z^N-computable. For each $s \in T_N$ let h_s be given by Proposition 2.19 for $\preccurlyeq^{z^N}_{\xi_s}$; define a Γ-name M by letting $f^M_s(\sigma) = f^N_s(h_s(\sigma))$ and $\beta^M_s(\sigma) = \beta^N_s(h_s(\sigma))$. Then $F^M = F^N \circ \Phi$. □

Next, we build a universal set. The following lemma implies that we can effectively list all (z, ξ)-approximations with convergence witnesses of the appropriate form. The key is the existence of the default value at the beginning of the approximation.

Lemma 3.11. Uniformly, given an oracle z, z-computable ordinals ξ and η, and an index for partial z-computable functions $f\colon \omega^{<\omega} \to \mathbb{N}$ and $\beta\colon \omega^{<\omega} \to \eta + 1$, and a "default value" $n^* \in \mathbb{N}$, we can compute an index for total z-computable functions $g\colon \omega^{<\omega} \to \mathbb{N}$ and $\gamma\colon \omega^{<\omega} \to \eta + 1$ satisfying:

(i) g is a (z, ξ)-approximation of a function $G\colon \mathcal{N} \to \mathbb{N}$;
(ii) γ is a witness for the convergence of g;
(iii) For all σ, if $\gamma(\sigma) = \eta$ then $g(\sigma) = n^*$;
(iv) If f and β are total, f is a (z, ξ)-approximation of a function $F\colon \mathcal{N} \to \mathbb{N}$, β is a witness for the convergence of f, and for all σ, if $\beta(\sigma) = \eta$ then $f(\sigma) = n^*$, then $G = F$.

Proof. This is fairly standard. We proceed recursively. Define $f(\langle\rangle) = n^*$ and $\gamma(\langle\rangle) = \eta$.

Given $\sigma \in \omega^{<\omega}$ with $\sigma \neq \langle\rangle$, let $k = |\sigma|^z_\xi$, and let τ be σ's $\preccurlyeq^z_\xi$-predecessor (i.e., $\tau \preccurlyeq^z_\xi \sigma$ and $|\tau|^z_\xi = k - 1$.) Let $\rho \preccurlyeq^z_\xi \sigma$ be longest with $f(\rho)$ and $\beta(\rho)$ both converging within k steps, and $\beta(\rho) < \gamma(\tau)$. If there is no such ρ, define $g(\sigma) = g(\tau)$ and $\gamma(\sigma) = \gamma(\tau)$. Otherwise, define $g(\sigma) = f(\sigma)$ and $\gamma(\sigma) = \beta(\sigma)$. □

As a result, for a class description Γ, we can list all total Γ-names. For simplicity, suppose that Γ is computable. Then there are, for each oracle z, a list $N^z_0, N^z_1, \ldots$ of total $\Gamma(z)$-names such that N^z_e is z-computable, uniformly in e and z, and such that $\Gamma(z) = \{F^{N^z_e} : e < \omega\}$. Using Proposition 2.18 for $w = e\hat{\ }z$ we get:

Lemma 3.12. Let Γ be a computable class description. There is, for each oracle z, a total $\Gamma(z)$-name N^z, uniformly z-computable, such that $\{F^{N^z} : z \in \mathcal{N}\}$ lists all the total functions in $\mathbf{\Gamma}$.

The join of these is a universal set for Γ. Fix a pairing function $\langle z, x\rangle$ as discussed before Proposition 2.21.

Lemma 3.13. Let Γ be a computable class description and let $\langle N^z\rangle$ be as in Lemma 3.12. Then $F(\langle z, x\rangle) = F^{N^z}(x)$ is in Γ.

Proof. For each non-leaf $s \in T_\Gamma$, let h_s be given by Proposition 2.21 for the ordinal ξ_s. By slowing approximations (as in the proof of Lemma 3.11), we may assume that for all s, σ and z, the values $f_s^z(\sigma) = f_s^{N^z}(\sigma)$ and $\beta_s^z(\sigma) = \beta_s^{N^z}(\sigma)$ can be determined by consulting only $z \upharpoonright |\sigma|$; we write $f_s^\rho(\sigma)$ and $\beta_s^\rho(\sigma)$ when $|\rho| \geqslant |\sigma|$. Now define a computable Γ-name M by letting $f_s^M(\sigma) = f_s^{(\sigma)_0}(h_s(\sigma))$ and $\beta_s^M(\sigma) = \beta_s^{(\sigma)_0}(h_s(\sigma))$; F^M is as required. □

Relativising to y^Γ we get:

Proposition 3.14. *For any class description Γ, there is a function in $\Gamma(y^\Gamma)$ which is universal for total $\mathbf{\Gamma}$ functions. Hence, $\mathbf{\Gamma}$ is a non-self-dual Wadge class.*

Proof. The relativisation of Lemmas 3.12 and 3.13 is that there is a total $\Gamma(\langle y^\Gamma, z\rangle)$-name $N^{\langle y^\Gamma, z\rangle}$, uniformly $\langle y^\Gamma, z\rangle$-computable, such that $\{F^{N^{\langle y^\Gamma, z\rangle}} : z \in \mathcal{N}\}$ lists all the total functions in $\mathbf{\Gamma}$, and further that $F(\langle z, x\rangle) = F^{N^{\langle y^\Gamma, z\rangle}}(x)$ is in $\mathbf{\Gamma}$. This is the universal function.

Now suppose towards a contradiction that $1 - F \in \mathbf{\Gamma}$. By Proposition 3.10, $x \mapsto 1 - F(\langle x, x\rangle) \in \mathbf{\Gamma}$, so fix z with $F^{N^{\langle y^\Gamma, z\rangle}}(x) = 1 - F(\langle x, x\rangle)$. Then

$$F(\langle z, z\rangle) = F^{N^{\langle y^\Gamma, z\rangle}}(z) = 1 - F(\langle z, z\rangle),$$

a contradiction. □

3.4. **Some closure properties.** Let Γ be a class description, and let $z \geqslant_{\mathrm{T}} y^\Gamma$.

Proposition 3.15. *If F is in $\Gamma(z)$ and $X \subseteq \operatorname{dom} F$ is $\Sigma^0_{1+o(\Gamma)}(z) \wedge \Pi^0_{1+o(\Gamma)}(x)$, then $F \upharpoonright X \in \Gamma(z)$.*

Proof. Let N be a $\Gamma(z)$-name of F. By Lemma 2.7(c), there is a $(z, o(\Gamma))$-forest $T \subseteq S^N$ with $[T] = X$. Define the required $\Gamma(z)$-name M by restricting each approximation (f_s^N, β_s^N) to $\sigma \in T$. □

Proposition 3.16. *Every $F \in \Gamma(z)$ can be extended to a total function in $\Gamma(z)$.*

Proof. Let N be a $\Gamma(z)$-name; let $S = S^N$. For $s \in T_\Gamma$ a non-leaf, extend $(f_s, \beta_s) = (f_s^N, \beta_s^N)$ to functions (f_s, β_s) on all of $\omega^{<\omega}$ as follows: for $\sigma \notin S$,

- If σ has no $\prec^z_{\xi_s}$-predecessor in S, let $f_s(\sigma)$ be the leftmost child of s on T_Γ, and let $\beta_s(\sigma) = \eta_s$.
- If σ has a $\prec^z_{\xi_s}$-predecessor in S, let τ be the longest such; set $f_s(\sigma) = f_s(\tau)$ and $\beta_s(\sigma) = \beta_s(\tau)$. □

Proposition 3.17. *Let X be $\Sigma^0_{1+o(\Gamma)}(z) \wedge \Pi^0_{1+o(\Gamma)}(x)$. Suppose that (X_n) is a partition of X into sets which are $\Sigma^0_{1+o(\Gamma)}(z)$ within X, uniformly in n. Let $F\colon X \to \{0, 1\}$ and suppose that for all n, $F \upharpoonright X_n \in \Gamma(z)$, uniformly in n. Then $F \in \Gamma(z)$.*

Proof. Let S be a z-computable $(z, o(\Gamma))$-forest such that $[S]^z_\alpha = X$. By Lemma 2.11, let (S_n) be a uniformly z-computable sequence of sets, $o(\Gamma)$-open in S, pairwise $(z, o(\Gamma))$-orthogonal, such that $X_n = [S_n]^z_\alpha$, and such that $\bigcup_n S_n$ is also z-computable.

Let N_n be uniformly z-computable $\Gamma(z)$-names such that $F \restriction X_n = F^{N_n}$. Define a $\Gamma(z)$-name N by taking the "disjoint union" of the N_n's: set $S^N = S$. For non-leaf $s \in T_\Gamma$ we define $(f_s, \beta_s) = (f^N_s, \beta^N_s)$ as follows: for $\sigma \in S$,

- If there is a $\rho \preccurlyeq^z_{o(\Gamma)} \sigma$ and an n with $\sigma \in S_n \cap S^{N_n}$, fix the longest such ρ and define $f_s(\sigma) = f^{N_n}_s(\rho)$ and $\beta_s(\sigma) = \beta^{N_n}_s(\rho)$.
- If there is no such ρ, define $f_s(\sigma)$ to be the leftmost child of s in T_Γ, and $\beta_s(\sigma) = \eta_s$. □

We state a corollary, that will be used in conjunction with Proposition 2.14. It implies that $\Gamma(z)$ is closed under taking approximations at levels lower than $o(\Gamma)$.

Corollary 3.18. *Let $\xi < o(\Gamma)$. Suppose that X is $\Sigma^0_{1+\xi}(z) \wedge \Pi^0_{1+\xi}(z)$, and that $g\colon X \to \mathbb{N}$ is $\Delta^0_{1+\xi+1}(z)$-measurable. Suppose that $F_n\colon X \to \{0,1\}$ are uniformly in $\Gamma(z)$. Define $G\colon X \to \{0,1\}$ by $G(x) = F_{g(x)}(x)$. Then $G \in \Gamma(z)$.*

Proof. For $n \in \mathbb{N}$ let $X_n = g^{-1}\{n\}$. Since $\xi + 1 \leqslant o(\Gamma)$, the sets X_n are uniformly $\Sigma^0_{1+o(\Gamma)}$ within X. By Proposition 3.15, the functions $G \restriction X_n = F_n \restriction X_n$ are uniformly in $\Gamma(z)$. By Proposition 3.17, G is in $\Gamma(z)$. □

The propositions above were stated for functions on reals. As usual, their proofs reveal uniformity in terms of manipulating names, and this uniformity will be used. For example, in Proposition 3.17, observe that from the sequence $\langle N_n \rangle$ of names for $F \restriction X_n$, we can z-computably construct the name N for F.

3.5. **Effective containment.**

Definition 3.19. Let Γ and Λ be two class descriptions; let N be a Γ-name and let M be a Λ-name. We say that N and M are *equivalent* (and write $N \equiv M$) if $z^N = z^M$, $S^N = S^M$, and $F^N = F^M$.

In that definition, we implicitly assume that $S = S^N = S^M$ is a (z, ξ)-forest for some $\xi \leqslant o(\Gamma), o(\Lambda)$.

Definition 3.20. Let Λ and Γ be class descriptions.

(a) We write
$$\Lambda \subseteq \Gamma$$
if $y^\Lambda \geqslant_{\mathrm{T}} y^\Gamma$, and $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}$, uniformly above y^Λ: there is a y^Λ-computable function which, given a total Λ-name N, outputs a total Γ-name M equivalent to N.

(b) We write
$$\Lambda \equiv \Gamma$$
if $\Lambda \subseteq \Gamma$ and $\Gamma \subseteq \Lambda$.

(c) We write
$$\Lambda < \Gamma$$
if $\Lambda \subseteq \Gamma$ and $\check{\Lambda} \subseteq \Gamma$.

Note that $\Lambda < \Gamma$ iff $\check{\Lambda} < \Gamma$ iff $\Lambda < \check{\Gamma}$.

Lemma 3.21. The relations $\subseteq$ and $<$ on class descriptions are transitive. The relation $\equiv$ is an equivalence relation.

The requirement in Definition 3.20 that the names be total comes from the fact that if $o(\Lambda) \neq o(\Gamma)$ then the partial Λ-functions and the Γ-functions are not defined on the same domains. However, this is not a serious restriction. If $\Lambda \subseteq \Gamma$ and $\xi \leqslant o(\Lambda), o(\Gamma)$ then every $F \in \mathbf{\Lambda}$ whose domain is $\mathbf{\Sigma}^0_{1+\xi} \wedge \mathbf{\Pi}^0_{1+\xi}$ is also in $\mathbf{\Gamma}$, effectively; this follows from Propositions 3.15 and 3.16.

In practice, we will always have $o(\Gamma)$ and $o(\Lambda)$ comparable (but both $o(\Lambda) > o(\Gamma)$ and $o(\Gamma) < o(\Lambda)$ will occur in nature). So we can take $\xi = \min\{o(\Lambda), o(\Gamma)\}$. However, incorporating this requirement into the definition would make the relation $\subseteq$ not transitive.

Definition 3.22. A sequence $\bar{\Theta} = \Theta_0, \Theta_1, \dots$ of class descriptions is *uniform* if $y^{\Theta_0}, y^{\Theta_1}, \dots$ is constant y, and the class descriptions are uniformly y-computable. We write $y^{\bar{\Theta}}$ for y.

Note that if Γ is a class description and for all n, $(n) \in T_\Gamma$, then the sequence $\Gamma_{(0)}, \Gamma_{(1)}, \dots$ is uniform.

Definition 3.23. Let Γ be a class description and let $\bar{\Theta} = \Theta_0, \Theta_1, \dots$ be a uniform sequence of class descriptions. We write

$$\bar{\Theta} \subseteq \Gamma$$

if $\Theta_n \subseteq \Gamma$ for all n, uniformly in n. Similarly, we write $\bar{\Theta} < \Gamma$ if $\Theta_n < \Gamma$, uniformly.

3.6. **Notation for constructed classes.**

Notation 3.24. If $\bar{\Theta}$ is a uniform sequence of class descriptions and ξ is an ordinal, then we write $\xi \leqslant o(\bar{\Theta})$ if for all n, $\xi \leqslant o(\Theta_n)$.

Note that we are not requiring that $o(\Theta_n)$ is the same for all n. Also note that if $\xi \leqslant o(\bar{\Theta})$ then as $o(\Theta_n)$ is $y^{\bar{\Theta}}$-computable, so is ξ.

Definition 3.25. Suppose that:

(i) $\bar{\Theta} = \Theta_0, \Theta_1, \dots$ is a uniform sequence of class descriptions;
(ii) ξ is an ordinal and $\xi \leqslant o(\bar{\Theta})$; and
(iii) η is a nonzero $y^{\bar{\Theta}}$-computable ordinal.

Then we let

$$\mathrm{SU}^\eta_\xi(\Theta_0, \Theta_1, \dots)$$

be the class description Γ determined by setting:

- $y^\Gamma = y^{\bar{\Theta}}$;
- $o(\Gamma) = \xi$;
- $\eta^\Gamma = \eta$; and
- for all n, $\Gamma_{(n)} = \Theta_n$.

If $\eta = 1$ then we write $\mathrm{SU}_\xi(\Theta_0, \Theta_1, \dots)$.

The notation is derived from the notion of *separated unions*. If $\mathbf{\Theta}_0 = \{\varnothing\}$ and $\eta = 1$, then the sets in $\mathbf{\Gamma}$ are the ones of the form $\bigcup_{n \geqslant 1}(C_n \cap A_n)$, where $A_n \in \mathbf{\Theta}_n$, and $C_1, C_2, \dots$ are pairwise disjoint $\mathbf{\Sigma}^0_{1+\xi}$ sets. In the more general case, we add $B \setminus \bigcup_n C_n$, where $B \in \mathbf{\Theta}_0$. When $\eta > 1$, under some restrictions, $\mathbf{\Gamma}$ is the class of separated unions based on $D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ sets; see [GTb]. We observe that if Γ is a class

description and for all n, $(n) \in T_\Gamma$, then $\Gamma = \mathrm{SU}^\eta_\xi(\Gamma_{(0)}, \Gamma_{(1)}, \dots)$, where $\xi = o(\Gamma)$ and $\eta = \eta^\Gamma$.

For the following lemma and below, we write $\neg\mathrm{SU}^\eta_\xi(\bar\Theta)$ for the dual of the class.

Lemma 3.26. Suppose that $\bar\Theta$ is a uniform sequence, $\xi \leqslant o(\bar\Theta)$, and $\eta > 0$ is $y^{\bar\Theta}$-computable.

(a) $\neg\mathrm{SU}^\eta_\xi(\bar\Theta) \equiv \mathrm{SU}^\eta_\xi(\check\Theta_0, \check\Theta_1, \dots)$.
(b) $\bar\Theta \subseteq \mathrm{SU}^\eta_\xi(\bar\Theta)$.
(c) If $\Lambda_n \subseteq \Theta_n$, uniformly, and $\xi \leqslant o(\bar\Lambda)$, then $\mathrm{SU}^\eta_\xi(\bar\Lambda) \subseteq \mathrm{SU}^\eta_\xi(\bar\Theta)$.

Lemma 3.27. Suppose that $o(\Gamma) > \xi$, $\bar\Theta < \Gamma$, $\xi \leqslant o(\bar\Theta)$, and η is $y^{\bar\Theta}$-computable. Then $\mathrm{SU}^\eta_\xi(\bar\Theta) < \Gamma$.

Proof. Let N be a total $\mathrm{SU}^\eta_\xi(\bar\Theta)$-name; let $z = z^N$. Let $F = F^N$ and $F_n = F^{N_{(n)}}$. So $F_n \in \Theta_n(z)$, uniformly, and so, as $\bar\Theta < \Gamma$, $F_n \in \Gamma(z)$ and $F_n \in \check\Gamma(z)$, uniformly. By Proposition 2.14 and Corollary 3.18, as $\xi < o(\Gamma)$, $F \in \Gamma(z)$ and $F \in \check\Gamma(z)$. This is uniform in everything. □

3.7. **The case $\eta = 1$.** Let Γ be a class description and let N be a Γ-name. For $s \in T_\Gamma$ let X^N_s be the collection of $x \in [S^N]$ such that $s \preccurlyeq \ell(x)$, where we use the notation from Definition 3.3. That is, X^N_s consists of those x for which the procedure computing $F^N(x)$ "passes through" the node s.

In general, X^N_s may be complicated. Suppose, however, that for all non-leaf $s \in T_\Gamma$, $\eta_s = 1$. In this case, if t is a child of s on T_Γ then X^N_t is $\Sigma^0_{1+\xi_s}(z) \wedge \Pi^0_{1+\xi_s}(z)$ (where $z = z^N$). In fact, X^N_t is either $\Sigma^0_{1+\xi_s}(z)$ or $\Pi^0_{1+\xi_s}(z)$ within X^N_s; the latter holds for the default (leftmost) child.

Definition 3.28. Let Γ be a class description satisfying $\eta_s = 1$ for all non-leaf $s \in T_\Gamma$; let N be a Γ-name. By induction on $|s|$, for $s \in T_\Gamma$ we define $S^N_s \subseteq S^N$ as follows:

- $S^N_{\langle\rangle} = S^N$;
- If t is a child of s on T_Γ, then $S^N_t = \{\sigma \in S^N_s \,:\, f^N_s(\sigma) = t\}$.

If t is the default child of s then S^N_t is (z, ξ_s)-closed in S^N_s; otherwise, it is (z, ξ_s)-open in S^N_s. Since $\xi_s \leqslant \xi_t$, by induction on $|s|$ we see that for non-leaf s, S^N_s is a (z, ξ_s)-forest; and $X^N_s = [S^N_s]^z_{\xi_s}$.

Thus, in the case that each η_s is 1, for each s, we do not need to define the approximation f^N_s on all of S^N, but rather, only on S^N_s. In the general case, since X^N_s may fail to be $\Sigma^0_{1+\xi_s}(z) \wedge \Pi^0_{1+\xi_s}(z)$, we need to define our approximations on all of S^N at each node s.

3.8. **Montone sequences.**

Definition 3.29. A uniform sequence $\Theta_0, \Theta_1, \dots$ is *monotone* if $\Theta_n \subseteq \check\Theta_{n+1}$, uniformly in n.

There are two main cases: either $\Theta_{n+1} = \check\Theta_n$ for all n, or $\Theta_n < \Theta_{n+1}$ for all n. It will be a consequence of our analysis that by passing to infinite subsequences, these are the only two possibilities.

Proposition 3.30. *Suppose that $\bar\Theta = \langle\Theta_0, \Theta_1, \dots\rangle$ is monotone, $\xi \leqslant o(\bar\Theta)$, and η is $y^{\bar\Theta}$-computable.*

(a) $\bar{\Theta} < \mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$.
(b) $\mathrm{SU}^{\eta}_{\xi}(\Theta_0, \Theta_1, \dots) \subseteq \neg\mathrm{SU}^{\eta}_{\xi}(\Theta_1, \Theta_2, \dots)$.
(c) *If* $\Theta_0 < \Theta_1$ *then* $\mathrm{SU}^{\eta}_{\xi}(\Theta_0, \Theta_1, \dots) < \mathrm{SU}^{\eta}_{\xi}(\Theta_1, \Theta_2, \dots)$.
(d) *If* $\eta' < \eta$ *then* $\mathrm{SU}^{\eta'}_{\xi}(\bar{\Theta}) < \mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$.
(e) *If* $\xi' < \xi$ *then* $\mathrm{SU}_{\xi'}(\bar{\Theta}) < \mathrm{SU}_{\xi}(\bar{\Theta})$.

Proof. (a) follows from Lemma 3.26(b) and $\bar{\Theta}$ being monotone.

(b) follows from Lemma 3.26(a),(c).

For (c), in light of (b), it suffices to show that $\mathrm{SU}^{\eta}_{\xi}(\Theta_0, \Theta_1, \dots) \subseteq \mathrm{SU}^{\eta}_{\xi}(\Theta_1, \Theta_2, \dots)$. Let N be a $\mathrm{SU}^{\eta}_{\xi}(\Theta_0, \Theta_1, \dots)$-name. Define a $\mathrm{SU}^{\eta}_{\xi}(\Theta_1, \Theta_2, \dots)$-name M equivalent to N by letting $M_{(0)}$ be a Θ_1-name equivalent to $N_{(0)}$, and for $n \geqslant 2$, let $M_{(n)}$ be a Θ_{n+1}-name equivalent to $N_{(n-1)}$. It doesn't matter how we define $M_{(1)}$. We define $\beta^M = \beta^N$ and $f^M(\sigma) = 0$ if $f^N(\sigma) = 0$, otherwise $f^M(\sigma) = f^N(\sigma) + 1$.

For (d), $\mathrm{SU}^{\eta'}_{\xi}(\bar{\Theta}) \subseteq \mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$ is clear, as every $\mathrm{SU}^{\eta'}_{\xi}(\bar{\Theta})$-name is also a $\mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$-name, and has the same interpretation. To see that $\mathrm{SU}^{\eta'}_{\xi}(\bar{\Theta}) \subseteq \neg\mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$, let N be a $\mathrm{SU}^{\eta'}_{\xi}(\bar{\Theta})$-name. We define a $\mathrm{SU}^{\eta}_{\xi}(\Theta_0, \Theta_1, \dots)$ name M equivalent to N by letting $M_{(n+1)} \equiv \check{N}_{(n)}$ and setting $\beta^M = \beta^N$ and $f^M = f^N + 1$. The point is that $\eta > \eta'$ so even if $\beta^N(\sigma) = \eta'$, in M, we are allowed to take a non-default outcome.

(e) follows from (a) and Lemma 3.27. □

Notice that for any class description Γ and any $\xi < o(\Gamma)$, $\mathrm{SU}^{\eta}_{\xi}(\Gamma, \Gamma, \dots) \equiv \Gamma$ (one direction relies on Corollary 3.18). So the ordinal $o(\Gamma)$ may vary between descriptions Γ of the same class. However, we can identify a largest ordinal among all such descriptions.

Lemma 3.31. If $\bar{\Theta}$ is monotone, $\xi \leqslant o(\bar{\Theta})$, and Γ is a description with $\mathbf{\Gamma} = \mathbf{SU}^{\eta}_{\xi}(\bar{\Theta})$, then $\mathrm{otp}(o(\Gamma)) \leqslant \mathrm{otp}(\xi)$.

Recall that when we write equality of boldface classes, we mean that they contain the same total functions.

Proof. For a contradiction, suppose $\mathrm{otp}(o(\Gamma)) > \mathrm{otp}(\xi)$, and fix $F \in \mathbf{SU}^{\eta}_{\xi}(\bar{\Theta})$ universal, by Proposition 3.14. It suffices to show that $1 - F \in \mathbf{\Gamma}$.

Fix $z = y^{\Gamma} \oplus y^{\bar{\Theta}}$ and N an $\mathrm{SU}^{\eta}_{\xi}(\bar{\Theta})$-name of F. Then $X_n = \{x \in \mathcal{N} : f^N_{\diamondsuit}(x) = n\}$ is $\Delta^0_{1+\xi+1}(z)$. By Proposition 3.30(a) and monotonicity, $1 - F^{N_{(n)}} \in \mathbf{SU}^{\eta}_{\xi}(\bar{\Theta}) = \mathbf{\Gamma}$, and so by Corollary 3.18, $1 - F \in \mathbf{\Gamma}$. □

3.9. **Acceptable descriptions.**

Definition 3.32.

(a) A class description Γ is *acceptable* if for all non-leaf $s \in T_{\Gamma}$,
  (i) $\eta^{\Gamma}_s = 1$; and
  (ii) for all n, $s\hat{\ }n \in T_{\Gamma}$, and the sequence $\Gamma_{s\hat{\ }0}, \Gamma_{s\hat{\ }1}, \dots$ is monotone, uniformly in s.
(b) A uniform sequence $\Theta_0, \Theta_1, \dots$ of class descriptions is acceptable if it is monotone, and the class descriptions Θ_n are uniformly acceptable.

Uniform acceptability implies that for all n and non-leaf $s \in T_{\Theta_n}$, the sequence $(\Theta_n)_{s\hat{\ }0}, (\Theta_n)_{s\hat{\ }1}, \dots$ is monotone, uniformly in s and n.

Note that some of the class descriptions given in Section 3.2 are not acceptable. Nevertheless, we can give acceptable descriptions for these classes.

b'. $\mathbf{\Sigma}^0_\alpha$: let T_Γ consist of the root, and children (0), (1), (2), and so on. Let the labels of the leaves alternate between 0 and 1, with even leaves labeled 0 and odd leaves labeled 1. Set $o(\Gamma)$ to be some ordinal ξ, and let $\eta^\Gamma = 1$. Then $\mathbf{\Gamma} = \mathbf{\Sigma}^0_{1+\xi}$.

c'. *Hausdorff difference hierarchy*: The class $D_\eta(\mathbf{\Sigma}^0_{1+\xi})$ has an acceptable description defined inductively on η as follows. First, suppose that η is a successor ordinal, $\eta = \gamma + 1$. Set $o(\Gamma)$ to be ξ, and let $\eta^\Gamma = 1$. Let $\Gamma_{(0)}$ be a leaf labeled 0, and let $\Gamma_{(i)}$ be an acceptable description for either $D_\gamma(\mathbf{\Sigma}^0_{1+\xi})$ (if i is even) or its dual (if i is odd). If η is a limit ordinal, let $\langle \eta_k \rangle$ be a computable increasing and cofinal sequence in η. We still set $o(\Gamma)$ to be ξ, and let $\eta^\Gamma = 1$. Let $\Gamma_{(0)}$ be a leaf labeled 0, and let $\Gamma_{(i)}$ be an acceptable description for $D_{\eta_i}(\mathbf{\Sigma}^0_{1+\xi})$.

Lemma 3.33.

(a) If Γ is acceptable, then the sequence $\Gamma_{(0)}, \Gamma_{(1)}, \dots$ is acceptable,
(b) If $\bar\Theta$ is acceptable and $\xi \leqslant o(\bar\Theta)$ then $\mathrm{SU}_\xi(\bar\Theta)$ is acceptable.

Proposition 3.34. *Suppose that $\bar\Theta$ is acceptable; $\xi \leqslant o(\bar\Theta)$, and η is $y^{\bar\Theta}$-computable. Then there is some acceptable $\Gamma \equiv \mathrm{SU}^\eta_\xi(\bar\Theta)$ with $o(\Gamma) = \xi$.*

Proof. This is done by induction on η. Since everything has to be uniformly computable, this is, in fact, $y^{\bar\Theta}$-effective transfinite recursion on η.

Suppose that $\eta > 1$ is a successor ordinal. By induction, for each n, let $\Lambda_n \equiv \mathrm{SU}^{\eta-1}_\xi(\Theta_n, \Theta_{n+1}, \dots)$ be acceptable. Then $\Theta_0, \Lambda_0, \Lambda_1, \Lambda_2, \dots$ is monotone: $\Theta_0 \subseteq \check\Lambda_1$ follows from Proposition 3.30(a); $\Lambda_n \subseteq \check\Lambda_{n+1}$ follows from Proposition 3.30(b). By Lemma 3.33, $\Gamma = \mathrm{SU}_\xi(\Theta_0, \Lambda_0, \Lambda_1, \Lambda_2, \dots)$ is acceptable.

We check that $\Gamma \equiv \mathrm{SU}^\eta_\xi(\bar\Theta)$. In one direction, suppose that N is a $\mathrm{SU}^\eta_\xi(\bar\Theta)$-name; we build a Γ-name M equivalent to N. Let $z = z^N$ and $S = S^N$. To define M, we use the notation of Definition 3.28. We let $S^M_{(0)} = \{\sigma \in S : \beta^N(\sigma) = \eta\}$; we let $M_{(0)}$ be the restriction of $N_{(0)}$ to $S^M_{(0)}$ (Proposition 3.15).

For simplicity, we may assume that for all $\sigma \in S$, if $\beta^N(\sigma) < \eta$ then $\beta^N(\tau) = \eta - 1$ for some $\tau \preccurlyeq^z_\xi \sigma$ in S. For all n we let W_n be the (z, ξ)-open subset of S generated by the strings σ with $\beta^N(\sigma) = \eta - 1$ and $f^N(\sigma) = n$. We will define a $\mathrm{SU}^{\eta-1}_\xi(\Theta_n, \Theta_{n+1}, \dots)$-name G_n with $S^{G_n} = W_n$; then, we let $S^M_{(n+1)} = W_n$ and let $M_{(n+1)}$ be a Λ_n-name equivalent to G_n. To define G_n, we let $\beta^{G_n} = \beta^N \restriction W_n$. We let $(G_n)_{(0)} = N_{(n)}$. Since $\bar\Theta$ is monotone, we can find a computable function $g\colon \mathbb{N} \to \mathbb{N}$ and for each m, let $(G_n)_{(m)}$ be a Θ_{n+m}-name, so that:

- $g(n) = 0$; and
- for all k, $(G_n)_{g(k)}$ is equivalent to $N_{(k)}$.

For $\sigma \in W_n$ we let $f^{G_n}(\sigma) = g(f^N(\sigma))$.

In the other direction, suppose that N is a Γ-name. We define an $\mathrm{SU}^\eta_\xi(\bar\Theta)$-name M equivalent to N. Again let $z = z^N$ and $S = S^N$. For each n, let K_n be an $\mathrm{SU}^{\eta-1}_\xi(\Theta_n, \Theta_{n+1}, \dots)$-name equivalent to $N_{(n+1)}$ (so $S^{K_n} = S^N_{(n+1)}$). Now, as $\bar\Theta$ is monotone, find a computable function g and for each m, let $M_{(m)}$ be a Θ_m-name such that:

- $M_{(0)} = N_{(0)}$; and

- for all n and k, $(K_n)_{(k)}$ is equivalent to $M_{(g(n,k))}$.

For $\sigma \in S^N_{(0)}$ let $f^M(\sigma) = 0$ and $\beta^N(\sigma) = \eta$. For each $n \geqslant 1$, for each $\sigma \in S^N_{(n)}$ let $\beta^M(\sigma) = \beta^{K_n}(\sigma)$ and $f^M(\sigma) = g(n, f^{K_n}(\sigma))$.

Suppose that η is a limit ordinal. Fix a computable increasing and cofinal sequence $\langle \eta_n \rangle$ in η. By induction, find acceptable $\Lambda_n \equiv \mathrm{SU}^{\eta_n}_\xi(\bar{\Theta})$. The sequence $\Theta_0, \Lambda_1, \Lambda_2, \Lambda_3, \dots$ is monotone: $\Theta_0 \subseteq \check{\Lambda}_1$ follows from Proposition 3.30(a); $\Lambda_n \subseteq \check{\Lambda}_{n+1}$ follows from Proposition 3.30(d). Hence $\Gamma = \mathrm{SU}_\xi(\Theta_0, \Lambda_1, \Lambda_2, \dots)$ is acceptable.

The argument that $\Gamma \equiv \mathrm{SU}^\eta_\xi(\bar{\Theta})$ is a little simpler than the successor case. To translate a $\mathrm{SU}^\eta_\xi(\bar{\Theta})$-name N to a Γ-name M, we of course keep the default outcome $M_{(0)} = N_{(0)}$. Suppose that $\sigma \in S$ is minimal with $\beta^N(\sigma) < \eta$. Then $\beta^N(\sigma) < \eta_n$ for some $n \geqslant 1$. Take least such, and set $\sigma \in S^M_{(n)}$ (putting all of its successors in $(S, \preccurlyeq^z_\xi)$ into $S^M_{(n)}$ as well). We then let $M_{(n)}$ be a Λ_n-name equivalent to a $\mathrm{SU}^{\eta_n}_\xi(\bar{\Theta})$-name G_n defined by $\beta^{G_n} = \beta^N \restriction S^M_{(n)}$, $f^{G_n} = \beta^N \restriction S^M_{(n)}$, and $(G_n)_{(k)} = N_{(k)}$. The point is that we don't have to worry about the default outcome of G_n, since we already have $\beta^N(\sigma) < \eta_n$.

The other direction is exactly as in the successor case. □

By Lemma 3.31, it follows that amongst the class descriptions of a given class $\mathbf{\Lambda}$, the acceptable descriptions have the largest von-Neumann ordinal (and we will eventually show that every Borel Wadge class has an acceptable description). We can take this to be the ordinal of the class. For equivalent definitions of the ordinal level of a Borel Wadge class see [LSR88a].

In light of Proposition 3.34, one might ask why we allow $\eta > 1$ in our general class descriptions in the first place. The answer is that classes $\mathrm{SU}^\eta_\xi(\bar{\Theta})$ with $\eta > 1$ show up naturally in our classification of simpler class descriptions, in particular, in the proof of Proposition 4.9; so we have designed our descriptions to easily accommodate them. Another reason is that natural descriptions of more complicated classes, such as $\mathrm{BiSep}(D_\eta(\mathbf{\Sigma}^0_{1+\xi}), \mathbf{\Gamma})$, and separated differences require $\eta > 1$. Descriptions with $\eta > 1$ are also used in [GTb] to characterise the effective reduction property.

4. Classification of acceptable classes

We now give our classification of acceptable classes, by analysing their ambiguous classes. This resembles the work in [Lou83], but is much simplified; it also relies on our effective methods.

Definition 4.1. For a class description Γ and $z \geqslant_{\mathrm{T}} y^\Gamma$, we let

$$\Delta(\Gamma(z)) = \Gamma(z) \cap \check{\Gamma}(z)$$

and

$$\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}.$$

Note that the first includes partial functions, while for the second we will usually only consider total functions.

Definition 4.2. Let Γ be an acceptable class description.

(1) Γ has *zero type* if $o(\Gamma) = \omega_1$, i.e., if $T_\Gamma = \{\langle\rangle\}$.
(2) Γ has *countable type* if there is an acceptable sequence $\bar{\Theta}$ such that:

- $\bar{\Theta} < \Gamma$;
- $y^{\bar{\Theta}} = y^{\Gamma}$;
- $o(\Theta_n) \geqslant o(\Gamma)$ for all n; and
- For any $z \geqslant_{\mathrm{T}} y^{\Gamma}$, for any $F \in \Delta(\Gamma(z))$, there is a partition of $X = \operatorname{dom} F$ into sets Y_n that are $\Sigma^0_{1+o(\Gamma)}(z)$ within X (uniformly), such that $F \restriction Y_n \in \Theta_n(z)$, uniformly.

(3) Γ has *uncountable type* if for every $z \geqslant_{\mathrm{T}} y^{\Gamma}$, for every $F \in \Delta(\Gamma(z))$ whose domain is $\Pi^0_{1+o(\Gamma)}(z)$, there is some acceptable Λ such that:
- $\Lambda < \Gamma$;
- $y^{\Lambda} = z$;
- $o(\Lambda) \geqslant o(\Gamma)$; and
- $F \in \Lambda(z)$.

Remark 4.3. Suppose that Γ has countable type, witnessed by $\bar{\Theta}$. Let $z \geqslant_{\mathrm{T}} y^{\Gamma}$, X be $\Sigma^0_{1+o(\Gamma)}(z) \wedge \Pi^0_{1+o(\Gamma)}(z)$, $\langle Y_n \rangle$ a partition of X into sets that are uniformly $\Sigma^0_{1+o(\Gamma)}(z)$ within X, and $F\colon X \to \{0,1\}$ be such that $F \restriction Y_n \in \Theta_n(z)$, uniformly. Then by Proposition 3.17, $F \in \Delta(\Gamma(z))$.

Hence, for both the countable and uncountable type, the condition described is actually a characterisation of $\Delta(\Gamma(z))$.

We say that an acceptable class description is *classified* if it has one of the three types. The main theorem of this section is:

Theorem 4.4. *Every acceptable class description is classified.*

Let us provide some examples. Fix $\bar{\Theta}$ the sequence $0, 1, 0, 1, \dots$. For any ξ, let $\Gamma = \mathrm{SU}_{\xi}(\bar{\Theta})$, so that $\mathbf{\Gamma} = \mathrm{BiSep}(\mathbf{\Sigma}^0_{1+\xi}, 1, 0) = \mathbf{\Sigma}^0_{1+\xi}$. Then Γ has countable type, where the witnessing sequence is this same $\bar{\Theta}$. This is immediate: let $F \in \Delta(\Gamma(z))$; for $i < 2$, let $X_i = F^{-1}[i]$. Then (X_0, X_1) partition $\operatorname{dom} F$ into sets that are relatively $\Delta^0_{1+\xi}(z)$, and F is constant on both these sets.

For a slightly more complicated example, consider $\mathbf{\Lambda} = D_2(\mathbf{\Sigma}^0_{1+\xi})$, which is described by an acceptable Λ constructed in example c' after Definition 3.32: if Γ is the standard acceptable description for $\mathbf{\Sigma}^0_{\xi}$ then $\Gamma = \mathrm{SU}_{\xi}(0, \Gamma, \check{\Gamma}, \Gamma, \check{\Gamma}, \dots)$. In this case we let $\Theta_n = \Gamma$ and $\Theta_{n+1} = \check{\Gamma}$. To see that this satisfies the definition, let $F \in \Delta(\Lambda(z))$. We can partition $X = \operatorname{dom} F$ into four sets, each $\Sigma^0_{1+\xi}(z)$ within X: those $z \in X$ for which the Λ-name for F moves off the default outcome before the $\check{\Lambda}$-name does, and it moves to an even outcome (n); those $z \in X$ for which the Λ-name moves off the default outcome first and moves to and odd outcome; and two similar sets for when the $\check{\Lambda}$-name is first to move. On each of these four sets, F is either in $\Gamma(z)$ or in $\check{\Gamma}(z)$, depending on the parity of the outcome chosen.

On the other hand, let $\Gamma = \mathrm{SU}_1(0, 1, 0, \dots)$ be the standard acceptable name for $\mathbf{\Sigma}^0_2$; and let $\Upsilon = \mathrm{SU}_0(\Gamma, \check{\Gamma}, \Gamma, \check{\Gamma}, \dots)$, so that $\mathbf{\Upsilon} = \mathrm{BiSep}(\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_2)$. It will be a consequence of Theorem 4.7 that Υ has uncountable type. Indeed, by examining the proofs of Proposition 4.8 and Proposition 4.9, we can see that for any $F \in \Delta(\mathbf{\Upsilon})$, $F \in \mathrm{BiSep}(\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_2, D_{\eta}(\mathbf{\Sigma}^0_1))$ for some sufficiently large $\eta < \omega_1$, and all of these classes are $< \Upsilon$; and so are as required for uncountable type.

4.1. **How classification helps.** The purpose of the notion of type is to tell us about the ambiguous classes, and the cofinality of the class in the Wadge degrees. If Γ is an acceptable description, there will be three cases:

(i) $o(\Gamma) = \omega_1$;
(ii) $\Delta(\mathbf{\Gamma})$ is a principal pointclass, the least upper bound in the Wadge degrees of a countable sequence of degrees;
(iii) $\Delta(\mathbf{\Gamma})$ is the (necessarily uncountable) union of those $\mathbf{\Lambda} \subset \Delta(\mathbf{\Gamma})$.

These three cases will *not* exactly correspond to the three sorts of types we have just defined. It will turn out that case (b) happens only when Γ has countable type and $o(\Gamma) = 0$; if Γ has uncountable type or $0 < o(\Gamma) < \omega_1$, then case (c) will hold. We summarise this in the following two propositions.

Proposition 4.5. *Let Γ be an acceptable class description. If Γ has countable type, witnessed by $\bar{\Theta}$, and $o(\Gamma) = 0$, then $\Delta(\mathbf{\Gamma})$ is principal, and is the least principal pointclass containing $\bigcup_n \mathbf{\Theta}_n$.*

Proposition 4.6. *Let Γ be an acceptable class description. If Γ has uncountable type, or $0 < o(\Gamma) < \omega_1$, then $\Delta(\mathbf{\Gamma}) = \bigcup\{\mathbf{\Lambda} : \Lambda$ is acceptable and $\Lambda < \Gamma\}$, and is non-principal.*

One can think of Proposition 4.5 as covering two cases depending on the character of the sequence $\bar{\Theta}$. As mentioned above, the separation theorem will imply the semi-linear-ordering property for described classes: for any two class descriptions Γ and Λ, either $\mathbf{\Lambda} \subseteq \mathbf{\Gamma}$ or $\mathbf{\Gamma} \subseteq \check{\mathbf{\Lambda}}$. This will imply that if $\bar{\Theta}$ is monotone then it has a subsequence which is $<$-increasing, or it eventually alternates between a class and its dual. In the first case, $\Delta(\mathbf{\Gamma})$ above will have cofinality ω in the Wadge degrees; in the latter, it will be a successor of the two dual classes which form a tail of $\bar{\Theta}$.

The discrepancy between the type of a class and its cofinality in the Wadge degrees would lead one to ask why we have chosen to define countable and uncountable type as we have. What is countable about a class of uncountable cofinality? The heart of the matter is in our general plan for proving Theorem 4.4 (and in particular, the proof of Proposition 4.9 below). Indeed, given an acceptable description, we can easily tell its type.

Theorem 4.7. *Let Γ be an acceptable class description, and suppose that $o(\Gamma) < \omega_1$. Let s^* be the leftmost leaf of T_Γ (necessarily of the form 0^n for some n).*

(a) *If for all $s \prec s^*$, $\xi_s^\Gamma = o(\Gamma)$, then Γ has countable type.*
(b) *Otherwise, Γ has uncountable type.*

That is, Γ has countable type if climbing from the root to the leftmost leaf s^*, the ordinal labels are constant; it has uncountable type if the ordinal labels increase at some point. The class $\mathbf{\Gamma}$ has countable cofinality (among non-self dual pairs) if the ordinal labels along s^* are all 0.

The real answer to the question "what is countable about a countable type with uncountable cofinality" relies on an alternative understanding of the ordinal level of a class. Let Γ be a class description. Ignoring computability issues, we can define a class description $\Lambda = \Gamma^{-o(\Gamma)}$ by subtracting $o(\Gamma)$ from all ξ-ordinal labels: $o(\Gamma) + \xi_s^\Lambda = \xi_s^\Gamma$ for all non-leaf $s \in T_\Gamma$. The class $\mathbf{\Lambda}$ is the "$o(\Gamma)$-jump inversion" of $\mathbf{\Gamma}$: $F \in \mathbf{\Gamma}$ if and only if $F = H \circ g$ where $H \in \mathbf{\Lambda}$ and g is $\mathbf{\Sigma}^0_{1+o(\Gamma)}$-measurable. Equivaletly, g can be taken to be the $o(\Gamma)$-iterated jump function, relative to some oracle (see Lemma 7.11 below). Indeed, $o(\Gamma)$ can be characterised as the greatest ordinal α such that $\mathbf{\Gamma}$ is the "α-jump" of a class $\mathbf{\Lambda}$ ([LSR88a]). We remark that taking the α-jumps of classes is one of the main staples of all analyses in print of Borel Wadge classes, from Wadge's [Wad84] to [KM19]. The true stage relations

allow us to argue about classes directly, without needing to transform them using iterated jumps.

For our matter, Theorem 4.7 implies that jump and jump inversion preserve the type of a class. An acceptable Γ has countable type if and only if its $o(\Gamma)$-jump inversion $\mathbf{\Lambda}$ has countable cofinality in the Wadge degrees. Equivalently, if it has countable cofinality *among all classes of level* $\geqslant o(\Gamma)$. For example, $\mathbf{\Sigma}^0_{1+\xi}$ has countable type because among classes of level $\geqslant \xi$, it is the successor of the pair $\{\varnothing\}, \{\mathcal{N}\}$; and $D_2(\mathbf{\Sigma}^0_{1+\xi})$ is the successor of the pair $\mathbf{\Sigma}^0_{1+\xi}, \mathbf{\Pi}^0_{1+\xi}$ among such classes.

4.2. **Proving the classification theorem.** We turn to the proof of Theorem 4.7, which clearly implies Theorem 4.4. As one would guess, the proof is by induction on the length of the leftmost leaf s^* of Γ. Note that this is traditional induction on $\mathbb{N}$, not transfinite induction on the rank of T_Γ. The induction has three parts, depending on the type and ordinal level of the default sub-class $\Gamma_{(0)}$:

Proposition 4.8. *Let Γ be an acceptable class description, and suppose that $o(\Gamma) < \omega_1$.*

(a) *If $o(\Gamma_{(0)}) = \omega_1$ then Γ has countable type.*
(b) *If $o(\Gamma) < o(\Gamma_{(0)}) < \omega_1$, or if $\Gamma_{(0)}$ has uncountable type, then Γ has uncountable type.*
(c) *If $o(\Gamma_{(0)}) = o(\Gamma)$ and $\Gamma_{(0)}$ has countable type then Γ has countable type.*

To prove Proposition 4.8 we will use:

Proposition 4.9. *Suppose that Γ is classified and that $o(\Gamma) < \omega_1$. Then for all $\xi < o(\Gamma)$, for all $z \geqslant_{\mathrm{T}} y^\Gamma$, if $F \in \Delta(\Gamma(z))$ and $\operatorname{dom} F$ is $\Pi^0_{1+\xi}(z)$, then there is some acceptable Λ such that:*

- *$\Lambda < \Gamma$;*
- *$y^\Lambda = z$;*
- *$o(\Lambda) \geqslant \xi$; and*
- *$F \in \Lambda(z)$.*

We note that for any Γ, if the conclusion holds for some $\xi \leqslant o(\Gamma)$ then it holds for all $\xi' \leqslant \xi$. Hence the proposition holds for all Γ which have uncountable type. Indeed, what it says, in some sense, is that if $o(\Gamma) > 0$ then Γ has "weak uncountable type", even if it has countable type. This will imply Proposition 4.6 (apart from being used for Proposition 4.8). Due to its length, we delay the proof of Proposition 4.9 to the end of this section.

The starred class. Toward proving Proposition 4.8, we define a sub-class $\mathbf{\Gamma}^*$ of $\mathbf{\Gamma}$, consisting of functions that never take the default outcome at the root. This is a subclass of $\Delta(\mathbf{\Gamma})$ which behaves nicely. For example, it will always have countable type (Proposition 4.12). The main step of the argument is Proposition 4.13, which says that we can understand a set in $\Delta(\mathbf{\Gamma})$ by separating into two parts: on a relative closed set, we stay on the default outcome, and so in $\Delta(\mathbf{\Gamma}_{(0)})$; on the rest, we're in $\mathbf{\Gamma}^*$.

Definition 4.10. Let Γ be a class description. A Γ^*-*name* is a Γ-name N for which for all $\sigma \in S^N$, $\beta^N(\sigma) < \eta^\Gamma$.

In particular, if $\eta^\Gamma = 1$ this means that $f^N = f^N_\diamondsuit$ never changes: for all $\sigma \preccurlyeq^z_{o(\Gamma)} \tau$ in S^N, $f^N(\tau) = f^N(\sigma)$.

We let $\Gamma^*(z)$ denote the collection of all functions F^N, where N is a Γ^*-name and $z^N = z$. We extend the relations of Definition 3.20 to Γ^* as expected.

Lemma 4.11. If Γ is acceptable, then for all $z \geqslant_{\mathrm{T}} y^\Gamma$, $\check{\Gamma}^*(z) \equiv \Gamma^*(z)$.

As usual, everything is as uniform as possible. Since $\Gamma^*(z) \subseteq \Gamma(z)$, it follows that $\Gamma^*(z) \subseteq \Delta(\Gamma(z))$.

Proof. Suppose that N is a $\check{\Gamma}^*$-name. For each n let M_{n+1} be a $\Gamma_{(n+1)}$-name equivalent to $N_{(n)}$; set $f^M(\sigma) = f^N(\sigma)+1$ and $\beta^M(\sigma) = \beta^N(\sigma)$, which is permitted since $\beta^N(\sigma) < \eta^\Gamma$. □

As a result, when Γ is acceptable, $\Lambda \subseteq \Gamma^*$ implies $\Lambda < \Gamma^*$. Informally, the following proposition says that Γ^* has countable type.

Proposition 4.12. *For any acceptable Γ with $o(\Gamma) < \omega_1$ there is an acceptable sequence $\bar{\Theta}$ satisfying:*

- $\bar{\Theta} < \Gamma^*$*;*
- $y^{\bar{\Theta}} = y^\Gamma$*;*
- $o(\Theta_n) \geqslant o(\Gamma)$ *for all n; and*
- *For any $z \geqslant_{\mathrm{T}} y^\Gamma$, for any $F \in \Gamma^*(z)$, there is a partition of $X = \operatorname{dom} F$ into sets Y_n that are $\Sigma^0_{1+o(\Gamma)}(z)$ within X (uniformly), such that $F \restriction Y_n \in \Theta_n(z)$, uniformly.*

Proof. Let $\Theta_n = \Gamma_{(n)}$. Lemma 3.33 says that $\bar{\Theta}$ is acceptable. The (easy) proof of Lemma 3.26(b) shows that $\bar{\Theta} < \Gamma^*$. Let N be a Γ^*-name; recall that $S^N_{(n)} = \{\sigma \in S^N : f^N(\sigma) = n\}$. The sets $S^N_{(n)}$ are pairwise $(z, o(\Gamma))$-orthogonal, and $F^N \restriction [S_n] \in \Theta_n(z)$. □

Proposition 4.13. *Suppose that Γ is acceptable, $z \geqslant_{\mathrm{T}} y^\Gamma$, and $F \in \Delta(\Gamma(z))$. Then there is partition of $X = \operatorname{dom} F$ into a set Y, $\Pi^0_{1+o(\Gamma)}(z)$ within X, and a set Z, $\Sigma^0_{1+o(\Gamma)}(z)$ within X, such that:*

- $F \restriction Y \in \Delta(\Gamma_{(0)}(z))$*; and*
- $F \restriction Z \in \Gamma^*(z)$*.*

Proof. This is a "stage comparison" argument. Let N be a $\Gamma(z)$-name and M be a $\check{\Gamma}(z)$-name such that $F = F^N = F^M$. We may assume that $S^N = S^M$, call it S.

Define

$$T = \{\sigma \in S : \beta^N(\sigma) = \beta^M(\sigma) = 1\};$$

then T is $(z, o(\Gamma))$-closed in S, so $Y = [T]$ is $\Pi^0_{1+o(\Gamma)}(z)$ within X. Let $W = S \backslash T$, which is $(z, o(\Gamma))$-open in S; we let $Z = [W]$. We decompose W into two parts:

$$W^+ = \left\{\sigma \in W : (\exists \tau \preccurlyeq^z_{o(\Gamma)} \sigma) \;\; \beta^N(\tau) = 0 \;\&\; \beta^M(\tau) = 1\right\},$$

and $W^- = W \backslash W^+$. Let $Z^+ = [W^+]$ and $Z^- = [W^-]$. The sets Z^- and Z^+ partition Z, and are both $\Sigma^0_{1+o(\Gamma)}(z)$ in Z, and so in X. Note that W^+ and W^- are $(z, o(\Gamma))$-orthogonal.

Since $f^N(\sigma) = f^M(\sigma) = 0$ for all $\sigma \in T$, $N_{(0)}$ and $M_{(0)}$ with $S^{N_{(0)}} = S^{M_{(0)}} = T$ are $\Gamma_{(0)}$- and $\check{\Gamma}_{(0)}$-names for $F \restriction Y$.

The restriction of N to W^+ shows that $F \upharpoonright Z^+ \in \Gamma^*(z)$. Similarly, the restriction of M to W^- shows that $F \upharpoonright Z^- \in \check{\Gamma}^*(z)$; by Lemma 4.11, $F \upharpoonright Z^- \in \Gamma^*(z)$. Since W^+ and W^- are orthogonal, the proof of Proposition 3.17 shows that $F \upharpoonright Z \in \Gamma^*(z)$. □

Ignoring our major debt (Proposition 4.9), we can now finish the proof.

Proof of Proposition 4.8. (a): The main point in this case is that for any z, $\Delta(\Gamma_{(0)}(z))$ contains only the empty partial function, whence by Proposition 4.13, $\Delta(\Gamma(z)) = \Gamma^*(z)$. Let $\bar{\Theta}$ be as provided by Proposition 4.12 for Γ; it is as required.

(b): Let $z \geqslant_{\mathrm{T}} y^\Gamma$ and let $F \in \Delta(\Gamma(z))$, with $X = \operatorname{dom} F$ being $\Pi^0_{1+o(\Gamma)}(z)$.

Obtain a decomposition $X = Y \cup Z$ from Proposition 4.13; then Y is also $\Pi^0_{1+o(\Gamma)}(z)$.

In either case, we can find some acceptable $\Lambda < \Gamma_{(0)}$ with $y^\Lambda = z$, $o(\Lambda) \geqslant o(\Gamma)$ and $F \upharpoonright Y \in \Lambda(z)$. If $\Gamma_{(0)}$ has uncountable type this is by definition; if $o(\Gamma) < o(\Gamma_{(0)}) < \omega_1$ this is by Proposition 4.9.

Now let

$$\Upsilon = \mathrm{SU}_{o(\Gamma)}(\Lambda, \Gamma_{(0)}, \Gamma_{(1)}, \Gamma_{(2)}, \dots);$$

except that we update the oracle of all classes to be z (as $y^\Lambda = z$). This is a class description since $o(\Lambda) \geqslant o(\Gamma)$. The sequence $\Lambda, \Gamma_{(0)}, \Gamma_{(1)}, \Gamma_{(2)}, \dots$ is acceptable since $\Lambda < \Gamma_{(0)}$. By Lemma 3.33, Υ is acceptable. By Proposition 3.30(c), $\Upsilon < \Gamma$.

The fact that $F \upharpoonright Y \in \Lambda(z)$ and $F \upharpoonright Z \in \Gamma^*(z)$ implies $F \in \Upsilon(z)$.

(c): Let $\bar{\Theta}$ witness that $\Gamma_{(0)}$ has countable type. Let

$$\Lambda_n = \mathrm{SU}_{o(\Gamma)}(\Theta_n, \Gamma_{(0)}, \Gamma_{(1)}, \Gamma_{(2)}, \dots).$$

Note that $o(\Theta_n) \geqslant o(\Gamma_{(0)}) = o(\Gamma)$. Since $\Theta_n < \Gamma_{(0)}$, each Λ_n is acceptable. By Proposition 3.30(c), $\bar{\Lambda} < \Gamma$. Monotonicity of $\bar{\Theta}$ implies that of $\bar{\Lambda}$: the argument is similar to that of Proposition 3.30(c).

Let $z \geqslant_{\mathrm{T}} y^\Gamma$ and $F \in \Delta(\Gamma(z))$. Obtain a partition $\operatorname{dom} F = Y \cup Z$ given by Proposition 4.13; say $Y = [T]$ and $Z = [W]$ where T is $(z, o(\Gamma))$-closed in S and W is $(z, o(\Gamma))$-open in S (where $\operatorname{dom} F = [S]$). By assumption (and because $o(\Gamma_{(0)}) = o(\Gamma)$), there are pairwise $(z, o(\Gamma))$-orthogonal $T_n \subseteq T$, $(z, o(\Gamma))$-open in T, such that $Y_n = [T_n]$ partition Y and $F \upharpoonright Y_n \in \Theta_n(z)$. Let S_n be the $(z, o(\Gamma))$-open subset of S generated by T_n. Add to S_0 those $\sigma \in W$ with no $\preccurlyeq^z_{o(\Gamma)}$-predecessor in T. Then the S_n are pairwise $(z, o(\Gamma))$-orthogonal and $X_n = [S_n]$ partition $\operatorname{dom} F$. It is not difficult to see that $F \upharpoonright [S_n] \in \Lambda_n(z)$. □

This completes the proof of Theorem 4.7, and thus of Theorem 4.4.

4.3. **Consequences of classification.** Before we prove Proposition 4.9, we verify the desired consequences of classification.

Proof of Proposition 4.5. That $\Delta(\mathbf{\Gamma})$ contains $\bigcup_n \mathbf{\Theta}_n$ follows from Remark 4.3. $\Delta(\mathbf{\Gamma})$ is closed under continuous pre-images because both $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$ are.

Suppose $\mathbf{\Lambda} \supseteq \bigcup_n \mathbf{\Theta}_n$ is a principal Wadge pointclass, i.e., there is some total $G \in 2^{\mathcal{N}}$ with

$$\mathbf{\Lambda} = \{G \circ f : f \text{ continuous}\}.$$

(Again, we are identifying subsets of $\mathcal{N}$ with their characteristic functions.)

Fix $F \in \Delta(\mathbf{\Gamma})$ total. By countable type for $o(\Gamma) = 0$, and because F is total, there is actually a *clopen* partition of $\mathcal{N}$ into sets Y_n such that $F \restriction Y_n \in \mathbf{\Theta}_n$. By Proposition 3.16, we may extend each $F \restriction Y_n$ to a total $F_n \in \mathbf{\Theta}_n$.

For each n, fix a continuous g_n such that $F_n = G \circ g_n$. Then define $g = \bigcup_n g_n \restriction Y_n$; g is continuous, with $F = G \circ g$, and so $\Delta(\mathbf{\Gamma}) \subseteq \mathbf{\Lambda}$.

It remains to show that $\Delta(\mathbf{\Gamma})$ is principal. For each n, fix H_n a universal Θ_n function. Define H by $H(n\hat{\ }x) = H_n(x)$. $H \in \Delta(\mathbf{\Gamma})$ by Remark 4.3. Every total function in $\mathbf{\Theta}_n$ is continuously reducible to H_n, and so to H by preppending n. By the previous argument, the continuous pre-images of H contains $\Delta(\mathbf{\Gamma})$. Thus $\Delta(\mathbf{\Gamma})$ is principal. □

Proposition 4.5 allows us to identify the successor of a dual pair of classes, or of an increasing ω-sequence of classes.

Proposition 4.14. *Let $\bar{\Theta}$ be an acceptable sequence. The least principal pointclass containing $\bigcup_n \mathbf{\Theta}_n$ is $\Delta(\mathbf{\Gamma})$, where $\Gamma = \mathrm{SU}_0(0, \bar{\Theta})$.*

Here again 0 is the standard description of $\{\varnothing\}$. In particular, when $\bar{\Theta}$ is an alternating sequence $\Theta, \check{\Theta}, \Theta, \ldots$, the next non-self-dual pair above the pair $\mathbf{\Theta}, \check{\mathbf{\Theta}}$ is $\Gamma = \mathrm{BiSep}(\mathbf{\Sigma}^0_1, \mathbf{\Theta}, \{\varnothing\})$ and its dual.

Proof. By the proof of Proposition 4.12, $\bar{\Theta}$ witnesses that Γ^* has countable type. Since $\mathbf{\Gamma}_0 = \{\varnothing\}$, $\bar{\Theta}$ witnesses that Γ has countable type (see the proof of Proposition 4.8(a)). The result now follows from Proposition 4.5. □

Next, to the uncountable case.

Proof of Proposition 4.6. The equality $\Delta(\mathbf{\Gamma}) = \bigcup\{\mathbf{\Lambda} : \Lambda \text{ is acceptable and } \Lambda < \Gamma\}$ follows from the definition of uncountable type, if Γ has uncountable type, and from Proposition 4.9, if $o(\Gamma) > 0$. Since each acceptably described $\mathbf{\Lambda}$ is non-self-dual, and $\Delta(\mathbf{\Gamma})$ is self-dual, it follows that $\Delta(\mathbf{\Gamma})$ is non-principal. □

Remark 4.15. Proposition 4.6 did not actually state that in these cases, the cofinality of $\Delta(\mathbf{\Gamma})$ is uncountable (under the Wadge ordering). This is because this fact requires Wadge's semi-linear-ordering principle, at least for acceptably described classes, which we will only be able to deduce later (it follows from Lemma 6.2 below; for a simpler proof, see [GTb]). Using this principle, suppose, for a contradiction, that there is a countable set Y of acceptable $\Lambda < \Gamma$ such that $\Delta(\mathbf{\Gamma}) = \bigcup\{\mathbf{\Lambda} : \Lambda \in Y\}$. We can then produce a sequence $\mathbf{\Theta}_0, \mathbf{\Theta}_1, \ldots$ with $\mathbf{\Theta}_n \subseteq \check{\mathbf{\Theta}}_{n+1}$ such that $\Delta(\mathbf{\Gamma}) = \bigcup_n \mathbf{\Theta}_n$. By Lemma 6.4 below, if we update to a sufficiently strong oracle, $\bar{\Theta}$ is an acceptable sequence. By Proposition 4.5, the sequence $\bar{\mathbf{\Theta}}$ has a least upper bound $\Delta(\mathbf{\Lambda})$, which is principal. Then $\Delta(\mathbf{\Gamma}) = \bigcup_n \mathbf{\Theta}_n \subsetneq \Delta(\mathbf{\Lambda})$. But the minimality of $\Delta(\mathbf{\Lambda})$ implies that $\Delta(\mathbf{\Lambda}) \subseteq \mathbf{\Gamma}$, and since $\Delta(\mathbf{\Lambda})$ is self-dual, it follows that $\Delta(\mathbf{\Lambda}) \subseteq \Delta(\mathbf{\Gamma})$.

The following is not needed at any point in our analysis, but it is included for the sake of aesthetics.

Proposition 4.16. *There is no acceptable description Γ which has both countable and uncountable type.*

Proof. Note that Propositions 4.5 and 4.6 together imply that no acceptable Γ with $o(\Gamma) = 0$ can have both countable and uncountable type. We can then use jump inversion to conclude the general case. Alternatively, we argue directly as follows.

Suppose that Γ has both countable and uncountable type. Let $\bar{\Theta}$ witness that Γ has countable type. By Proposition 4.14, let Υ be acceptable such that $\Delta(\boldsymbol{\Upsilon})$ is the least principal pointclass continaing $\bigcup_n \boldsymbol{\Theta}_n$. By minimality, $\Delta(\boldsymbol{\Upsilon}) \subseteq \Delta(\boldsymbol{\Gamma})$. Since $\Delta(\boldsymbol{\Upsilon})$ is principal and Γ has uncountable type, there is some acceptable $\Lambda < \Gamma$ with $o(\Lambda) \geqslant o(\Gamma)$ and $\Delta(\boldsymbol{\Upsilon}) \subseteq \boldsymbol{\Lambda}$, whence $\bigcup_n \boldsymbol{\Theta}_n \subseteq \Delta(\boldsymbol{\Lambda})$. Since $o(\Lambda) \geqslant o(\Gamma)$, the fact that $\bar{\Theta}$ witnesses that Γ has countable type, together with Proposition 3.17, shows that $\Delta(\boldsymbol{\Gamma}) \subseteq \Delta(\boldsymbol{\Lambda})$. This contradicts $\Lambda < \Gamma$. □

It follows that Theorem 4.7 gives an exact characterization of when an acceptable description has countable or uncountable type.

4.4. **Paying our debts.** It remains to give the proof of Proposition 4.9.

Proof of Proposition 4.9. We remark that the stipulation $o(\Gamma) < \omega_1$ is necessary, because if $o(\Gamma) = \omega_1$, then $\Delta(\Gamma(z)) = \{F\}$ where F is the empty partial function, and there can be no $\Lambda < \Gamma$.

Now as noted above, if the conclusion holds for some $\xi \leqslant o(\Gamma)$ then it holds for all $\xi' \leqslant \xi$. Hence the proposition holds for all Γ which have uncountable type. So it remains to show that the proposition holds for classes Γ which have countable type. Let Γ have countable type, witnessed by $\bar{\Theta}$; and let $\xi < o(\Gamma)$.

There are two main cases, depending on whether $o(\Gamma)$ is a successor or a limit.

The successor case generalises the Hausdorff-Kuratowski theorem. Suppose that $o(\Gamma)$ is a successor; by our observation above, we may assume that $o(\Gamma) = \xi + 1$.

Fix some $z \geqslant_{\mathrm{T}} y^{\Gamma}$, and let $F\colon X \to \{0,1\}$ be in $\Delta(\Gamma(z))$, where $X = [S]$ and S is a (z,ξ)-forest (in this case of the proof, we do not need to assume that S is (z,ξ)-closed in $\omega^{<\omega}$). Let (Y_n) be a partition of X into uniformly $\Sigma^0_{1+o(\Gamma)}(z)$ sets, such that $F \restriction Y_n \in \Theta_n(z)$, uniformly.

As X is $\Sigma^0_{1+\xi}(z) \wedge \Pi^0_{1+\xi}(z)$, and $\xi < o(\Gamma)$, X is $\Delta^0_{1+o(\Gamma)}(z)$. Thus the Y_n are uniformly $\Delta^0_{1+o(\Gamma)}(z)$ sets, and the function $x \mapsto n$ if $x \in Y_n$ is $\Delta^0_{1+o(\Gamma)}(z)$-measurable. By the "limit lemma" Proposition 2.14, since $o(\Gamma) = \xi + 1$, f has a z-computable (z,ξ)-approximation on S, which we also denote by f. Note that here we use that S is a (z,ξ)-forest rather than only a $(z, o(\Gamma))$-forest.

By Lemma 2.16, there is a z-computable ordinal η and a z-computable witness $\beta\colon S \to \eta$ for the convergence of f.

Let

$$\Upsilon = \mathrm{SU}^{\eta}_{\xi}(\bar{\Theta}),$$

except that we set $y^{\Upsilon} = z$ rather than $y^{\bar{\Theta}} = y^{\Gamma}$ (this is necessary since η is only z-computable). This class is well-defined since $\xi \leqslant o(\bar{\Theta})$ (as $o(\Gamma) \leqslant o(\bar{\Theta})$). By Lemma 3.27, $\Upsilon < \Gamma$. By Proposition 3.34, Υ is equivalent to some acceptable Λ with $o(\Lambda) = \xi$. So it remains to show that $F \in \Upsilon(z)$.

Define an $\Upsilon(z)$-name N with $S^N = S$, and at the root let $(f^N, \beta^N) = (f, \beta)$. For each n, let $N_{(n)}$ be a $\Theta_n(z)$-name for an extension of $F \restriction Y_n$ to all of X (Propositions 3.15 and 3.16). Then $F^N = F$, as required.

Next, we consider the case that $o(\Gamma) = \lambda$ is a limit ordinal. We perform an analysis similar to Wadge's analysis of $\boldsymbol{\Delta}^0_{\gamma}$ sets. Before we consider a particular $F \in \Delta(\boldsymbol{\Gamma})$, we first describe the possible classes we will be using for Λ. Fix some $z \geqslant_{\mathrm{T}} y^{\Gamma}$, and recall that we fixed some $\xi < \lambda$. By TSP(5), let

$$\xi < \lambda_0 < \lambda_1 < \lambda_2 < \dots,$$

be increasing, y^Γ-computable and cofinal in λ, such that for every $\sigma \in \omega^{<\omega}$, if $k = |\sigma|^z_\lambda$ is the height of σ in the tree $(\omega^{<\omega}, \preccurlyeq^z_\lambda)$, then $k = |\sigma|^z_{\lambda_k}$ and $[\sigma]^z_\lambda$ is $\Delta^0_{1+\lambda_k}(z)$, as for all τ, $\sigma \preccurlyeq^z_\lambda \tau \iff \sigma \preccurlyeq^z_{\lambda_k} \tau$.

Fix a z-computable ordinal δ. By recursion on $\alpha \leqslant \delta$ we define sequences $\bar{\Lambda}_\alpha = \Lambda_{\alpha,0}, \Lambda_{\alpha,1} \cdots$ of classes, with $y^{\bar{\Lambda}_\alpha} = z$ and $o(\Lambda_{\alpha,k}) = \lambda_k$, as follows:

- We let
$$\Lambda_{0,k} = \mathrm{SU}_{\lambda_k}(\bar{\Theta})$$
(except that as mentioned, we set the oracle to be z); this is well-defined since $\lambda_k \leqslant \lambda \leqslant o(\bar{\Theta})$.
- For any $\alpha < \delta$, we let
$$\Lambda_{\alpha+1,k} = \mathrm{SU}_{\lambda_k}(\Lambda_{\alpha,k}, \Lambda_{\alpha,k+1}, \Lambda_{\alpha,k+2}, \dots)$$
and again note that this is well-defined since $o(\Lambda_{\alpha,n}) = \lambda_n \geqslant \lambda_k$ when $n \geqslant k$.
- For limit $\alpha \leqslant \delta$, let $\alpha_0, \alpha_1, \dots$ be a z-computable, cofinal increasing sequence in α; we let
$$\Lambda_{\alpha,k} = \mathrm{SU}_{\lambda_k}(\Lambda_{\alpha_k,k}, \Lambda_{\alpha_{k+1},k+1}, \Lambda_{\alpha_{k+1},k+2}, \dots)$$

(As usual, technically, these class descriptions are defined by z-effective transfinite recursion.) We verify that:

(a) For all $\alpha \leqslant \delta$, $\bar{\Lambda}_\alpha$ is acceptable and strictly increasing: $\Lambda_{\alpha,k} < \Lambda_{\alpha,k+1}$ for all k; and
(b) For all $\beta < \alpha \leqslant \delta$, $\bar{\Lambda}_\beta < \Lambda_{\alpha,0}$.

(And again as usual, to make this work, everything is uniform; so for example, the computable functions that witness the monotonicity of $\bar{\Lambda}_\alpha$ are constructed by z-effective transfinite recursion.)

Both (a) and (b) are proved simultaneously by induction on $\alpha \leqslant \delta$. We start with $\alpha = 0$, where the classes $\Lambda_{0,k}$ are acceptable since $\bar{\Theta}$ is acceptable (Lemma 3.33). We have $\Lambda_{0,k} < \Lambda_{0,k+1}$ by Proposition 3.30(e).

For the successor case, let $\alpha < \delta$. Each $\Lambda_{\alpha+1,k}$ is acceptable by (a) and Lemma 3.33. To see that $\Lambda_{\alpha+1,k} < \Lambda_{\alpha+1,k+1}$ use Proposition 3.30(b),(e). For (b), $\bar{\Lambda}_\alpha < \Lambda_{\alpha+1,0}$ follows from Proposition 3.30(a).

For limit $\alpha \leqslant \delta$, the argument for (a) is the same as the successor case, noting that by induction, $\Lambda_{\alpha_0,0} < \Lambda_{\alpha_1,1} < \Lambda_{\alpha_2,2} < \cdots$. For (b), we again note that $\Lambda_{\alpha_k,k} < \Lambda_{\alpha,0}$ and use induction: for any $\beta < \alpha$, for sufficiently large k, $\Lambda_{\beta,m} < \Lambda_{\alpha_k,k}$.

Next, by induction on α, we show:

(c) For all $\alpha \leqslant k$, $\bar{\Lambda}_\alpha < \Gamma$.

To see this, we repeatedly use Lemma 3.27, using the fact that $o(\Gamma) > \lambda_k$.

Now again fix some $z \geqslant_{\mathrm{T}} y^\Gamma$, and let $F\colon X \to \{0,1\}$ be in $\Delta(\Gamma(z))$, where $X = [S]$ and S is a (z,ξ)-tree, i.e., is (z,ξ)-closed in $\omega^{<\omega}$. Let (Y_n) be a partition of X into sets which are uniformly $\Sigma^0_{1+\lambda}(z)$, such that $F \restriction Y_n \in \Theta_n(z)$, uniformly.

By Lemma 2.11, let S_n be pairwise (z,λ)-orthogonal subsets of S, (z,λ)-open in S, such that $Y_n = [S_n]^z_\lambda$, and such that the sets S_n are uniformly z-computable, and $\bigcup_n S_n$ also z-computable.

Let $T = S \setminus \bigcup_n S_n$. Then T is (z,λ)-closed in $\omega^{<\omega}$, and is well-founded, as X is covered by the Y_n's. By Lemma 2.17, let δ be a z-computable ordinal and let $r\colon T \to \delta$ be a rank function for $(T, \preccurlyeq^z_\lambda)$. We make the following modification: we

assume that $r(\sigma) > 0$ for all $\sigma \in T$; we then extend r to all of S by letting $r(\sigma) = 0$ for all $\sigma \in S \backslash T$.

We will show that $F \in \Lambda_{\delta,1}(z)$. For each n, let K_n be a $\Theta_n(z)$-name for $F \restriction Y_n$, with $S^{K_n} = S_n$.

For brevity, for $\sigma \in S$ let $k(\sigma) = |\sigma|^z_\lambda$; let $S_\sigma = \{\tau \in S : \tau \geqslant^z_\lambda \sigma\}$ and $X_\sigma = [S_\sigma] = [S] \cap [\sigma]^z_\lambda$. Note that S_σ is $(z, \lambda_{k(\sigma)})$-open in S (recalling that S is a (z, ξ)-tree and $\xi < \lambda_0$). Let

$$\Lambda_\sigma = \Lambda_{r(\sigma),k(\sigma)+1}.$$

We will show that for all $\sigma \in S$, $F_\sigma = F \restriction X_\sigma \in \Lambda_\sigma(z)$. Which means that uniformly in $\sigma \in S$ we z-compute a $\Lambda_\sigma(z)$-name N_σ for F_σ, with $S^{N_\sigma} = S_\sigma$.

For $\sigma \in S \backslash T$, i.e., when $r(\sigma) = 0$, this is easy, since for such σ we can compute the n such that $\sigma \in S_n$; from K_n, we can obtain a $\Theta_n(z)$-name for F_σ (i.e., we restrict K_n to S_σ); since $\bar{\Theta} < \Lambda_{0,k(\sigma)+1}$, we obtain a name N_σ as required.

For $\sigma \in T$, we compute N_σ by effective transfinite recursion. For such σ, we have $r(\sigma) > 0$ and for every $\tau \in S_\sigma \backslash \{\sigma\}$, $r(\tau) < r(\sigma)$. So by recursion, for each such τ we have already computed N_τ.

We define N_σ by (using the notation of Definition 3.28) defining $(S^{N_\sigma})_{(n)}$ for each n, and defining $(N_\sigma)_{(n)}$ for all n.

Let $R(\sigma)$ be the collection of immediate successors of σ on S: those $\tau \in S_\sigma$ such that $k(\tau) = k(\sigma) + 1$.

With oracle z, compute an injective $f \colon R(\sigma) \to \mathbb{N} \backslash \{0\}$ such that:

(i) For all $\tau \in R(\sigma)$, $\Lambda_\tau \subseteq (\Lambda_\sigma)_{(f(\tau))}$; and
(ii) The range of f is z-computable.

(i) can be achieved since $r(\tau) < r(\sigma)$; (ii) can be done by padding. The point is that $R(\sigma)$ is merely z-c.e., and we cannot even tell if it is empty or not. But in all cases, we will ensure that N_σ is a well-defined Λ_σ-name.

Now for each $m > 0$, if $m = f(\tau)$ then let $(S^{N_\sigma})_{(m)} = S_\tau$ and let $(N_\sigma)_{(m)}$ be a $(\Lambda_\sigma)_{(m)}$-name equivalent to N_τ. If $m \notin \operatorname{range} f$ let $(S^{N_\sigma})_{(m)} = \varnothing$; we set $(S^{N_\sigma})_{(0)} = \{\sigma\}$. In either case, $(N_\sigma)_{(m)}$ is trivial. This defines N_σ. What's important to note is that each $(N_\sigma)_{(m)}$ is indeed $(z, o(\Lambda_\sigma))$-open in S_σ, as $o(\Lambda_\sigma) = \lambda_{k(\sigma)+1} = \lambda_{k(\tau)}$ for $\tau \in R(\sigma)$. Since $X_\sigma = \bigcup_{\tau \in R(\sigma)} X_\tau$, we have $F^{N_\sigma} = F \restriction X_\sigma$.

Since S is (z, ξ)-closed, we have $\langle\rangle \in S$. Since $r(\langle\rangle) \leqslant \delta$ and $k(\langle\rangle) = 0$ we conclude, as promised, that $F = F_{\langle\rangle} \in \Lambda_{\delta,1}(z)$. □

5. The separation theorem(s)

In this section we give a new, simplified proof of the separation theorem of Louveau and Saint-Raymond [LSR87, LSR88b], which will be essential for Borel Wadge determinacy.

Theorem 5.1 (Louveau & Saint-Raymond)**.** *Let r be an oracle. Let Γ be an r-computable, acceptable class description; let $F \in \Gamma(r)$ be total. Let $B_0, B_1 \in \Sigma^1_1(r)$ be disjoint. Then at least one of the following holds:*

- *There is a Lipschitz$_1$ function $g \colon \mathcal{N} \to \mathcal{N}$ such that for all $x \in \mathcal{N}$, $g(x) \in B_{F(x)}$.*
- *There is a $G \in \Gamma(\Delta^1_1(r))$ such that for both $j < 2$, for all $x \in B_j$, $G(x) = 1 - j$.*

Here Lipschitz$_1$ means that $g(x) \upharpoonright n$ is determined by $x \upharpoonright n$. Recall that $\Gamma(\Delta^1_1(r)) = \bigcup \{\Gamma(w) : w \in \Delta^1_1(r)\}$. In terms of sets, if $F = 1_A$, the first option says that g reduces the pair $(A^{\complement}, A)$ to (B_0, B_1). The second option, if $G = 1_C$, says that C separates B_0 from B_1: $B_0 \subseteq C$ and $B_1 \subseteq C^{\complement}$.

As in [DGHTT], and as mentioned in [LSR88b], this theorem implies a strong version of Louveau's separation theorem ([Lou80]), which was proved by different means in [Lou83].

Theorem 5.2 (Louveau)**.** *Let r be an oracle; let Γ be an r-computable acceptable class description. Let B_0 and B_1 be disjoint $\Sigma^1_1(r)$ sets. If there is a $\mathbf{\Gamma}$ separator between B_0 and B_1, then there is a $\Gamma(\Delta^1_1(r))$ separator between B_0 and B_1.*

Proof. Apply Theorem 5.1 to F, a univeral Γ-function (see Lemma 3.13). There cannot be a continuous reduction g of F to (B_0, B_1): if $G \in \mathbf{\Gamma}$ is a separator between B_0 and B_1, then $1-F = G \circ g$ would be in $\mathbf{\Gamma}$ (Proposition 3.10), contradicting Proposition 3.14. □

Corollary 5.3 (Louveau)**.** *Let r be an oracle, and let Γ be an r-computable, acceptable description. Suppose that $F \in 2^{\mathcal{N}}$ is in $\mathbf{\Gamma}$ and is $\Delta^1_1(r)$. Then $F \in \Gamma(\Delta^1_1(r))$.*

We now prove Theorem 5.1. We assume that $o(\Gamma) < \omega_1$, as otherwise the result is immediate. The argument generalises the proof we gave in [DGHTT], where $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\xi$. Hence, we mainly give the details of how to adapt that argument to this more general setting.

Fix a $\Gamma(r)$-name N for F and r-computable trees T_0 and T_1 such that for $j = 0, 1$, $B_j = \{y : (y, z) \in [T_j]\}$. For simplicity of notation, we assume that $r = \varnothing$.

Remark 5.4. Without loss of generality we may make the following assumptions about N by modifying the functions f^N_t. Recall the notation of Definition 3.3: the leaf $\ell^N(\sigma)$ of T_Γ is defined for finite strings σ as well as infinite ones.

(i) If $|\sigma| \leqslant 1$ then $\ell^N(\sigma)$ is the leftmost leaf of T_Γ, reached by always taking the default outcome 0.
(ii) For any non-leaf $t \in T_\Gamma$, $m > 0$, and $\sigma \in \omega^{<\omega}$, if $\ell^N(\sigma) \geqslant t\hat{\ }m$ and σ is $\leqslant_{\xi_t}$-minimal such that $f^N_t(\sigma) = t\hat{\ }m$, then $\ell^N(\sigma)$ is the leftmost leaf of T_Γ extending $t\hat{\ }m$.

Making such changes to the functions f^N_t does not change the leaf $\ell^N(x)$ for infinite x, and so does not change the function $F = F^N$ on $\mathcal{N}$.

5.1. **The game.** The game $G_\Gamma(N, B_0, B_1)$ is essentially the same as the one used in [DGHTT]. Two players take turns; at each step, player I plays a single natural number, and player II plays a pair of natural numbers. As in [DGHTT], the idea is that player I plays a real $x \in \mathcal{N}$, and player II plays $y, z \in \mathcal{N}$; y is player II's proposed value for $g(x)$, and a subsequence of z should witnesses that y is in the correct B_j, depending on the value $F(x)$. The appropriate subsequence is determined by the true stage machinery.

As in [DGHTT], the game is open for player I. We index the plays so that after n steps, player I has played $(x_0, \ldots, x_{n-1})$ and player II has played (y_1, z_1), (y_2, z_2), $\ldots$, (y_n, z_n). We write $\bar{x} = (x_0, \ldots, x_{n-1})$. We need to determine whether player I has already won the game. We do this by using N and $\bar{x}$ to approximate $F(x)$; we choose a subsequence of the z_i's and check whether player II is still on the tree T_j corresponding to the guess about $F(x)$. The new aspect of this calculation is that

the level δ which we use to choose the z_i's is determined by our approximation of $F(x)$.

Specifically, recall that $F^N(\bar{x})$ is defined to be the label $\Gamma(s)$ that Γ gives to the leaf $s = \ell^N(\bar{x})$, and is defined for finite sequences $\bar{x}$, as well as elements of Baire space. Our assumption that $o(\Gamma) < \omega_1$ means that s is not the root of T_Γ, so has a parent (immediate predecessor) s^- on T_Γ; we let $\delta = \delta(\bar{x}) = \xi^\Gamma_{s^-}$, the ordinal level at which the last choice $f^N_{s^-}(\bar{x}) = s$ was made.

We then let

$$D = \left\{ i \in \{1, \dots, n\} \,:\, \bar{x} \restriction i \preccurlyeq_{\delta(\bar{x})} \bar{x} \quad \& \quad \ell^N(\bar{x} \restriction i) = \ell^N(\bar{x}) \right\};$$

note that $n \in D$, but that $0 \notin D$. Enumerate D as $D = \{a_0 < a_1 < \dots < a_{k-1}\}$, so $k \geqslant 1$; let $\bar{y} = (y_1, \dots, y_k)$ and $\bar{z} = (z_{a_0}, z_{a_1}, \dots, z_{a_{k-1}})$. We declare that player II has not yet lost the game if $(\bar{y}, \bar{z}) \in T_j$, where $j = F^N(\bar{x})$.

As this is an open game for player I, it is determined. Note that the game is computable.

5.2. **If player II has a winning strategy.** Suppose that player II has a winning strategy. For $x \in \mathcal{N}$, let (y, z) be the sequences generated by player II playing according to this strategy when player I plays x; we set $g(x) = y$. Certainly g is continuous, indeed Lipschitz$_1$. We need to show that g reduces F to (B_0, B_1), i.e., that for all x, $g(x) \in B_{F(x)}$.

To do this, fix $x \in \mathcal{N}$; let $s = \ell^N(x)$ and $j = \Gamma(s) = F(x)$; let $\delta = \delta(x) = \xi^\Gamma_{s^-}$ for s^- the parent of s on T_Γ.

Lemma 5.5. For all but finitely many $\rho \prec_\delta x$, $\ell^N(\rho) = s$.

Proof. By induction on the length of $t \preccurlyeq s$ (on T_Γ) we show that for all but finitely many $\rho \prec_\delta x$, $\ell^N(\rho) \succcurlyeq t$. This is immediate for $t = \langle\rangle$. Suppose this holds for $t \prec s$, and let $t^+ \preccurlyeq s$ be the child of t on T_Γ in the direction of s. Since $t \preccurlyeq s^-$, $\xi_t = \xi^\Gamma_t \leqslant \delta$. For all but finitely many $\rho \prec_{\xi_t} x$ we have $f_t(\rho) = f_t(x) = t^+$; and $\rho \prec_\delta x$ implies $\rho \prec_{\xi_t} x$ (TSP(1)). □

As there are infinitely many δ-true initial segments of x (TSP(3)), let $a_0 < a_1 < \cdots$ enumerate those $i > 0$ for which $x \restriction i \prec_\delta x$ and $\ell(x \restriction i) = s$. Let $v = (z_{a_0}, z_{a_1}, \dots)$. For each $k < \omega$, observe the step at the end of which player player I has played $\bar{x} = x \restriction a_k$. Since $\bar{x} \prec_\delta x$, the set D computed at this step is $\{a_0, \dots, a_k\}$ (essentially, TSP(2)). Since player II did not lose, $(y \restriction k, v \restriction k) \in T_j$. Hence, $(y, v) \in [T_j]$, so $y \in B_j$, as required.

5.3. **If player I has a winning strategy.** Suppose now that player I has a winning strategy Q. Since the game is effectively open for player I, we may take $Q \in \Delta^1_1$. We use Q to compute a Γ-name M for a separator between B_0 and B_1. For each $y \in \mathcal{N}$, we need to find a leaf $\ell^M(y)$ which will of course determine $F^M(y)$. As in [DGHTT], the idea is to examine possible second-coordinate sequences σ for player II to play along y. We use Q to pull back the true stage relations among such strings σ, and so identify "correct" strings that will indicate where we should go on T_Γ.

In [DGHTT], we used ξ-correct strings to put y inside a $\Sigma^0_{1+\xi}(Q)$ separator. In the current proof, we need to apply the same idea at every non-leaf $t \in T_\Gamma$; correct strings will guide us towards non-default outcomes $t\hat{\ }m$ for $m > 0$. This is an inductive process (on the length of t). The key difficulty is that as we go

up the tree, the ordinals ξ_t may increase; but to define f_t^M, we are only allowed to check correctness up to level ξ_t, and not higher ordinals. That is, we cannot require a string ρ, that guides us to a non-default outcome $t\hat{\ }m$, to be more than ξ_t-correct. But if $\xi_{t\hat{\ }m} > \xi_t$, such a string ρ may fail to be $\xi_{t\hat{\ }m}$-correct, and the inducive procedure of producing longer and longer correct extensions may stall. For this reason, once we get to $t\hat{\ }m$, we may need to ignore ρ and its predecessors, and use the more elaborate notion of ξ-correctness *above* ρ.

To the details. For $y \in \mathcal{N}$ and $\sigma \in \omega^{<\omega}$, let $Q(y, \sigma)$ be the sequence $\langle x_0, \dots, x_n \rangle$ which results from player I playing according to Q and player II playing $(y \restriction |\sigma|, \sigma)$. Note that $|Q(y, \sigma)| = |\sigma| + 1$.[2]

Definition 5.6. Let α be a computable ordinal, and let $y \in \mathcal{N}$. For $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma \trianglelefteq_\alpha^y \tau$ when $\sigma \preccurlyeq \tau$ and $Q(y, \sigma) \preccurlyeq_\alpha Q(y, \tau)$.

Observe that $\trianglelefteq_\alpha^y$ is Q-computable, uniformly in y and α.

Definition 5.7. Let $y \in \mathcal{N}$. A string $\sigma \in \omega^{<\omega}$ is *legal for* y if in the play of the game in which player I played $Q(y, \sigma) \restriction |\sigma|$ and player II played $(y \restriction |\sigma|, \sigma)$, player I has not yet won.[3]

Note that if σ is legal for y then every $\eta \preccurlyeq \sigma$ is also legal for y.

Definition 5.8. For $y \in \mathcal{N}$, a computable ordinal α, and $\rho \prec \sigma \in \omega^{<\omega}$, we define what it means for σ to be *α-correct* or *strongly α-correct for y above ρ*.

- σ is 0-correct for y above ρ if it is legal for y.
- σ is $(\alpha + 1)$-correct for y above ρ if it is strongly α-correct for y above ρ, and for every $\tau \trianglerighteq_\alpha^y \sigma$ which is strongly α-correct for y above ρ, we have $\tau \trianglerighteq_{\alpha+1}^y \sigma$.
- For λ a limit, σ is λ-correct for y above ρ if it is strongly β-correct for y above ρ for all $\beta < \lambda$.
- σ is strongly α-correct for y above ρ if for every τ with $\rho \prec \tau \trianglelefteq_\alpha^y \sigma$, τ is α-correct for y above ρ.

Note that only the definition of strong correctness involves ρ; the first three items are identical to [DGHTT, Def. 4.2.3]. In the definition of strong correctness, note that the extension $\rho \prec \tau$ is proper. Indeed, we do not define the notion of a string being correct above itself.

This definition of correctness may seem odd. Let us outline the plan of the construction, so that we may give a bit of motivation for it. After establishing some results about how correctness behaves, we will use this definition to construct a set G, which is our intended separator. If G fails to be a separator, witnessed by some y, then we will use y to construct a series of plays for player II which defeats Q, contrary to assumption. Fix v with $(y, v) \in [T_j]$ for the appropriate j.

Suppose $\delta(Q(y, \sigma)) = \alpha$, meaning that α is the largest ordinal used by N to send $Q(y, \sigma)$ to $\ell^N(Q(y, \sigma))$. If player II has played $(y \restriction |\sigma|, \sigma)$, and as their next

[2]In [DGHTT], we used a "string" π of length -1 so that $Q(y, \pi) = \langle\rangle$. This was needed as a base case for generating a sequence of ξ-correct strings; the string π was ξ-correct, whereas the string $\langle\rangle$ was not necessarily ξ-correct. In the current construction, as we use the more elaborate notion of correctness above a string ρ, the extension Lemma 5.12 below does not require such a base case, so we do not need the "string" π.

[3]Note that it is still possible that after player I plays the last entry of $Q(y, \sigma)$, player II has no response that would prevent I from winning; we still consider σ to be legal for y.

move they play $(y(|\sigma|), b)$, then the sequence $\bar{z}$ which is constructed to determine if they have lost the game is precisely $(\sigma(|\rho_0|), \sigma(|\rho_1|), \ldots, b)$, where (ρ_i) enumerates those $\rho \lhd^y_\alpha \sigma$, in increasing order. We would like to arrange that $\bar{z} \prec v$, so b should be $v(n)$ for the correct n. Then, by some sequence of legal plays, we would hope to return to $\ell^N(Q(y,\sigma))$. That is, we would like to find a $\tau \supseteq \sigma\hat{\ }v(n)$ with $\ell^N(Q(y,\tau)) = \ell^N(Q(y,\sigma))$.

α-correctness turns out to be sufficient to ensure this. If σ is α-correct above some ρ, there will be a $\tau \supseteq \sigma\hat{\ }v(n)$ with $\sigma \unlhd^y_\alpha \tau$ and τ is strongly α-correct above σ. So we may have player II play according to (y, τ), and then play $(y(|\tau|), v(n+1))$ after that. Then we can repeat. In this fashion we build an infinite sequence of plays such that player II never loses, always playing elements of v at the appropriate moment.

One concern is that if we have just found τ as described above, there may be ν with $\sigma \lhd^y_\alpha \nu \lhd^y_\alpha \tau$. Then when we play $v(n+1)$, the sequence $\bar{z}$ which is constructed to check if our position is legal will contain more than just the elements of v; it will also contain the extraneous element $\tau(|\nu|)$. This is why it is important that τ is *strongly* α-correct above σ: this implies that ν is α-correct above σ, and so we could have taken ν instead of τ.

We now continue with the details.

Lemma 5.9. Let $y \in \mathcal{N}$, α a computable ordinal, and $\rho \prec \sigma \in \omega^{<\omega}$.

(a) If σ is strongly α-correct for y above ρ then it is α-correct for y above ρ.
(b) σ is strongly 0-correct for y above ρ if and only if it is legal for y.
(c) If σ is α-correct for y above ρ, then it is strongly β-correct for y above ρ, for every $\beta < \alpha$.
(d) If σ is strongly α-correct for y above ρ, then any ν with $\rho \prec \nu \unlhd^y_\alpha \sigma$ is also strongly α-correct for y above ρ.
(e) If $\sigma \preccurlyeq \tau$ and both σ and τ are α-correct for y above ρ, then $\sigma \unlhd^y_\alpha \tau$.

Proof. The same as [DGHTT, Claim 4.2.4]. □

The following claim has the same proof as [DGHTT, Claim 4.2.5].

Lemma 5.10. Let α be a computable ordinal. The relations "σ is α-correct for y above ρ" and "σ is strongly α-correct for y above ρ" are:

(i) $\Delta^0_1(Q)$ if $\alpha = 0$;
(ii) $\Pi^0_\alpha(Q)$ if $\alpha \neq 0$ is finite;
(iii) $\Delta^0_\alpha(Q)$ for limit α; and
(iv) $\Pi^0_{\alpha-1}(Q)$ if α is a successor and infinite.

The following is new.

Lemma 5.11. Let α be a computable ordinal and let $y \in \mathcal{N}$. Let $\rho_0 \prec \rho_1$ and suppose that ρ_1 is strongly α-correct for y above ρ_0. Then the following are equivalent for all $\sigma \succ \rho_1$:

(1) σ is strongly α-correct for y above ρ_0.
(2) σ is strongly α-correct for y above ρ_1.

Proof. We prove the claim by induction on α. For $\alpha = 0$ the claim is immediate (see Lemma 5.9(b)). Suppose that $\alpha > 0$ and that the claim holds for all $\beta < \alpha$; and suppose that ρ_1 is strongly α-correct for y above ρ_0.

First, we observe that any $\sigma \succ \rho_1$ is α-correct for y above ρ_0 if and only if it is α-correct for y above ρ_1. This is not difficult to check using Definition 5.8, whether α is limit or a successor.

Given this equivalence, (1) $\Longrightarrow$ (2) follows easily. For (2) $\Longrightarrow$ (1), we need to consider strings ν such that $\rho_0 \prec \nu \preccurlyeq \rho_1$ and $\nu \trianglelefteq^y_\alpha \sigma$; we need to show that such ν are α-correct for y above ρ_0. We have just observed that σ is α-correct for y above ρ_0; and we are assuming that ρ_1 is (strongly) α-correct for y above ρ_0. By Lemma 5.9(e), $\rho_1 \trianglelefteq^y_\alpha \sigma$. By TSP(2), $\nu \trianglelefteq^y_\alpha \rho_1$. Since ρ_1 is strongly α-correct for y above ρ_0, ν is α-correct for y above ρ_0, as required. □

Lemma 5.12. Let $y \in \mathcal{N}$ and $\rho \in \omega^{<\omega}$. If σ is a one element extension of ρ that is legal for y, then for all computable α there is a $\tau \succcurlyeq \sigma$ that is strongly α-correct for y above ρ.

Proof. The proof of [DGHTT, Claim 4.2.6] gives this lemma, and in fact, is slightly simplified, since we do not need to consider $\nu \preccurlyeq \rho$. In the limit case, since we may have $\rho \not\trianglelefteq^y_\alpha \tau$, we take $k = |\rho| + 2$; this (and the minimality of τ) ensures that $|Q(y, \tau)|_\alpha \leqslant k$. □

We now work towards defining the Γ-name M. By induction on the length of $t \in T_\Gamma$ we define a set $U_t \subseteq \mathcal{N} \times \omega^{<\omega}$. We start by letting, for $t = \langle\rangle$ the root of T_Γ,

- $U_{\langle\rangle} = \{(y, \langle\rangle) : y \in \mathcal{N}\}$.

Suppose that U_t has been defined, and that t is not a leaf of T_Γ. We let

- $U_{t\hat{\ }0} = U_t$;
- For $m > 0$, $(y, \tau) \in U_{t\hat{\ }m}$ if there is some $\sigma \prec \tau$ such that:
  - (i) $(y, \sigma) \in U_t$;
  - (ii) τ is strongly ξ_t-correct for y above σ;
  - (iii) $\ell^N(Q(y, \tau)) \succcurlyeq t\hat{\ }m$; and
  - (iv) τ is $\trianglelefteq^y_{\xi_t}$-minimal with respect to (ii)&(iii).

By induction on the length of t, using Lemma 5.10, we observe that for non-leaf t, for all $m > 0$, the set

$$Y_{t\hat{\ }m} = \{y \in \mathcal{N} : (\exists \sigma)\ (y, \sigma) \in U_{t\hat{\ }m}\}$$

is $\Sigma^0_{1+\xi_t}(Q)$. By the effective reduction property for the class $\Sigma^0_{1+\xi_t}(Q)$, we can find, for $m > 0$, $\Sigma^0_{1+\xi_t}(Q)$ sets $X_{t\hat{\ }m}$ that reduce the sets $Y_{t\hat{\ }m}$: $X_{t\hat{\ }m} \subseteq Y_{t\hat{\ }m}$ for all m, $\bigcup_{m>0} X_{t\hat{\ }m} = \bigcup_{m>0} Y_{t\hat{\ }m}$, but the sets $X_{t\hat{\ }m}$ are pairwise disjoint.

Since $\eta_t = 1$ for all t (as the class description Γ is acceptable), we can use the sets $X_{t\hat{\ }m}$ to define a Γ-name M so that for all t, y and $m > 0$,

$$\ell^M(y) \succ t \implies [y \in X_{t\hat{\ }m} \iff \ell^M(y) \succcurlyeq t\hat{\ }m].$$

See Section 3.7. Technically, to define M (with $z^M = Q$ and $S^M = \omega^{<\omega}$), for each non-leaf t, by Lemma 2.11, we find, for $m > 0$, pairwise (Q, ξ_t)-orthogonal sets $W_{t\hat{\ }m} \subseteq \omega^{<\omega}$, uniformly Q-computable, such that $\bigcup_{m>0} W_{t\hat{\ }m}$ is Q-computable as well (and all of this is uniform in t), and such that $[W_{t\hat{\ }m}]^Q_{\xi_t} = X_{t\hat{\ }m}$; and we define f^M_t by letting $f^M_t(\nu) = m$ if $\nu \in W_{t\hat{\ }m}$, $f^M_t(\nu) = 0$ if $\nu \notin \bigcup_{m>0} W_{t\hat{\ }m}$. This defines M.[4]

[4]Note that N is a *computable* name, and so the game, the relations $\trianglelefteq^y_\alpha$, and the notions of α-correctness, all used the unrelativised true stage relations $\preccurlyeq_{\xi_t}$. In contrast, M is only Q-computable, and so needs to use the relativised relations $\preccurlyeq^Q_{\xi_t}$.

We let $G = F^M$, so $G \in \Gamma(Q)$. The following lemma concludes the proof of Theorem 5.1.

Lemma 5.13. G defines a separator between B_0 and B_1: For both $j = 0, 1$, for all $y \in B_j$, $G(y) = 1 - j$.

Proof. Suppose not; fix $j \in \{0, 1\}$ and $y \in B_j$ such that $G(y) = j$. Let $s = \ell^M(y)$ (so $j = \Gamma(s)$). Also, fix $v \in \mathcal{N}$ such that $(y, v) \in [T_j]$; we write $v = (v_0, v_1, \dots)$. Let $\delta = \xi_{s^-}$, where s^- is the parent of s on T_Γ.

Since y is fixed, for simplicity of notation:

- We omit "for y" when we talk about α-correctness and legality;
- We write $\trianglelefteq_\alpha$ for $\trianglelefteq_\alpha^y$;
- For $\sigma \in \omega^{<\omega}$, we let $\ell(\sigma) = \ell^N(Q(y, \sigma))$.

For legal $\sigma \in \omega^{<\omega}$ let

$$E = \{i < |\sigma| \,:\, (\sigma \restriction i) \lhd_\delta \sigma \quad \& \quad \ell(\sigma \restriction i) = \ell(\sigma)\}\,,$$

and let $\bar{z}(\sigma) = (\sigma(a_0), \sigma(a_1), \dots, \sigma(a_{k-1}))$ where $E = \{a_0 < a_1 < \dots < a_{k-1}\}$; we can have $E = \varnothing$, in which case $\bar{z}(\sigma)$ is the empty string. Let

$$K = \{\sigma \in \omega^{<\omega} \,:\, \sigma \text{ is legal},\ \ell(\sigma) = s,\ \&\ \bar{z}(\sigma) \prec v\}\,.$$

The point is that $\{i + 1 \,:\, i \in E\} \cup \{|\sigma| + 1\}$ is the set D of lengths which is used to determine whether I has won the game after I plays $Q(y, \sigma)$ and II plays $(y \restriction (|\sigma| + 1), \sigma\hat{\ }b)$ for any $b \in \omega$. So if σ is legal and $\Gamma(\ell(\sigma)) = j$ (in particular, if $\ell(\sigma) = s$), then for any $b \in \omega$, $\sigma\hat{\ }b$ is legal if and only if $(y \restriction (k + 1), \bar{z}(\sigma)\hat{\ }b) \in T_j$. Hence:

Claim 5.13.1. Let $\sigma \in K$; let $k = |\bar{z}(\sigma)|$. Then $\sigma\hat{\ }v_k$ is legal. Further, if $\tau \succcurlyeq \sigma\hat{\ }v_k$ is legal, $\sigma \lhd_\delta \tau$, $\ell(\tau) = s$, and there is no ν with $\sigma \lhd_\delta \nu \lhd_\delta \tau$, then $\tau \in K$.

The plan is to find some "nice" string in K from which we can keep extending to longer strings in K, as described by the claim, and thus build a winning play for II against Q, contradicting Q being a winning strategy for I. Indeed, we will show:

Claim 5.13.2. There are $\sigma \prec \theta$ such that:

(i) $\theta \in K$;
(ii) θ is strongly δ-correct over σ; and
(iii) for every $\eta \succ \sigma$ that is strongly δ-correct over σ, $\ell(\eta) = s$.

Claim 5.13.2 suffices to complete the proof of Lemma 5.13. Indeed, starting with $\theta_0 = \theta$ and σ given by the claim, we define an increasing sequence $\theta_0 \prec \theta_1 \prec \theta_2 \prec \cdots$ of strings $\theta_e \in K$ that are each strongly δ-correct over θ. Given θ_e, let $k = |\bar{z}(\theta_e)|$. As $\theta_e\hat{\ }v_k$ is legal (Claim 5.13.1), Lemma 5.12 gives us some θ_{e+1}, an extension of $\theta_e\hat{\ }v_k$, that is strongly δ-correct over θ_e; we choose a $\preccurlyeq$-minimal such string. By Lemma 5.11, θ_{e+1} is strongly δ-correct above σ, and so by choice of θ, $\ell(\theta_{e+1}) = s$. By Lemma 5.9(e), $\theta_e \lhd_\delta \theta_{e+1}$. By minimality of θ_{e+1} (and Lemma 5.9(d)), there is no ν with $\theta_e \lhd_\delta \nu \lhd_\delta \theta_{e+1}$. Hence, by Claim 5.13.1, $\theta_{e+1} \in K$.

It suffices, then, to prove Claim 5.13.2.

Let $t_1 \prec t_2 \prec \cdots \prec t_n$ list those nodes $t \prec s$ at which s takes a non-default outcome: $t\hat{\ }m \preccurlyeq s$ for some $m > 0$. Let t_i^+ denote the child of t_i extended by s. Note that $n = 0$ is possible: when s is the leftmost node on T_Γ.

If $n > 0$, then for all $i \in \{1, \dots, n\}$, $y \in Y_{t_i^+}$, and so, starting at t_n and working our way down, we can choose strings σ_i such that $(y, \sigma_i) \in U_{t_i^+}$ and such that σ_{i-1} witnesses this fact; here $\sigma_0 = \langle\rangle$. We let $\xi_i = \xi_{t_i}$. If $n = 0$ then we just choose $\sigma_0 = \langle\rangle$.

Claim 5.13.3. For all $1 \leqslant i \leqslant j \leqslant n$, $\sigma_i \trianglelefteq_{\xi_i} \sigma_j$.

Proof. We may assume that $i < j$. Repeated applications of Lemma 5.11 give that σ_j is ξ_i-correct above σ_{i-1}. The claim then follows from Lemma 5.9(e). □

Claim 5.13.4. For all $i = 1, \dots, n$, for all $\tau \triangleleft_{\xi_i} \sigma_i$, $\ell(\tau) \not\geqslant t_i^+$.

Proof. We prove the claim by induction on i. Suppose that it has been proved for all $j < i$. Let $\tau \triangleleft_{\xi_i} \sigma_i$. Let $j \leqslant i$ be least such that $\tau \prec \sigma_j$ (so $j \geqslant 1$). By Claim 5.13.3 and TSP(2), $\tau \triangleleft_{\xi_j} \sigma_j$. Since $t_i^+ \geqslant t_j^+$, it suffices to show that $\ell(\tau) \not\geqslant t_j^+$.

Suppose that $\tau = \sigma_{j-1}$. If $j = 1$, then by (i) of Remark 5.4, $\ell(\tau)$ is the leftmost leaf of T_Γ and so does not extend t_1^+. If $j > 1$ then by the induction assumption for $j - 1$, and by (ii) of Remark 5.4, $\ell(\tau)$ is the leftmost leaf of T_Γ extending t_{j-1}^+, and so does not extend t_j^+.

Suppose that $\sigma_{j-1} \prec \tau$. Then by Lemma 5.9(d), τ is ξ_j-correct above σ_{j-1}. If $\ell(\tau) \geqslant t_j^+$ then τ would contradict the minimality of σ_j (part (iv) of the definition of $U_{t_j^+}$). □

Claim 5.13.5. For all $\eta \succ \sigma_n$ that is strongly δ-correct above σ_n, $\ell(\eta) = s$.

Proof. Let $\eta \succ \sigma_n$ be strongly δ-correct above σ_n. By induction on the length of $r \preccurlyeq s$ we show that $r \preccurlyeq \ell(\eta)$. Suppose that $r \preccurlyeq s$ and $r \preccurlyeq \ell(\eta)$. There are two cases.

First, suppose that $r = t_i$ for some $i \geqslant 1$. By Lemma 5.11, η is strongly ξ_i-correct above σ_{i-1}. By Lemma 5.9(e), $\sigma_i \triangleleft_{\xi_i} \eta$. Since $f_{t_i}^N(Q(y, \sigma_i)) = t_i^+$ is not the default outcome of t_i and $Q(y, \sigma_i) \prec_{\xi_i} Q(y, \eta)$, we must have $f_{t_i}^N(Q(y, \eta)) = t_i^+$ as well, i.e., $t_i^+ \preccurlyeq \ell(\eta)$.

Otherwise, $r\hat{\ }0 \preccurlyeq s$. Let $i \in \{1, \dots, n\}$ be greatest such that $t_i \prec r$; let $i = 0$ if there is no such i. As $\delta \geqslant \xi_r$, by Lemma 5.11, η is ξ_r-correct above σ_i. If $\ell(\eta) \geqslant r\hat{\ }m$ for some $m > 0$, then η would show that there is some $\tau \succ \sigma_i$ such that $(y, \tau) \in U_{r\hat{\ }m}$, so $y \in Y_{r\hat{\ }m}$, in which case, we would not have $r\hat{\ }0 \preccurlyeq \ell^M(y) = s$. □

Claim 5.13.4 and Remark 5.4 imply that $\ell(\sigma_n) = s$, and further, that $\bar{z}(\sigma_n) = \langle\rangle$. By Claim 5.13.1, $\sigma_n\hat{\ }v_0$ is legal. By Lemma 5.12, let $\theta \geqslant \sigma_n\hat{\ }v_0$ be strongly δ-correct above σ_n, and choose minimal such.

Claim 5.13.6. $\sigma_n \triangleleft_{\xi_n} \theta$.

Proof. Same as the proof of Claim 5.13.3. □

Claim 5.13.7. If $\tau \triangleleft_\delta \theta$ and $\ell(\tau) = s$ then $\tau = \sigma_n$.

Proof. Let $\tau \triangleleft_\delta \theta$. By minimality of θ, we have $\tau \preccurlyeq \sigma_n$. Suppose that $\tau \prec \sigma_n$ (and in particular, $\sigma_n \neq \langle\rangle$, so $n > 0$). By Claim 5.13.6 and TSP(2), $\tau \triangleleft_{\xi_n} \sigma_n$, so the claim follows by Claim 5.13.4. □

By Claim 5.13.5, $\ell(\theta) = s$. Now there are two possibilities. If $\sigma_n \triangleleft_\delta \theta$ then $\bar{z}(\theta) = \langle v_0 \rangle$. If not, then $\bar{z}(\theta) = \langle\rangle$. In either case, $\bar{z}(\theta) \prec v$. Hence $\theta \in K$. This completes the proof of Claim 5.13.2, and so of Lemma 5.13, and so of Theorem 5.1. □

Remark 5.14. The proof of Theorem 4.4 relied heavily on the acceptability of the class description Γ. In contrast, the proof of Theorem 5.1 did not at all use the fact that sequences $\langle \Gamma_{t\hat{}m} \rangle$ are monotone. In the construction of the Γ-name M and its analysis, we used the fact that $\eta_t^\Gamma = 1$ for all non-leaf $t \in T_\Gamma$. This was mostly for convenience. We could have proved the theorem for all class descriptions Γ; in defining M, we would need to define level sets $Y_{t\hat{}m,\beta}$ for all $\beta < \eta_t^\Gamma$, and then use the reduction property for the classes $D_{\eta_t}(\Sigma^0_{1+\xi_t})$. The current formulation of Theorem 5.1 suffices, as we will use both it and Theorem 4.4 in the next section to show that every non-self-dual Borel Wadge class has an acceptable description.

6. Borel Wadge determinacy

As mentioned in the introduction, Louveau and Saint-Raymond [LSR88b] used Theorem 5.1 and Louveau's version of Theorem 4.4 from [Lou83] to give a proof of Borel Wadge determinacy in second-order arithmetic. In this section we use our work above to give a proof in $\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$, the subsystem of second-order arithmetic consisting, in addition to the base system RCA_0 of computable mathematics, both the principle of arithmetic transfinite recursion, and induction on Π^1_1 subsets of $\mathbb{N}$.

Working in the restricted setting of ATR_0, we need to be careful about how exactly we formalise set-theoretic notions in the language second-order arithmetic. In this setting, it is also important to fill in the details of some steps that are missing in [LSR88b].

6.1. **Formalisation.** So far in this paper, all of our reasoning has occurred within ATR_0, including the construction in Section 7. In particular, the true stage relations are constructed via arithmetic transfinite recursion, and their various properties are provable in ATR_0. As a result, for a Γ-name N (for some class description Γ) and $x \in \mathcal{N}$, the value $F^N(x)$ is well-defined in ATR_0 (without the need for evaluation maps). ATR_0 implies open determinacy, and so Theorem 5.1 is provable in ATR_0. For the rest of this section, unless otherwise stated, we work in ATR_0.

For a Borel code B and a class description Γ, we write $B \in \mathbf{\Gamma}$ to mean that there is some Γ-name N with $F^N = 1_B$. So that we can use familiar notation, we henceforth identify sets and their characteristic functions. We also use "Borel set" to mean a Borel code. For class descriptions Γ and Λ, we write $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if $B \in \mathbf{\Gamma} \implies B \in \mathbf{\Lambda}$ for all Borel B; we write $\mathbf{\Gamma} < \mathbf{\Lambda}$ if $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ and $\mathbf{\Gamma} \subseteq \check{\mathbf{\Lambda}}$.

TSP(7), together with effective transfinite recursion, shows that we can effectively pass from an acceptable class description Γ and a Γ-name N to a Borel code of F^N.

Note that the construction of a name N_Γ of a universal function F_Γ for a class description Γ is effective (given Γ).

Lemma 6.1. Let B be a Borel set, and let Γ be an acceptable class description. Then $B \in \mathbf{\Gamma}$ if and only if player I has a winning strategy in the game $G_\Gamma(N_\Gamma, B, B^\complement)$.

Proof. If player I has a winning strategy then $B \in \mathbf{\Gamma}$. If player II has a winning strategy then we get a Wadge reduction of F_Γ to $B^\complement$; by Proposition 3.14 (and Proposition 3.10), $B \notin \mathbf{\Gamma}$. □

For a Borel set B, we let

- $[B]_W$ be the Wadge class of B: the collection of pullbacks $1_B \circ g$ for continuous functions g;
- $[B]_L$ be the Lipschitz class of B: the collection of pullbacks $1_B \circ g$ for $\mathrm{Lipschitz}_1$ functions g.

An immediate corollary of Theorem 5.1 and Lemma 6.1 is:

Lemma 6.2. Let B be a Borel set, and let Γ be an acceptable class description. If $B \notin \mathbf{\Gamma}$ then $\check{\mathbf{\Gamma}} \subseteq [B]_L$.

6.2. **Well-foundedness.** Our next goal is to show that every Borel Wadge class either has an acceptable description, or is the ambiguous class of a class with an acceptable description. The plan is to find a minimal acceptable Γ such that $B \in \mathbf{\Gamma}$, and then argue that $[B]_W$ is either $\mathbf{\Gamma}$ or $\Delta(\mathbf{\Gamma})$. Thus, we need the relation $<$ on acceptably-described Wadge classes to be well-founded.

Proposition 6.3 (Martin)**.** *There is no infinite sequence $\langle \Gamma_n \rangle$ of acceptable class descriptions such that for all n, $\mathbf{\Gamma}_{n+1} < \mathbf{\Gamma}_n$.*

We can follow the standard argument (see for example [Kec95, (21.15)] or [LSR88b, Thm. 3.5]). Since the map $\Gamma \mapsto N_\Gamma$ is computable, the sequence $\langle N_{\Gamma_n} \rangle$ exists. Let $F_n = F_{\Gamma_n}$. The arguments in print use the determinacy of the games $G_L(F_n, F_{n+1})$ and $G_L(F_n^{\complement}, F_{n+1})$. This determinacy shows that there are $\mathrm{Lipschitz}_{1/2}$ reductions of F_{n+1} to F_n and to $F_n^{\complement}$. Here a function $g \colon \mathcal{N} \to \mathcal{N}$ being $\mathrm{Lipschitz}_{1/2}$ means that $g(x) \restriction k+1$ is determined by $x \restriction k$; such reductions are precisely winning strategies for player I in $G_L(F_n, F_{n+1})$.

We can show the existence of such reductions, indeed that the sequence of these reductions exists, without proving Wadge determinacy in general. We use our construction of the universal functions F_Γ. We note that $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ does not imply $\Gamma \subseteq \Lambda$ (as defined in Definition 3.20). However, this can be obtained by relativising to a not-too-complicated oracle. For a class description Γ and an oracle $z \geqslant_{\mathrm{T}} y^\Gamma$, let Γ^z be the class description which is exactly the same as Γ, except that we let $y^{\Gamma^z} = z$.

Lemma 6.4. For two acceptable class descriptions Γ and Λ, $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if and only if for some oracle $w \geqslant_{\mathrm{T}} y^\Gamma, y^\Lambda$ with $w \in \Delta^1_1(\langle y^\Gamma, y^\Lambda \rangle)$, $\Gamma^w \subseteq \Lambda^w$.

Proof. We apply Theorem 5.1 with the game $G_\Lambda(N_\Lambda, F_\Gamma, F_\Gamma^{\complement})$. Player II cannot have a winning strategy since $\mathbf{\Lambda} \nsubseteq \check{\mathbf{\Gamma}}$, (since $\mathbf{\Gamma} \nsubseteq \check{\mathbf{\Gamma}}$). Hence, there is some $w \in \Delta^1_1(y^\Gamma, y^\Lambda)$ (which we may assume computes both y^Γ and y^Λ) such that $F_\Gamma \in \Lambda(w)$. Now, given some $\Gamma(z)$-name M, where $z \geqslant_{\mathrm{T}} w$, we can effectively find $a \leqslant_{\mathrm{T}} z$ such that F^M is the a-section of F_Γ; using a $\Lambda(w)$-name of F_Γ, we find a $\Lambda(z)$-name equivalent to M (we use the effectivity of Proposition 3.10). $\square$

Recall that the complete function F_Γ was defined using an effective pairing function $(x, y) \mapsto \langle x, y \rangle$ from $\mathcal{N}^2$ to $\mathcal{N}$. The standard pairing function $(x, y) \mapsto x_0 y_0 x_1 y_1 \ldots$ has the property that for all a, $y \mapsto \langle a, y \rangle$ is $\mathrm{Lipschitz}_{1/2}$. Using this pairing function, we get:

Lemma 6.5. Let Γ be a class description. For all $B \in \mathbf{\Gamma}$, there is a $\mathrm{Lipschitz}_{1/2}$ reduction of B to F_Γ.

Lemma 6.4 holds for infinite sequences. In particular, if $\langle \Gamma_n \rangle$ is a sequence of acceptable class descriptions with $\mathbf{\Gamma}_0 > \mathbf{\Gamma}_1 > \cdots$, then there is a single oracle w such

that $\Gamma^w_0 > \Gamma^w_1 > \cdots$, uniformly. This is because ATR_0 implies open determinacy for infinite sequences of open games.[5] Thus, after relativising to an oracle, in the proof of Proposition 6.3, we may assume that $\Gamma_{n+1} < \Gamma_n$, uniformly. By Lemma 6.5, then, we get the desired Lipschitz$_{1/2}$ reductions of F_{n+1} to F_n and $F_n^{\complement}$. The last step of the proof is the fact that Borel sets have the property of Baire; this is provable in ATR_0 (the standard construction can be carried out in ATR_0; see, for example, [DFSW21]). Hence, Proposition 6.3 is provable in ATR_0.

Minimal elements. The relation $\{(\Gamma, \Lambda) : \mathbf{\Gamma} < \mathbf{\Lambda}\}$ is a relation on reals. Thus, from Proposition 6.3 we cannot immediately deduce stronger formulations of this relation being well-founded. Indeed, very strong forms of dependent choice on reals are not even provable in second-order arithmetic. To find minimal elements, we restrict to a countable set.

Lemma 6.6 ($\mathsf{ATR}_0 + \Pi^1_1$-IND)**.** Let r be an oracle, and let B be a Borel set with an r-computable code. There is an acceptable description $\Gamma \in \Delta^1_1(r)$ with $B \in \mathbf{\Gamma}$, such that for any acceptable description $\Lambda \in \Delta^1_1(r)$, if $\mathbf{\Lambda} < \mathbf{\Gamma}$ then $B \notin \mathbf{\Lambda}$.

We will shortly see, though, that the class description $\mathbf{\Gamma}$ is actually minimal among all acceptable class descriptions: the condition $\Lambda \in \Delta^1_1(r)$ can be omitted. We need the following complexity calculations.

Lemma 6.7. The following relations and sets are all Π^1_1:

(i) The collection of acceptable class descriptions;
(ii) The relation $B \in \mathbf{\Gamma}$ between a Borel code B and an acceptable class description Γ;
(iii) The relation $\mathbf{\Gamma} < \mathbf{\Lambda}$ between acceptable class descriptions Γ and Λ.

Proof. First, we observe that the relation $F^N(x) = F^M(x)$ for a real $x \in \mathcal{N}$ and Γ- and Λ-names N and M (for any class descriptions Γ and Λ, not necessarily acceptable), is Π^1_1. Hence, the relation $N \equiv M$ (Definition 3.19) is also Π^1_1. Hence, the relation $\Gamma \subseteq \Lambda$ (Definition 3.20) is Π^1_1. This implies that being a monotone sequence of class descriptions (Definition 3.29) is Π^1_1. This gives (i). Throughout, we used that the class Π^1_1 is closed under number quantifiers (effective countable unions and intersections); this is provable in ATR_0, as ATR_0 implies Σ^1_1-choice (see [Sim99, VIII.3.21]).

(ii) follows from Lemma 6.1 and open determinacy: $B \in \mathbf{\Gamma}$ if and only if player II does not have a winning strategy in the game $G(N_\Gamma, B, B^{\complement})$. (Since player II's side of the game is closed, a given strategy being a winning strategy for them is arithmetic.)

(iii) follows from the map $\Gamma \mapsto N_\Gamma$ being computable. $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if and only if $F_\Gamma \in \mathbf{\Lambda}$ (and recall that we effectively obtain a Borel name for F_Γ). □

Proof of Lemma 6.6. For simplicity, assume that r is computable. Let $\mathcal{C}$ be the collection of all $e \in \mathbb{N}$ which are Δ^1_1-codes of an acceptable class descriptions Γ such that $B \in \mathbf{\Gamma}$.

$\mathcal{C}$ is not empty: for some computable ordinal ξ, we have $B \in \Sigma^0_\xi$, and there is a computable acceptable description of this class.

[5] Specifically, a winning strategy for I in an open game is (uniformly) computed from the rank function on the well-founded tree of positions at which I has not yet won; we can join infinitely many well-founded trees to get ranking functions for all.

$\mathcal{C}$ is Π^1_1. This follows from Lemma 6.7, and the fact that being a Δ^1_1-code is Π^1_1. Note that we are not assuming that $\mathcal{C}$ exists; it is merely definable.

For $a \in \mathcal{C}$, let Γ_a be the class description coded by a. For $a, b \in \mathcal{C}$, let $a \prec b$ if $\mathbf{\Gamma}_a < \mathbf{\Gamma}_b$. By Lemma 6.7, this relation is Π^1_1.

The principle of Π^1_1-dependent choice for relations on numbers is provable in $\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$ ([Sim99, VIII.4.10]). Hence, if $\mathcal{C}$ does not have a $\prec$-minimal element, then Proposition 6.3 fails. □

6.3. **All classes are described; SLO; determinacy.** We can now obtain all the results we are after.

Theorem 6.8 ($\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$)**.** *Every Borel Wadge class is either* $\mathbf{\Gamma}$ *or* $\Delta(\mathbf{\Gamma})$ *for some accptable* Γ*.*

Proof. Let B be a Borel set. Let r be an oracle that computes a Borel code for B. Let Γ be supplied by Lemma 6.6. We show that $[B]_W = \mathbf{\Gamma}$ or $[B]_W = \Delta(\mathbf{\Gamma})$. For simplicity, assume that r is computable.

If $B \notin \check{\mathbf{\Gamma}}$ then $[B]_W = \mathbf{\Gamma}$ follows from Lemma 6.2. Indeed, in this case, we get $\mathbf{\Gamma} = [B]_W = [B]_L$.

Suppose, then, that $B \in \Delta(\mathbf{\Gamma})$. We now use Theorem 4.4. As $\Delta(\mathbf{\Gamma}) \neq \varnothing$, Γ does not have zero type. By Corollary 5.3, $B \in \Delta(\Gamma(\Delta^1_1))$. If Γ has uncountable type, or $o(\Gamma) > 0$, then by definition, or by Proposition 4.9, $B \in \mathbf{\Lambda}$ for some acceptable $\Lambda \in \Delta^1_1$ with $\Lambda < \Gamma$, contrary to the minimality of Γ. Hence, $o(\Gamma) = 0$ and Γ has countable type; let $\langle \Theta_n \rangle$ witness this fact. Since $\Theta_n \in \Delta^1_1$ and $\Theta_n < \Gamma$ for all n, the minimality of Γ implies that $B \notin \bigcup_n \mathbf{\Theta}_n$. By Lemma 6.2, and since the sequence $\langle \Theta_n \rangle$ is monotone, $\bigcup_n \mathbf{\Theta}_n \subseteq [B]_W$. By Proposition 4.5, $\Delta(\mathbf{\Gamma}) \subseteq [B]_W$. □

We can deduce the semi-linear ordering property for Borel Wadge classes:

Theorem 6.9 ($\mathsf{ATR}_0 + \Pi^1_1\text{-}\mathsf{IND}$)**.** *For any two Borel sets* A *and* B*,* $A \leqslant_W B$ *or* $B^{\complement} \leqslant_W A$*.*

Proof. If $[B]_W$ is non-self-dual, then the result follows from Lemma 6.2.

Suppose that $[B]_W = \Delta(\mathbf{\Gamma})$ for some non-self-dual $\mathbf{\Gamma}$. If $A \nleqslant_W B$ then either $A \notin \mathbf{\Gamma}$ or $A \notin \check{\mathbf{\Gamma}}$; in either case, the result again follows from Lemma 6.2. □

Finally, for Borel Wadge determinacy. The proof in the non-self-dual case is simplified by our construction of complete sets.

Theorem 6.10. *For any two Borel sets* A *and* B*, the game* $G_L(A, B)$ *is determined.*

Proof. Recall that this means: either $A \leqslant_W B$ by a Lipschitz$_1$ reduction, or $B^{\complement} \leqslant_W A$ by a Lipshitz$_{1/2}$ reduction.

If $[B]_W = \mathbf{\Gamma}$ is non-self dual, then we have already observed that $\mathbf{\Gamma} = [B]_L$. Hence, if $A \leqslant_W B$ then $A \leqslant_W B$ by a Lipschitz$_1$ reduction. Otherwise, by Lemma 6.2, $F^{\complement}_{\Gamma} \leqslant_W A$ by a Lipschitz$_1$ reduction. By Lemma 6.5, $B \leqslant_W F_{\Gamma}$ via a Lipschitz$_{1/2}$ reduction. Composing, we get $B^{\complement} \leqslant_W A$ via a Lipschitz$_{1/2}$ reduction.

Suppose that $[B]_W = \Delta(\mathbf{\Gamma})$. Hence, Γ has countable type and $o(\Gamma) = 0$; let $\bar{\Theta}$ witness this. If $A \in \bigcup_n \mathbf{\Theta}_n$ then as observed above, as $\bar{\Theta}$ is monotone, $\bigcup_n \mathbf{\Theta}_n \subseteq [B]_L$. Thus, the last case we need to consider is $[A]_W = [B]_W = \Delta(\mathbf{\Gamma})$.

Let T_A be the collection of $\sigma \in \omega^{<\omega}$ such that $A \cap [\sigma] \notin \bigcup_n \mathbf{\Theta}_n$; similarly define T_B. By Proposition 3.15, T_A and T_B are trees. Since there is a clopen

partition of Baire space into sets C with $A \cap C \in \bigcup_n \mathbf{\Theta}_n$, the tree T_A is well-founded; similarly, so is T_B. By assumption, the empty string is on both T_A and T_B. The trees exist by Δ^1_1-comprehension.

Now players I and II play an auxiliary game, taking turns choosing natural numbers, and trying to stay on their respective tree: player II wins if there is a step of the game at which player I played $\tau \notin T_A$ and player II played $\sigma \in T_B$. This is a clopen game and so one of the players has a winning strategy. We claim that the same player has a winning strategy in $G_L(A, B)$.

Suppose, for example, that II has a winning strategy in the auxiliary game. At the winning step of the auxiliary game, say player I has played $\tau \notin T_A$ and player II has played $\sigma \in T_B$; note that $|\tau| = |\sigma| + 1$. There is some n such that $A \cap [\tau] \in \mathbf{\Theta}_n$. By Proposition 3.17, there is some m such that $B \cap [\sigma\hat{\ }m] \notin \check{\mathbf{\Theta}}_n$. Player II chooses this m. Now by Lemma 6.2, $A \cap [\tau] \in [B \cap [\sigma\hat{\ }m]]_L$, so player II can use this reduction to win the rest of $G_L(A, B)$. The same argument works when player I wins the auxiliary game. $\square$

7. Developing the true stage machinery

In this section we define the true stage relations and prove they have the various desirable properties we have been using. Our development closely follows the one in [GTa]. In that paper, however, we did not use the machinery to construct subsets of Baire space; rather, we used it to build a computable countable structure. We therefore defined relations $s \leqslant_\alpha t$ for $s, t \leqslant \omega$; in the notation of the current paper, we used the restriction of $\preccurlyeq_\alpha$ to the initial segments of $x = 0^\infty$. The first step to extending that machinery to all of Baire space is uniform relativisation: $s \leqslant_{\alpha,x} t$ means that s appears α-true to t relative to x. However, it is soon observed that this is a "Lipschitz" relation: the x-use is the identity, i.e., $s \leqslant_{\alpha,x} t$ only depends on $x \restriction t$. We can therefore dispense with the oracle and write $x \restriction s \preccurlyeq_\alpha x \restriction t$, giving the notation that we used in [DGHTT] and here.

7.1. **Definition and properties of the true stages relations.** We fix a computable ordinal δ (see Section 2.2). To define the relation $\preccurlyeq_\delta$, we will need to define all relations $\preccurlyeq_\alpha$ for all $\alpha \leqslant \delta$. For $\alpha < \delta$, we let n_α be the element in the field of δ which is the least upper bound of α in δ. When we write $n_\alpha < n_\beta$ we mean the natural ordering $(\mathbb{N}, <)$, so this is different from $\alpha < \beta$.

Jumps of strings. For a string $\sigma \in \omega^{\leqslant\omega}$, let σ' denote the collection of inputs on which a universal Turing machine with oracle sequence σ halts in fewer than $|\sigma|$ many steps. Thus the jump of the empty string is empty. If $\sigma \preccurlyeq \tau$ then $\sigma' \subseteq \tau'$. We assume that if σ is a one-entry extension of τ, then σ' contains at most one more element than τ'. Thus, for every string σ we get an enumeration of the elements of σ' in order of which converged earlier (i.e. with shorter oracle). For $\sigma \in \omega^{<\omega}$ we let $p(\sigma)$ be the last element enumerated into σ' according to this ordering. In other words, $\sigma' = \tau' \cup \{p(\sigma)\}$ for some $\tau \prec \sigma$. If $\sigma' = \varnothing$ then we let $p(\sigma) = -1$. If $x \in \omega^\omega$ is infinite then we let $p(x) = \infty$.

Definition. By induction on $\alpha \leqslant \delta$ we define the relation $\sigma \preccurlyeq_\alpha \tau$. Simultaneously, for $\alpha < \delta$ we also define strings $\tau^{(\alpha)}$. Suppose that $\preccurlyeq_\alpha$ is already defined. We will show shortly that $\sigma \preccurlyeq_\alpha \tau$ implies $\sigma \preccurlyeq \tau$, so that for each τ, $\{\sigma : \sigma \prec_\alpha \tau\}$ is naturally linearly ordered.

- The string $\tau^{(\alpha)}$ is defined to be the increasing enumeration of all strings $\sigma \prec_\alpha \tau$, *excluding the first n_α many such strings.*

We will see that this exclusion is what makes the machinery work at limit levels. If τ has at most n_α-many $\prec_\alpha$-predecessors, then $\tau^{(\alpha)}$ is the empty sequence. Note that $\sigma \preccurlyeq_\alpha \tau$ implying $\sigma \preccurlyeq \tau$ also ensures that $\tau^{(\alpha)}$ is finite whenever τ is finite; we will work to show that $x^{(\alpha)}$ is infinite when x is infinite.

Each $\tau^{(\alpha)}$ is a sequence of finite sequences, i.e., an element of $(\omega^{<\omega})^{\leqslant\omega}$. However, after using some computable coding of finite sequences by natural numbers (a computable bijection between $\omega^{<\omega}$ and ω), we can consider each $\tau^{(\alpha)}$ as a string in $\omega^{\leqslant\omega}$, and so use it as an oracle in computations.

For infinite $x \in \mathcal{N}$, $x^{(\alpha)}$ will be Turing equivalent to the iteration of the Turing jump of length α, starting with x. For finite $\tau \in \omega^{<\omega}$, $\tau^{(\alpha)}$ is τ's guess about $x^{(\alpha)}$ for x extending α. The guess will be correct when $\tau \prec_\alpha x$.

The definition of $\preccurlyeq_\alpha$ is as follows. Let $\sigma, \tau \in \omega^{\leqslant\omega}$.

- $\sigma \preccurlyeq_0 \tau \iff \sigma \preccurlyeq \tau$.
- If α is a limit ordinal, then $\sigma \preccurlyeq_\alpha \tau$ if for all $\beta < \alpha$, $\sigma \preccurlyeq_\beta \tau$.
- If $\alpha < \delta$ then $\sigma \preccurlyeq_{\alpha+1} \tau$ if $\sigma \preccurlyeq_\alpha \tau$, and there is no $e < p(\sigma^{(\alpha)})$ in $(\tau^{(\alpha)})' \setminus (\sigma^{(\alpha)})'$.

In the successor case, $\alpha+1$ denotes the ordinal $\beta \leqslant \delta$ with order-type $\mathrm{otp}(\alpha)+1$. The motivation for the successor step was discussed in [DGHTT]. In brief, the definition says that when trying to compute $(\sigma^{(\alpha)})'$ (which for infinite σ will be Turing equivalent to $\sigma^{(\alpha+1)}$), the string σ only commits to numbers below $p(\sigma^{(\alpha)})$, the last number enumerated into the jump of $\sigma^{(\alpha)}$. If $\sigma \not\preccurlyeq_\alpha \tau$ then τ thinks that σ was likely wrong about the oracle $\sigma^{(\alpha)}$, and so there is no meaningful comparison between their jumps $(\sigma^{(\alpha)})'$ and $(\tau^{(\alpha)})'$. Suppose, however, that $\sigma \preccurlyeq_\alpha \tau$. We will shortly show that $\sigma^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$, and so $(\sigma^{(\alpha)})' \subseteq (\tau^{(\alpha)})'$. So σ's commitment about $(\sigma^{(\alpha)})'$ is discovered to be false by τ exactly when some number $e < p(\sigma^{(\alpha)})$, which σ claims is not in $(\sigma^{(\alpha)})'$, is enumerated into $(\tau^{(\alpha)})'$.

Basic properties. We gradually verify the properties of the relations $\preccurlyeq_\alpha$ as listed in Section 2. First, we remark that for each $\alpha < \delta$, $\preccurlyeq_\alpha$ as defined above does not depend on δ, only on α. In other words, if we only started with α and performed the same construction, we would arrive with the same relation $\preccurlyeq_\alpha$. (Note though that $\sigma^{(\alpha)}$ depends on $\alpha+1$, not just α, because its definition involves n_α.) Now, a simple induction shows that $\alpha \leqslant \beta$ and $\sigma \preccurlyeq_\beta \tau$ implies $\sigma \preccurlyeq_\alpha \tau$. This, together with the definition of $\preccurlyeq_0$, establishes TSP(1).

The following lemma includes the property (♣). In Section 2 we derived this property from TSP(6), but in fact, we need it to establish all the other properties.

Lemma 7.1. For all $\alpha \leqslant \delta$,

(a) The relation $\preccurlyeq_\alpha$ is a partial ordering.
(♢) For all $\sigma \preccurlyeq \rho \preccurlyeq \tau \in \omega^{\leqslant\omega}$, if $\sigma, \rho \preccurlyeq_\alpha \tau$, then $\sigma \preccurlyeq_\alpha \rho$.
(♣) For all $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau \in \omega^{\leqslant\omega}$, if $\sigma \preccurlyeq_{\alpha+1} \tau$ then $\sigma \preccurlyeq_{\alpha+1} \rho$.
(b) If $\sigma \preccurlyeq_\alpha \tau$ then $\sigma^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$.
(c) If $\sigma \preccurlyeq_{\alpha+1} \tau$ then $p(\sigma^{(\alpha)}) \leqslant p(\tau^{(\alpha)})$.

The converse of (b) may fail because of the exclusion of the first n_α many $\prec_\alpha$-predecessors in the definition of $\tau^{(\alpha)}$. Nonetheless, this converse is "close" to the truth, and this gives an informal motivation for two of the properties illustrated in

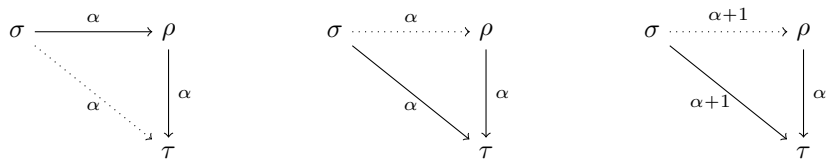

FIGURE 1. From left to right: the transitivity of $\preccurlyeq_\alpha$, the property $(\diamondsuit)_\alpha$, and the property $(\clubsuit)_\alpha$ (given $(\diamondsuit)_\alpha$).

Fig. 1. The relation $\preccurlyeq_\alpha$ is transitive because if $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$ then $\sigma^{(\alpha)} \preccurlyeq \rho^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$, and so $\sigma^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$. The property $(\diamondsuit)$ is similar: if $\sigma^{(\alpha)}, \rho^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$ then $\sigma^{(\alpha)}$ and $\rho^{(\alpha)}$ are comparable.

Proof. First, we observe that $(\mathrm{a})_\alpha$ and $(\diamondsuit)_\alpha$ imply $(\mathrm{b})_\alpha$, $(\mathrm{c})_\alpha$, and $(\clubsuit)_\alpha$.

By the definition of $\tau^{(\alpha)}$, to show $(\mathrm{b})_\alpha$ we need to show that if $\rho \prec \sigma \preccurlyeq_\alpha \tau$ then $\rho \preccurlyeq_\alpha \sigma \leftrightarrow \rho \preccurlyeq_\alpha \tau$. One direction follows from $(\mathrm{a})_\alpha$, the other from $(\diamondsuit)_\alpha$.

For $(\mathrm{c})_\alpha$, suppose that $\sigma \preccurlyeq_{\alpha+1} \tau$. Then $\sigma \preccurlyeq_\alpha \tau$, which by $(\mathrm{b})_\alpha$ implies that $\sigma^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$, and so $(\sigma^{(\alpha)})' \subseteq (\tau^{(\alpha)})'$. Suppose that $p(\tau^{(\alpha)}) < p(\sigma^{(\alpha)})$. Then $\tau^{(\alpha)}$ is finite, and $p(\tau^{(\alpha)}) \notin (\sigma^{(\alpha)})'$, so $e = p(\tau^{(\alpha)})$ violates the definition of $\sigma \preccurlyeq_{\alpha+1} \tau$.

For $(\clubsuit)_\alpha$, suppose that $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$ and $\sigma \preccurlyeq_{\alpha+1} \tau$. By $(\mathrm{b})_\alpha$, $\sigma^{(\alpha)} \preccurlyeq \rho^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$, and so $(\sigma^{(\alpha)})' \subseteq (\rho^{(\alpha)})' \subseteq (\tau^{(\alpha)})'$. Suppose that $e \in (\rho^{(\alpha)})' \backslash (\sigma^{(\alpha)})'$. Then $e \in (\tau^{(\alpha)})' \backslash (\sigma^{(\alpha)})'$; since $\sigma \preccurlyeq_{\alpha+1} \tau$, $e > p(\sigma^{(\alpha)})$, so $\sigma \preccurlyeq_{\alpha+1} \rho$.

By induction on $\alpha \leqslant \delta$ we prove that both $(\mathrm{a})_\alpha$ and $(\diamondsuit)_\alpha$ hold.

For $\alpha = 0$ this is easy. For limit α, both $(\mathrm{a})_\alpha$ and $(\diamondsuit)_\alpha$ follow from the inductive assumption and the fact that the relation $\preccurlyeq_\alpha$ is the intersection of the relations $\preccurlyeq_\beta$ for $\beta < \alpha$.

It remains to check the successor case. Let $\alpha < \delta$, and suppose that $(\mathrm{a})_\alpha$ and $(\diamondsuit)_\alpha$ hold (and so also $(\mathrm{b})_\alpha$, $(\mathrm{c})_\alpha$ and $(\clubsuit)_\alpha$). We verify that $(\mathrm{a})_{\alpha+1}$ and $(\diamondsuit)_{\alpha+1}$ hold as well.

For $(\mathrm{a})_{\alpha+1}$, first observe that $\tau \preccurlyeq_{\alpha+1} \tau$ follows from $\tau \preccurlyeq_\alpha \tau$ and $(\tau^{(\alpha)})' = (\tau^{(\alpha)})'$. We check transitivity of $\preccurlyeq_{\alpha+1}$. Suppose that $\sigma \preccurlyeq_{\alpha+1} \rho \preccurlyeq_{\alpha+1} \tau$. Then $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$, and so by $(\mathrm{a})_\alpha$, $\sigma \preccurlyeq_\alpha \tau$. By $(\mathrm{b})_\alpha$, $\sigma^{(\alpha)} \preccurlyeq \rho^{(\alpha)} \preccurlyeq \tau^{(\alpha)}$, and so $(\sigma^{(\alpha)})' \subseteq (\rho^{(\alpha)})' \subseteq (\tau^{(\alpha)})'$. Suppose that $e \in (\tau^{(\alpha)})' \backslash (\sigma^{(\alpha)})'$; we need to show that $e > p(\sigma^{(\alpha)})$. There are two cases. If $e \in (\rho^{(\alpha)})'$ then $e > p(\sigma^{(\alpha)})$ by the assumption $\sigma \preccurlyeq_{\alpha+1} \rho$. Otherwise, $e \in (\tau^{(\alpha)})' \backslash (\rho^{(\alpha)})'$ and so $e > p(\rho^{(\alpha)})$ by the assumption $\rho \preccurlyeq_{\alpha+1} \tau$. By $(\mathrm{c})_\alpha$ and the assumption $\sigma \preccurlyeq_{\alpha+1} \rho$ we get $p(\rho^{(\alpha)}) \geqslant p(\sigma^{(\alpha)})$.

$(\diamondsuit)_{\alpha+1}$ follows from $(\diamondsuit)_\alpha$ and $(\clubsuit)_\alpha$. Suppose that $\sigma \preccurlyeq \rho \preccurlyeq \tau$ and that $\sigma, \rho \preccurlyeq_{\alpha+1} \tau$. By $(\diamondsuit)_\alpha$, $\sigma \preccurlyeq_\alpha \rho$. By $(\clubsuit)_\alpha$, $\sigma \preccurlyeq_{\alpha+1} \rho$. □

We can conclude:

Lemma 7.2. For all $\alpha < \delta$, if $\sigma \preccurlyeq_\alpha \tau$ then $\sigma \preccurlyeq_{\alpha+1} \tau$ if and only if $p(\rho^{(\alpha)}) \geqslant p(\sigma^{(\alpha)})$ for all finite ρ with $\sigma \preccurlyeq_\alpha \rho \preccurlyeq_\alpha \tau$.

Note that this almost gives us TSP(6), by letting $p_{\alpha+1}(\sigma) = p(\sigma^{(\alpha)})$; what remains is showing that $\sigma \mapsto \sigma^{(\alpha)}$ is computable.

Proof. We have required that convergence of the universal machine is delayed so that when we extend an oracle by one entry, at most one halting computation is

added to the jump. If ρ_0 is an immediate $\prec_\alpha$-predecessor of ρ_1, then $|\rho_1^{(\alpha)}| \leqslant |\rho_0^{(\alpha)}| + 1$. Hence,

$$(\tau^{(\alpha)})' \backslash (\sigma^{(\alpha)})' = \big\{p(\rho^{(\alpha)}) \,:\, \sigma \prec_\alpha \rho \preccurlyeq_\alpha \tau,\ \rho \text{ is finite, and } p(\rho^{(\alpha)}) \neq p(\sigma^{(\alpha)})\big\}. \qquad \square$$

Structurally, the transitivity of $\preccurlyeq_\alpha$ and the property $(\diamondsuit)_\alpha$ together say that $(\omega^{\leqslant\omega}, \preccurlyeq_\alpha)$ is a forest; every σ has height at most $|\sigma|$ in that forest. We shall soon verify that this forest is in fact a tree (it has a single root), and that each infinite x has height ω in this tree.

When we say nothing. For a string σ and $\alpha \leqslant \delta$ we let $|\sigma|_\alpha$ be the number of (proper) $\prec_\alpha$-predecessors of σ, in other words, its height in the forest $(\omega^{\leqslant\omega}, \preccurlyeq_\alpha)$.

Lemma 7.3. Suppose that $\alpha < \delta$, $\sigma \preccurlyeq_\alpha \tau$, and $|\sigma|_\alpha \leqslant n_\alpha$. Then $\sigma \preccurlyeq_{\alpha+1} \tau$.

Proof. The assumption implies that $\sigma^{(\alpha)}$ is the empty string, and so that $p(\sigma^{(\alpha)}) = -1$. $\square$

By induction on $\alpha \leqslant \delta$, we see that for all $\tau \in \omega^{\leqslant\omega}$, $\langle\rangle \preccurlyeq_\alpha \tau$, where again $\langle\rangle$ denotes the empty string. Together with $(\diamondsuit)_\alpha$ of Lemma 7.1, this gives TSP(2). We can also deal with the limit case:

Proof of TSP(5). Let $\lambda \leqslant \delta$ be a limit ordinal. The first part is by definition of $\preccurlyeq_\lambda$. We verify the rest. Starting with any $\lambda_0 < \lambda$, recursively define λ_k by letting

$$\lambda_k = \max\{\lambda_{k-1} + 1, \alpha \,:\, \alpha < \lambda \ \&\ n_\alpha \leqslant k\}.$$

The definition ensures that $\langle\lambda_k\rangle$ is cofinal in λ and computable. Now we show:

$(*)$ for all $\sigma \in \omega^{<\omega}$, if $|\sigma|_{\lambda_k} \leqslant k$, then for all τ, $\sigma \preccurlyeq_\lambda \tau \iff \sigma \preccurlyeq_{\lambda_k} \tau$.

To see this, let $k < \omega$ and suppose that $|\sigma|_{\lambda_k} \leqslant k$ and $\sigma \preccurlyeq_{\lambda_k} \tau$. By induction on $\alpha \in [\lambda_k, \lambda]$, we show that $\sigma \preccurlyeq_\alpha \tau$. If α is limit we use the definition of $\preccurlyeq_\alpha$. Suppose that $\lambda_k \leqslant \alpha < \lambda$ and $\sigma \preccurlyeq_\alpha \tau$. By choice of λ_k, $n_\alpha \geqslant k$. Since $\alpha \geqslant \lambda_k$, $|\sigma|_\alpha \leqslant |\sigma|_{\lambda_k} \leqslant k$. Hence, by Lemma 7.3, $\sigma \preccurlyeq_{\alpha+1} \tau$. This establishes $(*)$.

To prove TSP(5), let $k < \omega$; we show that if $|\sigma|_{\lambda_k} > k$ then $|\sigma|_\lambda > k$. For suppose that $|\sigma|_{\lambda_k} > k$. Then there are $k+1$ distinct $\rho \prec_{\lambda_k} \sigma$ with $|\rho|_{\lambda_k} \leqslant k$. For each such ρ, by $(*)$, we have $\rho \prec_\lambda \sigma$. Hence $|\sigma|_\lambda > k$. $\square$

Computability. Restricted to finite strings, the true stage relations are computable. As discussed above, this is uniform in α.

Proof of TSP(4). The relations $\sigma \preccurlyeq_\alpha \tau$ for $\alpha \leqslant \delta$ and the functions $\tau \mapsto \tau^{(\alpha)}$ for $\alpha < \delta$ are computed by simultaneous recursion on $|\tau|$ (note, this is not effective transfinite recursion on α). If we have decided, for all $\sigma \preccurlyeq \tau$, whether $\sigma \preccurlyeq_\alpha \tau$, then we can compute $\tau^{(\alpha)}$. The definition of the relation $\preccurlyeq_{\alpha+1}$ shows that we can then decide whether $\sigma \preccurlyeq_{\alpha+1} \tau$ or not.

Now given $\sigma \preccurlyeq \tau$, the algorithm for computing whether $\sigma \preccurlyeq_\alpha \tau$ is given with the aid of Lemma 7.3. Enumerate the set $\{\beta < \delta \,:\, n_\beta \leqslant |\sigma|\} \cup \{0, \delta\}$ as $0 = \beta_0 < \beta_1 < \cdots < \beta_k = \delta$. Then $\sigma \preccurlyeq_{\beta_0} \tau$. If $i < k$ and we have decided that $\sigma \preccurlyeq_{\beta_i} \tau$, then we check if $\sigma \preccurlyeq_{\beta_i+1} \tau$. If so, then we can conclude that $\sigma \preccurlyeq_{\beta_{i+1}} \tau$: this uses $|\sigma| \geqslant |\sigma|_\beta$ for all β, so for all $\beta \in (\beta_i, \beta_{i+1})$ we have $|\sigma|_\beta \leqslant n_\beta$. $\square$

As mentioned above (after Lemma 7.2), TSP(6) follows as well.

Remark 7.4. The continuity of the relations $\preccurlyeq_\alpha$ implies that for all $\sigma \preccurlyeq \tau \in \omega^{\leqslant\omega}$, $\max\{\gamma \leqslant \delta : \sigma \preccurlyeq_\gamma \tau\}$ must exist. The function $(\sigma, \tau) \mapsto \max\{\gamma \leqslant \delta : \sigma \preccurlyeq_\gamma \tau\}$ is computable for finite σ and τ (by a search over α, or by following the algorithm described in the proof above).

True stages. For $x \in \mathcal{N}$ and $\alpha \leqslant \delta$ let

$$D_x^\alpha = \{\sigma : \sigma \prec_\alpha x\}.$$

So $x^{(\alpha)}$ is the increasing enumeration of D_x^α after removing the first n_α many strings. If α is a limit ordinal then $D_x^\alpha = \bigcap_{\beta<\alpha} D_x^\beta$. If D_x^α is infinite, then $x^{(\alpha)} = \bigcup_{\sigma \in D_x^\alpha} \sigma^{(\alpha)}$.

Lemma 7.5. Suppose that D_x^α is infinite, and let $\sigma \in D_x^\alpha$. The following are equivalent:

(1) $\sigma \in D_x^{\alpha+1}$;
(2) for all $\tau \succcurlyeq_\alpha \sigma$ in D_x^α, $\sigma \preccurlyeq_{\alpha+1} \tau$;
(3) for infinitely many $\tau \succcurlyeq_\alpha \sigma$ in D_x^α, $\sigma \preccurlyeq_{\alpha+1} \tau$.

Proof. The implication (1) → (2), follows from the property (♣). For (2) → (3), note that if $\tau \in D_x^\alpha$ and $\sigma \preccurlyeq \tau$ then $\sigma \preccurlyeq_\alpha \tau$ (TSP(2)). For (3) → (1), we use the fact that $(x^{(\alpha)})' = \bigcup_{\tau \in D_x^\alpha} (\tau^{(\alpha)})'$. So if $\sigma \not\prec_{\alpha+1} x$ then there is some $e \in (x^{(\alpha)})' \setminus (\sigma^{(\alpha)})'$ with $e < p(\sigma^{(\alpha)})$; for large enough τ in D_x^α, $e \in (\tau^{(\alpha)})'$ and so $\sigma \not\preccurlyeq_{\alpha+1} \tau$. □

Proposition 7.6. *For every $x \in \mathcal{N}$ and $\alpha \leqslant \delta$, D_x^α is infinite.*

Proof. By induction on α. D_x^0 is the set of all finite initial segments of x. At successor levels we use non-deficiency stages (following Dekker in [Dek54]). Namely, let $m \in \mathbb{N}$; let σ in D_x^α be such that $|\sigma| \geqslant m$ and $p(\sigma^{(\alpha)})$ is minimal among $p(\tau^{(\alpha)})$ for all $\tau \in D_x^\alpha$ with $|\tau| \geqslant m$. By Lemma 7.2, $\sigma \in D_x^{\alpha+1}$.

Suppose that α is a limit ordinal, and suppose that for all $\beta < \alpha$, D_x^β is infinite. Given $m \in \mathbb{N}$, we find some $\gamma < \alpha$ such that $n_\gamma \geqslant m$ and for all $\beta \in (\gamma, \alpha)$, $n_\beta > n_\gamma$. This can be done since α is a limit ordinal (and $\mathbb{N}$ is well-founded). We then let $\sigma \prec x$ be the string in D_x^γ with $|\sigma|_\gamma = n_\gamma$. Then $|\sigma| \geqslant m$ (as $|\sigma| \geqslant |\sigma|_\gamma$). For all $\beta \in [\gamma, \alpha)$, $|\sigma|_\beta \leqslant n_\beta$ (as $|\sigma|_\beta \leqslant |\sigma|_\gamma$ and $n_\gamma \leqslant n_\beta$). So by induction on such β, aided by Lemma 7.3, we see that $\sigma \in D_x^\beta$. □

We obtain:

Proof of TSP(3). This is proved by induction on α. At the successor step, if $\sigma \in D_x^\alpha \setminus D_x^{\alpha+1}$, then any infinite path above σ in $(\{\rho : \rho \prec x\}, \preccurlyeq_{\alpha+1})$ must also induce an infinite path in $(\{\sigma : \sigma \prec x\}, \preccurlyeq_\alpha)$, and thus must be a subset of D_x^α; but by Lemma 7.5, $\sigma \preccurlyeq_{\alpha+1} \tau$ for only finitely many $\tau \in D_x^\alpha$. □

Remark 7.7. Proposition 7.6 implies that for all $x \in \mathcal{N}$ and $\alpha < \delta$ $x^{(\alpha)}$, is an infinite sequence, and so $p(x^{(\alpha)}) = \infty$. As a result, in TSP(6), we do not need to restrict to finite ρ.

The effective Borel hierarchy. Recall that the Σ^0_α sets, for $1 \leqslant \alpha \leqslant \delta$, are defined by induction on α. The Σ^0_1 sets are the effectively open sets. For $\alpha > 1$, the Σ^0_α sets are those which are effective unions of sets which are Π^0_β for some $\beta < \alpha$.[6]

We can now verify the last of the properties listed in Section 2. We use the notation of that section regarding α-forests and the subsets of Baire space they define. For $\sigma \in \omega^{<\omega}$ we let $[\sigma]_\alpha = \{x \in \mathcal{N} : \sigma \prec_\alpha x\}$; this is $[W]_\alpha$, where W consists of the $\preccurlyeq_\alpha$-extensions of σ.

Proof of TSP(7). We prove TSP(7) by effective transfinite induction on α. Thus, it will be important that we work uniformly: by recursion on α, we show how to effectively translate between indices of $\Sigma^0_{1+\alpha}$ sets and c.e. indices of subsets of $\omega^{<\omega}$.

For $\alpha = 0$, TSP(7) is immediate, as $\preccurlyeq_0$ is $\preccurlyeq$, and the Σ^0_1 sets are the effectively open ones.

Suppose that $\alpha \leqslant \delta$ is a limit ordinal.

In one direction, since $\Sigma^0_{1+\alpha}$ is closed under effective unions, it suffices to show that for all $\sigma \in \omega^{<\omega}$, $[\sigma]_\alpha$ is $\Sigma^0_{1+\alpha}$. TSP(5) implies that it is in fact $\Sigma^0_{<\alpha}$.

In the other direction, since $1 + \alpha = \alpha$, and since c.e. sets are closed under effective unions, it suffices to show that every $\Sigma^0_{<\alpha}$ set is $[U]_\alpha$ for some c.e., $\preccurlyeq_\alpha$-upwards closed U. Let A be $\Sigma^0_{1+\beta}$ for some $\beta < \alpha$. By induction, there is a c.e. set U, upwards closed in $\preccurlyeq_\beta$, such that $A = [U]_\beta$. Then U is also $\preccurlyeq_\alpha$-upwards closed, and $[U]_\alpha = [U]_\beta = A$ (see Remark 2.8).

For the successor case, suppose that $\alpha < \delta$. In one direction, it suffices to show that for all $\sigma \in \omega^{<\omega}$, $[\sigma]_{\alpha+1}$ is $\Sigma^0_{1+\alpha+1}$. In fact, it is $\Pi^0_{1+\alpha}$. To see this, let T be the collection of $\tau \in \omega^{<\omega}$ such that either $\tau \preccurlyeq_\alpha \sigma$ or $\sigma \preccurlyeq_{\alpha+1} \tau$. Then T is computable; (♣) implies that T is $\preccurlyeq_\alpha$-downwards closed (it is a subtree of $(\omega^{<\omega}, \preccurlyeq_\alpha)$), so $[T]_\alpha$ is $\Pi^0_{1+\alpha}$. Lemma 7.5 implies that $[\sigma]_{\alpha+1} = [T]_\alpha$.

In the other direction, it suffices to show that every $\Pi^0_{1+\alpha}$ set A is $[U]^{\prec}_{\alpha+1}$ for some c.e. U, upwards closed in $\preccurlyeq_{\alpha+1}$. Given such A, by induction, let T be a computable subtree of $(\omega^{<\omega}, \preccurlyeq_\alpha)$ with $A = [T]_\alpha$. Note that for all $x \in \mathcal{N}$, $x \in A$ if and only if for all σ listed in $x^{(\alpha)}$, $\sigma \in T$. This is a Π^0_1 property of $x^{(\alpha)}$, so there is some $e \in \mathbb{N}$ such that for all $x \in \mathcal{N}$, $x \in A \iff e \notin (x^{(\alpha)})'$. Let $W = \{\sigma \in \omega^{<\omega} : e \notin (\sigma^{(\alpha)})' \ \&\ p(\sigma^{(\alpha)}) > e\}$. Then W is computable, and is $\preccurlyeq_{\alpha+1}$-upwards closed. We claim that $A = [W]_{\alpha+1}$. For all $\sigma \in W$, for all $x \in [\sigma]_{\alpha+1}$, $e \notin (x^{(\alpha)})'$ by definition of $\preccurlyeq_{\alpha+1}$. On the other hand, if $e \notin (x^{(\alpha)})'$, find $\sigma \prec_{\alpha+1} x$ sufficiently long so that $p(\sigma^{(\alpha)}) > e$; then $\sigma \in W$. □

Relativising true stages. Given an oracle z and a z-computable ordinal α, we define the relation $\preccurlyeq^z_\alpha$ precisely as above; however, the universal machine which calculates the jumps of strings is given an extra oracle tape, containing z. Write $(z, \sigma)^{(\alpha)}$ instead of $\sigma^{(\alpha)}$.

For finite σ, the universal machine computing $(z, \sigma)^{(\alpha)}$ is stopped after $|\sigma|$ many steps, and so consults only $z \restriction |\sigma|$. If α is computable (rather than z-computable),

[6] To make the notion of an effective union of $\Pi^0_{<\alpha}$ sets precise, we need to define an effective indexing of each class. For each $\alpha \leqslant \delta$ with $\alpha > 0$, let $\langle W_{e,\alpha} \rangle$ be an effective enumeration of the c.e. subsets of $\omega^{<\omega} \times \alpha$. We let $A_{(e,1)} = \{x \in \mathcal{N} : (\exists \sigma \prec x)\ (\sigma, n_0) \in W_{e,1}\}$. For $\alpha > 1$ we let $A_{(e,\alpha)} = \bigcup \{\mathcal{N} \setminus A_{(i,\beta)} : (i, n_\beta) \in W_{(e,\alpha)}\}$. The sets $A_{e,\alpha}$ form an effective enumeration of the Σ^0_α sets.

this shows that for finite τ, determining whether $\sigma \preccurlyeq^z_\alpha \tau$ depends only on $z \restriction |\tau|$, as promised in Section 2.

Further, the easier directions in the proof of TSP(7) give:

Lemma 7.8. Let α be a computable ordinal. For any $\tau \in \omega^{<\omega}$, the set

$$\{(z, x) \in \mathcal{N}^2 \,:\, \tau \prec^z_\alpha x\}$$

is $\Sigma^0_{1+\alpha}$.

As usual, this is uniform in α, z and τ.

7.2. **Interlude: the Kuratowski-Sierpiński theorem.** We still need to prove Propositions 2.14 and 2.18 to 2.21 and Lemma 2.16. Before we do so, we briefly comment on iterated Turing jumps and the Kuratowski-Sierpiński characterisation of the classes $\mathbf{\Sigma}^0_\xi$.

In [DGHTT], we justified TSP(7) as follows. We started with the aforesaid characterisation.

Theorem 7.9 (Kuratowski [Kur33], Sierpiński [Sie33])**.** *Let $\xi \geqslant 1$ be a countable ordinal. A set $A \subseteq \mathcal{N}$ is $\mathbf{\Sigma}^0_\xi$ if and only if there is a closed set $K \subseteq \mathcal{N}$, a bijection $h \colon \mathcal{N} \to K$, and an open set $O \subseteq \mathcal{N}$, such that:*

(i) h is $\mathbf{\Sigma}^0_\xi$-measurable;
(ii) h^{-1} is continuous; and
(iii) $A = h^{-1}[O]$.

This theorem allows us to prove facts about $\mathbf{\Sigma}^0_\xi$ sets by changing the topology using a "generalised homeomorphism" such as h. For example, this allowed Kuratowski to extend Hausdorff's analysis of the $\mathbf{\Delta}^0_2$ sets to the classes $\mathbf{\Delta}^0_{\xi+1}$.

In [DGHTT], we stated that Theorem 7.9 can be effectivised; we can take h to be an iteration of the Turing jump of length α, where $\xi = 1 + \alpha$. We then argue that D^α_x, as defined above, is (uniformly) Turing equivalent to this iteration of the jump starting with x. We can then pull back the open set O to obtain a c.e. set U satisfying $A = [U]_\alpha$.

Here, rather, we directly derived TSP(7) from the other basic properties of the true stage relations, and the definition of the classes $\Sigma^0_{1+\alpha}$. We can now argue in the other direction, deriving an effective version of Theorem 7.9, which in turn, by relativisation, implies Theorem 7.9.

For a computable ordinal α and $x \in \mathcal{N}$, let $x^{\{\alpha\}}$ be the increasing enumeration of $D^\alpha_x = \{\sigma \,:\, \sigma \prec_\alpha x\}$.[7]

Lemma 7.10. Let α be a computable ordinal. A set $A \subseteq \mathcal{N}$ is $\Sigma^0_{1+\alpha}$ if and only if there is an effectively open set $O \subseteq (\omega^{<\omega})^\omega$ such that for all x, $x \in A \iff x^{\{\alpha\}} \in O$.

Proof. We of course use TSP(7). Suppose that $A = [U]_\alpha$ for some c.e., $\preccurlyeq_\alpha$-upwards-closed U. Let $O = \{y \in (\omega^{<\omega})^\omega \,:\, \operatorname{range} y \cap U \neq 0\}$. Then $x \in A \iff x^{\{\alpha\}} \in O$.

In the other direction, let O be an effectively open subset of $(\omega^{<\omega})^\omega$; so the set of $\mu \in (\omega^{<\omega})^{<\omega}$ such that $[\mu] \subseteq O$ is c.e. Let U be the set of $\sigma \in \omega^{<\omega}$

[7]The reason we use $x^{\{\alpha\}}$ rather than $x^{(\alpha)}$ is that the latter does not depend solely on α, as it requires n_α. That is, it also depends on the upper bound of α in an ordinal $\alpha + 1 > \alpha$. Observe, though, that once such $\alpha + 1$ is chosen, we get $x^{(\alpha)} \equiv_{\mathrm{T}} x^{\{\alpha\}}$, uniformly: given the least element of $x^{(\alpha)}$ we can computably list its n_α-many $\preccurlyeq_\alpha$-predecessors.

such that the increasing enumeration μ of $\{\rho : \rho \preccurlyeq_\alpha \sigma\}$ satisfies $[\mu] \subseteq O$. Then $x^{\{\alpha\}} \in O \iff x \in [U]_\alpha$. □

Lemma 7.11. Let α be a computable ordinal. The map $x \mapsto x^{\{\alpha\}}$:

(a) has Π^0_1 graph and Π^0_1 image;
(b) has computable (effectively continuous) inverse;
(c) is $\Sigma^0_{1+\alpha}$-measurable.

Indeed, the map $x \mapsto x^{\{\alpha\}}$ is universal for $\Sigma^0_{1+\alpha}$-measurable functions: for any computable Polish space X and any $\Sigma^0_{1+\alpha}$-measurable function $h\colon \mathcal{N} \to X$, there is a partial computable map $g\colon (\omega^{<\omega})^\omega \to X$ such that $h(x) = g(x^{\{\alpha\}})$.

Proof. (a) follows from TSP(3): Let $K = \{x^{\{\alpha\}} : x \in \mathcal{N}\}$. Then $y \in K$ if and only if y is an infinite path in $(\omega^{<\omega}, \preccurlyeq_\alpha)$; $y = x^{\{\alpha\}}$ if and only if $y \in K$ and $x = \bigcup_n y(n)$.[8] (b) is immediate. (c) follows from Lemma 7.10.

For universality, let $h\colon \mathcal{N} \to X$ be $\Sigma^0_{1+\alpha}$-measurable. Let $W_0, W_1, \dots$ be an effective listing of the effectively open subsets of X; so $h^{-1}[W_e]$ are uniformly $\Sigma^0_{1+\alpha}$; by Lemma 7.10, there are uniformly effectively open sets O_e with $h(x) \in W_e \iff x^{\{\alpha\}} \in O_e$. For $y \in (\omega^{<\omega})^\omega$, let $g(y)$ be the unique element of $\bigcap \{W_e : y \in O_e\}$ if there is such an element, undefined otherwise. □

Theorem 7.9 now follows by relativising to an oracle.

The line of argument just given does not mention the concept of an iteration of the Turing jump, but of course, this concept was the original motivation for the entire machinery. For a computable ordinal δ and $x \in \mathcal{N}$, define $H_\alpha(x)$ in Cantor space by induction on $\alpha \leqslant \delta$: $H_0(x)$ is the set of (codes of) the finite initial segments of x; $H_{\alpha+1}(x) = (H_\alpha(x))'$; for limit $\alpha \leqslant \delta$, $H_\alpha(x) = \bigoplus_{\beta<\alpha} H_\beta(x)$.[9]

Proposition 7.12. *For all computable α and all $x \in \mathcal{N}$, $H_\alpha(x) \equiv_{\mathrm{T}} x^{\{\alpha\}}$.*

This is of course uniform in α and x.

Proof. We use effective transfinite recursion on α. Since $x^{\{\alpha\}} \equiv_{\mathrm{T}} D^\alpha_x$, we show that $D^\alpha_x \equiv_{\mathrm{T}} H_\alpha(x)$.

If α is a limit, we need to show that D^α_x is Turing equivalent to $\bigoplus_{\beta<\alpha} D^\beta_x$. If $\beta < \alpha$ then $D^\beta_x \leqslant_{\mathrm{T}} D^\alpha_x$: $\sigma \prec_\beta x$ if $\sigma \prec_\beta \tau$ for some or all $\tau \succcurlyeq \sigma$ in D^α_x. On the other hand, given D^β_x for all $\beta < \alpha$, we can tell whether a given σ is in D^α_x by using Lemma 7.3: we let $\beta = \max\{\gamma < \alpha : n_\gamma \leqslant |\sigma|\}$, and ask whether $\sigma \in D^\beta_x$.

For the successor case, we need to check that $D^{\alpha+1}_x \equiv_{\mathrm{T}} (D^\alpha_x)'$. Note that $(x^{(\alpha)})' \equiv_{\mathrm{T}} (D^\alpha_x)'$ (as $x^{(\alpha)} \equiv_{\mathrm{T}} D^\alpha_x$). Given $(x^{(\alpha)})'$ we first compute D^α_x. For $\sigma \in D^\alpha_x$ we can check if there is some $e < p(\sigma^{(\alpha)})$ in $(x^{(\alpha)})' \backslash (\sigma^{(\alpha)})'$ and so tell if $\sigma \prec_{\alpha+1} x$. In the other direction, given $D^{\alpha+1}_x$, we can compute $(x^{(\alpha)})'$ by observing $(\sigma^{(\alpha)})' \restriction p(\sigma^{(\alpha)})$ for $\sigma \in D^{\alpha+1}_x$. □

7.3. **Final debts.** We give proofs of the remaining propositions of Section 2.

Proof of Proposition 2.14. In one direction, suppose that $f\colon S \to \mathbb{N}$ is a computable α-approximation of F. For each n and k, let

$$A_{n,k} = \bigcup \{[\sigma]_\alpha : \sigma \in S,\ f(\sigma) = n \ \&\ |\sigma|_\alpha = k\}.$$

[8] Note that $y = D^\alpha_x$ is Π^0_2: we need to say that it is infinite.
[9] To be precise, $H_\alpha(x) = \{(n_\beta, m) : \beta < \alpha \ \&\ m \in H_\beta(x)\}$.

By TSP(7), and since f and S are computable, the sets $A_{n,k}$ are uniformly $\Delta^0_{1+\alpha}$. For $x \in [S]_\alpha$, $F(x) = n$ if and only if $x \in A_{n,k}$ for almost all k, hence $F^{-1}\{n\}$ is $\Sigma^0_{1+\alpha+1}$ within $[S]_\alpha$.

In the other direction, let $F\colon [S] \to \mathbb{N}$ be $\Sigma^0_{1+\alpha+1}$-measurable. By TSP(7) and Lemma 2.11, let $V_0, V_1, \dots$ be uniformly computable subsets of S, $\preccurlyeq_{\alpha+1}$-upwards closed in S, such that $F^{-1}\{n\} = [V_n]_{\alpha+1}$ and $\bigcup_n V_n$ computable.

Define $f\colon S \to \mathbb{N}$ by letting $f(\sigma) = n$ if $\sigma \in V_n$, $f(\sigma) = 0$ if $\sigma \notin \bigcup_n V_n$. Then f is computable. Let $x \in [S]$; let $n = F(x)$. There is some $\sigma \prec_{\alpha+1} x$ with $\sigma \in V_n$. Let τ be such that $\sigma \prec_\alpha \tau \prec_\alpha x$. By (♣), $\sigma \preccurlyeq_{\alpha+1} \tau$. Since V_n is $\preccurlyeq_{\alpha+1}$-upwards closed, $\tau \in V_n$, so $f(\tau) = n$. □

Proof of Lemma 2.17. Let $<_{\mathrm{KB}}$ denote the Kleene-Brouwer ordering on $\omega^{<\omega}$. If T is a well-founded tree, then the restriction of $<_{\mathrm{KB}}$ to T is a T-computable well-ordering, and the identity function on T is a rank function. This well-ordering may fail to be a T-computable ordinal, as T may fail to compute the collection of limit points and the successor relation. This is overcome by multiplying on the left by ω, that is, replacing every point by a copy of ω. □

Proof of Lemma 2.16. For nonempty $\sigma \in \omega^{<\omega}$, let σ^- denote the immediate predecessor of σ on the tree $(\omega^{<\omega}, \preccurlyeq_\alpha)$. Let R be the collection of all $\sigma \in S$ such that either:

- $\sigma = \langle\rangle$ or $\sigma^- \notin S$; or
- $\sigma^- \in S$ and $f(\sigma^-) \neq f(\sigma)$.

Then $(R, \preccurlyeq_\alpha)$ is a computable well-founded tree. By Lemma 2.17, let η be a computable ordinal and $r\colon R \to \eta$ be a computable rank function. For $\sigma \in S$, let $\beta(\sigma) = r(\tau)$ for the longest $\tau \preccurlyeq_\alpha \sigma$ in R. □

Proof of Proposition 2.18. Since $w \geqslant_{\mathrm{T}} z$, every z-c.e. set is w-c.e., uniformly in their indices. This, together with effective transfinite recursion on α, shows that every $\Sigma^0_{1+\alpha}(z)$ set is $\Sigma^0_{1+\alpha}(w)$, again uniformly in the indices. Using TSP(7), we see that we can obtain uniformly computable, $\preccurlyeq^w_\alpha$-upwards closed sets U_σ for $\sigma \in \omega^{<\omega}$, such that $[U_\sigma]^w_\alpha = [\sigma]^z_\alpha$ for all σ. We may assume that $U_{\langle\rangle} = \omega^{<\omega}$.

We define the function $h(\sigma)$ by recursion on $\preccurlyeq^w_\alpha$. Of course we set $h(\langle\rangle) = \langle\rangle$. Suppose that $\sigma \neq \langle\rangle$; let σ^- be σ's immediate $\preccurlyeq^w_\alpha$-predecessor, and suppose that $h(\sigma^-)$ has already been defined.

If there is some $\rho \preccurlyeq \sigma$ such that $\sigma \in U_\rho$ and ρ is a $\preccurlyeq^z_\alpha$-immediate successor of $h(\sigma^-)$ then we let $h(\sigma) = \rho$ for some such ρ (if there is more than one such ρ then $[\sigma]^w_\alpha = \varnothing$, so it doesn't matter which one we choose). If there is no such ρ we let $h(\sigma) = h(\sigma^-)$.

Note that by our construction, $h(\sigma) \preccurlyeq \sigma$ and $\sigma \in U_{h(\sigma)}$ for all σ, i.e., $[\sigma]^w_\alpha \subseteq [h(\sigma)]^z_\alpha$. Let $x \in \mathcal{N}$. We need to argue that for all $\tau \prec^z_\alpha x$ there is some $\sigma \prec^w_\alpha x$ such that $\tau = h(\sigma)$. This is done by induction on τ. Let τ^- be τ's $\preccurlyeq^z_\alpha$-immedate predecessor, and let $\rho \prec^w_\alpha x$ such that $h(\rho) = \tau^-$. Since $x \in [\tau]^z_\alpha$, there is some $\sigma \prec^w_\alpha x$ with $\sigma \in U_\tau$. If we take σ minimal such that $\rho \prec^w_\alpha \sigma \prec^w_\alpha x$ and $\sigma \in U_\tau$, then $h(\sigma) = \tau$. □

Propositions 2.19 and 2.20 are proved in exactly the same way. For the former, we use the fact that the pull-back of a $\Sigma^0_{1+\alpha}$ set by a computable function is $\Sigma^0_{1+\alpha}$, uniformly. For the latter, we use the fact that if α and β are computably isomorphic, then $\Sigma^0_{1+\alpha} = \Sigma^0_{1+\beta}$, uniformly in the indices. (We know that if α and β are

isomorphic, then $\Sigma^0_{1+\alpha} = \Sigma^0_{1+\beta}$; but if the isomorphism is not computable, then this will not be uniform.) Proposition 2.21 is proved similarly, using Lemma 7.8.

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