ALGORITHMICALLY RANDOM SERIES

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Abstract. Rademacher [Rad22], Steinhaus [Ste30] and Paley and Zygmund [PZ30a, PZ30b, PZ32] initiated the extensive study of random series. Using the theory of algorithmic randomness, which is a mix of computability theory and probability theory, we investigate the effective content of some classical theorems. We discuss how this is related to an old question of Kahane and Bollobás. We also discuss how considerations of such algorithmic questions about random series seems to lead to new notions of algorithmic randomness.

1. Introduction

The concern of this paper is the area of random trigonometric series, an area going back to seminal papers of Paley and Zygmund in the 1930's, and subsequently having a rich history, with applications to ergodic theory and Brownian motion (see, for instance, Angst and Poly [AP21], Cohen and Cuny [CC06], Filip et. al. [FJT19], Hill [Hi12], or Salem and Zygmund [SZ54]). The goal of the present paper is to use ideas from the theory of *algorithmic randomness*, or *algorithmic information theory* (see Li and Vitanyi [LV19], Downey and Hirschfeldt [DH10], or Nies [Nie12]), to examine natural questions which grew from the theory of such series. The area enables us to quantify, or calibrate, *the amount* of randomness (in the sense of [DHNT06]) needed to prove classical theorems involving almost everywhere behaviour. In turn, this allows us to address questions about algorithmic aspects of random series, which so far have been stated informally.

In particular, the original motivation for the present paper was an intriguing comment by Bollobás in the introduction to his book [Bol01], originally in 1985. In this introduction, Bollobás motivates the use of probabilistic ideas in graph theory. He mentioned that earlier probabilistic application had been found in analysis via three famous papers of Paley and Zygmund [PZ30a, PZ30b, PZ32]:

> "Paley and Zygmund (1930a,b,1932) had investigated random series of functions. One of their results was that if the real numbers c_n satisfy $\sum_{n=0}^{\infty} c_n^2 = \infty$ then $\sum_{n=0}^{\infty} \pm c_n \cos nx$ fails to be a Fourier-Lebesgue series for almost all choices of the signs. To exhibit a sequence of signs with this property is surprisingly difficult: indeed there is no algorithm known which constructs an appropriate sequence of signs from any sequence c_n with $\sum_{n=0}^{\infty} c_n^2 = \infty$."

An almost identical question can be found even earlier in the 1968 version of Kahane's book (most recently, [Kah03], page 47), on random trigonometric series:

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> "If $\sum c_n^2 = \infty$, there exists a choice of signs $\pm$ such that $\sum \pm c_n \cos(nt+\varphi_n)$ is not a Fourier-Stieltjes series. A surprising fact is that nobody knows how to construct these signs explicitly, but a random choice works."

Thus, this natural question is now at least 50 years old.

The first thing we need to do in answering such a question is to understand how to formulate it mathematicially. Fortunately, we can use computability theory to do this. The natural tool to use is Turing's "oracle machine". A positive solution to Bollobás's problem would consist of an algorithm which runs on Turing's idealised machine. On an "input tape" of the machine is written the sequence of reals $\langle c_n \rangle$. The machine runs indefinitely, and on an "output tape" is gradually written a solution: a sequence $\langle x_n \rangle \in \{-1, 1\}^\infty$ such that $\sum x_n c_n \cos nt$ is not a Fourier-Lebesgue series. The main point is that there is a single algorithm which given the input $\langle c_n \rangle$ produces a desired output $\langle x_n \rangle$. We say that the outputs are *uniformly computable* from the inputs.

Paradoxically, a positive solution to the Bollobás / Kahane question can be given using the Paley-Zygmund almost everywhere result. Given an instance $\langle c_n \rangle$ of the problem (with $\sum c_n^2 = \infty$), we know that the collection of "untypical" $x = \langle x_n \rangle \in \{-1, 1\}^\infty$, those for which $\sum x_n c_n \cos nt$ is a Fourier-Lebesgue series, is null. The theory of algorithmic randomness allows us to inquire into how *effectively null* it is. It turns out that the null set in this case is particularly simple.

Computability theory gives us the notion of an *enumerable* open set (also called an *effectively open* set). Since we allow non-computable inputs, we give a definition that can be relativised. For the following, a sequence of sets $U_0, U_1, \ldots$ is called *nested* if $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$.

Definition 1.1.

(a) A *name of an open set* U is a list $\langle V_0, V_1, V_2, \ldots \rangle$ of basic open sets such that $U = \bigcup_n V_n$.[1]
(b) A *name of a sequence of open sets* $U_0, U_1, \ldots$ is a sequence consisting of a name of U_0, a name of U_1, ...
(c) A *name* of a G_δ set G is a name of a nested sequence of open sets $U_0, U_1, U_2, \ldots$ such that $G = \bigcap_n U_n$.
(d) A name of an F_σ set is a name of its complement.

Potgieter [Pot18] first studied the complexity of the null sets arising from the Paley-Zygmund theorem. Implicit in his calculations is the following:

Theorem 1.2. *Given $\langle c_n \rangle$ and $\langle \varphi_n \rangle$ with $\sum c_n^2 = \infty$, we can compute a name of a null F_σ set containing all $x \in \{-1, 1\}^\infty$ for which $\sum x_n c_n \cos(nt + \varphi_n)$ is a Fourier-Stieltjes series.*

We can then quote a standard result from computability theory:

Fact 1.3. Given a name of a null F_σ set H, we can compute a point $x \notin H$.

Theorem 1.2 and Fact 1.3 together give a positive answer to Kahane's question:

[1]For a closed interval, we can take the basis consisting of rational open intervals; in Cantor space, the basis of clopen sets, each determined by finitely many values.

Theorem 1.4. *There is an algorithm which, given $\langle c_n\rangle$ and $\langle \varphi_n\rangle$ with $\sum c_n^2 = \infty$, outputs a sequence $x \in \{-1,1\}^\infty$ for which $\sum x_n c_n \cos(nt+\varphi_n)$ is not a Fourier-Stieltjes series.*

We elaborate on Theorem 1.2 and Fact 1.3 in Section 2.

1.1. **Algorithmic randomness.** Algorithmic randomness ([DH10, Nie12, LV19]) seeks to give meaning to randomness of individual sequences. We say that a point x in a computable measure space is random if it passes all "appropriately computable tests" for randomness. The idea is that if only a specified kind of computable testing processes are available to us, then we cannot distinguish x from one classically chosen at random. For the reader unfamiliar with these concepts, we refer to Downey and Hirschfeldt [DH19b, DH19a] for recent surveys, aimed at a lay audience, expositing the ideas of this area. For our purposes, to give this a formal meaning, a *notion of randomness* is determined by specifying a countable collection of null sets; a point is then declared to be random if it belongs to none of these null sets. Here is an example:

Definition 1.5 (Kurtz [Ku81], Wang [Wa96])**.**
- (a) A set A is *Kurtz null* if it is contained in a null F_σ set which has a computable name.
- (b) A point is *Kurtz random* if it is not an element of any Kurtz null set.

Since there are only countably many algorithms, there are only countably many computable names of F_σ sets. It follows that almost every point is Kurtz random. Since we allow noncomputable instances $\langle c_n\rangle$ of theorems such as Paley and Zygmund's, we can use the *relativised* notion of randomness as well:

Definition 1.6. Let y be an element of Baire space ω^ω.[2]
- (a) A set A is *y-Kurtz null* (or *Kurtz null relative to y*) if it is contained in a null F_σ set which has a y-computable name.
- (b) A point is *y-Kurtz random* (or *Kurtz random relative to y*) if it is not an element of any y-Kurtz null set.

Again, for all y, almost every x is Kurtz random relative to y. With this terminology, a consequence of Theorem 1.2 is:

Theorem 1.7. *Let $\langle c_n\rangle$ and $\langle \varphi_n\rangle$ be sequences of real numbers with $\sum c_n^2 = \infty$. If $x \in \{-1,1\}^\infty$ is Kurtz random relative to $(\langle c_n\rangle, \langle \varphi_n\rangle)$, then $\sum x_n c_n \cos(nt+\varphi_n)$ is not a Fourier-Stieltjes series.*

A different selection of naming of null sets results in possibly different notions of randomness. For example, the most commonly used notion of randomness is named after Martin-Löf [ML66]:

Definition 1.8. An *ML-name* of a null G_δ set G is a name of a sequence $\langle U_n\rangle$ of open sets satisfying $G = \bigcap_n U_n$ and $\mu(U_n) \leqslant 2^{-n}$.[3]

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[2] The point y is often referred to as an "oracle" in a computation. For our purposes, y will usually be a code for a sequence $\langle c_n\rangle$. Instead of Baire space we could take any other 0-dimensional computable topological space, for example Cantor space.

[3] Implicit in the definition here is that we are working with a computable measure space (X,μ). We will only need three examples of these, so do not give a general definition.

We can then similarly define the notion of an ML-null set (being contained in a set with a computable ML-name), ML-randomness (not an element of any ML-null set), and the relativised version when an oracle y is present. This notion of randomness is strictly stronger than Kurtz's (Kurtz [Ku81]). Given a name of a null F_σ set, we can computably produce an ML-name of the same set. Hence, every Kurtz null set is ML-null, and so, every ML-random point is Kurtz random. The converse fails. The distance between these notion is so large, that it is reflected in the non-effective theory: there is a null set which is not contained in a null F_σ set (whereas every null set is contained in a null G_δ set). This is witnessed computably, in that there is an ML-null set which is not Kurtz null, and indeed, we can find a point in an ML-null set which avoids all Kurtz null sets (See Downey and Hirschfeldt [DH10], Ch.7, for instance).

We remark that Potgieter's statement of Theorem 1.7 ([Pot18, Thm.4.1]) refers only to ML-randomness. However, the "computable avoidance" property of Kurtz null sets (Fact 1.3) fails for ML-null sets. In fact, there is a single ML-null set which contains all computable points. Thus, ML-null sets do not suffice to answer Kahane's question.

1.2. **Rademacher series.** The Paley-Zygmund theorems were motivated by questions of Rademacher, who, along with Steinhaus [Ste30], seem to be the original people to study random series. Quite aside from their intrinsic interest, random trigonometric series arise quite naturally in, for example, Brownian motion, and random noise in image processing (see for example [FJT19]). Since the seminal Paley-Zygmund papers, the area has flowered into a significant area of analysis (see, for example, [BP95]).

Rademacher [Rad22] studied the series $\sum x_n c_n$, for a given sequence of reals $\langle c_n \rangle$ and randomly chosen $x_n \in \{-1, 1\}$. Such a series is called a *Rademacher series*. Rademacher's insight was that the convergence or divergence of the random Rademacher series depended on the sum $\sum_{n=0}^{\infty} c_n^2$:

Theorem 1.9 ([Rad22])**.** *Let $\langle c_n \rangle$ be a sequence of real numbers.*

(a) *If $\sum c_n^2 = \infty$ then $\sum x_n c_n$ diverges for almost all $x \in \{-1,1\}^\infty$.*
(b) *If $\sum c_n^2 < \infty$ then $\sum x_n c_n$ converges for almost all $x \in \{-1,1\}^\infty$.*

Clearly, if $\sum c_n^2 = \infty$, then choosing x_n so as to make $x_n c_n > 0$ will cause divergence of the Rademacher series. Nevertheless, it seems an interesting project to understand the level of algorithmic randomness needed for convergence / divergence of Rademacher series. To give an answer in the convergent case, we use the following notion of randomness which lies between Kurtz and Martin-Löf randomness:

Definition 1.10. A *Schnorr name* of a null G_δ set G is a name of a nested sequence $\langle U_n \rangle$ of open sets such that $G = \bigcap_n U_n$ and $\mu(U_n) = 2^{-n}$.

As above, we obtain the notions of Schnorr null sets and Schnorr random points. From a name of a null F_σ set we can compute a Schnorr name for the set; every Schnorr name of a null set is also an ML-name. Hence ML randomness implies Schnorr randomness implies Kurtz randomness. Unlike with Kurtz, the difference between ML- and Schnorr randomness cannot be expressed clasically: both ML-null and Schnorr null sets are types of null G_δ sets. Here, the difference is purely computational. A Schnorr name tells us what $\mu(U_n)$ is, while an ML-name witholds that information: we only get an upper bound.

The following holds:

Theorem 1.11. *Let $\langle c_n\rangle$ be a sequence of real numbers and let $x = \langle x_n\rangle \in \{-1,1\}^\infty$.*

(a) *If $\sum c_n^2 = \infty$ and x is Kurtz random relative to $\langle c_n\rangle$ then $\sum x_n c_n$ diverges.*
(b) *If $\sum c_n^2 < \infty$ and x is Schnorr random relative to $(\langle c_n\rangle, \sum c_n^2)$ then $\sum x_n c_n$ converges.*

Part (b) was first shown by Ongay-Valverde and Tveite [OVT21]. Potgieter [Pot18] showed that ML-randomness suffices for both cases. In Section 3 we give simplified proofs of both parts.

Note that for part (b), to compute a Schnorr name for the appropriate null set, the information required is not only the sequence $\langle c_n\rangle$, but also, the value of the sum $\sum c_n^2$. In Section 3 we also enquire what happens if this information is not supplied: there, we show that a notion of randomness stronger than Martin-Löf's suffices. We also consider the question of a "reversal" – is it possible that some level of randomness not only suffices but is actually required?

1.3. **Pointwise convergence.** Paley and Zygmund also considered pointwise convergence of trigonometric series. They showed:

Theorem 1.12. *Let $\langle c_n\rangle$ and $\langle \varphi_n\rangle$ be a sequences of real numbers.*

(a) *If $\sum c_n^2 < \infty$, then for almost all $x \in \{-1,1\}^\infty$, $\sum x_n c_n \cos(nt + \varphi_n)$ converges for almost all $t \in [0, 2\pi]$.*
(b) *If $\sum c_n^2 = \infty$, then for almost all $x \in \{-1,1\}^\infty$, $\sum x_n c_n \cos(nt+\varphi_n)$ diverges for almost all $t \in [0, 2\pi]$.*

In Section 4 we study the effective content of these theorems.

1.4. **Preliminaries.** We follow standard notation and terminology for computability and randomness; standard references are [Soa87, DH10, Nie12]. We use λ to denote Lebesgue measure on $[0, 2\pi]$. We use μ to denote the "fair-coin" measure on Cantor space, or in general, a computable measure on a space. The spaces we will use are $([0, 2\pi], \lambda)$; $(\{-1,1\}^\infty, \mu)$; and their product.

We have not given formal details about the coding of real numbers and sequences of real numbers into objects that can be manipulated by Turing machines (usually, elements of Cantor space). The reason is that for our purposes, it makes no difference what particular coding we use. Turing, for example, used binary expansions to define computable real numbers [Tur36]. A more modern approach uses fast-converging Cauchy sequences of rational numbers (see for example [PER17, Wei00]). It is recognised as a more versatile aproach, for example, because it makes addition of real numbers computable.[4] However, for the purposes of convergence or divergence of random series, small perturbations are immaterial. For example, if $|c_n - d_n| \leqslant 2^{-n}$, then for all $x \in \{-1,1\}^\infty$, $\sum x_n c_n$ converges if and only if $\sum x_n d_n$ converges; this is because $\sum 2^{-n}$ converges absolutely. A similar phenomenon holds for being a Fourier-Stieltjes series. Hence, when manipulating an oracle such as a sequence $\langle c_n\rangle$ of real numbers, we may assume that we are actually working with a

[4] In the correction to [Tur36], Turing realised that binary expansions were a poor model, and essentially used Cauchy sequences. However, in this seminal work he only considered functions acting on the *computable reals*, whereas the modern "type 2" approach considers realitivised computations, and so functions defined on all reals.

rational approximation of the input. For instance, we can consider the input series $\{c_n \mid n \in \mathbb{N}\}$ to consist of rationals represented by some simple coding.

2. Fourier-Stieltjes series

We give a proof of Theorem 1.2.

Proof of Theorem 1.2: We are given $\langle c_n \rangle$ and $\langle \varphi_n \rangle$. For each finite binary string $\tau = (\tau_0, \tau_1, \ldots, \tau_m) \in \{-1,1\}^{m+1}$, let the corresponding *Fejér sum* be

$$\sigma_\tau(t) = \sum_{n \leqslant m} \left(1 - \frac{n}{m}\right) \tau_n c_n \cos\left(nt + \varphi_n\right).$$

This is a continuous function on $[0, 2\pi]$ and the functions σ_τ for $\tau \in \{-1,1\}^{<\infty}$ are *uniformly computable* relative to $(\langle c_n \rangle, \langle \varphi_n \rangle)$. By [Kah03, Chap.5,Prop.1] (who refers to [Zyg59]), for all $x \in \{-1,1\}^\infty$, $\sum x_n c_n \cos(nt + \varphi_n)$ is Fourier-Stieltjes if and only if

$$\sup_m \left\|\sigma_{x \restriction m}\right\|_1 < \infty,$$

where recall that $\|f\|_1 = \int_0^{2\pi} |f(t)|\, dt$. By [PER17, Ch.0,Thm.5], the values $\|\sigma_\tau\|_1$ are uniformly computable relative to the data. For each K, let

$$C_K = \left\{x \in \{-1,1\}^\infty \, : \, (\forall m) \ \left\|\sigma_{x \restriction m}\right\|_1 \leqslant K\right\}.$$

Then each C_K is closed, effectively so given the data. The required F_σ set is thus $\bigcup_K C_K$; this set is null by the classical result that under the assumption, $\sum x_n c_n \cos(nt + \varphi_n)$ is not Fourier-Stieltjes for almost all x (see [Kah03, Ch.5,Prop.6]).[5] □

For completeness, we provide a proof of Fact 1.3.

Proof of Fact 1.3: We are given a name of $H = \bigcup_n F_n$, where each F_n is closed and null. We construct a point $x \notin H$ by open approximations. For simplicity, we consider the case that the underlying space is Cantor space $\{0,1\}^\infty$. In that case we construct x by specifying ever-longer initial segments of x. We define a sequence $\langle \tau_n \rangle$, starting with τ_{-1} being the empty string. Given τ_{n-1}, we let τ_n be a proper extension of τ_{n-1} such that $[\tau_n] \cap F_n = \varnothing$. This we can do because the name of F_n allows us to enumerate the clopen subsets of the complement of F_n; the fact that this complement is co-null implies that it is *dense*, i.e., every clopen set contains a clopen set disjoint from F_n. We let $x = \bigcup_n \tau_n$ be the unique point in the intersection of the clopen sets $[\tau_n]$. □

We remark that a stronger result holds: Schnorr null sets have the same "computable escaping" property. Here the idea is that since we know $\mu(U_n)$ (where $G = \bigcap_n U_n$ is the null set named), and since we know that $\mu(U_1) < 1$, we can construct a point $x \notin U_1$ (and so not in G) by keeping $\mu(U_1 | (x \restriction n)) < 1$ for all initial segments $x \restriction n$ of x (here $\mu(U|\tau)$ denotes the conditional measure). Finding some i such that $\mu(U_1 | (x \restriction n)\hat{\ } i) < 1$ can be done since we know $\mu(U_1)$.

[5] We remark that Potgieter [Pot18, Thm.4.1] follows the same argument. However, he skips the fact that integration of functions is computable, and as a result his sets B_K (the intended complements of our sets C_K) may be too large: having some Riemann sum being greater than K does not ensure that the integral is greater than K; we need to make use of the fact that the functions are uniformly continuous (as a family of functions) to obtain error bounds for the Riemann sums, as is done in [PER17].

3. Rademacher series

3.1. **Divergence of Rademacher series.** If $\sum c_n^2 = \infty$ then for almost all $x \in \{-1,1\}^\infty$, $\sum x_n c_n$ diverges. Theorem 1.11(a) says that Kurtz randomnes is sufficient. It follows from the following:

Proposition 3.1. *Given $\langle c_n \rangle$ for which $\sum c_n^2 = \infty$ we can (uniformly) compute a name of a null F_σ set containing all $x \in \{-1,1\}^\infty$ for which $\sum x_n c_n$ converges.*

Following [Pot18, Thm.3.4], we make use of the following *Paley-Zygmund* inequality (see [Kah03, Ch.3,Thm.3]): for any natural N and sequence of reals $a_0, a_1, \dots, a_{N-1}$, if $\sum_{n<N} a_n^2 > 1/4$ then

$$\mathbb{P}\left\{\tau \in \{-1,1\}^N \,:\, \left|\sum_{n<N} \tau_n a_n\right| > \frac{1}{2}\right\} > \frac{1}{6}, \tag{1}$$

where $\mathbb{P}$ denotes the fair-coin probability measure on $\{-1,1\}^N$.

Proof. Given $\langle c_n \rangle$ with $\sum c_n^2 = \infty$ we can compute a partition of $\mathbb{N}$ into intervals $I_0 < I_1 < \cdots$ (so $\min I_{k+1} = \max I_k + 1$), with each interval I_i sufficiently long so that

$$\sum_{n \in I_i} c_n^2 > \frac{1}{4}.$$

For each i let

$$C_i = \left\{x \in \{-1,1\}^\infty \,:\, (\forall j \geqslant i) \left|\sum_{n \in I_j} x_n c_n\right| \leqslant \frac{1}{2}\right\}.$$

Then each C_i is closed and null (it is the product of infinitely many independent clopen sets, each with measure at most 1/6). Hence, $H = \bigcup_i C_i$ is a null F_σ set with $\langle c_n \rangle$-computable name, that contains every x for which $\sum x_n c_n$ converges. ☐

3.2. **Convergence of Rademacher series.** For convergence, we use the following *Kolmogorov equality* (see for example [Kah03, Ch.3,Thm.1]): for any N, sequence of real numbers $\langle a_n \rangle_{n<N}$ and any $\varepsilon > 0$,

$$\mathbb{P}\left\{\tau \in \{-1,1\}^N \,:\, \max_{m<N} \left|\sum_{n \leqslant m} \tau_n a_n\right| > \varepsilon\right\} \leqslant \frac{1}{\varepsilon^2} \sum_{n<N} a_n^2. \tag{2}$$

The inequality holds for $N = \infty$ as well, in which case we need of course to replace max with sup. With the triangle inequality, we can deduce the following:

$$\mathbb{P}\left\{\tau \in \{-1,1\}^N \,:\, \max_{k \leqslant m < N} \left|\sum_{n=k}^{m} \tau_n a_n\right| > \varepsilon\right\} \leqslant \frac{4}{\varepsilon^2} \sum_{n<N} a_n^2. \tag{3}$$

(In fact, the proof of Kolmogorov's inequality gives the bound $\sum a_n^2/\varepsilon^2$.)

Toward building Schnorr null sets, we use the following:

Fact 3.2. Given both a name of a nested sequence $\langle U_n \rangle$ of open sets such that $\mu(U_n) \to 0$, and the sequence $\langle \mu(U_n) \rangle$, we can compute a Schnorr name of $\bigcap_n U_n$.

Proof. We enumerate the components $\langle V_n \rangle$ of a Schnorr name inductively. Given the algorithm for V_{n-1}, we search for $m = m(n)$ sufficiently large so that $\mu(U_m) < 2^{-n}$. We declare that $U_{m(n)} \subseteq V_n$. Once we have enumerated U_m up to some small ε of measure, we can add some parts of V_{n-1} not currently in V_n so that the total measure enumerated into V_n is $2^{-n} - \varepsilon$. ☐

As with divergence, Theorem 1.11(b) follows from:

Proposition 3.3. *Given $\langle c_n \rangle$ for which $\sum c_n^2 < \infty$, and the value of that sum, we can (uniformly) compute a Schnorr name of a null set containing all $x \in \{-1,1\}^\infty$ for which $\sum x_n c_n$ diverges.*

Proof. Given $\langle c_n \rangle$ and $\sum c_n^2$, we can compute a partition of $\mathbb{N}$ into intervals $I_0 < I_1 < \cdots$ such that for all $k \geqslant 1$, $\sum_{n \in I_k} c_n^2 < 2^{-3k-2}$, and so by the extended Kolmogorov inequality (3), $\mu(A_k) \leqslant 2^{-k}$, where

$$A_k = \left\{ x \in \{-1,1\}^\infty \ : \ \max_{J \subseteq I_k} \left| \sum_{n \in J} x_n c_n \right| > 2^{-k} \right\},$$

where the quantification is over all sub-intervals $J \subseteq I_k$. Let $U_m = \bigcup_{k>m} A_k$. A name of $\langle U_m \rangle$ can be obtained computably given the data, and $\mu(U_m)$ is computable as well given the data ($U_{m,s} = \bigcup_{k=m+1}^{s} A_k$ is a clopen set approximating U_m to within 2^{-s}). If $x \in \{-1,1\}^\infty$ and $\sum x_n c_n$ diverges then $x \in A_k$ for infinitely many k, so $x \in \bigcap_m U_m$. □

Remark 3.4. Potgieter [Pot18, Thm.3.2] uses, for each $\varepsilon > 0$, the intersection of the sets $V_m = \{x \ : \ \sup_{k \geqslant m} |\sum_{n=m}^{k} x_n c_n| > \varepsilon\}$. Using Kolmogorov's inequality, we can compute a bound on the measure of each V_m, and so Potgieter shows that every ML-random x makes $\sum x_n c_n$ converge. It is not clear how to compute the measure of V_m though, so the proof does not give Schnorr randomness. Further, note that this argument does not give a single ML-null (relative to $\langle c_n \rangle$) set which captures all "deviant" x's making $\sum x_n c_n$ diverge; rather, for each $\varepsilon > 0$, we have an ML-null set G_ε, and their union captures all such x's. In the Schnorr context also, this reminds us that Proposition 3.3 is stronger than Theorem 1.11(b); to prove the latter, we could use infinitely many null sets rather than just one. The union of infinitely many Schnorr null sets may fail to be Schnorr null because it has worse descriptive complexity: it is $G_{\delta\sigma}$ (or Σ^0_3 in the notation of computability / set theory). In terms of convergence, this emphasises that the null set given by Proposition 3.3 captures some x for which $\sum x_n c_n$ converges (the set of diverging x is again Σ^0_3, not G_δ). The proof shows that if x is Schnorr random relative to $\langle c_n \rangle$, then not only does $\sum x_n c_n$ converge, but we can put an effective upper bound on how quickly this convergence happens.

We also remark that Ongay-Valverde and Tveite [OVT21, Lem.6.7] claim to prove Theorem 1.11(b). They use sophisticated machinery developped by Rute in an unpublished manuscript, rather than directly producing Schnorr null sets. However, it appears that they only prove convegrence of a subsequence of the partial sums $\sum_{n \leqslant k} x_n c_n$.

What if we are given a sequence $\langle c_n \rangle$ with $\sum c_n^2 < \infty$, but we are not told what the sum is? It appears that Schnorr randomness will not suffices in this case. For an upper bound, we use a notion of randomness slightly stronger than ML-randomness. The following definition uses the notion of a *left-c.e.* (or *lower semicomputable*) real number: one which is approximable from below, as a limit of an increasing computable sequence of rational numbers; but which may fail to be computable itself. We use the notion of OW-randomess, first defined in [BGK+16].

Definition 3.5. An *OW-null set* is a set contained in an intersection $\bigcap_n U_n$, where $\langle U_n \rangle$ is a nested sequence of uniformly enumerable open sets such that for some left-c.e. real α and some increasing computable rational approximation $\langle \alpha_n \rangle$ of α, we have $\mu(U_n) \leqslant \alpha - \alpha_n$ for all n.

The idea again is that the intersection is a G_δ set with a computable name, but in this case we cannot compute $\mu(U_n)$, and may not even have any computable upper bound on that measure. Rather, the fact that $\mu(U_n) \to 0$ is witnessed by the fact that the approximation $\alpha_n \to \alpha$ converges. Computably, at very late stages s, we discover that the sets U_n for $n < s$ are "allowed to grow" by a large amount (much larger than 2^{-s}). This "amount of growing" eventually goes to 0, but we cannot tell computably how quickly.

Proposition 3.6. *Let $\langle c_n \rangle$ be such that $\sum c_n^2 < \infty$. If $x \in \{-1, 1\}^\infty$ is OW-random relative to $\langle c_n \rangle$, then $\sum x_n c_n$ converges.*

Proof. The simpler proof by Potgieter works. For each $\varepsilon > 0$ and m, let

$$U_m^\varepsilon = \left\{ x \in \{-1,1\}^\infty \;:\; \sup_{k \geqslant m} \left| \sum_{n=m}^{k} x_n c_n \right| > \varepsilon \right\}.$$

These sets are uniformly effectively open given $\langle c_n \rangle$. Let $\alpha_\varepsilon = \frac{1}{\varepsilon^2} \sum_n c_n^2$ and $\alpha_{\varepsilon,m} = \frac{1}{\varepsilon^2} \sum_{n<m} c_n^2$. Then $\langle \alpha_{\varepsilon,m} \rangle$ is an increasing approximation of α_ε, and by Kolmogorov's inequality (2), $\mu(U_m^\varepsilon) \leqslant \alpha_\varepsilon - \alpha_{\varepsilon,m}$, hence $\bigcap_m U_m^\varepsilon$ is OW-null relative to $\langle c_n \rangle$. If x is OW-random relative to $\langle c_n \rangle$, then $x \notin \bigcup_{\varepsilon > 0} \bigcap_m U_m^\varepsilon$; this shows that $\sum x_n c_n$ converges. ☐

3.3. **Lower bounds.** The upper bounds proved in this section raise the natural question: is randomness necessary for typical behavious for Rademacher series? Here we have in mind results in the literature which characterise notions of randomness using almost-everywhere theorems of analysis, for example:

Theorem 3.7 (Bratkka, Miller, Nies [BMN16])**.** *A point $x \in [0,1]$ is ML-random if and only if every computable function $f\colon [0,1] \to \mathbb{R}$ of bounded variation is differentiable at x.*

This is the effective version of Lebesgue's theorem that every function of bounded variation is differentiable almost everywhere. Similarly, the following is the effective version of Birkhoff's ergodic theorem:

Theorem 3.8 (Gács, Hoyrup, Rojas [GHR11])**.** *Let (X, μ) be a computable measure space, and let $T\colon X \to X$ be computable and ergodic. A point $x \in X$ is Schnorr random if and only if for every computable function $f\colon X \to \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i<n} f(T^i x) = \int f \, d\mu.$$

Is it possible, for example, that Theorem 1.11(a) characterises Kurtz randomness? To show that, a natural approach would be to take a Kurtz null set A and somehow produce a computable sequence $\langle c_n \rangle$ with $\sum c_n^2 = \infty$ and $\sum x_n c_n$ convergent for all $x \in A$. Currently, such a "reversal" is not known, and it is suspected that typicality with respect to convergence and divergence of Rademacher series is in fact a *new phenomenon* in computability theory, not equivalent to any known randomness notion. We have the following partial converse.

Proposition 3.9. *Suppose that $P \subset \{-1,1\}^\infty$ is effectively closed, and that there is a computable tree $T \subset \{-1,1\}^{<\infty}$ such that $P = [T]$ and for all n, T contains fewer than $\log_2 n$ many strings of length n. Then there is a computable sequence $\langle c_n \rangle$ such that $\sum c_n^2 = \infty$, but $\sum x_n c_n$ converges for all $x \in P$.*

Such an effectively closed set must be null, as $\log_2 n/2^n \to 0$, and so (as is necessary) no $x \in P$ is Kurtz random. We note that *very small* effectively closed sets of [Bin05] have this property.

Proof. Let $I_k = [2^k, 2^{k+1})$. For each k, since there are at most k strings of length 2^{k+1} in T, there is some $n_k \in I_k$ such that τ_{n_k} is a constant value i_k for all $\tau \in T$ of length 2^{k+1}. We let $c_{n_k} = (-1)^k i_k/\sqrt{k}$ and $c_n = 0$ if $n \neq n_k$ for all k. □

The following lower bound is also weaker than randomness. A sequence $x \in \{-1,1\}^\infty$ is *bi-immune* if neither $\{n : x_n = 1\}$ nor its complement $\{n : x_n = -1\}$ contain an infinite computable set (equivalently, an infinite computably enumerable set). All Kurtz random sequences are bi-immune.

Proposition 3.10. *If x is not bi-immune then there is a computable sequence $\langle c_n \rangle$ with $\sum c_n^2 = \infty$ but $\sum x_n c_n$ converges.*

Proof. Let A be an infinite computable set such that either $x_n = 1$ for all $n \in A$, or $x_n = -1$ for all $n \in A$. Let $n_1, n_2, \dots$ be the increasing enumeration of the elements of A. Let $c_{n_k} = (-1)^k/\sqrt{k}$; if $n \neq n_k$ for any k let $c_n = 0$. □

A similar approach in both cases (say setting $c_{n_k} = 1/k$) also shows atypicality with respect to convergence: for all x in P as in Proposition 3.9, and all x which are not bi-immune, we can find a computable $\langle c_n \rangle$ with $\sum c_n^2 < \infty$ and computable, but $\sum x_n c_n$ divergent.

4. Pointwise convergence and divergence of trigonometric series

Paley and Zygmund studied the pointwise convergence and divergence of random trigonometric series. As mentioned, they showed, for example, that if $\sum c_n^2 = \infty$ then for almost all $x \in \{-1,1\}^\infty$, $\sum x_n c_n \cos(nt + \varphi_n)$ diverges for almost all $t \in [0, 2\pi]$. The first natural question in the effective realm is to ask, how much randomness of x ensures that $\sum x_n c_n \cos(nt+\varphi_n)$ diverges almost everywhere. This was addressed by Potgieter [Pot18, Lem.4.1], stating that ML-randomness suffices. While being a little opaque, his proof seems to extend to Kurtz randomness.

We can refine the question by asking not only for almost everywhere divergence, but also, what level of randomness *of t* ensures this divergence. This leads us to consider randomness in the product space $\{-1,1\}^\infty \times [0, 2\pi]$, which is defined as expected, using the product measure $\mu \times \lambda$.

Theorem 4.1. *Let $\langle c_n \rangle$ and $\langle \varphi_n \rangle$ be sequences of real numbers, and suppose that $\sum c_n^2 = \infty$. If $(x,t) \in \{-1,1\}^\infty \times [0, 2\pi]$ is Schnorr random relative to $(\langle c_n \rangle, \langle \varphi_n \rangle)$ then $\sum x_n c_n \cos(nt + \varphi_n)$ diverges.*

We do not know as yet whether Kurtz randomness suffices. We note that this theorem implies that if x is Schnorr random then $\sum x_n c_n \cos(nt + \varphi_n)$ diverges almost everywhere.

Proof. For brevity, let $\xi_n(t) = c_n \cos(nt+\varphi_n)$. By [Kah03, Ch.5,Prop.4], for almost all $t \in [0, 2\pi]$, $\sum \xi_n^2(t) = \infty$. The set R of t for which this fails is F_σ, with a name computable in the data $(\langle c_n\rangle, \langle \varphi_n\rangle)$, and so, if $t \in [0, 2\pi]$ is Kurtz random relative to the data then $t \notin R$.

Let $\Phi\colon \{0,1\}^\omega \to [0, 2\pi]$ be the stretching by a factor of 2π of the usual binary representation of reals in the unit interval: formally, $\Phi(y) = 2\pi \sum_{n\geqslant 0} y_n 2^{-(n+1)}$. For each finite binary string $\sigma \in \{0,1\}^{<\infty}$, we let $[\![\sigma]\!] = \Phi[\sigma]$ be the image under Φ of the clopen set $[\sigma]$; it is a closed interval of length $2\pi 2^{-|\sigma|}$, where $|\sigma|$ denotes the length of σ.

For each σ and $m \leqslant |\sigma|$ we can compute $\min_{t\in[\![\sigma]\!]} \xi_m^2(t)$: these functions are uniformly computable (given the data $(\langle c_n\rangle, \langle \varphi_n\rangle)$), and as continuous functions on these closed intervals obtain minima; these minima are uniformly computable, see [PER17, Ch.0,Thm.7]. Based on these minima, we can inductively (on σ) compute intervals $I_{0,\sigma} < I_{1,\sigma} < \cdots < I_{k(\sigma),\sigma}$ which for each σ partition an initial segment of $\mathbb{N}$, and have the following properties:

(a) For all σ, all $k \leqslant k(\sigma)$ and all $t \in [\![\sigma]\!]$, $\sum_{n\in I_{k,\sigma}} \xi_n^2(t) > 1/4$;
(b) If $\sigma \preccurlyeq \tau$ (τ extends σ) then $k(\sigma) \leqslant k(\tau)$ and for all $k \leqslant k(\sigma)$, $I_{k,\sigma} = I_{k,\tau}$;
(c) If $t = \Phi(y)$ and $t \notin R$ (so $\sum \xi_n^2(t) = \infty$) then $\lim_n k(y{\upharpoonright}n) = \infty$.

Now, for each pair $m \leqslant N$, let

$$C_{[m,N]} = \left\{ (x, \Phi(y)) \,:\, (\exists \sigma \prec y)\, k(\sigma) \geqslant N \;\;\&\;\; (\forall k \in [m, N]) \left| \sum_{n\in I_{k,\sigma}} x_n \xi_n(t) \right| \leqslant \frac{1}{2} \right\}.$$

Each $C_{[m,N]}$ is open (with name uniformly computable in the data), the $(\mu \times \lambda)$-measure of $C_{[m,N]}$ is bounded by $(1/6)^{N-m}$ (using (1)), and this measure is uniformly computable from the data. Hence, for each m, $\bigcap_N C_{[m,N]}$ is Schnorr null relative to the data.

Suppose that (x, t) is Schnorr random relative to the data. Then t is Schnorr random, hence Kurtz random, so $t \notin R$, and t is not a "binary rational" $2\pi k 2^{-n}$ of the interval $[0, 2\pi]$, i.e., $y = \Phi^{-1}(t)$ is well-defined, and $k(y{\upharpoonright}n) \to \infty$. This, together with $(x,t) \notin \bigcup_m \bigcap_N C_{[m,N]}$ implies the divergence of $\sum x_n \xi_n(t)$. □

For convergence, the situation is much simpler.

Theorem 4.2. *Let $\langle c_n\rangle$ and $\langle \varphi_n\rangle$ be sequences of real numbers, and suppose that $\sum c_n^2 < \infty$. If $(x, t) \in \{-1, 1\}^\infty \times [0, 2\pi]$ is Schnorr random relative to $(\langle c_n\rangle, \langle \varphi_n\rangle, \sum c_n^2)$ then $\sum x_n c_n \cos(nt + \varphi_n)$ converges.*

Proof. Define ξ_n as above. The main point is that since $|\cos(nt + \varphi_n)| \leqslant 1$, for *all* $t \in [0, 2\pi]$ we have $\sum \xi_n^2(t) < \infty$, indeed these are uniformly bounded by $\sum c_n^2$. Hence, we can apply the proof of Proposition 3.3. We define the intervals $I_0 < I_1 < \cdots$ in the same way, and unlike the previous proof, they do not depend on t. The sets

$$A_k = \left\{ (x, t) \,:\, \max_{J\subseteq I_k} \left| \sum_{n\in J} x_n \xi_n(t) \right| > 2^{-k} \right\}$$

are open and have $(\mu \times \lambda)$-measue bounded by 2^{-k} (by Fubini's theorem). The rest of the proof follows that of Proposition 3.3. □

5. Further lines of investigation

It seems to us that there are many opportunities for further research in this area. To begin, one can follow the rich literature on random series. One can discuss convergence in L_p, or convergence to continuous functions (Billard's theorem, see [Kah03, Ch.5]).

There are other aspects of computability theory which pertain to this topic. One can, for example, ask about the complexity of the collection of $x \in \{-1,1\}^\infty$ which display typical behaviour with respect to convergence or divergence of Rademacher series or trigonometric series. This complexity could be measured by the Medvedev or Muchnik lattice of sets of reals (see for example [Sim05, Sor96]); here we would consider typical behaviour with respect to computable series.

A more nuanced approach involves the Weihrauch lattice [BG11, GM09]. Here we formulate "problems", which are binary relations between "instances" and "solutions". For example, one such problem could be that of Rademacher convergence: an instance is a sequence $\langle c_n \rangle$ (not necessarily computable) such that $\sum c_n^2 < \infty$; a solution is any $x \in \{-1,1\}^\infty$ which makes $\sum x_n c_n$ converge. *Weihrauch reducibility* (along with its strong form) is a tool for comparing the complexity of such problems.

Framing the study in terms of Weihrauch problems is related to that of the Medvedev lattice: every Weihrauch problem is associated with both a "highness class" and a "nonlowness class" in the Medvedev lattice. For example, for the Rademacher convergence problem, the highness class is the collection of oracles computing $x \in \{-1,1\}^\infty$ which make every computable instance $\langle c_n \rangle$ converge. The non-lowness class is the collection of oracles computing a series $\langle c_n \rangle$ with no computable solution x.

The same view is related to the study of cardinal characteristics of the continuum in set theory. Indeed, this was the motivation in [OVT21]: Theorem 1.11(b) is used there to build a strong Weihrauch reduction from the "Schnorr capturing" problem to the "rearrangement problem". This immediately implies two theorems, one in set theory and one in computability: the null covering number $\mathbf{cov}(\mathcal{N})$ is bounded by the "rearrangement number" $\mathfrak{rr}$ (see [BBB+20]); every Schnorr random degree is "imperturbable". In the case of Rademacher convergence, for example, the associated cardinal is the smallest size of a subset $A \subseteq \{-1,1\}^\infty$ such that whenever $\sum c_n^2$, there is some $x \in A$ which makes $\sum x_n c_n$ converge. For a detailed discussion of the connection between (strong) Weihrauch reducibility, cardinal characteristics, and non-lowness notions, see [GKT19].

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