COMPLETENESS OF THE HYPERARITMETIC ISOMORPHISM EQUIVALENCE RELATION

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Abstract. We show that the existence of hyperarithmetic isomorphisms between computable structures is complete for Π^1_1 equivalence relations under computable reductions. This uses Montalbán's true stages machinery for iterated priority arguments, of which we give a new development.

1. Introduction

Does a classification problem have good invariants? This is a fundamental question, which encompases endeavors across mathematics. To give a positive answer, it suffices to exhibit simple invariants. To take canonical examples, vector spaces over a fixed field are classified by their dimension, while algebraically closed fields of a fixed characteristic are classified by their transcendence degree. One of the main insights of mathematical logic is that tools in set theory and computability theory can be used to formally state and prove that certain classification problems do *not* have simple invariants.

In descriptive set theory, this is formalised by studying definable equivalence relations up to Borel reducibility (see [Gao09, Kan08, Kec99]). This notion of reducibility allows us to compare the complexity of equivalence relations on Polish spaces. An *anti-classification* result is given by completeness. A collection Γ of equivalence relations is specifed by either syntactic complexity or a structural property; for example the collections of Borel or analytic equivalence relations; or the relations whose equivalence classes are all countable. An equivalence relation $E \in \Gamma$ is complete if every equivalence relation in Γ reduces to E. This says that E is as complicated as possible within the class Γ. Invariants, on the other hand, serve to simplify, and so are incompatible with completeness. In the examples of vector spaces and fields, the invariants give a reduction from the isomorphism relation for these structures to equality of natural numbers, one of the simplest equivalence relations in existence.

Computability theory gives a different setting for the same general project. It studies equivalence relations on $\mathbb{N}$ under computable reductions. A computable reduction of an equivalence relation E to an equivalence relation F is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $i, j \in \mathbb{N}$, $iEj \iff f(i)Ff(j)$. If there is such a reduction we write $E \leqslant F$. This is a modification of computable reducibility on sets of natural numbers, first studied by Post [Pos44], who showed that the halting set is complete for c.e. sets under computable reducibility. The study of computable reducibility of equivalence relations has origins in work of Ershov's ([Ers77a] , see [Ers77b, Ers99]), and began in ernest with Bernardi and Sorbi, who

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studied c.e. equivalence relations [BS83]. Gao and Gerdes, and separately Andrews, Lempp, J. Miller, Ng, San Mauro and Sorbi, have also studied c.e. equivalence relations [GG01, ALM+14], while Fokina, S. Friedman and Nies have studied Σ^0_3 equivalence relations [FFN12], and Fokina, S. Friedman, Harizanov, Knight, McCoy and Montalbán have studied Σ^1_1 equivalence relations [FFH+12], all under computable reductions.

Set theory and computability theory give complementary views of the complexity of classification. Set theoretic methods allow topological and measure-theoretic techniques, and study definability on a large scale. Computability studies simpler objects (relations on countable rather than continuum-sized domains), however new distinctions come from the considerations of effectiveness. An example of this is the complexity of isomorphism relations. In the setting of descriptive set theory, H. Friedman and Stanley [FS89] initiated the study of isomorphism problems for classes of countable structures under Borel reducibility. One of their central results is that the isomorphism relation for all countable structures, while being analytic ($\boldsymbol{\Sigma}^1_1$), is not complete for that class. In contrast, Fokina et al. showed that isomorphism of computable structures is computably complete for effectively analytic (Σ^1_1) equivalence relations on $\mathbb{N}$ [FFH+12]. More precisely, if $(\mathcal{M}_i)_{i\in\omega}$ is an admissible numbering of all partial computable structures, consider the relation

$$iE_{\cong}j \Leftrightarrow [\mathcal{M}_i \text{ and } \mathcal{M}_j \text{ are total and } \mathcal{M}_i \cong \mathcal{M}_j].$$

This is a Σ^1_1 equivalence relation on $\mathbb{N}$[1], and for every Σ^1_1 equivalence relation E on $\mathbb{N}$, $E \leqslant E_{\cong}$.

In contrast with descriptive set theory, which does not differentiate between complexities of functions on countable sets, in computability theory it is natural to ask not about the existence of arbitrary isomorphisms but about definable, or concrete ones. At the simplest level, Fokina, Friedman and Nies examined a related equivalence relation by restricting to computable isomorphisms rather than allowing arbitrary isomorphisms; they showed that computable isomorphism of computable structures is computably complete for Σ^0_3 equivalence relations on $\mathbb{N}$ [FFN12], where this is understood in the same manner as the previous result.

Rather than restrict to computable isomorphisms, in this paper we look at the class of effectively Borel functions, also known as the hyperarithmetic functions. A function is hyperaithmetic if it can be built from very simple functions by iterating the operation of the Turing jump and closing under relative computability — much like the Borel sets are built from the open ones by taking countable unions and complements. Similarly, the hyperarithmetic functions are those which are Δ^1_1 — effectively analytic and effectively co-analytic, which again mirrors a characterisation of Borel sets. We answer the question (which again, cannot be asked in the set-theretic setting), what is the complexity of the existence of effectively Borel isomorphisms between computable structures? We prove:

Theorem 1.1. *Hyperarithmetic isomorphism on computable structures is computably complete for Π^1_1 equivalence relations on $\mathbb{N}$.*

In more detail: fix an admissible numbering of partial computable structures $(\mathcal{M}_i)_{i\in\omega}$. Consider the equivalence relation $i \sim j \Leftrightarrow$ both $\mathcal{M}_i$ and $\mathcal{M}_j$ are total,

[1] Strictly speaking, we must define $E_{\cong}$ to be the reflexive closure of the above relation to make it an equivalence relation.

and there is a hyperarthmetic isomorphism between $\mathcal{M}_i$ and $\mathcal{M}_j$.[2] The universe of this relation is $\mathbb{N}$. The Spector-Gandy theorem [Spe60, Gan60] implies that this relation is Π^1_1 (effectively co-analytic). The precise statement of Theorem 1.1 is that the equivalence relation $\sim$ is computably complete for Π^1_1 equivalence relations on $\mathbb{N}$: for every Π^1_1 equivalence relation E there is a computable function f such that for all $i, j \in \mathbb{N}$, $iEj \iff f(i) \sim f(j)$.

1.1. **Our approach, informally.** Let E be a Π^1_1 equivalence relation on $\mathbb{N}$. To prove Theorem 1.1 we construct an array $\langle \mathcal{N}_k \rangle$ of uniformly computable structures such that for all $i, j \in \mathbb{N}$,

$$iEj \iff \text{there is a hyperarithmetic isomorphism between } \mathcal{N}_i \text{ and } \mathcal{N}_j.$$

What do these structures look like? Each structure will have disjoint components indexed by pairs $(e, i) \in \mathbb{N}$. The component indexed by (e, i) in the structure $\mathcal{N}_k$ will be used to diagonalise against the e^{th} potential hyperarithmetic isomorphism between $\mathcal{N}_i$ and $\mathcal{N}_k$. We may assume that the universes of the component indexed by (e, i) are the same in all structures. Each component (e, i) has two parts. One is a tag, consisting of two elements $a^{e,i}$ and $b^{e,i}$ (which we may assume are the same elements in all the structures). These elements are not distinguished in the signature of these structures. To each one we attach a linear ordering: $A^{e,i}$ and $B^{e,i}$. Again we may assume that the universes are the same.

So formally, the signature will contain:

- Unary relations $T^{e,i}$;
- Binary relations $M^{e,i}$;
- Binary relations $L^{e,i}$.

Each structure, as mentioned, will contain disjoint sets $A^{e,i}$, $B^{e,i}$, and $\{a^{e,i}, b^{e,i} : (e, i) \in \mathbb{N}^2\}$, with each $A^{e,i}$ and $B^{e,i}$ infinite. These sets are uniformly computable, and the functions $(e, i) \mapsto a^{e,i}, b^{e,i}$ are computable.

- $T^{e,i}(x)$ holds in $\mathcal{N}_k$ if and only if $x = a^{e,i}$ or $x = b^{e,i}$;
- $(M^{e,i})^{\mathcal{N}_k}$ is a directed graph. $M^{e,i}(x, y)$ holds in $\mathcal{N}_k$ if and only if $x \in A^{e,i}$ and $y = a^{e,i}$, or $x \in B^{e,i}$ and $y = b^{e,i}$.
- $(L^{e,i})^{\mathcal{N}_k}$ will be a disjoint union of two linear orderings, one on $A^{e,i}$, the other on $B^{e,i}$.

We will denote these linear orderings by $A^{k,e,i}$ and $B^{k,e,i}$. Now the main point of these definitions is the following:

Proposition 1.2. *Let $m, k \in \mathbb{N}$. Suppose that $F \colon \mathcal{N}_m \to \mathcal{N}_k$ is an isomorphism. Then:*

(a) $F(a^{e,i}) \in \{a^{e,i}, b^{e,i}\}$.
(b) *If $F(a^{e,i}) = a^{e,i}$ then $F[A^{m,e,i}] = A^{k,e,i}$ and $F[B^{m,e,i}] = B^{k,e,i}$ (and of course preserves their linear orderings).*
(c) *If $F(a^{e,i}) = b^{e,i}$ then $F[A^{m,e,i}] = B^{k,e,i}$ and $F[B^{m,e,i}] = A^{k,e,i}$.*

Both are potentially possible, depending on the order-types of the relevant linear orderings.

We need to ensure two things:

[2]Again, we mean $\sim$ to be reflexively closed so as to be an equivalence relation.

- If mEk then for some computable ordinal β, $\varnothing^{(\beta)}$ computes an isomorphism between $\mathcal{N}_m$ and $\mathcal{N}_k$.
- If $\neg mEk$ then for all e, F_e is not an isomorphism between $\mathcal{N}_m$ and $\mathcal{N}_k$.

Here $\langle F_e \rangle$ is a uniform list of all Π^1_1 partial functions from $\mathbb{N}$ to $\mathbb{N}$; note that a function is hyperarithmetic if and only if it is total Π^1_1.

Let us consider first how to diagonalise against some F_e. As mentioned above, we reserved the component (e,i) of $\mathcal{N}_k$ to diagonalise against $F_e\colon \mathcal{N}_i \to \mathcal{N}_k$ being an isomorphism. The rough plan is as follows. Suppose that $\neg iEk$, so we want to diagonalise. We are only worried if $F(a^{e,i}) \in \{a^{e,i}, b^{e,i}\}$.

- If $F_e(a^{e,i}) = a^{e,i}$ then we ensure that $A^{i,e,i} \not\cong A^{k,e,i}$;
- If $F_e(a^{e,i}) = b^{e,i}$ then we ensure that $A^{i,e,i} \not\cong B^{k,e,i}$.

In either case this ensures that F_e cannot be an isomorphism from $\mathcal{N}_i$ to $\mathcal{N}_k$; this follows from Proposition 1.2.

On the other hand, if iEk then regardless of what $F_e(a^{e,i})$ may be, we need to ensure that $\mathcal{N}_i \cong \mathcal{N}_k$. An important point is that this doesn't mean that we must have $A^{i,e,i} \cong A^{k,e,i}$ and $B^{i,e,i} \cong B^{k,e,i}$; this could be flipped, with the isomorphism matching A's to B's. We will make sure that one of the two always happens, even if $\neg iEk$ — and so all the structures $\mathcal{N}_m$ we construct will be pairwise isomorphic. The only question is whether a hyperarithmetic oracle can correctly match the linear orderings. That is, an oracle constructing the isomorphism will need to know whether to send $a^{e,i}$ to itself or to $b^{e,i}$, and then construct the isomorphisms of the linear ordering.

Π^1_1 sets are $\Sigma_1(L_{\omega_1^{\text{ck}}})$. This means that they are analogous to c.e. sets, except that they take ω_1^{ck} many steps to enumerate. Here recall that ω_1^{ck} is the least non-computable ordinal, which is also the least admissible ordinal beyond ω. As a result, each true Π^1_1 fact φ can be attached a computable ordinal, below denoted by $[\![\varphi]\!]$, which is its "ordinal height": at time (or level) $\alpha = [\![\varphi]\!]$ we discover that φ is true. Roughly, this corresponds to $\varnothing^{(\alpha)}$, the α^{th} iteration of the Turing jump, computing the fact that φ is true. Recalling that it takes (about) α jumps to compute isomorphisms between copies of the linear ordering ω^α, the more involved plan is now as follows.

First, we present the equivalence relation E as the union $\bigcup_{\alpha<\omega_1^{\text{ck}}} E(\alpha)$ of hyperarithmetic equivalence relations which get coarser with time: for a time we believe that $\neg iEk$, but at some point α we may discover that iEk.

Now suppose that at time $\alpha < \omega_1^{\text{ck}}$, we discover that $F_e(a^{e,i})\downarrow$ (and is either $a^{e,i}$ or $b^{e,i}$). We will then make, for all $m \in \mathbb{N}$,

$$\left\{A^{m,e,i}, B^{m,e,i}\right\} \cong \{\omega^\alpha, \omega^\alpha + \omega^\alpha\},$$

and note that $\omega^\alpha \not\cong \omega^\alpha + \omega^\alpha = \omega^\alpha \cdot 2$. For all $k \in \mathbb{N}$,

- if $iE(\alpha)k$, then we let $A^{k,e,i} \cong \omega^\alpha$ and $B^{k,e,i} \cong \omega^\alpha \cdot 2$.
- if $\neg iE(\alpha)k$, then we may or may not flip, depending on $F_e(a^{e,i})$:
  - if $F_e(a^{e,i}) = b^{e,i}$ then we choose the same: $A^{k,e,i} \cong \omega^\alpha$ and $B^{k,e,i} \cong \omega^\alpha \cdot 2$.
  - if $F_e(a^{e,i}) = a^{e,i}$ then we flip, letting $A^{k,e,i} \cong \omega^\alpha \cdot 2$ and $B^{k,e,i} \cong \omega^\alpha$.

If we manage to do this, we will have diagonalised successfully: if $\neg iEk$ then for all α, $\neg iE(\alpha)k$, and so whenever some F_e converges, we will diagonalise. But now we need to address the other requirement: what happens when iEk? This is discovered at some stage β, and roughly (give or take some jumps), we will want $\varnothing^{(\beta)}$ to compute an isomorphism between $\mathcal{N}_i$ and $\mathcal{N}_k$. We work on each component separately. For a fixed (e, i), there are two possibilities:

- At stage $\alpha < \beta$, we discovered that $F_e(a^{e,i})\downarrow$ and so determined that the orderings are ω^α and $\omega^\alpha \cdot 2$. But $\alpha < \beta$ means that $\varnothing^{(\beta)}$ knows this, and knows which way we diagonalised, so it can correctly match $a^{e,i}$ to itself or to $b^{e,i}$. Then, again because $\alpha < \beta$, $\varnothing^{(\beta)}$ can construct isomorphisms between any two copies of ω^α (and of $\omega^\alpha \cdot 2$).
- Otherwise, suppose that $F_e(a^{e,i})\downarrow$ at some stage $\alpha \geqslant \beta$. By that stage we know that we want to make $\mathcal{N}_i$ and $\mathcal{N}_k$ the same. So we know that $A^{i,e,i} \cong A^{k,e,i}$ and $B^{i,e,i} \cong B^{k,e,i}$. In fact, we will ensure that the β^{th} Hausdorff derivative $A^{i,e,i}_\beta$ is the same linear ordering (not just isomorphic) as $A^{k,e,i}_\beta$, and the same for the B's. The oracle $\varnothing^{(\beta)}$ will compute these derivatives and match them by the identity function. Each point in the derivative is the image of a sub-ordering of type ω^β, and $\varnothing^{(\beta)}$, having computed the derivative process, can now match these point-by-point.

Roughly, that is the plan; but we have not addressed some questions:

(1) If $F_e(a^{e,i})\uparrow$, what do the orderings $A^{k,e,i}$ and $B^{k,e,i}$ look like? Unfortunately we cannot give a Π^1_1 enumeration (of order-type ω) of the total Π^1_1 functions, i.e. the hyperarithmetic, functions.

(2) It seems that the *construction* of the structures requires the oracles $\varnothing^{(\beta)}$ for all $\beta < \omega_1^{\text{ck}}$. But we want to make the structures $\mathcal{N}_k$ *computable*. And we can't enumerate all the computable ordinals, not even in a Δ^1_1 way.

There are two solutions that address both problems. To work computably, we use Montalbán's [Mon14] extension of the Ash machinery of iterated priority arguments (see [AK00]). That means that at every stage s, for every α, we have a finite, stage s approximation for $\varnothing^{(\alpha)}$, in a computable way, and we work with that oracle to construct our structures. Many of these approximations will be false, and so we need to delicately ensure that we can correct our mistakes when we discover them (or think that we discovered them, only to be recorrected again in the future). This is the machinery of α-true stages. In section 2 we give a redevelopment of this machinery, which we believe is simpler and more intuitive than that which has appeared in print so far.

We still need to address the question: what about getting all the ordinals up to ω_1^{ck}? This is a serious problem, because the Ash - Montalbán machinery works up to any chosen computable ordinal, not all the way up to ω_1^{ck}. The solution is to use a *pseudo-ordinal*: a Harrison linear ordering that supports a jump-hierarchy and other arithmetic structures such as the finite approximations to the jumps. Now, ω_1^{ck} is an initial segment – the well founded part – of a pseudo-ordinal δ_*, and we perform our construction imagining that the pseudo-ordinal is indeed well-founded. After the fact, we take a look and observe that the extra ill-founded layers (between ω_1^{ck} and δ_*) did not spoil our original plans. For example, it is possible that $\neg iEk$ but that we "discover" that $iE(\alpha)k$ at some level $\omega_1^{\text{ck}} < \alpha \leqslant \delta_*$. The same argument works: $\varnothing^{(\alpha)}$ will compute an isomorphism between $\mathcal{N}_i$ and $\mathcal{N}_k$. But that "iteration" of the jump is not hyperarithmetic. All true convergences of F_e appear at well-founded stages, so below α, and so we will have successfully diagonalised against those. And if $F_e(a^{e,i})\uparrow$ even at level δ_*, then we make both linear orderings isomorphic to $\omega^{\delta}*$, which is again a Harrison linear ordering. The argument that $\varnothing^{(\beta)}$, where we discover that iEk at level β, can compute an isomorphism between $\mathcal{N}_i$ and $\mathcal{N}_k$, works exactly the same: the β^{th} Hausdorff derivative is computed, and is identical in both structures — it will be ill founded, but that does not matter.

2. True Stages

We discovered a development of Montalbán's apparent true stages machinery that we believe is simpler and more intuitive. We present it here.

2.1. **True stages, including limits.**

We first develop a system of apparent true stages that includes limit iterations of the jump. Let δ^* be a computable ordinal. We fix a well-ordering $<^*$ of $\mathbb{N}$ of order-type δ^*+1 such that the successor function and collection of limits are computable (essentially, a notation for δ^*+1 in Kleene's system of ordinal notations $\mathcal{O}$). For $\alpha \leqslant \delta^*$ we let n_α denote the natural number in position α according to $<^*$.

Jumps of strings. For a string $\sigma \in \omega^{\leqslant\omega}$ let σ' denote the collection of inputs on which a universal Turing machine with oracle sequence σ halts in fewer than $|\sigma|$ many steps. Thus the jump of the empty string is empty. We assume that if σ is a one-entry extension of τ then $|\sigma'| \leqslant |\tau'|+1$. Thus, for every string σ we get an enumeration of the elements of σ' in order of which converged earlier (i.e. with shorter oracle). If σ is finite, we let $p(\sigma)$ be the last element enumerated into σ'; $p(\sigma) = -1$ if $\sigma' = \varnothing$.

Definition. By induction on $\alpha \leqslant \delta^*$ we define the relation $s \trianglelefteq^\alpha t$ (for $s \leqslant t \leqslant \omega$), which reads "$s$ appears α-true at stage t." We also define along the same induction strings σ_t^α.

- The string σ_t^α is defined to be the increasing enumeration of all the stages s such that $n_\alpha < s < t$ and $s \trianglelefteq^\alpha t$.

The aim is: σ_ω^α is our version of $\varnothing^{(\alpha)}$, and σ_t^α is the stage t approximation of σ_ω^α. The definition of $\trianglelefteq^\alpha$ is as follows.

- For $s,t \leqslant \omega$, $s \trianglelefteq^0 t$ if $s \leqslant t$.
- If α is a limit ordinal, then $s \trianglelefteq^\alpha t$ if for all $\beta < \alpha$, $s \trianglelefteq^\beta t$.
- If $\trianglelefteq^\alpha$ has been defined, then $s \trianglelefteq^{\alpha+1} t$ if $s \trianglelefteq^\alpha t$, and there is no $e < p(\sigma_s^\alpha)$ in $(\sigma_t^\alpha)' \backslash (\sigma_s^\alpha)'$.

The general idea for this definition is this: at stage $s < \omega$, with oracle σ_s^α we compute the jump set $(\sigma_s^\alpha)'$, but we only commit to the values of the jump up to the last element enumerated, namely $p(\sigma_s^\alpha)$. Suppose that $t > s$. If $s \ntrianglelefteq^\alpha t$ then t thinks that s was likely wrong about the oracle σ_s^α, and so there is no meaningful comparison between their jumps $(\sigma_s^\alpha)'$ and $(\sigma_t^\alpha)'$. Suppose, however, that $s \trianglelefteq^\alpha t$. We will shortly show that $\sigma_s^\alpha \preccurlyeq \sigma_t^\alpha$, and so $(\sigma_s^\alpha)' \subseteq (\sigma_t^\alpha)'$. Recall that s only commited to the jump up to $p(\sigma_s^\alpha)$. This commitment is discovered to be false by stage t if $(\sigma_s^\alpha)' \upharpoonright p(\sigma_s^\alpha) \neq (\sigma_t^\alpha)' \upharpoonright p(\sigma_s^\alpha)$, that is, if a number $e < p(\sigma_s^\alpha)$ was enumerated into $(\sigma_t^\alpha)'$ with use beyond σ_s^α.

Basic properties. First, it is easy to see: if $\alpha < \beta$ and $s \trianglelefteq^\beta t$ then $s \trianglelefteq^\alpha t$. In particular, as $\trianglelefteq^0$ is the natural ordering on $\omega+1$, if $s \trianglelefteq^\alpha t$ then $s \leqslant t$. Also, for all t and all α, $t \trianglelefteq^\alpha t$.

Lemma 2.1. For all $\alpha \leqslant \delta^*$,

(a) The relation $\trianglelefteq^\alpha$ is a partial ordering.
($\diamondsuit$) For all $s \leqslant r \leqslant t \leqslant \omega$, if $s, r \trianglelefteq^\alpha t$, then $s \trianglelefteq^\alpha r$.
($\clubsuit$) For all $s \trianglelefteq^\alpha r \trianglelefteq^\alpha t \leqslant \omega$, if $s \trianglelefteq^{\alpha+1} t$ then $s \trianglelefteq^{\alpha+1} r$.
(b) If $s \trianglelefteq^\alpha t$ then $\sigma_s^\alpha \preccurlyeq \sigma_t^\alpha$.
(c) If $s \trianglelefteq^{\alpha+1} t$ and t is finite, then $p(\sigma_s^\alpha) \leqslant p(\sigma_t^\alpha)$.

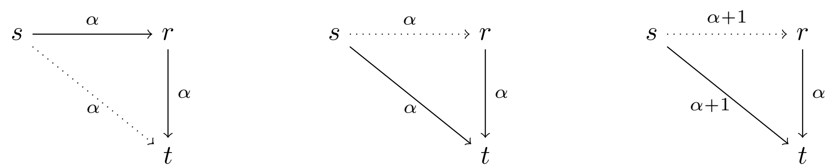


FIGURE 1. From left to right: the transitivity of $\trianglelefteq^\alpha$, the property $(\diamondsuit)_\alpha$, and the property $(\clubsuit)_\alpha$ (given $(\diamondsuit)_\alpha$).

While the converse of (b) may fail, it is "close" to the truth, and this gives an informal motivation for the three properties illustrated in fig. 1. The relation $\trianglelefteq^\alpha$ is transitive because if $s \trianglelefteq^\alpha r \trianglelefteq^\alpha t$ then $\sigma_s^\alpha \preccurlyeq \sigma_r^\alpha \preccurlyeq \sigma_t^\alpha$, and so $\sigma_s^\alpha \preccurlyeq \sigma_t^\alpha$. The property ($\diamondsuit$) is similar: if $\sigma_s^\alpha, \sigma_r^\alpha \preccurlyeq \sigma_t^\alpha$ then σ_s^α and σ_r^α are comparable. The property ($\clubsuit$) says that if $\sigma_s^\alpha \preccurlyeq \sigma_r^\alpha \preccurlyeq \sigma_t^\alpha$, so that $(\sigma_s^\alpha)' \subseteq (\sigma_r^\alpha)' \subseteq (\sigma_t^\alpha)'$, and no number $e < p(\sigma_s^\alpha)$ has been enumerated into the jump up to stage t, then certainly no such number was enumerated by the earlier stage r.

Proof. First, we observe that $(b)_\alpha$ follows from $(a)_\alpha$ and $(\diamondsuit)_\alpha$: by the definition of σ_t^α, for $(b)_\alpha$ we need to show that if $r < s \trianglelefteq^\alpha t$ then $r \trianglelefteq^\alpha s \iff r \trianglelefteq^\alpha t$. One direction follows from $(a)_\alpha$, the other from $(\diamondsuit)_\alpha$.

We also observe that $(c)_\alpha$ also follows from $(a)_\alpha$ and $(\diamondsuit)_\alpha$. Suppose that $s \trianglelefteq^{\alpha+1} t$ and t is finite. Then $s \trianglelefteq^\alpha t$, which by $(b)_\alpha$ implies that $\sigma_s^\alpha \preccurlyeq \sigma_t^\alpha$, and so $(\sigma_s^\alpha)' \subseteq (\sigma_t^\alpha)'$. If $p(\sigma_t^\alpha) \neq p(\sigma_s^\alpha)$ then $p(\sigma_t^\alpha) \notin (\sigma_s^\alpha)'$. If $p(\sigma_t^\alpha) < p(\sigma_s^\alpha)$ then $e = p(\sigma_t^\alpha)$ violates the definition of $s \trianglelefteq^{\alpha+1} t$.

By simultaneous induction on α we prove: $(a)_\alpha$, $(\diamondsuit)_\alpha$, and $(\forall \beta < \alpha)\,(\clubsuit)_\beta$.

For $\alpha = 0$ this is easy. Suppose that α is a limit and that the inductive assumption holds for all $\beta < \alpha$. The assumption certainly implies that for all $\beta < \alpha$, $(\clubsuit)_\beta$ holds. Since the relation $\trianglelefteq^\alpha$ is the intersection of the relations $\trianglelefteq^\beta$ for $\beta < \alpha$, we also get $(a)_\alpha$ and $(\diamondsuit)_\alpha$.

It remains to check the successor case. Let $\alpha < \delta^*$, and suppose that $(a)_\alpha$ and $(\diamondsuit)_\alpha$ hold (and so also $(b)_\alpha$ and $(c)_\alpha$). We verify that $(a)_{\alpha+1}$, $(\diamondsuit)_{\alpha+1}$ and $(\clubsuit)_\alpha$ hold.[3]

For $(a)_{\alpha+1}$, let $s \trianglelefteq^{\alpha+1} r \trianglelefteq^{\alpha+1} t$. Then $s \trianglelefteq^\alpha r \trianglelefteq^\alpha t$, and so by $(a)_\alpha$, $s \trianglelefteq^\alpha t$. By $(b)_\alpha$, $\sigma_s^\alpha \preccurlyeq \sigma_r^\alpha \preccurlyeq \sigma_t^\alpha$, and so $(\sigma_s^\alpha)' \subseteq (\sigma_r^\alpha)' \subseteq (\sigma_t^\alpha)'$. Suppose that $e \in (\sigma_t^\alpha)' \backslash (\sigma_s^\alpha)'$; we need to show that $e > p(\sigma_s^\alpha)$. There are two cases. If $e \in (\sigma_r^\alpha)'$ then $e > p(\sigma_s^\alpha)$

[3] We do not use the assumption that $(\clubsuit)_\beta$ holds for all $\beta < \alpha$.

by the assumption $s \trianglelefteq^{\alpha+1} r$. Otherwise, $e \in (\sigma_t^\alpha)'\backslash(\sigma_r^\alpha)'$ and so $e > p(\sigma_r^\alpha)$ by the assumption $r \trianglelefteq^{\alpha+1} t$. By (c)$_\alpha$ and the assumption $s \trianglelefteq^{\alpha+1} r$ we get $p(\sigma_r^\alpha) \geqslant p(\sigma_s^\alpha)$. Note that we may assume that $r < t$ and so that r is finite.

Next we verify $(\clubsuit)_\alpha$. Suppose that $s < r < t$, $s \trianglelefteq^\alpha r \trianglelefteq^\alpha t$, and $s \trianglelefteq^{\alpha+1} t$. We have observed that $(\sigma_r^\alpha)' \subseteq (\sigma_t^\alpha)'$. Let $e \in (\sigma_r^\alpha)'\backslash(\sigma_s^\alpha)'$. Then $e \in (\sigma_t^\alpha)'\backslash(\sigma_s^\alpha)'$, and so by $s \trianglelefteq^{\alpha+1} t$ we get $e > p(\sigma_s^\alpha)$; so $s \trianglelefteq^{\alpha+1} r$ as well.

Finally, we verify $(\diamondsuit)_{\alpha+1}$. In fact, it follows from $(\diamondsuit)_\alpha$ and $(\clubsuit)_\alpha$. Suppose that $s < r < t$ and that $s, r \trianglelefteq^{\alpha+1} t$. By $(\diamondsuit)_\alpha$, $s \trianglelefteq^\alpha r$. By $(\clubsuit)_\alpha$, $s \trianglelefteq^{\alpha+1} r$. ☐

Corollary 2.2. For finite t, $s \trianglelefteq^{\alpha+1} t$ if and only if $s \trianglelefteq^\alpha t$, and for all $r \in (s, t]$ such that $s \triangleleft^\alpha r \trianglelefteq^\alpha t$, $p(\sigma_r^\alpha) \geqslant p(\sigma_s^\alpha)$.

Proof. Each successive r such that $s \triangleleft^\alpha r \trianglelefteq^\alpha t$ extends σ^α by one bit, and so $(\sigma_t^\alpha)'\backslash(\sigma_s^\alpha)'$ equals $\{p(\sigma_r^\alpha) : s \triangleleft^\alpha r \trianglelefteq^\alpha t \;\&\; p(\sigma_r^\alpha) \neq p(\sigma_s^\alpha)\}$. ☐

Structurally, the transitivity of $\trianglelefteq^\alpha$ and the property $(\diamondsuit)_\alpha$ together say that $(\omega + 1, \trianglelefteq^\alpha)$ is a tree; every $s < \omega$ has height at most s in that tree. We shall soon see that ω has height ω in this tree.

When we say nothing.

Lemma 2.3. Suppose that $s \trianglelefteq^\alpha t$, and there is no stage $r \triangleleft^\alpha s$ such that $r > n_\alpha$. Then $s \trianglelefteq^{\alpha+1} t$.

Proof. The assumption implies that σ_s^α is the empty string, and so that $p(\sigma_s^\alpha) = -1$. ☐

The continuity of the relations $\trianglelefteq^\alpha$ (namely for limit α, $\trianglelefteq^\alpha$ is the intersection of $\trianglelefteq^\beta$ for $\beta < \alpha$) implies that for all $s \leqslant t \leqslant \omega$, $\max\{\gamma \leqslant \delta^* : s \trianglelefteq^\gamma t\}$ must exist.

Lemma 2.4. Let $s \leqslant t \leqslant \omega$; let $\gamma = \max\{\alpha \leqslant \delta^* : s \trianglelefteq^\alpha t\}$. If $\gamma < \delta^*$ then $s > n_\gamma$.

Proof. If $n_\gamma \geqslant s$ then there certainly is no $r \triangleleft^\gamma s$ such that $r > n_\gamma$, whence by Lemma 2.3, $s \trianglelefteq^{\gamma+1} t$, contrary to the definition of γ. ☐

The lemma says that $s \trianglelefteq^{\gamma+1} t$ can first fail only at a level γ which is "at play" at stage s. Another way of stating Lemma 2.4:

Corollary 2.5. Suppose that $s \trianglelefteq^{\beta+1} t$, that $\gamma > \beta$, and that for all $\alpha \in (\beta, \gamma)$, $n_\alpha \geqslant s$. Then $s \trianglelefteq^\gamma t$.

An application is:

Lemma 2.6. For all $t \leqslant \omega$, $0 \trianglelefteq^{\delta^*} t$.

Computability.

Proposition 2.7.

(a) *The relations $\trianglelefteq^\alpha$, restricted to $\mathbb{N}$, are uniformly computable.*
(b) *The functions $s \mapsto \sigma_s^\alpha$, restricted to $\mathbb{N}$, are uniformly computable.*
(c) *The function $(s, t) \mapsto \max\{\alpha \leqslant \delta^* : s \trianglelefteq^\alpha t\}$ (for $s \leqslant t < \omega$) is computable.*

When discussing computability, we ignore the difference between α and n_α. So for example, part (a) of the proposition means that the set $\{(n_\alpha, s, t) \in \mathbb{N}^3 : s \trianglelefteq^\alpha t\}$ is computable.

Proof. The relations $s \trianglelefteq^\alpha t$ and the functions $s \mapsto \sigma_t^\alpha$ are computed by simultaneous recursion on t. The latter are computed from the former. The definition of the relations shows that if we can decide $s \trianglelefteq^\alpha t$, and so know σ_s^α and σ_t^α, then we can also decide whether $s \trianglelefteq^{\alpha+1} t$. The algorithm is given with the aid of corollary 2.5. Enumerate the set $\{\beta \leqslant \delta^* : n_\beta < s\} \cup \{0\}$ as $0 = \beta_0 < \beta_1 < \cdots < \beta_k$. Then $s \trianglelefteq^{\beta_0} t$. If we have decided that $s \trianglelefteq^{\beta_i} t$ and $\beta_i < \delta^*$, then we check if $s \trianglelefteq^{\beta_i+1} t$; if so, and $i < k$, then $s \trianglelefteq^{\beta_{i+1}} t$. If $\beta_k < \delta_*$ and $s \trianglelefteq^{\beta_k+1} t$ then $s \trianglelefteq^{\delta^*} t$. This decision procedure also gives part (c). □

True stages. For brevity, for all $\alpha \leqslant \delta^*$, let

$$D^\alpha = \{s \in \mathbb{N} : s \triangleleft^\alpha \omega\}.$$

This is the set of the α-true stages. If α is a limit ordinal then $D^\alpha = \bigcap_{\beta<\alpha} D^\beta$. Note that σ_ω^α is the increasing enumeration of the stages $s > n_\alpha$ in D^α. If D^α is infinite then $\sigma_\omega^\alpha = \bigcup_{s \in D^\alpha} \sigma_s^\alpha$.

Lemma 2.8. Suppose that D^α is infinite. The following are equivalent:

(1) $s \in D^{\alpha+1}$;
(2) for all $t \geqslant s$ in D^α, $s \trianglelefteq^{\alpha+1} t$;
(3) for infinitely many $t \geqslant s$ in D^α, $s \trianglelefteq^{\alpha+1} t$.

Proof. For (1) → (2), we use the property (♣) (if $s \trianglelefteq^{\alpha+1} \omega$ and $s \trianglelefteq^\alpha t \trianglelefteq^\alpha \omega$ then $s \trianglelefteq^{\alpha+1} t$). For (3) → (1), we use the fact that $(\sigma_\omega^\alpha)' = \bigcup_{t \in D^\alpha} (\sigma_t^\alpha)'$. So if $s \ntrianglelefteq^{\alpha+1} \omega$ then there is some $e \in (\sigma_\omega^\alpha)' \backslash (\sigma_s^\alpha)'$ with $e < p(\sigma_s^\alpha)$; for large enough t in D^α, $e \in (\sigma_t^\alpha)'$ and so $s \ntrianglelefteq^{\alpha+1} t$. □

Proposition 2.9. *For every $\alpha \leqslant \delta^*$, D^α is infinite.*

Proof. By induction on α. $D^0 = \mathbb{N}$. At successor levels we use non-deficiency stages. Namely, let $s_0 < \omega$; let $s > s_0$ in D^α such that $p(\sigma_s^\alpha)$ is minimal among $p(\sigma_t^\alpha)$ for $t > s_0$ in D^α. By corollary 2.2 and property (◇), for all $t \geqslant s$ in D^α, $s \trianglelefteq^{\alpha+1} t$. By Lemma 2.8, $s \in D^{\alpha+1}$.

Suppose that α is a limit ordinal, and suppose that for all $\beta < \alpha$, D^β is infinite. Given $s_0 \in \mathbb{N}$ we find some $\gamma < \alpha$ such that $n_\gamma > s_0$ and for all $\beta \in (\gamma, \alpha)$, $n_\beta > n_\gamma$. This can be done since α is a limit ordinal[4]. We then let s be the least stage in D^γ greater than n_γ.

We claim that $s \in D^\alpha$. By induction on $\beta \in [\gamma, \alpha]$ we show that $s \in D^\beta$. Limit $\beta \leqslant \alpha$ we get for free. If $\beta \in [\gamma, \alpha)$ and $s \in D^\beta$, then since $n_\beta \geqslant n_\gamma$, and $D^\beta \subseteq D^\gamma$, either $s \leqslant n_\beta$ or s is the least stage in D^β greater than n_β. In either case there is no $r \triangleleft^\beta s$ such that $r > n_\beta$. Then $s \in D^{\beta+1}$ by Lemma 2.3. □

The argument just given for limit levels has some resemblence to taking the diagonal intersection of closed and unbounded sets. Indeed, it is inspired by an analogous construction in the context of uncountable admissible computability, in which each D^α is closed and unbounded, and at limit levels of uncountable cofinality we use diagonal intersections.

Remark 2.10. For every $\alpha \leqslant \delta^*$, D^α is the unique infinite path through the tree $(\mathbb{N}, \trianglelefteq^\alpha)$. This is proved by induction. At the successor step, if $t \in D^\alpha \backslash D^{\alpha+1}$, then any infinite path above t in $(\mathbb{N}, \trianglelefteq^{\alpha+1})$ must also induce an infinite path in $(\mathbb{N}, \trianglelefteq^\alpha)$,

[4]And $\mathbb{N}$ is well-founded

and thus must be a subset of D^α; but by Lemma 2.8, $t \trianglelefteq^{\alpha+1} s$ for only finitely many $s \in D^\alpha$.

The jumps. We show that $\sigma^\alpha_\omega \in \mathbf{0}^{(\alpha)}$, in a uniform way.

Proposition 2.11.

(a) *For every $\alpha < \delta^*$, $(\sigma^\alpha_\omega)' \equiv_{\mathrm{T}} \sigma^{\alpha+1}_\omega$, uniformly.*
(b) *For every limit $\alpha \leqslant \delta^*$, $\sigma^\alpha_\omega \equiv_{\mathrm{T}} \bigoplus_{\beta<\alpha} \sigma^\beta_\omega$, uniformly.*
(c) *For every infinite $X \subseteq D^\alpha$, $X \geqslant_{\mathrm{T}} \sigma^\alpha_\omega$, uniformly.*

The uniformity means, for example, that there is a single reduction procedure Ψ such that for all $\alpha < \delta^*$, $\Psi(\sigma^{\alpha+1}_\omega, \alpha) = (\sigma^\alpha_\omega)'$.

Proof. First observe that D^α and σ^α_ω are Turing equivalent, uniformly. Parts (a) and (b) of the proposition are proved by simultaneous effective transfinite recursion. For the successor step, suppose that this has been done up to and including level $\alpha < \delta^*$. Given $(\sigma^\alpha_\omega)'$ we first compute σ^α_ω and so D^α. Then, to decide if some $s \in D^\alpha$ is in $D^{\alpha+1}$, we compare $(\sigma^\alpha_s)'$ and $(\sigma^\alpha_\omega)'$ and see whether they agree below $p(\sigma^\alpha_s)$. In the other direction, given $D^{\alpha+1}$ we compute $(\sigma^\alpha_\omega)'$: for $s \in D^{\alpha+1}$ we output $(\sigma^\alpha_s)' \upharpoonright p(\sigma^\alpha_s)$.

Now suppose that α is a limit ordinal. The set D^α computes each D^β for $\beta < \alpha$, uniformly: $s \in D^\beta$ if and only if $s \trianglelefteq^\beta t$ for some or all $t \geqslant s$ in D^α. In the other direction, we use Lemma 2.4 (which applies to $t = \omega$): to decide if $s \in D^\alpha$ we check if $s \in D^{\beta+1}$ for all $\beta < \alpha$ such that $n_\beta < s$.

Finally, to show (c), note that $s \in D^\alpha$ if and only if $s \trianglelefteq^\alpha t$ for some or all $t \geqslant s$ in X. ☐

Together with the fact that D^0 is computable, we get that $\sigma^\alpha_\omega \in \mathbf{0}^{(\alpha)}$ as promised. Indeed, if $\left\langle \varnothing^{(\alpha)} \right\rangle_{\alpha \leqslant \delta^*}$ is the jump hierarchy along δ^*, defined in any reasonable way, then $\sigma^\alpha_\omega \equiv_{\mathrm{T}} \varnothing^{(\alpha)}$, uniformly in α.

2.2. **True stages, redux.** The system of approximations σ^α_s and apparent true stages $s \trianglelefteq^\alpha t$ defined above is possibly the most natural development of this machinery. Unfortunately, it is not the most useful for applications. The problem is that for α a limit, $\varnothing^{(\alpha)}$ is not as useful an oracle as we would like: the question of whether some event happens at some level $\beta < \alpha$ is c.e. in $\varnothing^{(\alpha)}$ but not computable from it.

It turns out, as was discovered by Montalbán, that we can modify the relations $\trianglelefteq^\alpha$ to get $\varnothing^{(\alpha+1)}$ at limit levels. The idea is to "shift the blame" to a lower level. Suppose that $\lambda < \delta^*$ is a limit ordinal, $s \trianglelefteq^\lambda t$ but $s \ntrianglelefteq^{\lambda+1} t$. Some $e \in (\sigma^\lambda_t)' \backslash (\sigma^\lambda_s)'$ is smaller than $p(\sigma^\lambda_s)$. Let $\gamma < \lambda$ be maximal such that $n_\gamma < s$. Between γ and λ, nothing is happening at stage s, and so morally speaking, σ^λ_s is essentially σ^γ_s, the only real difference coming from the fact that possibly $n_\lambda < n_\gamma$. Thus we should blame level γ and declare that s does not appear to be $(\gamma+1)$-true at stage t after all. This is what we do, except for one snag: if γ itself is a limit ordinal, then we have just repeated the problem at a lower level. We could repeat; alternatively, we can just shift the blame one level up to $\gamma + 2$, which is what we choose to do.

We therefore define relations $\leqslant^\alpha$ as follows. We assume henceforth that δ^* is a *successor* ordinal.

Definition 2.12. Let $s \leqslant t \leqslant \omega$; let $\lambda = \max\{\alpha \leqslant \delta^* : s \trianglelefteq^\alpha t\}$.

- If λ is a successor ordinal (including the case $\lambda = \delta^*$), then for all $\alpha \leqslant \delta^*$, $s \leqslant^\alpha t$ if and only if $s \trianglelefteq^\alpha t$ (if and only if $\alpha \leqslant \lambda$).
- Suppose that $\lambda < \delta^*$ is a limit ordinal. Let t_0 be the least stage in the interval $(s, t]$ such that $s \triangleleft^\lambda t_0 \trianglelefteq^\lambda t$ and $s \ntriangleleft^{\lambda+1} t_0$. By Lemma 2.8, t_0 is finite. We can therefore let γ be the greatest ordinal below λ such that $n_\gamma < t_0$.[5] Then for all $\alpha \leqslant \delta^*$, we let $s \leqslant^\alpha t$ if and only if $\alpha \leqslant \gamma + 1$.

Note that $\leqslant^\alpha$ implies $\trianglelefteq^\alpha$.

Remark 2.13. Suppose that $\lambda < \delta^*$ is a limit ordinal, that $s \triangleleft^\lambda t_0 \trianglelefteq^\lambda t$ but $s \ntriangleleft^{\lambda+1} t_0$; and that t_0 is finite. Let γ be the greatest ordinal $\alpha < \lambda$ such that $n_\alpha < t_0$. Then $s \nleqslant^{\gamma+2} t$. To see this, first observe that $s \triangleleft^\lambda t$ but by (♣), $s \ntriangleleft^{\lambda+1} t$. Let t_1 be least such that $s \triangleleft^\lambda t_1 \trianglelefteq^\lambda t$ and $s \ntriangleleft^{\lambda+1} t_1$. Then $t_1 \leqslant t_0$. Let γ' be the greatest $\alpha < \lambda$ such that $n_\alpha < t_1$. Then $\gamma' \leqslant \gamma$ and $s \nleqslant^{\gamma'+2} t$.

Properties. We show that $\leqslant^\alpha$ has the same nice properties as $\trianglelefteq^\alpha$.

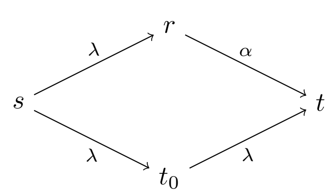
Lemma 2.14. For each $\alpha \leqslant \delta^*$, $\leqslant^\alpha$ is a partial ordering.

Proof. For all t, $t \trianglelefteq^{\delta^*} t$, and so $t \leqslant^\alpha t$ for all α.

Suppose that $s < r < t$ and that $s \nleqslant^\alpha t$. We need to show that either $s \nleqslant^\alpha r$ or $r \nleqslant^\alpha t$. We may assume that $s \trianglelefteq^\alpha r \trianglelefteq^\alpha t$, and so that $s \trianglelefteq^\alpha t$.

So there must be some limit ordinal $\lambda < \delta^*$ such that $s \trianglelefteq^\lambda t$ but $s \ntrianglelefteq^{\lambda+1} t$. Note that $\alpha \leqslant \lambda$.

Let t_0 be minimal such that $s \triangleleft^\lambda t_0 \trianglelefteq^\lambda t$ and $s \ntrianglelefteq^{\lambda+1} t_0$. As mentioned above, t_0 is finite. Let γ be greatest below λ such that $n_\gamma < t_0$. Then $s \leqslant^{\gamma+1} t$. Hence $\alpha \in (\gamma + 1, \lambda]$. The fact that $\alpha \geqslant \gamma + 1$ implies that for any u and v, if $u \leqslant t_0$ and $u \trianglelefteq^\alpha v$ then $u \trianglelefteq^\lambda v$ (corollary 2.5). This, for example, implies that $s \trianglelefteq^\lambda r$.



Now there are two cases, depending on the order between r and t_0. Suppose first that $t_0 \leqslant r$. By (♢) (with respect to r and t), we get $t_0 \trianglelefteq^\alpha r$. As just noticed, this implies that $t_0 \trianglelefteq^\lambda r$. So $s \triangleleft^\lambda t_0 \trianglelefteq^\lambda r$ but $s \ntriangleleft^{\lambda+1} t_0$. By Remark 2.13, $s \nleqslant^{\gamma+2} r$, so $s \nleqslant^\alpha r$.

Next suppose that $r < t_0$. As $r < t_0$ and $r \trianglelefteq^\alpha t$, we get $r \trianglelefteq^\lambda t$. By (♢), $r \trianglelefteq^\lambda t_0$. The minimality of t_0 shows that $s \trianglelefteq^{\lambda+1} r$, and so $r \ntrianglelefteq^{\lambda+1} t_0$. So again by Remark 2.13, $r \nleqslant^{\gamma+2} t$, whence $r \nleqslant^\alpha t$. □

The following is immediate from the definition of $\leqslant^\alpha$:

Lemma 2.15. For all $s \leqslant t \leqslant \omega$:

(a) $s \leqslant^0 t$; and
(b) $0 \leqslant^{\delta^*} t$.

The following as well:

---

[5]We may assume that $n_0 = 0$; since $t_0 > 0$, such γ must exist.

Lemma 2.16. The orderings $\leqslant^\alpha$ are nested: if $\alpha < \beta \leqslant \delta^*$ and $s \leqslant^\beta t$ then $s \leqslant^\alpha t$.

And the main point of this modification also follows from the definition.

Lemma 2.17. Let $\lambda < \delta^*$ be a limit ordinal, and let $s \leqslant t \leqslant \omega$.

(a) $s \leqslant^\lambda t$ if and only if $s \leqslant^\alpha t$ for all $\alpha < \lambda$.
(b) If $s \leqslant^\lambda t$ then $s \leqslant^{\lambda+1} t$.

Lemma 2.18. The property (♣) holds for the relations $\leqslant^\alpha$: if $s \leqslant^\alpha r \leqslant^\alpha t$ and $s \leqslant^{\alpha+1} t$ then $s \leqslant^{\alpha+1} r$.

Proof. Suppose that $s \leqslant^\alpha r \leqslant^\alpha t$ but that $s \nleqslant^{\alpha+1} r$. We show that $s \nleqslant^{\alpha+1} t$. Since $\ntrianglelefteq^{\alpha+1}$ implies $\nleqslant^{\alpha+1}$, we may assume that $s \trianglelefteq^{\alpha+1} t$. By (♣) for the orderings $\trianglelefteq^\alpha$, we get $s \trianglelefteq^{\alpha+1} r$. So the failure of $s \leqslant^{\alpha+1} r$ is blamed on some limit ordinal $\lambda > \alpha$ such that $s \trianglelefteq^\lambda r$ but $s \ntrianglelefteq^{\lambda+1} r$; let r_0 be least such that $s \triangleleft^\lambda r_0 \trianglelefteq^\lambda r$ and $s \ntriangleleft^{\lambda+1} r_0$; let $\gamma < \lambda$ be greatest such that $n_\gamma < r_0$. So $\alpha + 1 = \gamma + 2$, i.e., $\alpha = \gamma + 1$.

We have $r_0 \trianglelefteq^\lambda r \trianglelefteq^\alpha t$ and since $\lambda > \alpha$, we get $r_0 \trianglelefteq^\alpha t$; by corollary 2.5, we get $r_0 \trianglelefteq^\lambda t$. By Remark 2.13, $s \nleqslant^{\gamma+2} t$, so $s \nleqslant^{\alpha+1} t$ as required. □

Lemma 2.19. The property (♢) holds for the relations $\leqslant^\alpha$: if $s, r \leqslant^\alpha t$ and $s < r$ then $s \leqslant^\alpha r$.

Proof. $(♢)_\alpha$ is proved by induction on α. As in the proof of Lemma 2.1, $(♢)_{\alpha+1}$ follows from $(♢)_\alpha$ and $(♣)_\alpha$. □

Lemma 2.20. If α is a successor, $s \trianglelefteq^\alpha t$ and $s \nleqslant^\alpha t$, then $s < \min\{n_\alpha, n_{\alpha-1}\}$.

Proof. By the definition of $\leqslant^\alpha$, there is a limit ordinal λ such that $\lambda \geqslant \alpha$, $s \triangleleft^\lambda t$ and $s \ntriangleleft^{\lambda+1} t$. Since α is not a limit, $\alpha < \lambda$.

Let t_0 and γ be as in the definition of $\leqslant^\alpha$. So $s < t_0$ and $\alpha \in [\gamma + 2, \lambda)$; so $n_\alpha \geqslant t_0$ and $n_{\alpha-1} \geqslant t_0$. It follows that $s < n_\alpha$ and $s < n_{\alpha-1}$. □

Computability follows from Definition 2.12 and Proposition 2.7:

Lemma 2.21.

(a) The relations $\leqslant^\alpha$, restricted to $\mathbb{N}$, are uniformly computable.
(b) The function $(s,t) \mapsto \max\{\alpha \leqslant \delta^* : s \leqslant^\alpha t\}$ (for $s \leqslant t < \omega$) is computable.

Next we consider true stages. We let

$$C^\alpha = \{s \in \mathbb{N} : s \leqslant^\alpha \omega\}.$$

Proposition 2.22. *For each $\alpha \leqslant \delta^*$, C^α is infinite.*

Proof. By Lemma 2.15, $C^0 = \mathbb{N}$. By Lemma 2.17, for limit α we have $C^\alpha = C^{\alpha+1}$. It thus suffices to prove the proposition for successor ordinals α.

But for α a successor, every $s \in D^\alpha$ beyond n_α is in C^α, by Lemma 2.20. □

The argument just given also shows that for successor α, C^α and D^α are Turing equivalent, uniformly. Given D^α, we only need to check $s \leqslant n_\alpha$ in D^α; we check whether $s \leqslant^\alpha t$ for some $t > n_\alpha$ in D^α. A similar process reduces D^α to C^α.

2.3. **Shifting by one.** The fact that for limit $\alpha < \delta^*$, $\leqslant^\alpha$ is the same as $\leqslant^{\alpha+1}$, means that we can ignore the limit levels. We can thus shift the infinite levels by one. We make the following definitions for $\alpha < \delta^*$:

$$\leqslant_\alpha = \begin{cases} \leqslant^\alpha, & \text{if } \alpha < \omega; \\ \leqslant^{\alpha+1}, & \text{if } \alpha \geqslant \omega. \end{cases}$$

$$C_\alpha = \begin{cases} C^\alpha, & \text{if } \alpha < \omega; \\ C^{\alpha+1}, & \text{if } \alpha \geqslant \omega. \end{cases}$$

$$\varnothing_{(\alpha)} = \begin{cases} \varnothing^{(\alpha)}, & \text{if } \alpha < \omega; \\ \varnothing^{(\alpha+1)}, & \text{if } \alpha \geqslant \omega. \end{cases}$$

$$\tau^\alpha_s = \begin{cases} \sigma^\alpha_s, & \text{if } \alpha < \omega; \\ \sigma^{\alpha+1}_s, & \text{if } \alpha \geqslant \omega. \end{cases}$$

The sets $\varnothing_{(\alpha)}$ can be given a concise description: $\varnothing_{(0)} = \varnothing$, and for $\alpha > 0$, $\varnothing_{(\alpha)} = \left(\bigoplus_{\beta<\alpha} \varnothing_{(\beta)}\right)'$.

It might seem that these strings τ are just remnants of $\trianglelefteq$, having little to do with our new $\leqslant$. In fact, the τ are listing true stages, just as the σ did. Recall that σ^α_t is an increasing enumeration of the stages r such that $n_\alpha < r \lhd^\alpha t$. By Lemma 2.20, if α is a successor, then for $r > n_\alpha$, for all $t \geqslant r$, $r \leqslant^\alpha t$ if and only if $r \trianglelefteq^\alpha t$. Hence σ^α_t is also the increasing enumeration of the stages r such that $n_\alpha < r <^\alpha t$. Shifting by one, let $m_\alpha = n_\alpha$ for $\alpha < \omega$ and otherwise let $m_\alpha = n_{\alpha+1}$; then for all $\alpha \leqslant \delta_*$, τ^α_t is the increasing enumeration of the stages r such that $m_\alpha < r <_\alpha t$.

Again assuming δ^* is a successor ordinal, we let $\delta_* = \delta^* - 1$. We summarise the properties of these objects.

Proposition 2.23. *Let $\alpha \leqslant \delta_*$.*

(a) *$\leqslant_\alpha$ is a partial ordering on $\omega + 1$.*
(b) *$s \leqslant_0 t$ if and only if $s \leqslant t$.*
(c) *For all $t \leqslant \omega$, $0 \leqslant_{\delta_*} t$.*
(d) *The orderings are nested: if $\alpha < \beta \leqslant \delta_*$ and $s \leqslant_\beta t$ then $s \leqslant_\alpha t$.*
(e) *Continuity: if $\lambda \leqslant \delta_*$ is a limit, then $s \leqslant_\lambda t \iff (\forall \beta < \lambda)\ s \leqslant_\beta t$.*
($\diamondsuit$) *If $s, r \leqslant_\alpha t$ and $s \leqslant r$ then $s \leqslant_\alpha r$.*
($\clubsuit$) *If $\alpha < \delta_*$, $s \leqslant_\alpha r \leqslant_\alpha t$ and $s \leqslant_{\alpha+1} t$ then $s \leqslant_{\alpha+1} r$.*
(f) *If $s \leqslant_\alpha t$ then $\tau^\alpha_s \preccurlyeq \tau^\alpha_t$.*
(g) *$C_\alpha = \{s \in \mathbb{N} : s \leqslant_\alpha \omega\}$ is infinite, and $\tau^\alpha_\omega = \bigcup_{s \in C_\alpha} \tau^\alpha_s$.*
(h) *$\tau^\alpha_\omega \equiv_{\mathrm{T}} C_\alpha \equiv_{\mathrm{T}} \varnothing_{(\alpha)}$, uniformly.*
(i) *The functions $s \mapsto \tau^\alpha_s$, restricted to $s \in \mathbb{N}$, are uniformly computable.*
(j) *The relations $s \leqslant_\alpha t$, restricted to $s, t \in \mathbb{N}$, are uniformly computable, and further, the function $(s,t) \mapsto \max\{\alpha \leqslant \delta_* : s \leqslant_\alpha t\}$ is computable.*
(k) *For every $s < \omega$, for only finitely many $\alpha \leqslant \delta_*$ is τ^α_s nonempty, and the collection of such α can be obtained computably from s.*

3. Further technology

3.1. **Relative to a pseudo-ordinal.** Σ^1_1 bounding — an "overspill" argument — allows us to extend the technology of apparent α-true stages to ill founded extensions of the least non-computable ordinal ω_1^{ck}. A unifying approach to all overspill arguments uses an ill founded model of ZFC. Let V^* be a countable ω-model of ZFC which omits ω_1^{ck}; this implies that the well-founded part of V^* has height ω_1^{ck}, and so V^* contains ill-founded computable "ordinals". These are *pseudo-ordinals*: they are ill-founded, but the corresponding computable orderings have no hyperarithmetic descending chains. This is because every hyperarithmetic set is in V^*.

The development of the previous section, including Proposition 2.23, is a theorem of ZFC, and so holds in V^*. Let $\delta_* \in V^*$ be a pseudo-ordinal. For $\alpha \leqslant \delta_*$ we obtain an array of objects $\varnothing_{(\alpha)}$, $\leqslant_\alpha$, τ^α_s, C_α, all in the sense of V^*, satisfying the inductive definition of the sets $\varnothing_{(\alpha)}$ and the properties described above, all in V^*.

However, V^*, being an ω-model, is arithmetically absolute. For well-founded $\alpha < \delta_*$, the fact that $(\varnothing_{(\alpha)})^{V^*}$ satisfies the Π^0_2 inductive definition of this iteration of the jump implies that $(\varnothing_{(\alpha)})^{V^*} = \varnothing_{(\alpha)}$. And similarly, for well-founded α, $(\leqslant_\alpha)^{V^*} = \leqslant_\alpha$, $(\tau^\alpha_s)^{V^*} = \tau^\alpha_s$, and $(C_\alpha)^{V^*} = C_\alpha$. We therefore omit the superscript V^* even for the objects at ill-founded levels.

Remark 3.1. Let $\alpha \leqslant \delta_*$ be ill-founded. In V, there will be many jump-hierarchies along α, and so writing $\varnothing_{(\alpha)}$ for $(\varnothing_{(\alpha)})^{V^*}$ may be a bit abusive. In V^*, though, there is only one jump-hierarchy along α. Similarly, in V, $C_\alpha = (C_\alpha)^{V^*}$ will not be the unique path through $\leqslant_\alpha = (\leqslant_\alpha)^{V^*}$, but it is the unique path in V^*.

In V, $\leqslant_\alpha$ cannot have an isolated path: the restriction of $\leqslant_\alpha$ to $\mathbb{N}$ is computable in V^*, and so computable. An isolated path would be hyperarithmetic. But any such path is an infinite subset of C_β for all well-founded $\beta < \delta_*$, and so computes every hyperarithmetic set.

3.2. **Checking Π^1_1 statements.** The ordinals $\alpha < \omega_1^{\text{ck}}$ provide a sort of clock for Π^1_1 sets; just as we think of Σ^0_1 sets as being enumerated by a computable process of length ω, we can think of Π^1_1 sets as being enumerated by a computable process of length ω_1^{ck}. If φ is a Π^1_1 sentence, then φ is true if and only if there is some well-founded α such that $\varnothing_{(\alpha)}$ knows that φ is true.

To make this precise, recall that Kleene's $\mathcal{O}$ is a complete Π^1_1 set. Thus there is a computable function h from Π^1_1 sentences to $\mathbb{N}$ which is a one-one reduction of the set of true Π^1_1 sentences to $\mathcal{O}$.

Every $d \in \mathcal{O}$ is a notation for a computable ordinal, denoted by $|d|_{\mathcal{O}}$. For a true Π^1_1 sentence φ, we let $[\![\varphi]\!]$, the *ordinal height of* φ, be

$$[\![\varphi]\!] = |h(\varphi)|_{\mathcal{O}} + 1.$$

For false Π^1_1 sentences φ we let $[\![\varphi]\!] = \infty$. As is common, we write $\infty > \alpha$ for every ordinal α. When $[\![\varphi]\!] < \infty$, i.e. when φ is true, then $[\![\varphi]\!]$ is a *successor* ordinal; we will make use of this fact.

For computable α, let

$$\mathcal{O}_{<\alpha} = \{d \in \mathcal{O} : |d|_{\mathcal{O}} < \alpha\}.$$

Fact 3.2. For every $\alpha < \omega_1^{\text{ck}}$, $\varnothing_{(\alpha)}$ computes $\mathcal{O}_{<\alpha}$, uniformly.

We remark that Fact 3.2 is the reason for using the shifted jump hierarchy $\varnothing_{(\alpha)}$. For limit λ, $\varnothing^{(\lambda)}$ does not compute $\mathcal{O}_{<\lambda}$, rather it can only enumerate it.

Proposition 3.3. *There is a Turing functional Γ such that for every computable ordinal δ_*, for every $\alpha \leqslant \delta_*$, $\Gamma(\tau^\alpha_\omega, \alpha, -)$ is total, and for every Π^1_1 sentence φ,*

- *if $[\![\varphi]\!] \leqslant \alpha$ then $\Gamma(\tau^\alpha_\omega, \alpha, \varphi) = [\![\varphi]\!]$;*
- *if $[\![\varphi]\!] > \alpha$ then $\Gamma(\tau^\alpha_\omega, \alpha, \varphi) = \infty$.*[6]

Proof. Compose Fact 3.2 with the uniform reduction of $\varnothing_{(\alpha)}$ to τ^α_ω; and use the uniform reductions of $\varnothing_{(\beta)}$ to $\varnothing_{(\alpha)}$ for $\beta < \alpha$. So given τ^α_ω, α and φ, we first check if $h(\varphi) \in \mathcal{O}_{<\alpha}$; if so, the functional Γ searches for $\beta < \alpha$ such that $h(\varphi) \in \mathcal{O}_{<\beta+1} \backslash \mathcal{O}_{<\beta}$, and when found, outputs $\beta + 1$. If not, then Γ outputs ∞. □

Remark 3.4. We have been loose in our identification of α and n_α when discussing computability, and so used $[\![\varphi]\!]$ in two senses. Fix a computable ordinal δ_* with a fixed computable notation-like presentation as used in the previous section to develop the system $\leqslant_\alpha$ and τ^α_ω. Let φ be a true Π^1_1 sentence with $[\![\varphi]\!] \leqslant \delta_*$.

Then $d = h(\varphi) \in \mathcal{O}$, but it is not one of the notations n_α, and in general, there is no computable way to obtain $\alpha \leqslant \delta_*$ (i.e. to obtain n_α) of the same height. However the output of $\Gamma(\tau^\alpha_\omega, n_\alpha, \varphi)$ is n_β with $\beta = [\![\varphi]\!]$: τ^α_ω can perform this translation.

The nonstandard clock. Kleene's $\mathcal{O}$ is not an element of V^*, but V^* has its own version $\mathcal{O}^*$. A Π^1_1 sentence φ is true in V^* if and only if $h(\varphi) \in \mathcal{O}^*$ if and only if $[\![\varphi]\!]^{V^*}$ is a computable ordinal in the sense of V^*. Again by arithmetic absoluteness, if φ is true then $[\![\varphi]\!]^{V^*} = [\![\varphi]\!]$. Hence φ is true if and only if $[\![\varphi]\!]^{V^*}$, which we henceforth denote simply by $[\![\varphi]\!]$, is in the well-founded part of δ_*.

Working with the same functional Γ in V^*, Proposition 3.3 holds in V^*.

Approximating the clock. We wish to approximate this evaluation of Π^1_1 sentences at every stage $s < \omega$, using τ^α_s as a stand-in for τ^α_ω. Since we have only a finite fragment of the oracle, we will have three possible outcomes. Either τ^α_s is sufficiently long to compute $\Gamma(\tau^\alpha_s, \alpha, \varphi)$, in which case it has an opinion of whether φ is already witnessed to be true by level α or not. Alternatively, $\Gamma(\tau^\alpha_s, \alpha, \varphi)\uparrow$ (which we can check computably, as we bound the oracle computation to $|\tau^\alpha_s|$ many steps) — which means that τ^α_s is unsure about the status of φ.

Notation 3.5. For brevity and clarity, for $s \leqslant \omega$, we let $\texttt{height}_s(\alpha, \varphi) = \Gamma(\tau^\alpha_s, \alpha, \varphi)$. To emphasise that these functions are total, we write $\texttt{height}_s(\alpha, \varphi) = \texttt{unsure}$ rather than writing $\texttt{height}_s(\alpha, \varphi)\uparrow$. Restricted to $s \in \mathbb{N}$, these functions are computable. We may assume that if $\texttt{height}_s(\alpha, \varphi) < \infty$ then it is a successor ordinal, and so $\texttt{height}_s(0, \varphi) \in \{\texttt{unsure}, \infty\}$ for all s and φ.

Remark 3.6. Suppose that $s \leqslant_\alpha t$ and that $\texttt{height}_s(\alpha, \varphi) \neq \texttt{unsure}$. Then $\texttt{height}_t(\alpha, \varphi) = \texttt{height}_s(\alpha, \varphi)$. This is because $\tau^\alpha_s \preccurlyeq \tau^\alpha_t$. Also, $\texttt{height}_\omega(\alpha, s) \neq \texttt{unsure}$ and equals $\lim_{s \in C_\alpha} \texttt{height}_s(\alpha, s)$.

Notation 3.7. We write $\texttt{present}_s(\alpha, \varphi)$ if $\texttt{height}_s(\alpha, \varphi) = \alpha$. We write $\texttt{past}_s(\alpha, \varphi)$ if $\texttt{height}_s(\alpha, \varphi) < \alpha$.

Lemma 3.8. For every $s < \omega$ there are only finitely many $\alpha \leqslant \delta_*$ such that for any φ, $\texttt{height}_s(\alpha, \varphi) \neq \texttt{unsure}$. The collection of all such α can be obtained computably from s.

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[6]Again, this include the case $[\![\varphi]\!] = \infty$.

Proof. We may assume that $\Gamma(\langle\rangle, \alpha, \varphi)\uparrow$ for any α and φ, so this follows from Proposition 2.23(k). ☐

We will use this machinery in V^*; again, at well-founded levels, V and V^* agree.

3.3. **An application: approximating Π^1_1 equivalence relations.** Fix a pseudo-ordinal $\delta_* \in V^*$. Let E be a Π^1_1 equivalence relation on ω.

Let $s \leqslant \omega$ and $\alpha \leqslant \delta_*$. We let

$$q_s(\alpha) = \max\{q \leqslant s \,:\, (\forall i, j < q)\ \texttt{height}_s(\alpha, \text{“}iEj\text{”}) \neq \texttt{unsure}\}.$$

For brevity we let

$$Q_s(\alpha) = [0, q_s(\alpha)).$$

We define an equivalence relation $E_s(\alpha)$ on $Q_s(\alpha)$. Naively, we would like to take the reflexive, transitive closure of the set

$$\{(i,j) \,:\, i, j \in Q_s(\alpha)\ \&\ \texttt{past}_s(\alpha, \text{“}iEj\text{”})\}.$$

However this would not have a nice property we are after: if $i, j \in Q_s(\alpha)$ and $s \leqslant_\alpha t$ then $iE_s(\alpha)j \iff iE_t(\alpha)j$. This is because we could at stage t discover some large k equivalent to both. To avert that, we define by recursion on $j \in Q_s(\alpha)$, $E_s(\alpha)\restriction[0,j]$. If $E_s(\alpha)\restriction[0,j-1]$ was defined (and is an equivalence relation), then we extend it to an equivalence relation $E_s(\alpha)\restriction[0,j]$ as follows:

- If $\texttt{past}_s(\alpha, \text{“}iEj\text{”})$ for some $i < j$, then we choose the $i < j$ which makes $\texttt{height}_s(\alpha, \text{“}iEj\text{”})$ smallest (if this is not unique, we choose the least i among those), and we add j to the equivalence class of i;
- Otherwise, we start a new equivalence class for j.

We summarise the properties of these relations. We write $E(\alpha)$ for $E_\omega(\alpha)$.

Proposition 3.9. *Let $\alpha \leqslant \delta_*$.*

(a) *$E_s(\alpha)$ is an equivalence relation on $Q_s(\alpha)$.*
(b) *$E_s(0)$ is equality.*
(c) *If $s \leqslant_\alpha t$ then $Q_s(\alpha) \subseteq Q_t(\alpha)$ and $E_s(\alpha)$ is the restriction of $E_t(\alpha)$ to $Q_s(\alpha)$.*
(d) *$Q(\alpha) = \mathbb{N} = \bigcup_{s\in C_\alpha} Q_s(\alpha)$, and $E(\alpha) = \bigcup_{s\in C_\alpha} E_s(\alpha)$.*
(e) *If $\alpha < \beta \leqslant \delta_*$ then $E(\alpha)$ refines $E(\beta)$. For limit $\lambda \leqslant \delta_*$, $E(\lambda) = \bigcup_{\alpha<\lambda} E(\alpha)$.*
(f) *$E = \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} E(\alpha)$.*
(g) *The functions $(s,\alpha) \mapsto Q_s(\alpha), E_s(\alpha)$, restricted to $s \in \mathbb{N}$, are computable.*
(h) *For every stage s, $Q_s(\alpha) \neq \varnothing$ for only finitely many α; the set of such α's is obtained computably from s.*

Proof. Most follow from the properties of the functions $\texttt{height}_s$. For (f), note that $E(\alpha) \subseteq E$; this uses the fact that E is an equivalence relation. In the other direction, we prove by induction that for all j, for sufficiently large α, $E\restriction[0,j] = E(\alpha)\restriction[0,j]$. (h) follows from Lemma 3.8. ☐

3.4. **Weeding out inconsistencies.** Suppose that $\alpha < \beta \leqslant \delta_*$. If φ is a true Π^1_1 statement and $[\![\varphi]\!] \leqslant \alpha$ then $\texttt{height}_\omega(\beta, \varphi) = \texttt{height}_\omega(\alpha, \varphi) = [\![\varphi]\!]$. And if $s < \omega$ is a β-true stage then we get similar consistency; we may have $\texttt{height}_s(\beta, \varphi) = \texttt{unsure}$ or $\texttt{height}_s(\alpha, \varphi) = \texttt{unsure}$, but if not, then we have $\texttt{height}_s(\beta, \varphi) = \texttt{height}_s(\alpha, \varphi)$. On the other hand, if s is not β-true, then τ^β_s may compute incorrectly, in which case we may get inconsistencies. For example, we could see that

$\mathtt{height}_s(\alpha,\varphi) = \infty$ but $\mathtt{height}_s(\beta,\varphi) \leqslant \alpha$. Observing this gives us a proof that $s \notin C_\beta$. During the construction, at every stage we would like consistent opinions between levels, and so we will simply skip inconsistent stages. This is done as follows.

We fix a Π^1_1 equivalence relation E and define $E_s(\alpha)$ as in the previous section. Also recall that $\langle F_e \rangle$ lists the partial Π^1_1 functions.

Definition 3.10. For $s < \omega$ we say that the following Π^1_1 statements are *relevant at stage s*:

- "iEj", for $i, j \leqslant s$;
- "$F_e(a^{e,i}) = c$" for $e, i \leqslant s$ and $c = a^{e,i}, b^{e,i}$.

We define consistency of guesses.

Definition 3.11. For a stage s, a Π^1_1 statement φ, and ordinals $\alpha < \beta \leqslant \delta_*$, we say that $\mathtt{height}_s(\alpha,\varphi)$ and $\mathtt{height}_s(\beta,\varphi)$ are *mutually consistent* if either $\mathtt{height}_s(\alpha,\varphi)$ or $\mathtt{height}_s(\beta,\varphi)$ are $\mathtt{unsure}$, or:

- if $\mathtt{height}_s(\alpha,\varphi) \leqslant \alpha$ then $\mathtt{height}_s(\beta,\varphi) = \mathtt{height}_s(\alpha,\varphi)$;
- if $\mathtt{height}_s(\alpha,\varphi) = \infty$ then $\mathtt{height}_s(\beta,\varphi) > \alpha$.[7]

We also say, for $e, i, s < \omega$ and ordinal $\alpha \leqslant \delta_*$ that $F_e(a^{e,i})$ *is internally consistent* at stage s and level α if it is not the case that for both $c \in \{a^{e,i}, b^{e,i}\}$, $\mathtt{height}_s(\alpha, \text{"}F_e(a^{e,i}) = c\text{"}) < \infty$.[8]

We define an increasing function $u\colon \omega \to \omega$. We start with $u(0) = 0$. Given $u(s-1)$, we define $u(s)$ to be the least stage $t > u(s-1)$ such that

(1) For all $\alpha < \beta \leqslant \delta_*$, for every Π^1_1 statement φ which is relevant at stage s, $\mathtt{height}_t(\alpha,\varphi)$ and $\mathtt{height}_t(\beta,\varphi)$ are mutually consistent;
(2) For all $e, i < s$ and all $\alpha \leqslant \delta_*$, $F_e(a^{e,i})$ is internally consistent at stage t and level α.

If $t \in C_{\delta_*}$ then for all $\alpha < \beta$, for *any* φ, $\mathtt{height}_t(\alpha,\varphi)$ and $\mathtt{height}_t(\beta,\varphi)$ are mutually consistent, and for *any* e and i, $F_e(a^{e,i})$ is internally consistent at stage t and level α. This implies:

Lemma 3.12. *u is total and $C_{\delta_*} \subseteq \operatorname{range} u$.*

We also observe that u is computable; this follows from Lemma 3.8. We therefore re-index all of our stages:

- We redefine $s \leqslant_\alpha t$ to mean $u(s) \leqslant_\alpha u(t)$;
- We redefine τ^α_s to be $\tau^\alpha_{u(s)}$;
- We replace C_α by $u^{-1}[C_\alpha]$;
- We redefine $\mathtt{height}_s(\alpha,\varphi)$ to be $\mathtt{height}_{u(s)}(\alpha,\varphi)$;
- We redefine $Q_s(\alpha)$ to be $Q_{u(s)}(\alpha) \cap [0, s)$;

and so on.

Lemma 3.13. *Proposition 2.23, and all the development of the current section (including Proposition 3.9), still hold after the re-indexing of stages.*

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[7]Or briefly: if either $\mathtt{height}_s(\alpha,\varphi) \leqslant \alpha$ or $\mathtt{height}_s(\beta,\varphi) \leqslant \alpha$ then $\mathtt{height}_s(\beta,\varphi) = \mathtt{height}_s(\alpha,\varphi)$.

[8]The point is that $F_e(a^{e,i})$ cannot have more than one value.

Proof. The only thing that requires any comment is the uniform equivalence $C_\alpha \equiv_T \varnothing_{(\alpha)}$. However, by Lemma 3.12, $\operatorname{range} u \cap C_\alpha$ is infinite, and since $\operatorname{range} u$ is computable, by Proposition 2.11(c), $\operatorname{range} u \cap C_\alpha \equiv_T \varnothing_{(\alpha)}$, uniformly. □

For future reference, we summarise the consistency properties of every stage after our speedup:

Lemma 3.14. For every $s < \omega$,

- For all $\alpha < \beta \leqslant \delta_*$, for every Π^1_1 statement φ which is relevant at stage s, $\texttt{height}_s(\alpha, \varphi)$ and $\texttt{height}_s(\beta, \varphi)$ are mutually consistent;
- For all $e, i < s$ and all $\alpha \leqslant \delta_*$, $F_e(a^{e,i})$ is internally consistent at stage s and level α.

We can apply this consistency to the approximations $E_s(\alpha)$.

Lemma 3.15. For all $s \leqslant \omega$ and $\alpha < \beta \leqslant \delta_*$, $E_s(\alpha)$ refines $E_s(\beta)$ on $Q_s(\alpha) \cap Q_s(\beta)$.

The proof is the same as that of Proposition 3.9(d), using consistency. Namely, by induction on $j < q_s(\alpha), q_s(\beta)$, we show that $E_s(\alpha) \upharpoonright [0, j]$ refines $E_s(\beta) \upharpoonright [0, j]$. We assume this holds up to $j - 1$ and prove it for j. If j starts a new $E_s(\alpha)$-equivalence then this is immeidate. Otherwise, let $i_0 < j$ be such that $\texttt{height}_s(\alpha, \text{“}iEj\text{”})$ is minimal, say $\gamma < \alpha$ (and i_0 minimal among those giving γ). So $i_0 E_s(\alpha) j$. Then for no $i < j$ can we have $\texttt{height}_s(\beta, \text{“}iEj\text{”}) < \gamma$, as that would entail $\texttt{height}_s(\beta, \text{“}iEj\text{”}) = \texttt{height}_s(\alpha, \text{“}iEj\text{”})$, contrary to the minimality of γ; and similarly, for no $i < i_0$ can we have $\texttt{height}_s(\beta, \text{“}iEj\text{”}) = \gamma$.

Remark 3.16. Since we made sure that beliefs about Π^1_1 statements are consistent across levels, it would appear that using the functions $\texttt{height}_s(\alpha, \varphi)$ for $\alpha < \delta_*$ is redundant. We could simply consult $\texttt{height}_s(\delta_*, \varphi)$. Taking a step back, why do we even need to approximate any oracle except for $\varnothing_{(\delta_*)}$?

The point is that if iEj at level β then we need $\varnothing_{(\beta)}$ to compute the isomorphism between $\mathcal{N}_i$ and $\mathcal{N}_j$. To make this happen, we need level β of our construction, namely the β^{th} Hausdorff drivative of each $A^{k,e,i}$ and $B^{k,e,i}$, to be computed from $\varnothing_{(\beta)}$ (uniformly, of course).

This means that if $s \leqslant_\alpha t$, then whatever we construct at level α at stage s must be respected at stage t. Now suppose that $\texttt{height}_s(\alpha, \varphi) = \texttt{unsure}$ but $\texttt{height}_s(\beta, \varphi) = \alpha$ for some $\beta > \alpha$ and some φ which would cause us to build something at level α. If $s \leqslant_\alpha t$ but $s \nleqslant_\beta t$, in particular if s is α-true but not β-true, then it would be a bad idea to take τ^β_s's word that $[\![\varphi]\!] = \alpha$. To build at level α, we need τ^α_s to give us this assurance.

We remark that guessing only at level δ_* and working at all levels uniformly is the difference between Ash's α-system technology and the added power introduced by Montalbán with his α-true stages.

4. Π^1_1 completeness

In this section we prove Theorem 1.1. Fix a Π^1_1 equivalence relation E. We will build structures $\mathcal{N}_k$ as described in Subsection 1.1. To specify the structure $\mathcal{N}_k$, it is enough, for each e and i, to define the linear orderings $A^{k,e,i}$ and $B^{k,e,i}$ on the sets $A^{e,i}$ and $B^{e,i}$.

Most of the proof takes place in V^*. In that model we fix:

- An enumeration $(F_e)_{e\in\omega}$ of the Π^1_1 partial functions from ω to ω (again, Π^1_1 in the sense of V^*);
- A pseudo ordinal δ_*, equipped with the relations $\leqslant_\alpha$, the sequences τ^α_s, and the functions $\texttt{height}_s(\alpha,\varphi)$ which we defined above, sped up so as to avoid inconsistencies;
- Our approximations $E_s(\alpha)$ to the equivalence classes $E(\alpha)$.

4.1. **The last step.** Below, in Proposition 4.1, we detail the properties of objects that we build in the construction, and then show that this suffices to prove Theorem 1.1. Recall from our discussion in section 1.1 that part of our strategy is to determine, in some cases, that a computable linear ordering $A^{k,e,i}$ that we are building should have order-type ω^α or $\omega^\alpha\cdot 2$ for some computable ordinal α. To ensure that for $\daleth\in\omega^3$, a linear ordering $A^\daleth$ has the intended order-type, we construct, along with these orderings, the iteration of their Hausdorff derivative. Recall that the Hausdorff derivative L' of a linear ordering L is formed by identifying two points in L if the interval between them is finite, and giving the result the induced ordering. There is then a natural map from L onto L'. We can define $L^{(n)}$ by iterating this process, and we can extend to limit levels by taking direct limits. Performed within V^*, in this way we get a directed system of linear orders of height up to δ_*. Note that "the" derivative or its iterates are an order-type rather than a particular ordering; however if the order-type of K is the step α Hausdorff derivative of L then there is a unique quotient map from L to K, and we call K "the" step α derivative of L.

In the construction, for each $\daleth\in\omega^3$, we construct not only the linear orderings $A^\daleth$ and $B^\daleth$ but for all $\alpha\leqslant\delta_*$, linear orderings $A^\daleth_\alpha$ and $B^\daleth_\alpha$, with $A^\daleth_0 = A^\daleth$ and $B^\daleth_0 = B^\daleth$. The ordering $A^\daleth_\alpha$ is the α^{th} iteration of the Hausdorff derivative of $A^\daleth$, and will be $\varnothing_{(\alpha)}$-computable, uniformly, and similarly for the B's. Together with the orderings we will need to build the associated quotient maps $f^\daleth_\alpha$ from $A^\daleth_0$ to $A^\daleth_\alpha$, and the quotient maps $g^\daleth_\alpha$ from $B^\daleth_0$ to $B^\daleth_\alpha$.

In two cases though we will make the orderings $A^\daleth_\alpha$ and $B^\daleth_\alpha$ empty. The first is when we decide to make their Hausdorff rank smaller than α — in our case, when we decide to make $A^\daleth_\alpha$ and $B^\daleth_\alpha$ isomorphic to ω^β and $\omega^\beta\cdot 2$ (or the other way round) for some $\beta<\alpha$. The other case is when $\daleth=(k,e,i)$, $\shin=(m,e,i)$ for some $m<k$, and for some $\beta<\alpha$ we have $kE(\beta)m$. In this case the triple $\shin$ will "take over" from $\daleth$ and ensure that the order-type is correct; we will have to ensure that in this case $A^\shin_0\cong A^\daleth_0$, and that $\varnothing_{(\beta)}$ can construct the isomorphism between them; the same for $B^\shin_0$ and $B^\daleth_0$.

For brevity of notation, for $\daleth=(k,e,i)$ and $\shin=(m,e,i)$ (note the same e and i), we write $\daleth E(\alpha)\shin$ if $kE(\alpha)m$. When $m<k$ we also write $\shin<\daleth$. Similarly, if m is the least element of its $E(\alpha)$-equivalence class, then we say that $\shin$ is the least element of its $E(\alpha)$-equivalence class.

The following proposition summarises the required properties of the systems of linear orderings and quotient maps that we construct. It takes place inside V^*.

Proposition 4.1. *There are, for $\alpha\leqslant\delta_*$ and each $\daleth\in\omega^3$, linear orderings $A^\daleth_\alpha$ and $B^\daleth_\alpha$, and maps $f^\daleth_\alpha$ and $g^\daleth_\alpha$, satisfying:*

(a) *Uniformly in α and $\daleth$, the orderings $A^\daleth_\alpha$ and $B^\daleth_\alpha$ and maps $f^\daleth_\alpha$ and $g^\daleth_\alpha$ are computable from $\varnothing_{(\alpha)}$.*

(b) *The universe of* $A^{\daleth}_\alpha$ *(and of* $B^{\daleth}_\alpha$*) is either* $\varnothing$, $\{0\}$, $\{0,1\}$ *or* ω. $A^{\daleth}_\alpha$ *is empty if and only if* $B^{\daleth}_\alpha$ *is empty, and* $|A^{\daleth}_\alpha| \leqslant 2$ *if and only if* $|B^{\daleth}_\alpha| \leqslant 2$.

(c) *If* $A^{\daleth}_\alpha$ *is nonempty, then it is the* α^{th} *Hausdorff derivative of* $A^{\daleth}_0$, *and* $f^{\daleth}_\alpha$ *is the associated quotient map. Further,* $A^{\daleth}_0 \cong \omega^\alpha \cdot A^{\daleth}_\alpha$. *The same holds for* B *and* g.

(d) *Suppose that* α *is least such that* $\daleth$ *is not the least element of its* $E(\alpha)$*-equivalence class. Let* צ *be the least element of* $\daleth$*'s* $E(\alpha)$*-equivalence class. Then* $A^{\daleth}_\alpha = A^{\text{צ}}_\alpha$ *and* $B^{\daleth}_\alpha = B^{\text{צ}}_\alpha$. *For all* $\beta > \alpha$, $A^{\daleth}_\beta$ *and* $B^{\daleth}_\beta$ *are empty.*

(e) *Suppose that* $\daleth = (k,e,i)$ *is the least element of its* $E(\alpha)$*-equivalence class.*
Suppose that there is some $c \in \{a^{e,i}, b^{e,i}\}$ *such that* $[\![\text{“}F_e(a^{e,i}) = c\text{”}]\!] \leqslant \alpha$. *Since each* F_e *is a function, there is exactly one such* c. *Then:*
 (i) If $[\![\text{“}F_e(a^{e,i}) = c\text{”}]\!] < \alpha$ *then* $A^{\daleth}_\alpha$ *and* $B^{\daleth}_\alpha$ *are empty.*
 (ii) If $[\![\text{“}F_e(a^{e,i}) = c\text{”}]\!] = \alpha$ *then:*
 - *if* $c = b^{e,i}$ *then* $|A^{\daleth}_\alpha| = 1$ *and* $|B^{\daleth}_\alpha| = 2$;
 - *if* $c = a^{e,i}$ *and* $kE(\alpha)i$ *then* $|A^{\daleth}_\alpha| = 1$ *and* $|B^{\daleth}_\alpha| = 2$;
 - *if* $c = a^{e,i}$ *and* $\neg(kE(\alpha)i)$ *then* $|A^{\daleth}_\alpha| = 2$ *and* $|B^{\daleth}_\alpha| = 1$.

On the other hand, if there is no such c, *then* $A^{\daleth}_\alpha$ *and* $B^{\daleth}_\alpha$ *are infinite.*

In (d), we emphasise that we require actual equality of the linear orderings. That is, we require that the identity map $x \mapsto x$ is an isomorphism between the linear orderings.[9]

Most of the work will be the proof of Proposition 4.1. Before we embark on that proof, we show how this proposition implies Theorem 1.1; this proof is in V.

Proof of Theorem 1.1, assuming Proposition 4.1. As discussed in section 1.1, we have already assumed that $(e,i) \mapsto (a^{e,i}, b^{e,i})$ is computable, with computable range, and that $\langle A^{e,i}, B^{e,i}\rangle$ are uniformly computable infinite sets, pairwise disjoint and disjoint from $\{a^{e,i}, b^{e,i}\}$. We also fix uniformly computable bijections $h^{e,i}_A \colon \omega \to A^{e,i}$ and $h^{e,i}_B \colon \omega \to B^{e,i}$.

We note that the assumptions imply that for every $\daleth \in \omega^3$, $A^{\daleth}_0$ and $B^{\daleth}_0$ are infinite; this is because $E(0)$ is equality (Proposition 3.9), so (e) applies; and because $[\![\varphi]\!] > 0$ for all φ.

Using the signature discussed in section 1.1, for each k, we define the structure $\mathcal{N}_k$ as follows: the linear ordering $(A^{e,i})^{\mathcal{N}_k}$ is isomorphic to $A^{k,e,i}_0$ via $h^{e,i}_A$, and $(B^{e,i})^{\mathcal{N}_k}$ is isomorphic to $B^{k,e,i}_0$ via $h^{e,i}_B$. Since $A^{\daleth}_0$ and $B^{\daleth}_0$ are uniformly computable, the structures $\mathcal{N}_k$ are uniformly computable.

We note that if $\alpha < \beta \leqslant \delta_*$ and $A^{\daleth}_\beta$ is nonempty, then the step $(\beta - \alpha)$ Hausdorff derivative quotient map $f^{\daleth}_{\alpha,\beta} \colon A^{\daleth}_\alpha \to A^{\daleth}_\beta$ is computable from $\varnothing_{(\beta)}$, uniformly; this is because $f^{\daleth}_\beta = f^{\daleth}_{\alpha,\beta} \circ f^{\daleth}_\alpha$, and $f^{\daleth}_\alpha$ is onto $A^{\daleth}_\alpha$.

Before we show the desired reduction, we make the following observation, within V^*. For $(e,i) \in \omega^2$, if $F_e(a^{e,i}) = c \in \{a^{e,i}, b^{e,i}\}$ then let $\theta(e,i) = [\![\text{“}F_e(a^{e,k}) = c\text{”}]\!]$; otherwise (including when $F_e(a^{e,i})\uparrow$) let $\theta(e,i) = \infty$.

Claim 1.1.1. Let $(e,i) \in \omega^2$ and $\alpha \leqslant \theta(e,i)$ ($\alpha \leqslant \delta_*$ if $\theta(e,i) = \infty$). Let $\daleth = (k,e,i)$ and let צ be the least element of $\daleth$'s $E(\alpha)$-equivalence class. Then $A^{\daleth}_0 \cong A^{\text{צ}}_0$ and $B^{\daleth}_0 \cong B^{\text{צ}}_0$, $A^{\text{צ}}_\alpha$ is nonempty, and $\varnothing_{(\alpha)}$ computes the α^{th} Hausdorff quotients from $A^{\daleth}_0$ to $A^{\text{צ}}_\alpha$ and from $B^{\daleth}_0$ to $B^{\text{צ}}_\alpha$. This is uniform in α, $\daleth$ and (e,i).

[9] We could have required an isomorphism, uniformly computed from $\varnothing_{(\alpha)}$.

Proof. We prove the claim by effective transfinite recursion on α. For $\alpha = 0$, we know that $\text{צ} = \daleth$ and $A_0^{\daleth}$ is infinite. Suppose that the claim has been shown for $\alpha < \theta(e,i), \delta_*$; we consider $\alpha+1$. Given $\daleth = (k,e,i)$, since $\varnothing_{(\alpha+1)}$ computes $E(\alpha+1)$ (uniformly), we can find the least element צ of $\daleth$'s $E(\alpha+1)$-equivalence class. Also let ש be the least element of $\daleth$'s $E(\alpha)$-equivalence class. By (d) in case $\text{ש} \neq \text{צ}$, we know that $A_{\alpha+1}^{\text{ש}} = A_{\alpha+1}^{\text{צ}}$. As observed, $\varnothing_{(\alpha+1)}$ computes the quotient map $f_{\alpha,\alpha+1}^{\text{צ}}$ from $A_\alpha^{\text{צ}}$ to $A_{\alpha+1}^{\text{צ}}$, and so we compose it with the α-step quotient map from $A_0^{\daleth}$ to $A_\alpha^{\text{ש}}$ to get the desired map from $A_0^{\daleth}$ to $A_{\alpha+1}^{\text{צ}}$.

For limit $\alpha \leqslant \theta(e,i), \delta_*$, since $E(\alpha)$ is the limit of $E(\beta)$ for $\beta < \alpha$ (Proposition 3.9(e)), there is some $\beta < \alpha$ such that צ is the least element of $\daleth$'s $E(\beta)$-equivalence class; by recursion, we already have the map from $A_0^{\daleth}$ to $A_\beta^{\text{צ}}$, and so we compose with $f_{\beta,\alpha}^{\text{צ}}$. □

Let $k, m < \omega$; suppose that $\neg(kEm)$. So for all $\alpha < \omega_1^{\mathrm{ck}}$, $\neg(kE(\alpha)m)$. We show that there is no hyperarithmetic isomorphism between $\mathcal{N}_k$ and $\mathcal{N}_m$. Suppose that $F\colon \mathcal{N}_k \to \mathcal{N}_m$ is hyperarithmetic and total. Then $F \in V^*$ and is hyperarithmetic in the sense of V^*. Hence there is some e such that $F = F_e$. By Proposition 1.2, we may assume that $F_e(a^{e,k}) \in \{a^{e,k}, b^{e,k}\}$. Let $\alpha = \theta(e,k)$; $\alpha < \omega_1^{\mathrm{ck}}$. Suppose, for example, that $F(a^{e,k}) = a^{e,k}$.

Let $\hat{k}$ be the least elements of k's $E(\alpha)$-equivalence class, and let $\hat{m}$ be the least element of m's $E(\alpha)$-equivalence class.

- Since $\hat{k}E(\alpha)k$, $|A_\alpha^{\hat{k},e,k}| = 1$;
- Since $\neg(\hat{m}E(\alpha)k)$, $|A_\alpha^{\hat{m},e,k}| = 2$.

Thus, $A_0^{\hat{k},e,k} \cong \omega^\alpha$ and $A_0^{\hat{m},e,k} \cong \omega^\alpha \cdot 2$, so they are not isomorphic. By Claim 1.1.1, $A_0^{k,e,k} \cong A_0^{\hat{k},e,k}$ and $A_0^{m,e,k} \cong A_0^{\hat{m},e,k}$, so $A_0^{k,e,k} \ncong A_0^{m,e,k}$, whence $(A^{e,k})^{\mathcal{N}_k}$ and $(A^{e,k})^{\mathcal{N}_m}$ are not isomorphic. By Proposition 1.2, F is not an isomorphism from $\mathcal{N}_k$ to $\mathcal{N}_m$. If $c = b^{e,k}$ then we run the same argument, as $A_0^{k,e,k} \cong \omega^\alpha$ and $B^{m,e,k} \cong \omega^\alpha \cdot 2$.

Suppose now that kEm. So there is some $\alpha < \omega_1^{\mathrm{ck}}$ such that $kE(\alpha)m$. We build a hyperarithmetic isomorphism from $\mathcal{N}_k$ to $\mathcal{N}_m$. To build such an isomorphism, as discussed above, we need, uniformly in (e,i), to find isomorphisms from $A_0^{k,e,i}$ to either $A_0^{m,e,i}$ or to $B_0^{m,e,i}$ (and to tell which one), and similarly from $B_0^{k,e,i}$. By taking compositions, we may assume that k is the least element of its $E(\alpha)$-equivalence class. Fix a pair (e,i); let $\text{צ} = (k,e,i)$ and $\daleth = (m,e,i)$. There are two possibilities.

If $\alpha \leqslant \theta(e,i)$, then by Claim 1.1.1, for $\beta \leqslant \alpha$ let $g\colon A_0^{\daleth} \to A_\alpha^{\text{צ}}$ be the step α Hausdorff quotient map, which we obtain uniformly from $\varnothing_{(\alpha)}$. For each $z \in A_\alpha^{\text{צ}}$, $\varnothing_{(2\alpha)}$ can compute the isomorphism between $\{x \in A_0^{\daleth} : g(x) = z\}$ and $\{x \in A_0^{\text{צ}}\ f_\alpha^{\text{צ}}(x) = z\}$, as the assumption implies that they are both isomorphic to ω^α. This is uniform in z (and (e,i)). Piecing these together, we obtain the desired isomorphism between $A_0^{\daleth}$ and $A_0^{\text{צ}}$. We do the same for the B's.

Suppose that $\theta = \theta(e,i) < \alpha$. By Claim 1.1.1, one of $A_0^{\daleth}$ and B_0^{θ} is isomorphic to ω^θ, the other to $\omega^\theta \cdot 2$; and the same for $A_0^{\text{צ}}$ and $B_0^{\text{צ}}$. Computing the iterated Haudorff derivative, $\varnothing_{(2\alpha)}$ can figure out which is which and compute the required isomorphisms. □

The rest of the paper takes place entirely within V^*.

4.2. **Describing the construction.** How would we build the objects $A^{\daleth}_\alpha$ etc. discussed in Proposition 4.1 and compute them from the correct oracles? The obvious approach would be to perform some kind of effective transfinite recursion. The difficulty though is that the construction at a level α depends on what happens *above* α, rather than below it. For example, if $\varnothing_{(\beta)}$ knows that we want $|A^{\daleth}_\beta| = 1$, then somehow $\varnothing$ needs to contrive to make $A^{\daleth}_0 \cong \omega^\beta$. What we do, following Ash's iterated priority argument method [Ash90] proving the iterated version of Watnick's theorem about pulling back complexity of linear orderings, is approximate the objects at finite stages. At each stage s we use the stage s approximation of $\varnothing_{(\alpha)}$ engineered above, and so approximate $[\![\varphi]\!]$ by using the functions $\texttt{height}_s$, rather than $\texttt{height}_\omega$, and use our stage s approximation $E_s(\alpha)$ of $E(\alpha)$. We will use this information to approximate the orderings and maps at each stage. At each stage, only finitely much information is given by the function $\texttt{height}_s$ (Lemma 3.8), and so only finitely many linear orderings and maps will be nonempty. For each α, we will need to arrange that for α-true stages s, the stage s version $A^{\daleth}_{\alpha,s}$ of $A^{\daleth}_\alpha$[10], for example, is correct (it is a sub-ordering of the final $A^{\daleth}_\alpha$). The main difficulty is arranging this between levels. If $\alpha < \beta$ and s is α-true but not β-true, then we need to ensure that $A^{\daleth}_{\alpha,s}$ is correct; however $A^{\daleth}_{\beta,s}$ at that stage may be incorrect, and the structure of $A^{\daleth}_{\alpha,s}$ is determined to some extent by $A^{\daleth}_{\beta,s}$, as the latter is intended to be an iterated derivative of the former.

As our first step, we will define, for each stage s, what our guesses are for the linear orderings that should be built at that stage. Replacing $E(\alpha)$ by $E_s(\alpha)$, we use the same notational conventions for equivalence of triples $\daleth \in \omega^3$: we write $(k,e,i)E_s(\alpha)(m,e,i)$ if $kE_s(\alpha)m$. The universe of this relation is $Q_s(\alpha) \times \omega^2$. We use the same partial ordering on triples: $(k,e,i) < (m,e,i)$ if $k < m$. Thus, for each $\daleth \in Q_s(\alpha) \times \omega^2$, we speak of the least element of $\daleth$'s $E_s(\alpha)$-equivalence class.

Requiring attention and the instruction functions. Recall that the possible "outcomes" for one of our linear orderings $A^{\daleth}_\alpha$ are:

- Have size 1 or 2 (if we diagonalise at level α);
- be infinite;
- be equal to some other ordering $A^{צ}_\alpha$;
- be empty (if we diagonalise at a level below α, or have copied another linear ordering at a level below α).

The following definition states when we have sufficiently much information to form an opinion on what the outcomes of $A^{\daleth}_\alpha$ and $B^{\daleth}_\alpha$ should be.

Definition 4.2. Let $s \in \omega$ be a stage. We say that a pair $(\alpha, \daleth)$ (with $\daleth = (k,e,i)$) *requires attention* at stage s if:

(1) $e < s$;
(2) $i, k \in Q_s(\alpha)$;[11]
(3) For both $c \in \{a^{e,i}, b^{e,i}\}$, $\texttt{height}_s(\alpha, \text{“}F_e(a^{e,i}) = c\text{”}) \neq \texttt{unsure}$.

We define the *instruction functions* $\texttt{instr}^A_s(\alpha,\daleth)$ and $\texttt{instr}^B_s(\alpha,\daleth)$ for all pairs $(\alpha,\daleth)$ which require attention at stage s; they tell us the required outcome (size or shape) of the associated linear ordering:

[10]Below we will denote the stage s version of $A^{\daleth}_\alpha$ by $(A^{\daleth}_\alpha)^{p_s}$.
[11]Recall that $Q_s(\alpha)$ is the domain of $E_s(\alpha)$.

(1) Suppose that for some $c \in \{a^{e,i}, b^{e,i}\}$, $\mathtt{height}_s(\alpha, \text{“}F_e(a^{e,i}) = c\text{”}) \leqslant \alpha$. We know (Lemma 3.14) that there is just one such c. Then:
   (i) If $\mathtt{height}_s(\alpha, \text{“}F_e(a^{e,i}) = c\text{”}) < \alpha$ then $\mathtt{instr}_s^A(\alpha, \daleth) = \mathtt{instr}_s^B(\alpha, \daleth) = 0$.
   (ii) Suppose that $\mathtt{height}_s(\alpha, \text{“}F_e(a^{e,i}) = c\text{”}) = \alpha$.
      (a) If $c = b^{e,i}$ then $\mathtt{instr}_s^A(\alpha, \daleth) = 1$ and $\mathtt{instr}_s^B(\alpha, \daleth) = 2$.
      (b) If $c = a^{e,i}$ and $kE_s(\alpha)i$, then $\mathtt{instr}_s^A(\alpha, \daleth) = 1$ and $\mathtt{instr}_s^B(\alpha, \daleth) = 2$.
      (c) If $c = a^{e,i}$ and $\neg(kE_s(\alpha)i)$, then $\mathtt{instr}_s^A(\alpha, \daleth) = 2$ and $\mathtt{instr}_s^B(\alpha, \daleth) = 1$.

(2) Otherwise, if there is some $\beta \leqslant \alpha$ such that $(\beta, \daleth)$ requires attention and β is not the least element of its $E_t(\beta)$-equivalence class, then we let β be the least such β; we let $\tsadi$ be the least element of $\daleth$'s $E_t(\beta)$-equivalence class.
   (i) If $\beta < \alpha$ then $\mathtt{instr}_s^A(\alpha, \daleth) = \mathtt{instr}_s^B(\alpha, \daleth) = 0$.
   (ii) If $\beta = \alpha$ then $\mathtt{instr}_s^A(\alpha, \daleth) = \mathtt{instr}_s^B(\alpha, \daleth) = \tsadi$.

(3) Otherwise, $\mathtt{instr}_s^A(\alpha, \daleth) = \mathtt{instr}_s^B(\alpha, \daleth) = \omega$.

Note that $\mathtt{instr}_s^A(\alpha, \daleth) \in \{1, 2\}$ if and only if $\mathtt{instr}_s^B(\alpha, \daleth) \in \{1, 2\}$. We write $\mathtt{instr}_s(\alpha, \daleth) = \{1, 2\}$. Similarly, we write $\mathtt{instr}_s(\alpha, \daleth) = 0$, $\mathtt{instr}_s(\alpha, \daleth) = \omega$ or $\mathtt{instr}_s(\alpha, \daleth) = \tsadi$. If $\mathtt{instr}_s(\alpha, \daleth) = 0$ or $\mathtt{instr}_s(\alpha, \daleth) = \{1, 2\}$ then we write “$\mathtt{instr}_s(\alpha, \daleth)$ is finite”. When we write $\mathtt{instr}_s(\alpha, \daleth) = \mathtt{instr}_t(\beta, \shin)$, we mean that $\mathtt{instr}_s^A(\alpha, \daleth) = \mathtt{instr}_t^A(\beta, \shin)$ and $\mathtt{instr}_s^B(\alpha, \daleth) = \mathtt{instr}_t^B(\beta, \shin)$.

Lemma 4.3.

(a) At any stage s, only finitely many pairs $(\alpha, \daleth)$ require attention, and the collection of such pairs can be obtained computably from s.
(b) For each α and $\daleth$, for sufficiently large $s \in C_\alpha$, the pair $(\alpha, \daleth)$ requires attention at stage s.
(c) If $s \leqslant_\alpha t$ and $(\alpha, \daleth)$ requires attention at stage s, then it also requires attention at stage t, and $\mathtt{instr}_s(\alpha, \daleth) = \mathtt{instr}_t(\alpha, \daleth)$.
(d) If $\mathtt{instr}_\omega(\alpha, \daleth) \neq 0, \omega$ then α is a successor ordinal.
(e) If $\daleth E_s(\alpha)\shin$, then $(\alpha, \daleth)$ requires attention at stage s if and only if $(\alpha, \shin)$ does.
(f) If $(\alpha, \daleth)$ requires attention at stage s, and $\tsadi < \daleth$ is the least element of $\daleth$'s $E_s(\alpha)$-equivalence class, then either $\mathtt{instr}_s(\alpha, \daleth) = \tsadi$ and $\mathtt{instr}_s(\alpha, \tsadi) = \omega$, or $\mathtt{instr}_s(\alpha, \daleth) = \mathtt{instr}_s(\alpha, \tsadi)$ is finite.
(g) Suppose that $\beta < \gamma$, both $(\beta, \daleth)$ and $(\gamma, \daleth)$ require attention at stage s, and $\mathtt{instr}_s(\beta, \daleth) \neq \omega$. Then $\mathtt{instr}_s(\gamma, \daleth) = 0$.

Proof. (a) follows from Proposition 3.9(h); (b) follows from Remark 3.6 and Proposition 3.9(d). (c) follows from Remark 3.6 and from Proposition 3.9(c). (d) follows from the fact that if $< \infty$, $[\![\varphi]\!]$ is a successor ordinal, and also from the fact that the least ordinal β for which $\daleth$ is not the least element of its $E(\beta)$-equivalence class is not a limit (Proposition 3.9(e)). For (e), by definition, $\daleth = (k, e, i)$, $\shin = (m, e, i)$ and $k, m \in Q_s(\alpha)$; the remainder of the definition of requiring attention does not mention k or m. (f) follows from examining the possible cases of Definition 4.2: again say $\daleth = (k, e, i)$ and $\tsadi = (m, e, i)$. By definition, if $\mathtt{instr}_s(\alpha, \tsadi)$ is finite, then so is $\mathtt{instr}_s(\alpha, \daleth)$, and further, because $\tsadi E_s(\alpha)\daleth$, i.e. $kE_s(\alpha)m$, and $E_s(\alpha)$ is an equivalence relation, we have $kE_s(\alpha)i$ if and only if $mE_s(\alpha)i$; so when $\mathtt{instr}_s(\alpha, \tsadi) = \{1, 2\}$ we have $\mathtt{instr}_s(\alpha, \daleth) = \mathtt{instr}_s(\alpha, \daleth)$, that is, we diagonalise the same way. For (g) we consider the two possibilities for $\mathtt{instr}_s(\beta, \daleth)$. Say $\daleth = (k, e, i)$. If $\mathtt{height}_s(\beta, \text{“}F_e(a^{e,i}) = c\text{”}) \leqslant \beta$, then by consistency of our beliefs (Lemma 3.14), $\mathtt{height}_s(\gamma, \text{“}F_e(a^{e,i}) = c\text{”}) \leqslant \beta < \gamma$, giving $\mathtt{instr}_s(\gamma, \daleth) = 0$. Similarly, if $\daleth$ is not the least element of its $E_s(\beta)$-equivalence class, then by instruction, as $\beta < \gamma$, again $\mathtt{instr}_s(\gamma, \daleth) = 0$. □

Permissible resets. The natural approach, as is the case in the Ash-Watnick construction, is to require that if $s \leqslant_\alpha t$ then $A^\daleth_{\alpha,t}$ is a linear order extending $A^\daleth_{\alpha,s}$ (we write $A^\daleth_{\alpha,s} \subseteq A^\daleth_{\alpha,t}$). Then it would be clear that $\varnothing_{(\alpha)}$ computes $A^\daleth_\alpha = \bigcup_{s \in C_\alpha} A^\daleth_{\alpha,s}$. Unfortunately, we will not be able to always get this extension relation. The reason for this is delicate.

Consider a limit ordinal $\lambda \leqslant \delta_*$. We need to ensure that $A^\daleth_\lambda$ is the step λ Hausdorff derivative of $A^\daleth_0$. This means that it is the direct limit of the system $(A^\daleth_\beta, f^\daleth_{\beta,\gamma})_{\beta \leqslant \gamma < \lambda}$[12]. To make sure that the linear ordering that we are building is indeed this direct limit, the crucial requirement is to ensure that if $x, y \in A^\daleth_0$ and $f^\daleth_\lambda(x) = f^\daleth_\lambda(y)$, then there is some $\beta < \lambda$ such that $f^\daleth_\beta(x) = f^\daleth_\beta(y)$.

We could hope that this property can be ensured at the limit, but in fact the only way we found to make this work is by requiring that this property holds at every stage of the construction. Thus if at some stage s we have decided that $f^\daleth_{\alpha,\lambda}(x) = f^\daleth_{\alpha,\lambda}(y)$ for some $x, y \in A^\daleth_{\alpha,s}$, then we need to build some linear ordering $A^\daleth_{\beta,s}$ for some $\beta \in (\alpha, \lambda)$ in which we could merge x and y. The complication is that possibly, for no $\beta \in (\alpha, \lambda)$ does $(\beta, \daleth)$ require attention at stage s. Thus, we will have to build $B^\daleth_{\beta,s}$ before we know what the instructions are for this linear ordering.

If s is λ-true, then this is not a problem: we will later discover that $\varnothing_{(\beta)}$ wants us to build an infinite linear ordering at level β. If s is not β-true, then what we do at level β at stage s does not really matter, it will be ignored when building the true $A^\daleth_\beta$. However, it is possible that s is β-true but λ-false. At a later β-true stage we will see (or at least think) that we were wrong about $A^\daleth_{\lambda,s}$ and that in fact $A^\daleth_\beta$ should be, for example, empty, or instructed to copy some other linear ordering. Thus, we will need to *reset* the linear ordering $A^\daleth_{\beta,t}$, and allow it to not extend $A^\daleth_{\beta,s}$, even though $s \leqslant_\beta t$.

If we do this, how can we ensure that $\varnothing_{(\alpha)}$ computes $A^\daleth_\alpha$ correctly? For that matter, how do we ensure that $\lim_{s \in C_\alpha} A^\daleth_{\alpha,s}$ exists? We will allow only a *single* reset for each object. That is, as long as we are ignorant of any instruction to the contrary, we are required to extend the previously constructed linear ordering; and once the pair $(\alpha, \daleth)$ requires attention, we know what the instructions are, and then we no longer allow any further resets.[13] And $\varnothing_{(\alpha)}$ can find an α-true stage at which $(\alpha, \daleth)$ requires attention and start constructing $A^\daleth_\alpha$ from there.

The following definition allows us to keep track of those pairs $(\alpha, \daleth)$ for which at stage s we believe that we have evidence that we will not be performing the usual Ash-Watnick construction, and thus possibly allow a reset.

[12]As we saw in the proof above using Proposition 4.1, when given only $f^\daleth_\alpha = f^\daleth_{0,\alpha}$ for all α, we can find the intermediate maps $f^\daleth_{\alpha,\beta}$ as the map $f^\daleth_\alpha$ is onto $A^\daleth_\alpha$. In our construction it will be more convenient to keep track of all the maps $f^\daleth_{\alpha,\beta}$, because at some steps during the construction, some of these maps may fail to be onto.

[13]Of course this description is relative to the α-true stages only, so only a single reset along each $\leqslant_\alpha$-path.

Definition 4.4. For a stage $s \leqslant \omega$ and a level $\alpha \leqslant \delta_*$, we define $V_s(\alpha)$ to be the set of $\daleth$ such that for some $\beta \leqslant \alpha$, the pair $(\beta, \daleth)$ requires attention at stage s and $\texttt{instr}_s(\beta, \daleth) \neq \omega$.

The idea is that if $\daleth \in V_s(\alpha)$, then perhaps $(\alpha, \daleth)$ does not require attention at stage s, because we have not seen enough convergence; but if stage s is correct about $\varnothing_{(\beta)}$ for all $\beta \leqslant \alpha$, then once we do see enough convergence we will see that we want to make $A^{\daleth}_{\alpha}$ empty. The following lemma summarises the properties of the sets $V_s(\alpha)$, which all follow from our analysis above of requiring attention and the instruction functions.

Lemma 4.5.

(a) $V_s(0) = \varnothing$ for all s and $V_0(\alpha) = \varnothing$ for all α.
(b) If $s \leqslant_\alpha t$ then $V_s(\alpha) \subseteq V_t(\alpha)$.
(c) If $\alpha < \gamma$ then $V_s(\alpha) \subseteq V_s(\gamma)$.
(d) If $(\alpha, \daleth)$ requires attention at stage s, then $\daleth \in V_s(\alpha)$ if and only if $\texttt{instr}_s(\alpha, \daleth) \neq \omega$.[14]
(e) If $s \leqslant_\alpha t$ and $(\alpha, \daleth)$ requires attention at stage s, then $\daleth \in V_s(\alpha) \iff \daleth \in V_t(\alpha)$.
(f) $V_\omega(\alpha) = \bigcup_{s \in C_\alpha} V_s(\alpha)$.
(g) For $s < \omega$, $V_s(\alpha)$ is finite and uniformly computable from α and s.
(h) $V_\omega(\alpha)$ is uniformly computable from $\varnothing_{(\alpha)}$.

4.3. **Objects of the construction.** We now define *objects* of height δ_*. The intention of an object is to be a potential finite fragment of the above described collection of directed systems: the state of the construction at some finite stage s.

Definition 4.6. An *object* is a tuple

$$\left(\langle G_\alpha \rangle_{\alpha \leqslant \delta_*}, \langle r^{\daleth}_\alpha \rangle_{\substack{\alpha \leqslant \delta_* \\ \daleth \in \omega^3}}, \langle A^{\daleth}_\alpha \rangle_{\substack{\alpha \leqslant \delta_* \\ \daleth \in \omega^3}}, \langle B^{\daleth}_\alpha \rangle_{\substack{\alpha \leqslant \delta_* \\ \daleth \in \omega^3}}, \langle f^{\daleth}_{\alpha,\beta} \rangle_{\substack{\alpha \leqslant \beta \leqslant \delta_* \\ \daleth \in \omega^3}}, \langle g^{\daleth}_{\alpha,\beta} \rangle_{\substack{\alpha \leqslant \beta \leqslant \delta_* \\ \daleth \in \omega^3}} \right)$$

satisfying the following:

(1) For each $\alpha \leqslant \delta_*$ and $\daleth \in \omega^3$, $r^{\daleth}_\alpha \in \{0, 1\}$.
  (i) If $\alpha \leqslant \beta$ then $r^{\daleth}_\alpha \leqslant r^{\daleth}_\beta$.
  (ii) $r^{\daleth}_{\delta_*} = 0$ for all but finitely many $\daleth$.
(2) For each $\alpha \leqslant \delta_*$, $G_\alpha \subset \omega^3$ is finite, and for all but finitely many α, G_α is empty.
(3) Each $A^{\daleth}_\alpha$ and $B^{\daleth}_\alpha$ is a finite linear order, and all but finitely many are empty.
  (i) $A^{\daleth}_\alpha$ is empty if and only if $B^{\daleth}_\alpha$ is empty.
  (ii) $|A^{\daleth}_\alpha| \leqslant 2$ if and only if $|B^{\daleth}_\alpha| \leqslant 2$.
(4) For each $\daleth$ and α, the universes of $A^{\daleth}_\alpha$ and $B^{\daleth}_\alpha$ are initial segments of ω, and if nonempty, then 0 is their leftmost point.
(5) For $\alpha \leqslant \beta \leqslant \delta_*$,
  (i) If either $A^{\daleth}_\alpha$ or $A^{\daleth}_\beta$ are empty, then the function $f^{\daleth}_{\alpha,\beta}$ is empty.
  (ii) If both $A^{\daleth}_\alpha$ and $A^{\daleth}_\beta$ are nonempty, then $f^{\daleth}_{\alpha,\beta}$ is an order-preserving function from $A^{\daleth}_\alpha$ onto an initial segment of $A^{\daleth}_\beta$.

[14] That is, if $(\alpha, \daleth)$ requires attention at stage s, then in the definition of $\daleth \in V_s(\alpha)$ we can always take $\beta = \alpha$.

Similarly for the B's and g's.

(6) For a fixed $\daleth$, restricting to those $\alpha \leqslant \beta$ with $A^{\daleth}_{\alpha}$ and $A^{\daleth}_{\beta}$ nonempty, $\langle f^{\daleth}_{\alpha,\beta}\rangle$ is a directed system. Namely:
   (i) $f^{\daleth}_{\alpha,\alpha}$ is the identity map;
   (ii) if $\alpha \leqslant \beta \leqslant \gamma \leqslant \delta_*$, and $A^{\daleth}_{\alpha}$, $A^{\daleth}_{\beta}$ and $A^{\daleth}_{\gamma}$ are nonempty, then $f^{\daleth}_{\alpha,\gamma} = f^{\daleth}_{\beta,\gamma} \circ f^{\daleth}_{\alpha,\beta}$.

   The same holds for $\langle g^{\daleth}_{\alpha,\beta}\rangle$.

Note that an object is finite, in that it can be completely described by finitely much information: the finitely many linear orderings; the finitely many nonempty finite sets G_α; the finitely many $\daleth$ with $r^{\daleth}_{\delta_*} = 1$; and for each such $\daleth$, the least α such that $r^{\daleth}_{\alpha} = 1$.

Before we proceed, we remark on some components of the definition. Many components of an object tell us about the intention behind setting some linear orderings and maps the way we do. For example, setting $r^{\daleth}_{\alpha} = 1$ means that we possibly have spent a reset, and will not allow further resets to $A^{\daleth}_{\alpha}$ and $B^{\daleth}_{\alpha}$; at stage s we will set $r^{\daleth}_{\alpha} = 1$ exactly when $\daleth \in V_s(\alpha)$. Similarly, the set G_β indicates which locations will not allow any resets even if none were taken so far; at stage s we will set $\daleth \in G_\beta$ exactly if $(\beta, \daleth)$ requires attention at stage s. The reason that we do not directly incorporate attention seeking and the sets $V_s(\alpha)$ into the definition of an object is to allow us further flexibility. The definition will apply, for example, to the result of performing only one step of several during any stage.

In the same way, regarding (3)(ii), by setting $|A^{\daleth}_{\alpha}| \in \{1,2\}$, the object tells us that we intend to diagonalise at this location; at stage s we will set $|A^{\daleth}_{\alpha}| \in \{1,2\}$ exactly when $(\alpha, \daleth)$ requires attention at that stage and $\mathtt{instr}_s(\beta, \daleth) = \{1,2\}$.

The purpose of (4) is threefold. First, of course, it ensures that the universe of the final linear ordering will be computable, in fact it will be either $\{0\}$, $\{0,1\}$ or ω as required by Proposition 4.1. Second, it ensures that if we specify that the size of a linear ordering is 1 or 2, we will have specified the linear ordering as well. Third, it ensures that when extending linear orderings, we do not add points to the left. This will be useful when glueing together objects. Regarding (5), we remark that the definition implies that when nonempty, $f^{\daleth}_{\alpha,\beta}$ maps the leftmost point of $A^{\daleth}_{\alpha}$ to the leftmost point of $A^{\daleth}_{\beta}$; that is, $f^{\daleth}_{\alpha,\beta}(0) = 0$.

Convention 4.7. If o is an object, we write $\left(A^{\daleth}_{\alpha}\right)^o$ to refer to the $A^{\daleth}_{\alpha}$ element of o. Similarly for each of the other elements of o.

Notation 4.8. Suppose that o is an object, $\alpha \leqslant \gamma \leqslant \delta_*$, and $\daleth \in \omega^3$. If $\left(A^{\daleth}_{\alpha}\right)^o$ is nonempty, then for $x, y \in \left(A^{\daleth}_{\alpha}\right)^o$, we write $(x \sim_\gamma y)^o$ if there is some $\beta \in [\alpha, \gamma]$ such that $\left(A^{\daleth}_{\beta}\right)^o$ is nonempty and $\left(f^{\daleth}_{\alpha,\beta}\right)^o(x) = \left(f^{\daleth}_{\alpha,\beta}\right)^o(y)$ (and similarly for B and g). This notation is sparse, as the notion obviously also depends on α, $\daleth$ and whether we are looking at A or B; but these will usually be clear from the context.

Terminology 4.9. Suppose that o is an object, $\alpha \leqslant \beta \leqslant \delta_*$, and $\daleth \in \omega^3$. We write $\left(r^{\daleth}_{[\alpha,\beta]}\right)^o$ for the sequence $\langle \left(r^{\daleth}_{\xi}\right)^o \rangle_{\xi \in [\alpha,\beta]}$.

Object extensions. The intention of the following definition is to ensure that the α^{th} level of the construction is computed by $\varnothing_{(\alpha)}$. Let o and p be objects. We define the relation $o \leqslant_\alpha p$, which says that decisions made at levels $\beta \leqslant \alpha$ when constructing o are preserved in p, except for when there is a permissible reset.

Definition 4.10. Let o and p be objects. For $\alpha \leqslant \delta_*$, we let $o \leqslant_\alpha p$ if for every $\beta \leqslant \alpha$:

(1) For every $\urcorner \in \omega^3$, $\left(r^{\urcorner}_{\beta}\right)^o \leqslant \left(r^{\urcorner}_{\beta}\right)^p$;
(2) $(G_\beta)^o \subseteq (G_\beta)^p$, and for all $\urcorner \in (G_\beta)^o$, $\left(r^{\urcorner}_{\beta}\right)^o = \left(r^{\urcorner}_{\beta}\right)^p$;

and for every $\urcorner \in \omega^3$ such that $\left(r^{\urcorner}_{\beta}\right)^o = \left(r^{\urcorner}_{\beta}\right)^p$,

(3) $\left(A^{\urcorner}_{\beta}\right)^o \subseteq \left(A^{\urcorner}_{\beta}\right)^p$ and $\left(B^{\urcorner}_{\beta}\right)^o \subseteq \left(B^{\urcorner}_{\beta}\right)^p$;
(4) If $1 \leqslant |\left(A^{\urcorner}_{\beta}\right)^o| \leqslant 2$ then $\left(A^{\urcorner}_{\beta}\right)^o = \left(A^{\urcorner}_{\beta}\right)^p$ and $\left(B^{\urcorner}_{\beta}\right)^o = \left(B^{\urcorner}_{\beta}\right)^p$;

and in addition, for every $\gamma \in [\beta, \alpha]$ such that $\left(r^{\urcorner}_{[\beta,\gamma]}\right)^o = \left(r^{\urcorner}_{[\beta,\gamma]}\right)^p$,

(5) $\left(f^{\urcorner}_{\beta,\gamma}\right)^o \subseteq \left(f^{\urcorner}_{\beta,\gamma}\right)^p$ and $\left(g^{\urcorner}_{\beta,\gamma}\right)^o \subseteq \left(g^{\urcorner}_{\beta,\gamma}\right)^p$;
(6) for all $z \in \text{range}\left(f^{\urcorner}_{\beta,\gamma}\right)^o$, the leftmost point of $\{x \in \left(A^{\urcorner}_{\beta}\right)^o : \left(f^{\urcorner}_{\beta,\gamma}\right)^o(x) = z\}$ is also the leftmost point of $\{x \in \left(A^{\urcorner}_{\beta}\right)^p : \left(f^{\urcorner}_{\beta,\gamma}\right)^p(x) = z\}$;
(7) if $\gamma = \beta + 1$ and $\urcorner \in (G_{\beta+1})^o$ then for all $z \in \text{range}\left(f^{\urcorner}_{\beta,\beta+1}\right)^o$, $\{x \in \left(A^{\urcorner}_{\beta}\right)^o : \left(f^{\urcorner}_{\beta,\beta+1}\right)^o(x) = z\}$ is an initial segment of $\{x \in \left(A^{\urcorner}_{\beta}\right)^p : \left(f^{\urcorner}_{\beta,\beta+1}\right)^p(x) = z\}$;

suppose also that there is no $\zeta \in [\beta, \gamma]$ with $1 \leqslant |\left(A^{\urcorner}_{\zeta}\right)^p| \leqslant 2$; then:

(8) If $x, y \in \left(A^{\urcorner}_{\beta}\right)^o$ and $(x \sim_\gamma y)^p$ then $(x \sim_\gamma y)^o$;
(9) letting $f = \left(f^{\urcorner}_{\beta,\gamma}\right)^p$, for all $z \in \text{range}(f \upharpoonright \left(A^{\urcorner}_{\beta}\right)^o)$, the leftmost point of $\{x \in \left(A^{\urcorner}_{\beta}\right)^o : f(x) = z\}$ is the leftmost point of $\{x \in \left(A^{\urcorner}_{\beta}\right)^p : f(x) = z\}$;

the same holds for B and g.

Let us discuss some aspects of this definition. Some of the items are directly related to achieving some parts of Proposition 4.1; but as above, some are used in the process of producing the next finite object, which involves some glueing of several previous objects. And also, some items are stated in a particular way to ensure that the relations $\leqslant_\alpha$ are transitive.

For example, (3) and (5) are required so that we can eventually define $A^{\urcorner}_{\alpha}$ to be the union of $A^{\urcorner}_{\alpha,s}$ for $s \in C_\alpha$ after the last reset. The condition $\left(r^{\urcorner}_{\beta}\right)^o = \left(r^{\urcorner}_{\beta}\right)^p$ precisely says that no reset for the pair $(\beta, \urcorner)$ was taken in passing from the object o to the object p. (2) forces no resets for pairs which o has stated cannot have future resets (even if $\left(r^{\urcorner}_{\beta}\right)^o = 0$).

In (7) we are requiring that every 1-step Hausdorff derivative equivalence class which o witnesses up to level α must be an initial segment of an equivalence class in p. The purpose of this requirement is to ensure that each equivalence class in the final system has order-type ω, so that $A^{\urcorner}_{\alpha} \cong \omega \cdot A^{\urcorner}_{\alpha+1}$. On the other hand, (8) is only used when glueing objects together. It says that unless we have a very good reason to merge points x and y (because when constructing p we have discovered that we need to need to build a finite linear ordering), we do not; this is a departure from the Ash-Watnick construction.

It would seem that there is some redundancy in stating both (6) and (9). However, they do not quite imply each other, and are needed for different purposes. (6) is needed for ensuring that the preimage of a point at a limit level λ has the correct order-type, namely ω^λ. The inductive argument for showing this needs the fact that this preimage has a least element, which is ensured by this item. (9), on the other hand, is required for glueing objects. The difference is that we allow $z \notin \left(A^{\urcorner}_{\gamma}\right)^o$, equivalently (in light of (5)), that $\left(A^{\urcorner}_{\gamma}\right)^o$ is empty.

The condition there is no $\zeta \in [\beta,\gamma]$ with $1 \leqslant |(A^{\urcorner}_{\zeta})^p| \leqslant 2$ is added to this item as well as (8), because in order to make $\leqslant_\alpha$ transitive, we need to rely on (8) as well, and the extra condition is necessary for (8). On the other hand, to make the final argument successful, we need (6) to hold even when this extra condition fails.

We make a few observations about this definition.

Lemma 4.11.

(1) Each $\leqslant_\alpha$ is a partial ordering;
(2) The relations are nested: for $\alpha \leqslant \beta$, $o \leqslant_\beta p \Rightarrow o \leqslant_\alpha p$;
(3) The empty object is $\leqslant_{\delta_*} p$ for every object p;
(4) The relations $\leqslant_\alpha$ are uniformly computable.

Proof. Most are straightforward. In showing that $\leqslant_\alpha$ is transitive, we need to justify items 8 and 9. Suppose that $o \leqslant_\alpha p \leqslant_\alpha q$, that $\beta < \gamma \leqslant \alpha$, and that $(r^{\urcorner}_{[\beta,\gamma]})^o = (r^{\urcorner}_{[\beta,\gamma]})^q$; so $(r^{\urcorner}_{[\beta,\gamma]})^o = (r^{\urcorner}_{[\beta,\gamma]})^p = (r^{\urcorner}_{[\beta,\gamma]})^q$, and so $(f^{\urcorner}_{\beta,\gamma})^o \subseteq (f^{\urcorner}_{\beta,\gamma})^p \subseteq (f^{\urcorner}_{\beta,\gamma})^q$. Let $f = (f^{\urcorner}_{\beta,\gamma})^q$. Suppose that there is no $\zeta \in [\beta,\gamma]$ with $1 \leqslant |(A^{\urcorner}_{\zeta})^q| \leqslant 2$; by (4), there is no $\zeta \in [\beta,\gamma]$ with $1 \leqslant |(A^{\urcorner}_{\zeta})^p| \leqslant 2$. This shows that (8) holds between o and q.

Let $z \in \operatorname{range}(f \restriction (A^{\urcorner}_{\beta})^o)$; let y be the leftmost point in $(A^{\urcorner}_{\beta})^o$ mapped by f to z. Let $x \in (A^{\urcorner}_{\beta})^q$ such that $f(x) = z$. Also let w be the leftmost point in $(A^{\urcorner}_{\beta})^p$ mapped to z by f. Since $p \leqslant_\alpha q$, $w \leqslant x$. Note that $y \in (A^{\urcorner}_{\beta})^p$ and that $(y \sim_\gamma w)^q$; by (8), $(y \sim_\gamma w)^p$. So there is some $\zeta \in [\beta,\gamma]$ such that $(f^{\urcorner}_{\beta,\zeta})^p(y) = (f^{\urcorner}_{\beta,\zeta})^p(w)$. Now y is the leftmost point in $(A^{\urcorner}_{\beta})^o$ mapped by $(f^{\urcorner}_{\beta,\zeta})^p$ to $(f^{\urcorner}_{\beta,\zeta})^p(w)$; since $o \leqslant_\alpha p$, we have $y \leqslant w$. This shows that (9) holds for $o \leqslant_\alpha q$. ☐

Note that it is not the case that $o \leqslant_0 p$ for all o and p. Also, the sequence of relations is not continuous: it is possible, for λ limit, to have $o \leqslant_\beta p$ for all $\beta < \lambda$ but $o \nleqslant_\lambda p$.

Stage-based objects. We will define the notion of an s-object, where s is a stage. This means that the object is eligible to be picked at stage s: the decisions made in constructing the object are consistent with what we currently guess about the universe (i.e., about the $\varnothing_{(\alpha)}$). Further, in order to make the limit objects total, by stage s we need to ensure that some levels are nonempty and have at least s many elements.

Definition 4.12. Let p be an object. For clarity, in this definition we write $A^{\urcorner}_\alpha$ for $(A^{\urcorner}_\alpha)^p$, and similarly for the B, f, g, r and G. For a stage $s < \omega$, we say that p is an *s-object* if the following additional conditions are satisfied for all $\alpha \leqslant \delta_*$ and for all $\urcorner \in \omega^3$:

(1) If $A^{\urcorner}_\alpha \neq \varnothing$ then for all $\beta \geqslant \alpha$, $f^{\urcorner}_{\alpha,\beta}$ is onto $A^{\urcorner}_\beta$, and the same holds for $g^{\urcorner}_{\alpha,\beta}$;
(2) $r^{\urcorner}_\alpha = 1 \iff \urcorner \in V_s(\alpha)$;
(3) $\urcorner \in G_\alpha$ if and only if $(\alpha,\urcorner)$ requires attention at stage s;

and if $(\alpha,\urcorner)$ does not require attention at stage s, then:

(4) If $\urcorner \in V_s(\alpha)$ then $A^{\urcorner}_\alpha$ and $B^{\urcorner}_\alpha$ are empty;
(5) If $A^{\urcorner}_\alpha$ (and $B^{\urcorner}_\alpha$) are nonempty then $|A^{\urcorner}_\alpha|, |B^{\urcorner}_\alpha| \geqslant 3$;

but if $(\alpha,\urcorner)$ does require attention at stage s, then:

(6) If $\mathtt{instr}_s(\alpha,\urcorner)$ is finite, then $|A^{\urcorner}_\alpha| = \mathtt{instr}^A_s(\alpha,\urcorner)$ and $|B^{\urcorner}_\alpha| = \mathtt{instr}^B_s(\alpha,\urcorner)$;

(7) If $\mathtt{instr}_s(\alpha, \daleth) = \omega$ then $|A_\alpha^\daleth|, |B_\alpha^\daleth| \geqslant s + 3$;
(8) If $\mathtt{instr}_s(\alpha, \daleth) = \tsadi$ then $A_\alpha^\daleth = A_\alpha^\tsadi$ and $B_\alpha^\daleth = B_\alpha^\tsadi$;
(9) If $\alpha < \delta_*$, $\mathtt{instr}_s(\alpha, \daleth) = \omega$, and $A_{\alpha+1}^\daleth$ is nonempty, then for each $z \in A_{\alpha+1}^\daleth$,

$$\{x \in A_\alpha^\daleth : f_{\alpha,\alpha+1}^\daleth(x) = z\}$$

has at least s elements, and similarly for each $z \in B_{\alpha+1}^\daleth$;
(10) If α is a limit ordinal and $\daleth \notin V_s(\alpha)$ then there is some successor ordinal $\beta < \alpha$ such that $f_{\beta,\alpha}^\daleth$ and $g_{\beta,\alpha}^\daleth$ are isomorphisms.

We again make some remarks. As in the statement of Proposition 4.1, we stress that in item 8, we are requiring literal equality of linear orders.[15]

As discussed above, (10) is used to show that for limit λ, $A_\lambda^\daleth$ is the direct limit of $(A_\beta^\daleth)_{\beta<\lambda}$, rather than a quotient of this direct limit. This item, as well as (1), were added to this definition rather than to Definition 4.6 because as we mentioned, the latter will be applied to partial objects constructed during the construction of an s-object.

When $(\alpha, \daleth)$ does not require attention but we know that for some $\beta < \alpha$, there are special instructions for $A_\beta^\daleth$, then the eventual instruction will be for $A_\alpha^\daleth$ to be empty; (4) ensures that we indeed keep this linear ordering empty, since we may have already spent our one reset. As we shortly show, (5) together with other items will ensure that $|A_\alpha^\daleth| \in \{1, 2\}$ only when we are directly instructed to do so. (6, 7, 8) say that instructions issued at stage s are obeyed. (9) ensures that at the limit, each one-step preimage is infinite, and so has order-type precisely ω.

We remark on the condition $\daleth \notin V_s(\alpha)$ in (10). We need this because when attempting to meet this item, we will choose some $\beta < \lambda$ and let $A_\beta^\daleth = A_\lambda^\daleth$. This will be a level at which $\daleth$ does not require attention, and so in light of (4) can only do this if $\daleth \notin V_s(\beta)$. By Lemma 4.3(d), this condition is also sufficient.

Finally, we remark on the connection between the relations $s \leqslant_\alpha t$ and $p \leqslant_\alpha q$. We have noticed differences in their behaviour, such as with continuity and the $\leqslant_0$ relation. However, we will connect these relations in at least one direction. As we will shortly see, if, in our construction, at stage s we pick object p and at stage t we pick object q, and $s \leqslant_\alpha t$, then we will require $p \leqslant_\alpha q$. Examining the definition of s- and t-objects, and the definition of $p \leqslant_\alpha q$, we observe that this does make sense. For example, if $(\alpha, \daleth)$ requires attention at stage s and $\mathtt{instr}_s(\alpha, \daleth) = \omega$, then we require $\left(A_\alpha^\daleth\right)^p$ to have many points; there will be no reset, and so $\left(A_\alpha^\daleth\right)^q$ will need to extend $\left(A_\alpha^\daleth\right)^p$, and so also have many points; this is fine because $s \leqslant_\alpha t$ implies that $\mathtt{instr}_s(\alpha, \daleth) = \mathtt{instr}_t(\alpha, \daleth)$ (Lemma 4.3(c)), and so the extension requirement does not conflict with our desire to obey the instruction at stage t. The same holds for finite instructions, or for a copying instruction; here we use Proposition 3.9(c).

Lemma 4.13. Suppose that p is an s-object, $\beta \leqslant \delta_*$ and $\daleth \in \omega^3$, and $1 \leqslant |\left(A_\beta^\daleth\right)^p| \leqslant 2$. Then $(\beta, \daleth)$ requires attention at stage s and $\mathtt{instr}_s(\beta, \daleth) = \{1, 2\}$.

Proof. By (4,5,6,7) of Definition 4.12, $(\beta, \daleth)$ must indeed require attention at stage s, and $\mathtt{instr}_s(\beta, \daleth) \neq \omega, 0$. So we just need to show that $\mathtt{instr}_s(\beta, \daleth) \neq \tsadi$ for some $\tsadi$. But in that case, by (8), $\left(A_\beta^\daleth\right)^p = \left(A_\beta^\tsadi\right)^p$, and so $1 \leqslant |\left(A_\beta^\tsadi\right)^p| \leqslant 2$. Since $\tsadi$ is the least element of its $E_s(\beta)$-equivalence class, $\mathtt{instr}_s(\beta, \tsadi)$ is not a copying instruction; rather, by the same analysis, it must be that $\mathtt{instr}_s(\beta, \tsadi) = \{1, 2\}$. By Lemma 4.3(f), $\mathtt{instr}_s(\beta, \daleth) = \{1, 2\}$ as well. □

[15] We use the literal meaning of the word "literally".

4.4. **Proof of Proposition 4.1.** Our strategy is now fully revealed: any computable sequence $\langle p_s \rangle_{s \in \omega}$ such that each p_s is an s-object and $s \leqslant_\alpha t \Rightarrow p_s \leqslant_\alpha p_t$ will suffice.

Proposition 4.14. *There is a computable sequence $\langle p_s \rangle_{s \in \omega}$ such that for each s, p_s is an s-object, and $s \leqslant_\alpha t \Rightarrow p_s \leqslant_\alpha p_t$.*

Before we construct it, we show that such a sequence suffices.

Proof of Proposition 4.1, based on Proposition 4.14. Let $\langle p_s \rangle$ be a sequence as guaranteed by Proposition 4.14. We start with the following.

Claim 4.1.1. For all α and $\daleth$ there is a stage $s^* = s^*(\alpha, \daleth) \in C_\alpha$ such that for all $t > s^*$ in C_α, $\left(r^{\daleth}_{[0,\alpha]}\right)^{p_{s^*}} = \left(r^{\daleth}_{[0,\alpha]}\right)^{p_t}$. Such a stage can be found by $\varnothing_{(\alpha)}$, uniformly in α and $\daleth$.

Proof. By Definition 4.12, we need to find some $s^* \in C_\alpha$ such that for all $\beta \leqslant \alpha$, $\daleth \in V_{s^*}(\beta)$ if and only if $\daleth \in V_\omega(\beta)$. Such a stage exists because for $s \leqslant_\alpha t$, the least β for which $\daleth \in V_t(\beta)$ is no greater than the least β such that $\daleth \in V_s(\beta)$. However this only shows that $\varnothing_{(\alpha+1)}$ can find s^*. But s^* can be found directly: the sets $V_\omega(\beta)$ for $\beta \leqslant \alpha$ are $\varnothing_{(\alpha)}$-computable, uniformly in β; we can find the least β such that $\daleth \in V_\omega(\beta)$ (if such exists), and if so, the least $s \in C_\alpha$ such that $\daleth \in V_s(\beta)$. □

For each α and $\daleth$, if $s^*(\alpha, \daleth) \leqslant_\alpha s \leqslant_\alpha t$ then $\left(A^{\daleth}_\alpha\right)^{p_s} \subseteq \left(A^{\daleth}_\alpha\right)^{p_t}$. We thus define

$$A^{\daleth}_\alpha = \bigcup \left\{ \left(A^{\daleth}_\alpha\right)^{p_s} \,:\, s \in C_\alpha, s \geqslant s^*(\alpha, \daleth) \right\}.$$

We similarly define $B^{\daleth}_\alpha$. Similarly, for $\gamma \leqslant \alpha \leqslant \delta_*$, we let

$$f^{\daleth}_{\gamma,\alpha} = \bigcup \left\{ \left(f^{\daleth}_{\gamma,\alpha}\right)^{p_s} \,:\, s \in C_\alpha, s \geqslant s^*(\alpha, \daleth) \right\},$$

and similarly define $g^{\daleth}_{\gamma,\alpha}$. We let $f^{\daleth}_\alpha = f^{\daleth}_{0,\alpha}$ and similarly define $g^{\daleth}_\alpha$. (a), (b), (d) and (e) of Proposition 4.1 follow from the definitions above. It remains to check (c). We do so in the following claims. We fix $\daleth \in \omega^3$. We focus on the A-side of the construction, as the B-side is identical.

Claim 4.1.2. For all $\alpha < \delta_*$, if $A^{\daleth}_{\alpha+1}$ is nonempty then for all $z \in A^{\daleth}_{\alpha+1}$, the order-type of $\{x \in A^{\daleth}_\alpha \,:\, f^{\daleth}_{\alpha,\alpha+1}(x) = z\}$ is ω.

Proof. At all but finitely many $s \in C_{\alpha+1}$, $(\alpha+1, \daleth)$ requires attention, and so $\daleth \in (G_{\alpha+1})^{p_s}$ for such s. By Definition 4.10(7), the order-type of the set in question is an ordinal $\leqslant \omega$. By Definition 4.12(9), the set is infinite. □

Recall that for a limit $\lambda \leqslant \delta_*$, the direct limit of the system $(A^{\daleth}_\alpha, f^{\daleth}_{\alpha,\beta})_{\alpha \leqslant \beta < \lambda}$ is defined to be the set of equivalence classes for the following relation: for $x \in A^{\daleth}_0$ and $y \in A^{\daleth}_0$, $x \sim y$ if there is a $\gamma < \lambda$ such that $f_{0,\gamma}(x) = f_{0,\gamma}(y)$. Note that we are implicitly using the fact that the maps $f^{\daleth}_{\alpha,\beta}$ are onto. The classes are ordered with the induced ordering, and this induced ordering being well-defined follows from the $f_{0,\gamma}$ being order-preserving.

Claim 4.1.3. If $\lambda \leqslant \delta_*$ is a limit and $A^{\daleth}_\lambda$ is nonempty, then it is the direct limit of the system $(A^{\daleth}_\alpha, f_{\alpha,\beta})_{\alpha \leqslant \beta < \lambda}$.

Proof. The isomorphism from the direct limit to $A^{\urcorner}_{\lambda}$ is $h([x]) = f^{\urcorner}_{0,\lambda}(x)$ for $x \in A^{\urcorner}_0$. That this is well-defined follows from $f^{\urcorner}_{0,\lambda} = f^{\urcorner}_{\gamma,\lambda} \circ f^{\urcorner}_{\gamma,0}$. That it is surjective follows from $f^{\urcorner}_{0,\lambda}$ being surjective. That it is order-preserving follows from $f^{\urcorner}_{0,\lambda}$ being order-preserving.

The only point of difficulty is showing injectivity. Let $x, y \in A^{\urcorner}_0$, and suppose that $f^{\urcorner}_{0,\lambda}(x) = f^{\urcorner}_{0,\lambda}(y)$. We need to show that there is some $\gamma < \lambda$ such that $f^{\urcorner}_{0,\gamma}(x) = f^{\urcorner}_{0,\gamma}(y)$. Let $s \in C_\lambda$, $s \geqslant s^*(\lambda, \urcorner)$. By Definition 4.12(10), there is some $\gamma < \lambda$ such that $\left(f^{\urcorner}_{0,\gamma}\right)^{p_s} = \left(f^{\urcorner}_{0,\gamma}\right)^{p_s}$. By our choice of s, $\left(f^{\urcorner}_{0,\gamma}\right)^{p_s} \subseteq f^{\urcorner}_{0,\gamma}$. □

Claim 4.1.4. For any $\urcorner$ and $\gamma \leqslant \delta_*$ such that $A^{\urcorner}_\gamma$ is nonempty, $A^{\urcorner}_0 \cong \omega^\gamma \cdot A^{\urcorner}_\gamma$ and $f^{\urcorner}_\gamma$ is the map induced by the Hausdorff derivative.

Proof. We prove the claim by induction on γ. The case $\gamma = 0$ is immediate. At successor stages we use Claim 4.1.2. Suppose that γ is a limit ordinal. Claim 4.1.3 implies that A_γ is the direct limit of the system $(A^{\urcorner}_\beta, f^{\urcorner}_{\beta,\zeta})_{\beta \leqslant \zeta < \gamma}$ and that $f^{\urcorner}_\gamma$ is the quotient map. We show that $A^{\urcorner}_0 \cong \omega^\gamma \cdot A^{\urcorner}_\gamma$. For $\beta \leqslant \gamma$, for $z \in A^{\urcorner}_\beta$, let $U_\beta(z) = \{x \in A^{\urcorner}_0 : f^{\urcorner}_\beta(x) = z\}$. We need to show that for all $z \in A^{\urcorner}_\gamma$, $U_\gamma(z) \cong \omega^\gamma$.

Let $z \in A^{\urcorner}_\gamma$. Let $s \geqslant s^*(\gamma, \urcorner)$ in C_γ such that $z \in \left(A^{\urcorner}_\gamma\right)^{p_s}$. Then Definition 4.10(6) shows that the leftmost point y of $\left(A^{\urcorner}_0\right)^{p_s}$ mapped to z by $\left(f^{\urcorner}_\gamma\right)^{p_s}$ is the leftmost point of $U_0(z)$. For $\beta < \gamma$ let $y_\beta = f^{\urcorner}_\beta(y)$. Then y is the leftmost point of $U_\beta(y_\beta)$. By induction, for all $\beta < \gamma$, $U_\beta(y_\beta) \cong \omega^\beta$. For $\beta < \zeta < \gamma$, $U_\beta(y_\beta)$ is an initial segment of $U_\zeta(y_\zeta)$; this is because y_β is the leftmost point in $A^{\urcorner}_\beta$ mapped by $f^{\urcorner}_{\beta,\zeta}$ to y_ζ. Finally, by Claim 4.1.3, $U_\gamma(z) = \bigcup_{\beta<\gamma} U_\beta(y_\beta)$. It follows that $U_\gamma(z) \cong \sup_{\beta<\gamma} \omega^\beta = \omega^\gamma$. □

This concludes the proof of Proposition 4.1. □

4.5. **The weak extendibility condition.** It remains to prove Proposition 4.14, the existence of the desired computable sequence $\langle p_s \rangle$. The existence will follow from what Montalbán called the *weak extendibility condition* [Mon14, Def.4.1]:

Proposition 4.15. *Let $k \geqslant 0$; suppose that $s_k \leqslant s_{k-1} \leqslant \ldots \leqslant s_1 \leqslant s_0 \leqslant t$ are stages, $\delta_* = \alpha_k > \alpha_{k-1} > \cdots > \alpha_1 > \alpha_0 \geqslant 0$ are ordinals, and $p_k, p_{k-1}, \ldots, p_0$ are objects such that:*

(i) for each $i \leqslant k$, p_i is an s_i-object;
(ii) for each $i \leqslant k$, $s_i \leqslant_{\alpha_i} t$; and
(iii) for each $i < k$, $p_{i+1} \leqslant_{\alpha_i+1} p_i$.

Then there is a t-object q such that for all $i \leqslant k$, $p_i \leqslant_{\alpha_i} q$.

Montalbán showed that the weak extendibility condition implies Proposition 4.14, which he called his *metatheorem* [Mon14, Thm.4.2]. We repeat his argument for completeness.

Proof of Proposition 4.14, given Proposition 4.15. We define the sequence by recursion. We start with p_0 being the empty object, which is a 0-object. Suppose that $t > 0$ and that we have already defined $p_0, p_1, \ldots, p_{t-1}$. We must construct a t-object p_t with $p_s \leqslant_\alpha p_t$ for all pairs (α, s) with $s \leqslant_\alpha t$.

First, we define a pair of finite sequences. Start with $\alpha_{-1} = -1$ and $s_0 = t - 1$. Given $s_i < t$, let

$$\alpha_i = \max\{\alpha \leqslant \delta_* : s_i \leqslant_\alpha t\}.$$

If $\alpha_i = \delta_*$ we are done; otherwise note that $s_i > 0$, and let

$$s_{i+1} = \max\{s < s_i : s \leqslant_{\alpha_i+1} t\},$$

again noticing that such s exists because $0 \leqslant_{\delta_*} t$.

Let k be such that $\alpha_k = \delta_*$. Note that $\alpha_{-1} < \alpha_0 < \alpha_1 < \cdots < \alpha_k = \delta_*$ and $s_k < s_{k-1} < \cdots < s_0 = t-1$. Also note that for each $i \leqslant k$, s_i is the greatest $s < t$ such that $s \leqslant_{\alpha_{i-1}+1} t$ (rather than only $s < s_{i-1}$). Indeed, for all $s \in (s_j, s_{j-1}]$, $\max\{\alpha : s \leqslant_\alpha t\} \leqslant \alpha_{j-1}$.

As mentioned above, ($\diamondsuit$) and ($\clubsuit$) imply that for each $i < k$, $s_{i+1} \leqslant_{\alpha_i+1} s_i$, and so, by our inductive hypothesis, $p_{s_{i+1}} \leqslant_{\alpha_i+1} p_{s_i}$. Thus the hypotheses of the weak extendibility condition, Proposition 4.15, hold, with p_{s_i} in the role of p_i; we let p_t be the t-object given by the proposition. We note that the existence of the required p_t implies that we can find it by a search, as the conditions defining it can be checked computably; however, the proof of the weak extendibility condition will be constructive.

Let $s < t$ and let $\alpha \leqslant \delta_*$ such that $s \leqslant_\alpha t$; we need to show that $p_s \leqslant_\alpha p_t$. Write $s_{k+1} = -1$; find $i \in \{0, 1, \ldots, k\}$ such that $s_{i+1} < s \leqslant s_i$. The choice of s_{i+1} in the case $i < k$ implies that $\alpha \leqslant \alpha_i$ (if $i = k$ this is immediate from $\alpha_k = \delta_*$). Thus $s_i \leqslant_\alpha t$. By ($\diamondsuit$), $s \leqslant_\alpha s_i$, and so by induction, $p_s \leqslant_\alpha p_{s_i}$. Now $p_{s_i} \leqslant_{\alpha_i} p_t$ gives $p_{s_i} \leqslant_\alpha p_t$, and thus by transitivity, $p_s \leqslant_\alpha p_t$, as desired. □

We remark that we can further break down the weak extendibility condition to two simpler "extension lemmas".

Lemma 4.16. If $s \leqslant_\alpha t$ and p is an s-object, then there exists a t-object q such that $p \leqslant_\alpha q$.

Lemma 4.17. Suppose that $\alpha_2 > \alpha_1 > \alpha_0$ and $s_2 \leqslant s_1 \leqslant s_0$ with $s_2 \leqslant_{\alpha_2} s_0$ and $s_1 \leqslant_{\alpha_1} s_0$. If p_i is an s_i-object, for $i < 3$, with $p_2 \leqslant_{\alpha_1+1} p_1 \leqslant_{\alpha_1} p_0$, then there exists an s_0-object q with $p_i \leqslant_{\alpha_i} q$ for $i < 3$.

We allow $\alpha_0 = -1$, in which case $p_{s_0} \leqslant_{-1} q$ is vacuous.

Note that under the hypothesis of Lemma 4.17, by ($\diamondsuit$) and ($\clubsuit$), $s_2 \leqslant_{\alpha_1+1} s_1$.

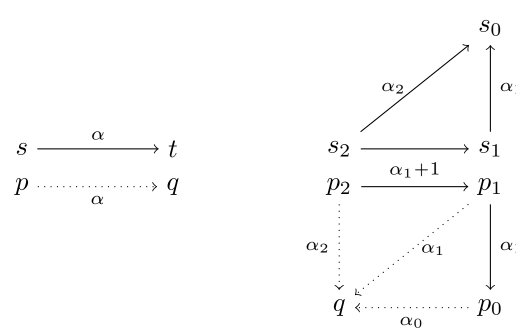

Figure 2. The two extension lemmas.

The weak extendibility condition is equivalent to the conjunction of the two extension lemmas (this equivalence is not restricted to our particular setting). In one direction, suppose that the weak extendibility condition Proposition 4.15 holds. We first note that we can deduce the apparently stronger version in which α_k is not required to equal δ_*. This is because we can always add $s_{k+1} = 0$, $\alpha_{k+1} = \delta_*$ and p_{k+1} being the empty object.

Using this more relaxed extendibility condition, for Lemma 4.16, take $k = 0$ with $\alpha_0 = \alpha$; and let $p_0 = p$ and $s_0 = s$. For Lemma 4.17, we let $t = s_0$ (so $s_0 \leqslant_{\alpha_0} t$ is immediate); if $\alpha_0 \geqslant 0$ we take $k = 2$; for $i = 0, 1, 2$ we are given s_i and p_i. The assumption $p_1 \leqslant_{\alpha_1} p_0$ certainly implies $p_1 \leqslant_{\alpha_0+1} p_0$; we are told that $p_2 \leqslant_{\alpha_1+1} p_1$. If $\alpha_0 = -1$ we omit p_0 and shift the indices by 1.

In the other direction, suppose that Lemmas 4.16 and 4.17 hold. We define a sequence $q_0, q_1, \ldots, q_k$ of t-objects as in fig. 3. By the first extension Lemma 4.16, we let q_0 be a t-object such that $p_0 \leqslant_{\alpha_0} q_0$. Suppose that $i < k$ and that q_i has been constructed, and that $p_i \leqslant_{\alpha_i} q_i$. Then by the second extension Lemma 4.17 applied to $p_2 := p_{i+1}$, $p_1 := p_i$ and $p_0 = q_i$ we obtain a t-object q_{i+1} such that:

- $q_i \leqslant_{\alpha_{i-1}} q_{i+1}$;
- $p_i \leqslant_{\alpha_i} q_{i+1}$; and
- $p_{i+1} \leqslant_{\alpha_{i+1}} q_{i+1}$.

Finally we let $q = q_k$. We observe that for $i = 0, 1, \ldots, k$, by transitivity, $p_i \leqslant_{\alpha_i} q$.

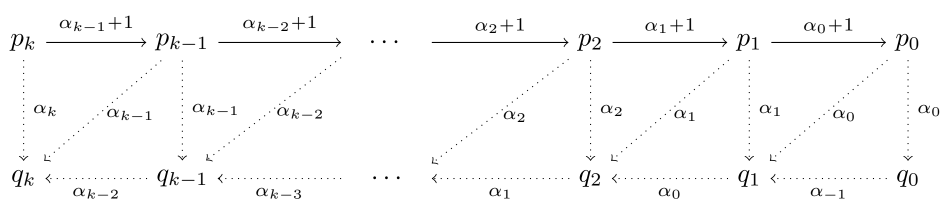

FIGURE 3. Constructing q

We remark that even though the statement of the two extension lemmas seems simpler than the weak extendibility condition, in fact, the proof more naturally gives the weak extendibility condition directly.

Remark 4.18. In Montalbán's application ([Mon14, Thm.5.3]), a simpler variant of the second extension Lemma 4.17 is used: an object q is obtained with $p_0 \leqslant_{\alpha_1} q$ and $p_2 \leqslant_{\delta_*} q$. This simpler extension lemma does not hold in our construction (as well as similar constructions such as for the proof of the Ash-Watnick theorem [Ash90]), so we require the more complicated Lemma 4.17.

We turn now to the proof of the weak extendibility condition. We break this proof up into two parts: we first take the objects $p_0, p_1, \ldots, p_k$ and "glue" them together in to some object o. This process will involve removing some linear orderings but will not require adding new ones. This is the step at which we spend resets if necessary. The object o produced will not quite be a t-object. The following definition lists properties of o that will enable us, in the second step, to add linear orderings to make a t-object.

Definition 4.19. An object o is called *admissible for stage t* if the following conditions hold for all $\beta \leqslant \delta^*$ and $\daleth \in \omega^3$: [16]

[16] As in Definition 4.12, we omit the superscript o, as o is the only object we will mention in this definition.

(1) $r^{\daleth}_{\beta} = 1 \iff \daleth \in V_t(\beta)$;
(2) $\daleth \in G_{\beta}$ if and only if $(\beta, \daleth)$ requires attention at stage t;
(3) If $1 \leqslant |A^{\daleth}_{\beta}| \leqslant 2$ then $(\beta, \daleth)$ requires attention at stage t and $\mathtt{instr}_t(\beta, \daleth) = \{1, 2\}$;

if $(\beta, \daleth)$ does not require attention at stage t, then:

(4) If $\daleth \in V_t(\beta)$ then $A^{\daleth}_{\beta}$ and $B^{\daleth}_{\beta}$ are empty;

but if $(\beta, \daleth)$ requires attention at stage t, *and $A^{\daleth}_{\beta}$ is nonempty*, then:

(5) If $\mathtt{instr}_t(\beta, \daleth)$ is finite, then $|A^{\daleth}_{\beta}| = \mathtt{instr}^A_t(\beta, \daleth)$ and $|B^{\daleth}_{\beta}| = \mathtt{instr}^B_t(\beta, \daleth)$[17];
(6) If $\mathtt{instr}_t(\beta, \daleth) = \text{צ}$ then $A^{\daleth}_{\beta} = A^{\text{ש}}_{\beta}$ and $B^{\daleth}_{\beta} = B^{\text{ש}}_{\beta}$.

We remark that we could have added (7) of Definition 4.12, in that it would hold for the object o that we build, but we will not need this condition when we extend o to a t-object. The same holds for Definition 4.12(5).

Using this definition, Proposition 4.15 will follow from the conjunction of the two following lemmas.

Lemma 4.20. Let $k \geqslant 0$; suppose that $s_k \leqslant s_{k-1} \leqslant \ldots \leqslant s_1 \leqslant s_0 \leqslant t$ are stages, $\delta_* = \alpha_k > \alpha_{k-1} > \cdots > \alpha_1 > \alpha_0 \geqslant 0$ are ordinals, and $p_k, p_{k-1}, \ldots, p_0$ are objects such that:

(i) for each $i \leqslant k$, p_i is an s_i-object;
(ii) for each $i \leqslant k$, $s_i \leqslant_{\alpha_i} t$; and
(iii) for each $i < k$, $p_{i+1} \leqslant_{\alpha_i+1} p_i$.

Then there is an object o, admissible for stage t, such that for all $i \leqslant k$, $p_i \leqslant_{\alpha_i} o$.

Lemma 4.21. If o is an object which is admissible for stage t, then there is a t-object q with $o \leqslant_{\delta_*} q$.

So it remains to prove Lemma 4.20 and Lemma 4.21. We start with the former.

4.6. **Proof of Lemma 4.20.** We build the object o as a combination of the objects p_i, except that we reset those linear orderings that we need removed.

We start, of course, by setting $\left(r^{\daleth}_{\beta}\right)^o = 1$ if and only if $\daleth \in V_t(\beta)$ for all $\beta \leqslant \delta_*$ and all $\daleth \in \omega^3$, and $\daleth \in (G_{\beta})^o$ if and only if $(\beta, \daleth)$ requires attention at stage t.

For all $\beta \leqslant \delta_*$, using the fact that $\alpha_k = \delta_*$, we let $i(\beta)$ be the smallest $i \leqslant k$ such that $\beta \leqslant \alpha_i$. For all $\beta \leqslant \gamma \leqslant \delta_*$, $i(\beta) \leqslant i(\gamma)$, and as $\beta \leqslant \alpha_{i(\beta)}$, we have

$$p_{i(\gamma)} \leqslant_{\beta+1} p_{i(\beta)}.$$

Let $\beta \leqslant \delta_*$ and let $\daleth \in \omega^3$. First, we define $\left(A^{\daleth}_{\beta}\right)^o$. To avoid excessive notation, we do not mention $\left(B^{\daleth}_{\beta}\right)^o$, but the definition is identical.

(a) If $(\beta, \daleth)$ does not require attention at stage $s_{i(\beta)}$ and $\daleth \in V_t(\beta)$ then we let $\left(A^{\daleth}_{\beta}\right)^o = \varnothing$.
(b) Otherwise, we let $\left(A^{\daleth}_{\beta}\right)^o = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$.

Now we define the maps $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ (as with the B's, we don't mention the g-maps but their definition is identical). The difficulty is with defining $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ when $i(\beta) < i(\gamma)$, because then we don't already have a map from $\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$ into $\left(A^{\daleth}_{\gamma}\right)^{p_{i(\gamma)}}$.

[17] In particular, $\mathtt{instr}_t(\beta, \daleth) \neq 0$

Fix $\urcorner$. Of course if either $\left(A^{\urcorner}_{\beta}\right)^{o}$ or $\left(A^{\urcorner}_{\gamma}\right)^{o}$ is empty then $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ is empty. We also let $\left(f^{\urcorner}_{\beta,\beta}\right)^{o}$ be the identity on $\left(A^{\urcorner}_{\beta}\right)^{o}$ for all β.

Suppose that $\beta < \gamma$ and both $\left(A^{\urcorner}_{\beta}\right)^{o}$ and $\left(A^{\urcorner}_{\gamma}\right)^{o}$ are nonempty. First we suppose that β and γ are *successive* $\urcorner$*-levels* of o, meaning that for all $\xi \in (\beta,\gamma)$, $\left(A^{\urcorner}_{\xi}\right)^{o}$ is empty. To define $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ we have two cases:

(i): If there is no $\zeta \leqslant \beta$ such that $\left(A^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}}$ is nonempty and $\left(r^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}} = \left(r^{\urcorner}_{\zeta}\right)^{p_{i(\beta)}}$, then we let $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ map every point in $\left(A^{\urcorner}_{\beta}\right)^{o}$ to 0.[18]

(ii): Otherwise, let $\zeta = \zeta^{\urcorner}_{\beta,\gamma}$ be the greatest $\zeta \leqslant \beta$ such that $\left(A^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}}$ is nonempty and $\left(r^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}} = \left(r^{\urcorner}_{\zeta}\right)^{p_{i(\beta)}}$. Since $\zeta \leqslant \beta$ and $p_{i(\gamma)} \leqslant_{\beta} p_{i(\beta)}$, we have $\left(A^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}} \subseteq \left(A^{\urcorner}_{\zeta}\right)^{p_{i(\beta)}}$, and so $\left(f^{\urcorner}_{\zeta,\beta}\right)^{p_{i(\beta)}}$ is defined on each $y \in \left(A^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}}$ and preserves their ordering.

For each $z \in \left(A^{\urcorner}_{\gamma}\right)^{p_{i(\gamma)}} = \left(A^{\urcorner}_{\gamma}\right)^{o}$ we let y_z be the leftmost element of $\left(A^{\urcorner}_{\zeta}\right)^{p_{i(\gamma)}}$ which is mapped to z by $\left(f^{\urcorner}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}$; and we let $x_z = \left(f^{\urcorner}_{\zeta,\beta}\right)^{p_{i(\beta)}}(y_z)$.[19] Note that $x_0 = y_0 = 0$. Hence, for all $w \in \left(A^{\urcorner}_{\beta}\right)^{o}$ there is some $z \in \left(A^{\urcorner}_{\gamma}\right)^{o}$ such that $x_z \leqslant w$ in $\left(A^{\urcorner}_{\beta}\right)^{o}$; we let $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ map w to the rightmost such z (rightmost in $\left(A^{\urcorner}_{\gamma}\right)^{o}$ of course).

Having defined $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ for successive $\urcorner$-levels $\beta < \gamma$, for any $\beta < \gamma$ with $\left(A^{\urcorner}_{\beta}\right)^{o}$ and $\left(A^{\urcorner}_{\gamma}\right)^{o}$ both nonempty we let $\beta = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_k = \gamma$ be a list of the $\urcorner$-levels of o between β and γ; we let

$$\left(f^{\urcorner}_{\beta,\gamma}\right)^{o} = \left(f^{\urcorner}_{\varepsilon_{k-1},\varepsilon_k}\right)^{o} \circ \cdots \circ \left(f^{\urcorner}_{\varepsilon_1,\varepsilon_2}\right)^{o} \circ \left(f^{\urcorner}_{\varepsilon_0,\varepsilon_1}\right)^{o};$$

this concludes the definition of o.

We start the verification with:

Claim 4.20.1. If $\beta < \gamma$ and $\left(A^{\urcorner}_{\beta}\right)^{o}$ and $\left(A^{\urcorner}_{\gamma}\right)^{o}$ are nonempty, then $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ is order-preserving.

Proof. By taking compositions, we may assume that β and γ are successive $\urcorner$-levels of o. We then consider how we defined $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$. In case (i) the claim is immediate. In case (ii), $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$ being order-preserving follows from the defintion, and the fact that $z \mapsto x_z$ (from the definition of $\left(f^{\urcorner}_{\beta,\gamma}\right)^{o}$) is order-preserving (as both $\left(f^{\urcorner}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}$ and $\left(f^{\urcorner}_{\zeta,\beta}\right)^{p_{i(\beta)}}$ are order-preserving). ☐

We also note that $\left(r^{\urcorner}_{\beta}\right)^{o} \leqslant \left(r^{\urcorner}_{\gamma}\right)^{o}$ follows from $V_t(\beta) \subseteq V_t(\gamma)$ (Lemma 4.5(c)). Similarly, for all but finitely many $\urcorner$, $\left(r^{\urcorner}_{\delta_*}\right)^{o} = 0$, as $V_t(\delta_*)$ is finite (Lemma 4.5(g)). Also, each $(G_\beta)^o$ is finite and all but finitely many are empty because only finitely many pairs require attention at any stage (Lemma 4.3(a)). There are only finitely many nonempty $\left(A^{\urcorner}_{\beta}\right)^{o}$ since each p_i is an object.

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[18]This is the only place in this paper in which we use the fact that the homomorphisms $f^{\urcorner}$ in objects do not have to be onto. The issue being that $\left(A^{\urcorner}_{\beta}\right)^{o}$ may have fewer points than $\left(A^{\urcorner}_{\gamma}\right)^{o}$.

[19]We use the fact that $p_{i(\gamma)}$ is an $s_{i(\gamma)}$-object, and so $\left(f^{\urcorner}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}$ is onto $\left(A^{\urcorner}_{\gamma}\right)^{o}$; but this is not really important, we could have defined x_z only for $z \in \operatorname{range}\left(f^{\urcorner}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}$.

Most other items of Definition 4.6 are immediate; to show that o is an object it remains to show that the range of each $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ is an initial segment of $\left(A^{\daleth}_{\gamma}\right)^o$. We need the following.

Claim 4.20.2. If $(\beta,\daleth)$ does not require attention at stage t and $\daleth \in V_t(\beta)$ then $\left(A^{\daleth}_{\beta}\right)^o$ and $\left(B^{\daleth}_{\beta}\right)^o$ are empty.

Proof. If $(\beta,\daleth)$ does not require attention at stage t, then since $s_{i(\beta)} \leqslant_\beta t$, the pair $(\beta,\daleth)$ does not require attention at stage $s_{i(\beta)}$ either (Lemma 4.3(c)). If $\daleth \in V_t(\beta)$ then our construction sets $\left(A^{\daleth}_{\beta}\right)^o$ to be empty. □

Claim 4.20.3. Suppose that $(\beta,\daleth)$ requires attention at stage t, and $\texttt{instr}_t(\beta,\daleth) \neq \omega$. Then for all $\gamma > \beta$, $\left(A^{\daleth}_{\gamma}\right)^o$ is empty.

Proof. Let $\gamma > \beta$. Then $\daleth \in V_t(\gamma)$. If $(\gamma,\daleth)$ does not require attention at stage t then $\left(A^{\daleth}_{\gamma}\right)^o$ is empty (Claim 4.20.2). If $(\gamma,\daleth)$ requires attention then $\texttt{instr}_t(\gamma,\daleth) = 0$ (Lemma 4.3(g)) and then $\left(A^{\daleth}_{\gamma}\right)^o = \varnothing$ by Definition 4.19(5). □

Claim 4.20.4. Suppose that $j \leqslant k$, $\beta \leqslant \alpha_j$, and $1 \leqslant |\left(A^{\daleth}_{\beta}\right)^{p_j}| \leqslant 2$. Then $(\beta,\daleth)$ requires attention at stage t, $\texttt{instr}_t(\beta,\daleth) = \{1,2\}$ and $\left(A^{\daleth}_{\beta}\right)^o = \left(A^{\daleth}_{\beta}\right)^{p_j}$.

Proof. Since p_j is an s_j-object, $(\beta,\daleth)$ requires attention at stage s_j and $\texttt{instr}_{s_j}(\beta,\daleth) = \{1,2\}$ (Lemma 4.13). Since $s_j \leqslant_\beta s_{i(\beta)} \leqslant_\beta t$, $(\beta,\daleth)$ requires attention at stages $s_{i(\beta)}$ and t and $\texttt{instr}_t(\beta,\daleth) = \texttt{instr}_{s_{i(\beta)}}(\beta,\daleth) = \texttt{instr}_{s_j}(\beta,\daleth)$.[20] Since $p_j \leqslant_\beta p_{i(\beta)}$, $\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = \left(A^{\daleth}_{\beta}\right)^{p_j}$; by construction, $\left(A^{\daleth}_{\beta}\right)^o = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$. □

Combining Claim 4.20.4 and Claim 4.20.3 we get:

Claim 4.20.5. Suppose that $j \leqslant k$, $\beta \leqslant \alpha_j$, and $1 \leqslant |\left(A^{\daleth}_{\beta}\right)^{p_j}| \leqslant 2$. Then for all $\gamma > \beta$, $\left(A^{\daleth}_{\gamma}\right)^o$ is empty.

Claim 4.20.6. Let $\beta < \gamma$ be successive $\daleth$-levels of o, suppose that $\zeta = \zeta^{\daleth}_{\beta,\gamma}$ is defined, and suppose that $\left(r^{\daleth}_{[\zeta,\beta]}\right)^{p_{i(\gamma)}} = \left(r^{\daleth}_{[\zeta,\beta]}\right)^{p_{i(\beta)}}$. Then $\left(f^{\daleth}_{\beta,\gamma}\right)^o \circ \left(f^{\daleth}_{\zeta,\beta}\right)^{p_{i(\beta)}}$ extends $\left(f^{\daleth}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}$.

Proof. Since $\left(A^{\daleth}_{\gamma}\right)^o$ is nonempty, by Claim 4.20.5, for no $\xi \leqslant \beta$ do we have $1 \leqslant |\left(A^{\daleth}_{\xi}\right)^{p_{i(\beta)}}| \leqslant 2$.

Let $y \in \left(A^{\daleth}_{\zeta}\right)^{p_{i(\gamma)}}$; let $z = \left(f^{\daleth}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}(y)$ and let $x = \left(f^{\daleth}_{\zeta,\beta}\right)^{p_{i(\beta)}}(y)$. Then $y_z \leqslant y$, and so $x_z \leqslant x$. Let $z' > z$ in $\left(A^{\daleth}_{\gamma}\right)^o$; then $y < y_{z'}$. As $(y \not\sim_\gamma y_{z'})^{p_{i(\gamma)}}$, we have $(y \not\sim_\beta y_{z'})^{p_{i(\gamma)}}$; since $p_{i(\gamma)} \leqslant_\beta p_{i(\beta)}$), $x < x_{z'}$ (Definition 4.10(8)), and so $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ maps x to z. □

It follows that if $\zeta = \zeta^{\daleth}_{\beta,\gamma}$ is defined, then $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ is onto $\left(A^{\daleth}_{\gamma}\right)^o$: for all z, since $\left(f^{\daleth}_{\zeta,\gamma}\right)^{p_{i(\gamma)}}(y_z) = z$, we have $\left(f^{\daleth}_{\beta,\gamma}\right)^o(x_z) = z$. Of course in case (i) of the definition of $\left(f^{\daleth}_{\beta,\gamma}\right)^o$, the range of this map is an initial segment of $\left(A^{\daleth}_{\gamma}\right)^o$, namely $\{0\}$. With compositions, we see that every map $\left(f^{\daleth}_{\beta,\gamma}\right)^o$ is onto an initial segment of $\left(A^{\daleth}_{\gamma}\right)^o$; we conclude that o is an object.

[20] Recall that this means that $\texttt{instr}^A_{s_{i(\beta)}}(\beta,\daleth) = \texttt{instr}^A_t(\beta,\daleth)$ and $\texttt{instr}^B_{s_{i(\beta)}}(\beta,\daleth) = \texttt{instr}^B_t(\beta,\daleth)$, and the same for s_j.

Next, we show that o is admissible for stage t. The first two items of Definition 4.19 are by definition. (4) is Claim 4.20.2. For (3), suppose that $1 \leqslant |\left(A^{\daleth}_{\beta}\right)^{o}| \leqslant 2$. By consturction, $\left(A^{\daleth}_{\beta}\right)^{o} = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$. Then (3) follows from Claim 4.20.4.

We check (5) and (6) of Definition 4.19. Suppose that $(\beta, \daleth)$ requires attention at stage t, that $\left(A^{\daleth}_{\beta}\right)^{o}$ is nonempty, and that $\mathtt{instr}_t(\beta, \daleth) \neq \omega$. Then $\daleth \in V_t(\beta)$. Since $\left(A^{\daleth}_{\beta}\right)^{o}$ is nonempty and $\daleth \in V_t(\beta)$, by construction, $(\beta, \daleth)$ requires attention at stage $s_{i(\beta)}$. Again, $\mathtt{instr}_{s_{i(\beta)}}(\beta, \daleth) = \mathtt{instr}_t(\beta, \daleth)$. Since $p_{i(\beta)}$ is an $s_{i(\beta)}$-object, $\left(A^{\daleth}_{\beta}\right)^{o} = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$ is of the right type:

- If $\mathtt{instr}^{A}_t(\beta, \daleth) \in \{0, 1, 2\}$ then $|\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}| = \mathtt{instr}^{A}_t(\beta, \daleth)$ (and the same for B);
- If $\mathtt{instr}^{A}_t(\beta, \daleth) = \text{צ}$ then $\left(A^{\daleth}_{\beta}\right)^{o} = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = \left(A^{\text{צ}}_{\beta}\right)^{p_{i(\beta)}}$. Since $(\beta, \text{צ})$ requires attention at stage $s_{i(\beta)}$ (Lemma 4.3(e)), $\left(A^{\text{צ}}_{\beta}\right)^{o} = \left(A^{\text{צ}}_{\beta}\right)^{p_{i(\beta)}}$.

It remains to show that $p_j \leqslant_{\alpha_j} o$ for all $j \leqslant k$. We need the following claims.

Claim 4.20.7. Let $\beta \leqslant \delta_*$ and $\daleth \in \omega^3$.

(a) $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} \leqslant \left(r^{\daleth}_{\beta}\right)^{o}$.

(b) $(G_\beta)^{p_{i(\beta)}} \subseteq (G_\beta)^{o}$, and if $\daleth \in (G_\beta)^{p_{i(\beta)}}$ then $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = \left(r^{\daleth}_{\beta}\right)^{o}$.

(c) If $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = \left(r^{\daleth}_{\beta}\right)^{o}$ then $\left(A^{\daleth}_{\beta}\right)^{o} = \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$.

Proof. (a): Suppose that $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = 1$. Since $p_{i(\beta)}$ is an $s_{i(\beta)}$-object, $\daleth \in V_{s_{i(\beta)}}(\beta)$. Since $s_{i(\beta)} \leqslant_\beta t$, $\daleth \in V_t(\beta)$ (Lemma 4.5(b)), and so $\left(r^{\daleth}_{\beta}\right)^{o} = 1$ as well.

(b): Again we use that $p_{i(\beta)}$ is an $s_{i(\beta)}$-object. So if $\daleth \in (G_\beta)^{p_{i(\beta)}}$ then $(\beta, \daleth)$ requires attention at stage $s_{i(\beta)}$, and so requires attention at stage t, and so by construction, $\daleth \in (G_\beta)^{o}$. Further, for such $\daleth$, $\daleth \in V_{s_{i(\beta)}}(\beta) \iff \daleth \in V_t(\beta)$ (Lemma 4.5(e)), and $\daleth \in V_{s_{i(\beta)}} \iff \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = 1$.

(c): Suppose that $\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}} \neq \left(A^{\daleth}_{\beta}\right)^{o}$. Then $\left(A^{\daleth}_{\beta}\right)^{o} = \varnothing$, $\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}} \neq \varnothing$, $\daleth \in V_t(\beta)$ and $(\beta, \daleth)$ does not require attention at stage $s_{i(\beta)}$. Then $\left(r^{\daleth}_{\beta}\right)^{o} = 1$ by construction, and $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = 0$ by Definition 4.12(4). ☐

We can now show that (1),(2) and (3) of Definition 4.10 hold between each p_j and o at the right levels. Namely, let $j \leqslant k$, and let $\beta \leqslant \alpha_j$. Since $p_j \leqslant_\beta p_{i(\beta)}$, we have $\left(r^{\daleth}_{\beta}\right)^{p_j} \leqslant \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$; with Claim 4.20.7(a), we conclude that $\left(r^{\daleth}_{\beta}\right)^{p_j} \leqslant \left(r^{\daleth}_{\beta}\right)^{o}$.

Similarly, if $\daleth \in (G_\beta)^{p_j}$ then $\daleth \in (G_\beta)^{p_{i(\beta)}}$ and so $\daleth \in (G_\beta)^{o}$, and further, $\left(r^{\daleth}_{\beta}\right)^{p_j} = \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$ and so also equals $\left(r^{\daleth}_{\beta}\right)^{o}$.

And if $\left(r^{\daleth}_{\beta}\right)^{p_j} = \left(r^{\daleth}_{\beta}\right)^{o}$ then as $\left(r^{\daleth}_{\beta}\right)^{p_j} \leqslant \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} \leqslant \left(r^{\daleth}_{\beta}\right)^{o}$, we have $\left(r^{\daleth}_{\beta}\right)^{p_j} = \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}} = \left(r^{\daleth}_{\beta}\right)^{o}$. Since $p_j \leqslant_\beta p_{i(\beta)}$, $\left(A^{\daleth}_{\beta}\right)^{p_j} \subseteq \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$; by Claim 4.20.7(c), we get $\left(A^{\daleth}_{\beta}\right)^{p_j} \subseteq \left(A^{\daleth}_{\beta}\right)^{o}$.

We also note that (4) between each p_j and o follows from Claim 4.20.4.

Claim 4.20.8. Suppose that $\beta < \gamma$, and that both $\left(A^{\daleth}_{\beta}\right)^{p_{i(\gamma)}}$ and $\left(A^{\daleth}_{\gamma}\right)^{o}$ are nonempty; suppose that $\left(r^{\daleth}_{[\beta,\gamma]}\right)^{p_{i(\gamma)}} = \left(r^{\daleth}_{[\beta,\gamma]}\right)^{o}$. Then:

(a) $\left(f^{\daleth}_{\beta,\gamma}\right)^{o}$ extends $\left(f^{\daleth}_{\beta,\gamma}\right)^{p_{i(\gamma)}}$.

(b) For every $z \in \left(A^{\daleth}_{\gamma}\right)^{o}$, the leftmost point of $\left(A^{\daleth}_{\beta}\right)^{p_{i(\gamma)}}$ mapped by $\left(f^{\daleth}_{\beta,\gamma}\right)^{p_{i(\gamma)}}$ to z is also the leftmost point of $\left(A^{\daleth}_{\beta}\right)^{o}$ mapped by $\left(f^{\daleth}_{\beta,\gamma}\right)^{o}$ to z.

Proof. For all $\varepsilon \in [\beta,\gamma]$, $p_{i(\gamma)} \leqslant_\varepsilon p_{i(\varepsilon)}$, so the assumption implies that $\big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{p_{i(\gamma)}} = \big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{p_{i(\varepsilon)}} = \big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{o}$. We prove the claim by induction on γ.

The base case is when γ is the next $\urcorner$-level above β. Then $\zeta^{\urcorner}_{\beta,\gamma}$ is defined and equals β. In this case both (a) and (b) follow directly from the construction (we do not need Claim 4.20.6); $y_z = x_z$ is the leftmost point in $\big(A^{\urcorner}_\beta\big)^o$ mapped to z by $\big(f^{\urcorner}_{\beta,\gamma}\big)^o$.

For the inductive case, let ε be the greatest $\urcorner$-level of o below γ. Then $\zeta = \zeta_{\varepsilon,\gamma}$ is defined and $\zeta \geqslant \beta$. For (a), let $w \in \big(A^{\urcorner}_\beta\big)^{p_{i(\gamma)}}$; let $y = \big(f^{\urcorner}_{\beta,\zeta}\big)^{p_{i(\gamma)}}$, $x = \big(f^{\urcorner}_{\zeta,\varepsilon}\big)^{p_{i(\varepsilon)}}(y)$ and $z = \big(f^{\urcorner}_{\zeta,\gamma}\big)^{p_{i(\gamma)}}(y) = \big(f^{\urcorner}_{\beta,\gamma}\big)^{p_{i(\gamma)}}(w)$.

- Since $p_{i(\gamma)} \leqslant_\varepsilon p_{i(\varepsilon)}$ and $\big(r^{\urcorner}_{[\beta,\zeta]}\big)^{p_{i(\gamma)}} = \big(r^{\urcorner}_{[\beta,\zeta]}\big)^{p_{i(\varepsilon)}}$, we have $y = \big(f^{\urcorner}_{\beta,\zeta}\big)^{p_{i(\varepsilon)}}(w)$, and so $x = \big(f^{\urcorner}_{\beta,\varepsilon}\big)^{p_{i(\varepsilon)}}(w)$.
- By Claim 4.20.6, $z = \big(f^{\urcorner}_{\varepsilon,\gamma}\big)^{o}(x)$.
- By induction, as $\big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{p_{i(\varepsilon)}} = \big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{o}$, $\big(f^{\urcorner}_{\beta,\varepsilon}\big)^{o}$ extends $\big(f^{\urcorner}_{\beta,\varepsilon}\big)^{p_{i(\varepsilon)}}$, and so $x = \big(f^{\urcorner}_{\beta,\varepsilon}\big)^{o}(w)$.

We conclude that $z = \big(f^{\urcorner}_{\beta,\gamma}\big)^{o}(w)$, establishing (a).

For (b), let $z \in \big(A^{\urcorner}_\gamma\big)^o$; let w be the leftmost in $\big(A^{\urcorner}_\beta\big)^{p_{i(\gamma)}}$ mapped to z by $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_{i(\gamma)}}$. Since $\big(f^{\urcorner}_{\beta,\zeta}\big)^{p_{i(\gamma)}}$ is onto an initial segment of $\big(A^{\urcorner}_\zeta\big)^{p_{i(\gamma)}}$ (in fact onto $\big(A^{\urcorner}_\zeta\big)^{p_{i(\gamma)}}$), y_z, which recall is the leftmost point in $\big(A^{\urcorner}_\zeta\big)^{p_{i(\gamma)}}$ mapped by $\big(A^{\urcorner}_{\zeta,\gamma}\big)^{p_{i(\gamma)}}$ to z, equals $\big(f^{\urcorner}_{\beta,\zeta}\big)^{p_{i(\gamma)}}(w)$. Then $x_z = \big(f^{\urcorner}_{\beta,\varepsilon}\big)^{p_{i(\varepsilon)}}(w)$; recall that this is the leftmost point in $\big(A^{\urcorner}_\varepsilon\big)^o$ mapped to z by $\big(f^{\urcorner}_{\varepsilon,\gamma}\big)^o$. Let $x \in \big(A^{\urcorner}_\beta\big)^o$ and suppose that $\big(f^{\urcorner}_{\beta,\gamma}\big)^o(x) = z$; let $x' = \big(f^{\urcorner}_{\beta,\varepsilon}\big)^o(x)$. Then $x_z \leqslant x'$. If $x_z < x'$ then as $\big(f^{\urcorner}_{\beta,\varepsilon}\big)^o$ is order-preserving, $w < x$.

Suppose that $x_z = x'$. By Claim 4.20.5, since $\big(A^{\urcorner}_\gamma\big)^o$ is nonempty, for no $\xi \leqslant \varepsilon$ do we have $1 \leqslant |\big(A^{\urcorner}_\xi\big)^{p_{i(\varepsilon)}}| \leqslant 2$. Since $p_{i(\gamma)} \leqslant_\varepsilon p_{i(\varepsilon)}$, w is also the leftmost point in $\big(A^{\urcorner}_\beta\big)^{p_{i(\varepsilon)}}$ mapped to $x' = x_z$ by $\big(f^{\urcorner}_{\beta,\varepsilon}\big)^{p_{i(\varepsilon)}}$. By induction applied at level ε, $w \leqslant x$. □

Again let $j \leqslant k$ and let $\beta \leqslant \gamma \leqslant \alpha_j$. Suppose that $\big(r^{\urcorner}_{[\beta,\gamma]}\big)^{p_j} = \big(r^{\urcorner}_{[\beta,\gamma]}\big)^{o}$. As above, for all $\varepsilon \in [\beta,\gamma]$, $\big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{p_j} = \big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{p_{i(\varepsilon)}} = \big(r^{\urcorner}_{[\beta,\varepsilon]}\big)^{o}$.

Since $p_j \leqslant_\gamma p_{i(\gamma)}$, $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_j} \subseteq \big(f^{\urcorner}_{\beta,\gamma}\big)^{p_{i(\gamma)}}$. By Claim 4.20.8(a), we also have $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_j} \subseteq \big(f^{\urcorner}_{\beta,\gamma}\big)^{o}$, establishing (5) of Definition 4.10 between p_j and o.

For (6), let $z \in \big(A^{\urcorner}_\gamma\big)^{p_j}$, and let y be the leftmost point in $\big(A^{\urcorner}_\beta\big)^{p_j}$ mapped by $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_j}$ to z. Since $p_j \leqslant_\gamma p_{i(\gamma)}$, y is the leftmost point in $\big(A^{\urcorner}_\beta\big)^{p_{i(\gamma)}}$ mapped by $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_{i(\gamma)}}$ to z. By Claim 4.20.8(b), y is also the leftmost point in $\big(A^{\urcorner}_\beta\big)^{o}$ mapped to z by $\big(f^{\urcorner}_{\beta,\gamma}\big)^{o}$.

Skipping (7) for now, suppose that in addition, for no $\xi \in [\beta,\gamma]$ do we have $1 \leqslant |\big(A^{\urcorner}_\xi\big)^{o}| \leqslant 2$. Let $f = \big(f^{\urcorner}_{\beta,\gamma}\big)^{o}$, and let $z \in \operatorname{range} f \upharpoonright \big(A^{\urcorner}_\beta\big)^{p_j}$; let $y \in \big(A^{\urcorner}_\beta\big)^{p_j}$ be leftmost mapped to z by f. Then $y \in \big(A^{\urcorner}_\beta\big)^{p_{i(\gamma)}}$, and $\big(f^{\urcorner}_{\beta,\gamma}\big)^{p_{i(\gamma)}}(y) = z$ (Claim 4.20.8(a)). We know that for no $\xi \in [\beta,\gamma]$ we have $1 \leqslant |\big(A^{\urcorner}_\xi\big)^{p_{i(\gamma)}}| \leqslant 2$ (Claim 4.20.4). Thus since $p_j \leqslant_\gamma p_{i(\gamma)}$, y is the leftmost point in $\big(A^{\urcorner}_\beta\big)^{p_{i(\gamma)}}$

mapped to z by $\left(f^{\daleth}_{\beta,\gamma}\right)^{p_{i(\gamma)}}$. Then as above, (9) of Definition 4.10 follows from Claim 4.20.8(b).

We verify (8). Let $y, y' \in \left(A^{\daleth}_{\beta}\right)^{p_j}$, and suppose that $(y \sim_\gamma y')^o$. By decreasing γ we may assume that $\left(A^{\daleth}_{\gamma}\right)^o$ is nonempty, so $\left(f^{\daleth}_{\beta,\gamma}\right)^o(y) = \left(f^{\daleth}_{\beta,\gamma}\right)^o(y')$. By Claim 4.20.8(a), $\left(f^{\daleth}_{\beta,\gamma}\right)^{p_{i(\gamma)}}(y) = \left(f^{\daleth}_{\beta,\gamma}\right)^{p_{i(\gamma)}}(y')$ and so $(y \sim_\gamma y')^{p_{i(\gamma)}}$. Since $p_j \leqslant_\gamma p_{i(\gamma)}$ and there is no $\xi \in [\beta,\gamma]$ with $1 \leqslant |\left(A^{\daleth}_{\xi}\right)^{p_{i(\gamma)}}| \leqslant 2$, $(y \sim_\gamma y')^{p_j}$ as well.

Finally, we show (7). We make use of:

Claim 4.20.9. Let $\beta < \delta_*$; suppose that $\left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{[\beta,\beta+1]}\right)^o$ and $\daleth \in (G_{\beta+1})^{p_{i(\beta+1)}}$. Suppose that $\left(A^{\daleth}_{\beta+1}\right)^o$ is nonempty, $w \in \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta+1)}}$, $x \in \left(A^{\daleth}_{\beta}\right)^o$, and $\left(f^{\daleth}_{\beta,\beta+1}\right)^o(x) = \left(f^{\daleth}_{\beta,\beta+1}\right)^o(w)$. Then $w \leqslant x$ in $\left(A^{\daleth}_{\beta}\right)^o$.

Proof. This is where we use the assumption that $p_{i(\beta+1)} \leqslant_{\beta+1} p_{i(\beta)}$, rather than just $\leqslant_\beta$. The difficulty is that when $i(\beta+1) > i(\beta)$ (i.e. when $\beta = \alpha_{i(\beta)}$) there needn't be much of a connection between $\left(f^{\daleth}_{\beta,\beta+1}\right)^o$ and $\left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta)}}$, as we do not make $p_{i(\beta)} \leqslant_{\beta+1} o$.

Since $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{\beta}\right)^o$, we have $\left(r^{\daleth}_{\beta}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{\beta}\right)^{p_{i(\beta)}}$. Since $\daleth \in (G_{\beta+1})^{p_{i(\beta+1)}}$ and $p_{i(\beta+1)} \leqslant_{\beta+1} p_{i(\beta)}$, we have $\left(r^{\daleth}_{\beta+1}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{\beta+1}\right)^{p_{i(\beta)}}$. That is, together, $\left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_{i(\beta)}}$.

Let $z = \left(f^{\daleth}_{\beta,\beta+1}\right)^o(x) = \left(f^{\daleth}_{\beta,\beta+1}\right)^o(w)$. We show that $\left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta)}}(x) \geqslant z$ (in $\left(A^{\daleth}_{\beta+1}\right)^{p_{i(\beta)}}$). We consider the definition of the map $\left(f^{\daleth}_{\beta,\beta+1}\right)^o$. Since $\left(A^{\daleth}_{\beta}\right)^{p_{i(\beta+1)}}$ is nonempty, we have $\zeta^{\daleth}_{\beta,\beta+1} = \beta$. Then $y_z = x_z \leqslant x$. Since $\left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta)}}$ is order-preserving and extends $\left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta+1)}}$, we must indeed have $\left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta)}} \geqslant z$.

By Claim 4.20.8(a), $z = \left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta+1)}}(w)$. Since $p_{i(\beta+1)} \leqslant_{\beta+1} p_{i(\beta)}$, by (7) applied between $p_{i(\beta+1)}$ and $p_{i(\beta)}$, we have $w \leqslant x$ as required. ☐

Let $j \leqslant k$ and $\beta < \alpha_j$; suppose that $\left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_j} = \left(r^{\daleth}_{[\beta,\beta+1]}\right)^o$ and that $\daleth \in (G_{\beta+1})^{p_j}$. Then $\left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_j} = \left(r^{\daleth}_{[\beta,\beta+1]}\right)^{p_{i(\beta+1)}} = \left(r^{\daleth}_{[\beta,\beta+1]}\right)^o$ and $\daleth \in (G_{\beta+1})^{p_{i(\beta+1)}}$. Let $z \in \left(A^{\daleth}_{\beta+1}\right)^{p_j}$. Since $p_j \leqslant_{\beta+1} p_{i(\beta+1)}$, $\left\{x \in \left(A^{\daleth}_{\beta}\right)^{p_j} : \left(f^{\daleth}_{\beta,\beta+1}\right)^{p_j}(x) = z\right\}$ is an initial segment of $\left\{x \in \left(A^{\daleth}_{\beta}\right)^{p_{i(\beta+1)}} : \left(f^{\daleth}_{\beta,\beta+1}\right)^{p_{i(\beta+1)}}(x) = z\right\}$. By Claim 4.20.9, that set is an initial segment of $\left\{x \in \left(A^{\daleth}_{\beta}\right)^o : \left(f^{\daleth}_{\beta,\beta+1}\right)^o(x) = z\right\}$, giving (7), and completing the proof of Lemma 4.20.

4.7. **Proof of Lemma 4.21.** We will split the construction giving Lemma 4.21 into two parts. We are given an object o which is admissible for stage t. In the first part we construct an object $p \geqslant_{\delta_*} o$ which satisfies all the conditions of Definition 4.12 except possibly for item 10. We then extend p to a t-object $q \geqslant_{\delta_*} p$.

Construction of an object $p \geqslant_{\delta_*} o$ satisfying all but item 10 of Definition 4.12. We start, of course, by declaring that for all α and $\daleth$, $\left(r^{\daleth}_{\alpha}\right)^p = 1$ if and only if $\daleth \in V_t(\alpha)$ and $\daleth \in (G_\beta)^p$ if and only if $(\beta, \daleth)$ requires attention at stage t. Since o is admissible for stage t, this means that $(G_\beta)^p = (G_\beta)^o$ for all β and $\left(r^{\daleth}_{\beta}\right)^p = \left(r^{\daleth}_{\beta}\right)^o$ for all β and $\daleth$.

We construct p “from the top down”. When we consider a level $\alpha \leqslant \delta^*$, we assume that we have already defined the linear orderings $\left(A^{\daleth}_{\zeta}\right)^p$ and $\left(B^{\daleth}_{\zeta}\right)^p$, and functions $\left(f^{\daleth}_{\zeta,\xi}\right)^p$ and $\left(g^{\daleth}_{\zeta,\xi}\right)^p$ for all $\alpha < \zeta \leqslant \xi \leqslant \delta_*$. We then define the α^{th} level of p, namely the orderings $\left(A^{\daleth}_{\alpha}\right)^p$ and $\left(B^{\daleth}_{\alpha}\right)^p$, and functions $\left(f^{\daleth}_{\alpha,\zeta}\right)^p$ and $\left(g^{\daleth}_{\alpha,\zeta}\right)^p$ for all $\zeta \geqslant \alpha$.

The reason that this reverse recursion on δ_* makes sense is that p will have only finitely many nonempty levels. A level α of p is empty if all linear orderings $\left(A^{\daleth}_{\alpha}\right)^p$ and $\left(B^{\daleth}_{\alpha}\right)^p$ are empty.

We commit that:

- A level α of p will be nonempty only if the α^{th} level of o is nonempty, or for some $\daleth \in \omega^3$, $(\alpha, \daleth)$ requires attention at stage t.

Since o is an object, only finitely many levels of o are nonempty, and since only finitely many pairs require attention at each stage (Lemma 4.3(a)), there are only finitely many such levels α.

The construction and some of its verification should be viewed as a grand induction. We state four claims now. At step α of the construction we assume that these claims hold at every level $\zeta > \alpha$.

Claim 4.21.1. $\left(A^{\daleth}_{\zeta}\right)^p$ and $\left(B^{\daleth}_{\zeta}\right)^p$ are finite linear orderings.

(a) $\left(A^{\daleth}_{\zeta}\right)^p$ is nonempty if and only if $\left(B^{\daleth}_{\zeta}\right)^p$ is nonempty.
(b) If nonempty, then the universe of $\left(A^{\daleth}_{\zeta}\right)^p$ is an initial segment of ω, and 0 is its leftmost point; similarly for B.

For the next two claims and further below, let

$$N = \max\{|\left(A^{\daleth}_{\varepsilon}\right)^o|, |\left(B^{\daleth}_{\varepsilon}\right)^o| \,:\, \varepsilon \leqslant \delta_* \ \&\ \daleth \in \omega^3\}.$$

Claim 4.21.2.

(a) If $|\left(A^{\daleth}_{\zeta}\right)^p| \geqslant 3$ or $|\left(B^{\daleth}_{\zeta}\right)^p| \geqslant 3$ then $|\left(A^{\daleth}_{\zeta}\right)^p|, |\left(B^{\daleth}_{\zeta}\right)^p| \geqslant N + t$.
(b) If $1 \leqslant |\left(A^{\daleth}_{\zeta}\right)^p| \leqslant 2$ or $1 \leqslant |\left(B^{\daleth}_{\zeta}\right)^p| \leqslant 2$ then $(\zeta, \daleth)$ requires attention at stage t and $\mathtt{instr}_t(\beta, \daleth) = \{1, 2\}$;
(c) If $(\zeta, \daleth)$ requires attention at stage t and $\mathtt{instr}_t(\zeta, \daleth)$ is finite, then $|\left(A^{\daleth}_{\zeta}\right)^p| = \mathtt{instr}^A_t(\zeta, \daleth)$ and $|\left(B^{\daleth}_{\zeta}\right)^p| = \mathtt{instr}^B_t(\zeta, \daleth)$;
(d) If $(\zeta, \daleth)$ does not require attention at stage t and $\daleth \in V_t(\zeta)$ then $\left(A^{\daleth}_{\zeta}\right)^p$ and $\left(B^{\daleth}_{\zeta}\right)^p$ are empty.

Henceforth we ignore the B-side of the construction, as it is identical to its A-side.

Claim 4.21.3. Let $\xi \geqslant \zeta$.

(a) If either $\left(A^{\daleth}_{\zeta}\right)^p$ or $\left(A^{\daleth}_{\xi}\right)^p$ are empty, then $\left(f^{\daleth}_{\zeta,\xi}\right)^p$ is empty.
(b) $\left(f^{\daleth}_{\zeta,\zeta}\right)^p$ is the identity on $\left(A^{\daleth}_{\zeta}\right)^p$.

Suppose that $\left(A^{\daleth}_{\zeta}\right)^p$ and $\left(A^{\daleth}_{\xi}\right)^p$ are nonempty and that $\xi > \zeta$.

(c) $\left(f^{\daleth}_{\zeta,\xi}\right)^p$ is an order-preserving function from $\left(A^{\daleth}_{\zeta}\right)^p$ onto $\left(A^{\daleth}_{\xi}\right)^p$.
(d) For all $\varepsilon \geqslant \xi$ for which $\left(A^{\daleth}_{\varepsilon}\right)^p$ is nonempty, $\left(f^{\daleth}_{\zeta,\varepsilon}\right)^p = \left(f^{\daleth}_{\xi,\varepsilon}\right)^p \circ \left(f^{\daleth}_{\zeta,\xi}\right)^p$.
(e) Every $z \in \left(A^{\daleth}_{\xi}\right)^p$ has at least N many $\left(f^{\daleth}_{\zeta,\xi}\right)^p$-pre-images in $\left(A^{\daleth}_{\zeta}\right)^p$.

Note that (c) implies that $\left(f^{\daleth}_{\zeta,\xi}\right)^p(0) = 0$.

Claim 4.21.4.
(a) $\left(A_\zeta^\daleth\right)^o \subseteq \left(A_\zeta^\daleth\right)^p$.
(b) For all $\xi > \zeta$, $\left(f_{\zeta,\xi}^\daleth\right)^o \subseteq \left(f_{\zeta,\xi}^\daleth\right)^p$.

Before we state the construction at level α, we draw one conclusion about level α:

Claim 4.21.5. If $(\alpha, \daleth)$ requires attention at stage t and $\daleth \in V_t(\alpha)$, then for all $\zeta > \alpha$, $\left(A_\zeta^\daleth\right)^p$ is empty.

Proof. We have $\daleth \in V_t(\zeta)$. By Claim 4.21.2(d) at level ζ, if $(\zeta, \daleth)$ does not require attention at stage t then $\left(A_\zeta^\daleth\right)^p$ is empty.

If $(\zeta, \daleth)$ requires attention at stage t then by Lemma 4.3(g), $\texttt{instr}_t(\zeta, \daleth) = 0$. The claim then follows from Claim 4.21.2(c) at level ζ. □

This also holds of course at all levels $\zeta > \alpha$; together with Claim 4.21.2(b), we get, for all $\zeta > \alpha$:

Claim 4.21.6. If $1 \leqslant |\left(A_\zeta^\daleth\right)^p| \leqslant 2$ then for all $\xi > \zeta$, $\left(A_\xi^\daleth\right)^p$ is empty.

So suppose that $\alpha \leqslant \delta_*$ and that the levels $\zeta > \alpha$ of p were already defined, and that the claims above hold at all levels $\zeta > \alpha$. Fix some $\daleth \in \omega^3$. We define two ordinals.

- If there is one, we let $\beta = \beta_\alpha^\daleth$ be the least $\beta > \alpha$ such that $\left(A_\beta^\daleth\right)^p$ is nonempty.
- If there is one, we let $\gamma = \gamma_\alpha^\daleth$ be the least $\gamma > \alpha$ such that $\left(A_\gamma^\daleth\right)^o$ is nonempty.

If γ is defined then by Claim 4.21.4(a) at level γ, $\left(A_\gamma^\daleth\right)^o \subseteq \left(A_\gamma^\daleth\right)^p$, and so β is defined and $\gamma \geqslant \beta$.

Claim 4.21.7. Suppose that $\beta = \beta_\alpha^\daleth$ is defined, and that $\left(A_\alpha^\daleth\right)^o$ is nonempty. There is an order-preserving map $h_\alpha^\daleth \colon \left(A_\alpha^\daleth\right)^o \to \left(A_\beta^\daleth\right)^p$ such that:

(i) $h_\alpha^\daleth(0) = 0$.[21]
(ii) If $\gamma = \gamma_\alpha^\daleth$ is defined, then $\left(f_{\alpha,\gamma}^\daleth\right)^o = \left(f_{\beta,\gamma}^\daleth\right)^p \circ h_\alpha^\daleth$;
(iii) For all ζ such that $|\left(A_\zeta^\daleth\right)^p| \geqslant 3$ and:
  - $\beta \leqslant \zeta$, if γ is undefined;
  - $\beta \leqslant \zeta < \gamma$, if γ is defined,

  $\left(f_{\beta,\zeta}^\daleth\right)^p \circ h_\alpha^\daleth$ is injective, and for all $x \in \left(A_\alpha^\daleth\right)^o$, $h_\alpha^\daleth(y)$ is the leftmost element of $\left(A_\beta^\daleth\right)^p$ which is mapped by $\left(f_{\beta,\zeta}^\daleth\right)^p$ to $\left(f_{\beta,\zeta}^\daleth\right)^p(h_\alpha^\daleth(x))$.
(iv) If γ is defined, then for all $z \in \operatorname{range}\left(f_{\alpha,\gamma}^\daleth\right)^o$, the leftmost point of
$$\left\{x \in \left(A_\alpha^\daleth\right)^o \;:\; \left(f_{\alpha,\gamma}^\daleth\right)^o(x) = z\right\}$$
is mapped by $h_\alpha^\daleth$ to the leftmost point of
$$\left\{y \in \left(A_\beta^\daleth\right)^p \;:\; \left(f_{\beta,\gamma}^\daleth\right)^p(y) = z\right\}.$$

Proof. If γ is not defined, if there is such, let ε be the greatest ordinal $\varepsilon \geqslant \beta$ such that $|\left(A_\varepsilon^\daleth\right)^p| \geqslant 3$. If γ is defined, if there is such, let ε be the greatest ordinal in the interval $[\beta, \gamma)$ such that $|\left(A_\varepsilon^\daleth\right)^p| \geqslant 3$.

[21] Note that since o is an object, 0 is the leftmost point of $\left(A_\alpha^\daleth\right)^o$; and by Claim 4.21.1, 0 is the leftmost point of $\left(A_\beta^\daleth\right)^p$

- If neither ε nor γ are defined, let $h^\urcorner_\alpha$ map every $x \in \left(A^\urcorner_\alpha\right)^o$ to 0.
- Suppose that γ is defined but ε is not. By Claim 4.21.6 we have $\gamma = \beta$, so we let $h^\urcorner_\alpha = \left(f^\urcorner_{\alpha,\beta}\right)^o$. (i) holds because o is an object. (iv) is immediate.

Suppose that ε is defined. We define an embedding $g\colon \left(A^\urcorner_\alpha\right)^o \to \left(A^\urcorner_\varepsilon\right)^p$ as follows.

- If γ is not defined then let g be the isomorphism from $\left(A^\urcorner_\alpha\right)^o$ to an initial segment of $\left(A^\urcorner_\varepsilon\right)^p$. This is possible because by Claim 4.21.2(a), $|\left(A^\urcorner_\varepsilon\right)^p| \geqslant N \geqslant |\left(A^\urcorner_\alpha\right)^o|$. Note that $g(0) = 0$.
- If γ is defined (and so $\varepsilon < \gamma$) we let $g\colon \left(A^\urcorner_\alpha\right)^o \to \left(A^\urcorner_\varepsilon\right)^p$ be an embedding such that $\left(f^\urcorner_{\alpha,\gamma}\right)^o = \left(f^\urcorner_{\varepsilon,\gamma}\right)^p \circ g$, and for all $z \in \operatorname{range}\left(f^\urcorner_{\alpha,\gamma}\right)^o$, if $x \in \left(A^\urcorner_\alpha\right)^o$ is leftmost such that $\left(f^\urcorner_{\alpha,\gamma}\right)^o(x) = z$, then $g(x)$ is the leftmost w in $\left(A^\urcorner_\varepsilon\right)^p$ such that $\left(f^\urcorner_{\varepsilon,\gamma}\right)^p(w) = z$. Note that such w exists because $\left(f^\urcorner_{\varepsilon,\gamma}\right)^p$ is onto $\left(A^\urcorner_\gamma\right)^p$; $g(0) = 0$ because $\left(f^\urcorner_{\varepsilon,\gamma}\right)^p(0) = 0 = \left(f^\urcorner_{\alpha,\gamma}\right)^o(0)$; and g can be made 1-1 because for each $z \in \left(A^\urcorner_\gamma\right)^o$, the $\left(f^\urcorner_{\varepsilon,\gamma}\right)^p$-preimage of z has size greater than $|\left(A^\urcorner_\alpha\right)^o|$ (Claim 4.21.3(e)).

In both cases, having defined g, we let, for $x \in \left(A^\urcorner_\alpha\right)^o$, $h^\urcorner_\alpha(x)$ be the leftmost y in $\left(A^\urcorner_\beta\right)^p$ such that $\left(f^\urcorner_{\beta,\varepsilon}\right)^p(y) = g(x)$ (so that $g = \left(f^\urcorner_{\beta,\varepsilon}\right)^p \circ h^\urcorner_\alpha$). Since 0 is the leftmost point in $\left(A^\urcorner_\beta\right)^p$ mapped by $\left(f^\urcorner_{\beta,\varepsilon}\right)^p$ to 0, and $g(0) = 0$, we get $h^\urcorner_\alpha(0) = 0$. (iii) holds by design, as $\zeta \leqslant \varepsilon$ for each such ζ: for any distinct $x, y \in \left(A^\urcorner_\alpha\right)^o$, as $g(x) \neq g(y)$ we must have $\left(f^\urcorner_{\beta,\zeta}\right)^p(h^\urcorner_\alpha(x)) \neq \left(f^\urcorner_{\beta,\zeta}\right)^p(h^\urcorner_\alpha(y))$. And since $h^\urcorner_\alpha(x)$ is leftmost in $\left(A^\urcorner_\beta\right)^p$ mapped by $\left(f^\urcorner_{\beta,\varepsilon}\right)^p$ to $g(x)$, and $\left(f^\urcorner_{\beta,\zeta}\right)^p$ is onto, it must be that $h^\urcorner_\alpha(x)$ is also leftmost in $\left(A^\urcorner_\beta\right)^p$ mapped by $\left(f^\urcorner_{\beta,\zeta}\right)^p$ to $\left(f^\urcorner_{\beta,\zeta}\right)^p(h^\urcorner_\alpha(x))$.

Suppose that γ is defined. Then $\left(f^\urcorner_{\beta,\gamma}\right)^p \circ h^\urcorner_\alpha = \left(f^\urcorner_{\varepsilon,\gamma}\right)^p \circ g$ which was designed to equal $\left(f^\urcorner_{\alpha,\gamma}\right)^o$, giving (ii). (iv) also holds by construction. □

Now, again concentrating on the A-side, we will build $\left(A^\urcorner_\alpha\right)^p$, and when β is defined, $\left(f^\urcorner_{\alpha,\beta}\right)^p$. Having done this, for all $\zeta > \alpha$ for which $\left(A^\urcorner_\zeta\right)^p$ is nonempty, we have $\zeta \geqslant \beta$ and so we define $\left(f^\urcorner_{\alpha,\zeta}\right)^p = \left(f^\urcorner_{\beta,\zeta}\right)^p \circ \left(f^\urcorner_{\alpha,\beta}\right)^p$. For other $\zeta > \alpha$ we leave $\left(f^\urcorner_{\alpha,\zeta}\right)^p$ empty. Of course we let $\left(f^\urcorner_{\alpha,\alpha}\right)^p$ be the identity on $\left(A^\urcorner_\alpha\right)^p$.

There are several cases.

(i): $(\alpha, \urcorner)$ does not require attention at stage t and $\left(A^\urcorner_\alpha\right)^o$ is empty: we let $\left(A^\urcorner_\alpha\right)^p$ be empty.

(ii): $\urcorner \notin V_t(\alpha)$ and either:

- $(\alpha, \urcorner)$ requires attention at stage t; or
- $(\alpha, \urcorner)$ does not require attention at stage t and $\left(A^\urcorner_\alpha\right)^o$ is nonempty.

In this case we define $\left(A^\urcorner_\alpha\right)^p$ to be a linear ordering extending $\left(A^\urcorner_\alpha\right)^o$.

(a) If β is undefined, then we let $\left(A^\urcorner_\alpha\right)^p$ be any end-extension of $\left(A^\urcorner_\alpha\right)^o$ of size at least $N + t + 3$ (with universe an initial segment of ω)[22]. If $\left(A^\urcorner_\alpha\right)^o$ is empty then we ensure that 0 is the leftmost element of $\left(A^\urcorner_\alpha\right)^p$. (If $\left(A^\urcorner_\alpha\right)^o$ is nonempty then 0 is its leftmost point as o is an object.)

[22] Recall that N is the bound on the size of all orderings in o.

(b) Suppose that β is defined. If $\left(A^{\daleth}_{\alpha}\right)^o$ is nonempty, let $h^{\daleth}_{\alpha}$ be the map given by Claim 4.21.7. Otherwise let $h^{\daleth}_{\alpha}$ be the empty map. We define $\left(A^{\daleth}_{\alpha}\right)^p \supseteq \left(A^{\daleth}_{\alpha}\right)^o$ (on an initial segment of ω) and $\left(f^{\daleth}_{\alpha,\beta}\right)^p$ extending $h^{\daleth}_{\alpha}$ as follows: for every $y \in \left(A^{\daleth}_{\beta}\right)^p$ we let

$$\left\{x \in \left(A^{\daleth}_{\alpha}\right)^p \,:\, (f^{\daleth}_{\alpha,\beta})^p(x) = y\right\}$$

have size at least $N+t+3$ and be an end-extension of

$$\left\{x \in \left(A^{\daleth}_{\alpha}\right)^o \,:\, h^{\daleth}_{\alpha}(x) = y\right\}.$$

See fig. 4. If $\left(A^{\daleth}_{\alpha}\right)^o$ is empty, then we also ensure that 0 is the leftmost point of $\left(A^{\daleth}_{\alpha}\right)^p$.

(iii): $(\alpha, \daleth)$ requires attention at stage t and $\daleth \in V_t(\alpha)$. In this case by Claim 4.21.5, β is not defined, so we only need to define $\left(A^{\daleth}_{\alpha}\right)^p$. We obey the instructions:

- If $\texttt{instr}_t(\alpha, \daleth)$ is finite: we determine $\left(A^{\daleth}_{\alpha}\right)^p$ by stipulating that $|\left(A^{\daleth}_{\alpha}\right)^p| = \texttt{instr}^A_t(\alpha, \daleth)$ and that if nonempty, 0 is the leftmost element of this linear ordering.
- If $\texttt{instr}_t(\alpha, \daleth) = \text{צ}$ then by Lemma 4.3(f), $\texttt{instr}_t(\alpha, \text{צ}) = \omega$ and so the pair $(\alpha, \text{צ})$ falls under case (ii), so $\left(A^{\text{צ}}_{\alpha}\right)^p$ has already been defined. We let $\left(A^{\daleth}_{\alpha}\right)^p = \left(A^{\text{צ}}_{\alpha}\right)^p$.

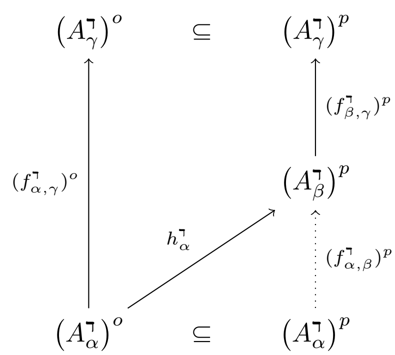

FIGURE 4. case (ii)(b)

These cases cover all possibilities, because if $(\alpha, \daleth)$ does not require attention and $\daleth \in V_t(\alpha)$ then $\left(A^{\daleth}_{\alpha}\right)^o$ is empty (Definition 4.19(4)).

Note that we abided by our promise to keep all but finitely many levels of p empty: if the α^{th} level of o is empty, then we only make the α^{th} level of p nonempty if $(\alpha, \daleth)$ requires attention for some $\daleth$. In fact, since o is an object and only finitely many pairs $(\alpha, \daleth)$ require attention at stage t, only finitely many linear orderings $\left(A^{\daleth}_{\beta}\right)^p$ are nonempty.

We turn to the verification. We give the proofs of the four claims stated above for $\zeta = \alpha$.

Proof of Claim 4.21.1. We need to check that 0 is the leftmost point of every nonempty $\left(A^{\daleth}_{\alpha}\right)^p$. Suppose that case (ii) holds for $(\alpha, \daleth)$. In sub-case (ii)(a), or when $\left(A^{\daleth}_{\alpha}\right)^{o}$ is empty, this is by construction, so suppose that β is defined and $\left(A^{\daleth}_{\alpha}\right)^{o}$ is nonempty. By Claim 4.21.7(i), $h^{\daleth}_{\alpha}(0) = 0$, i.e., $h^{\daleth}_{\alpha}$ maps the leftmost element of $\left(A^{\daleth}_{\alpha}\right)^{o}$ to the leftmost point of $\left(A^{\daleth}_{\beta}\right)^{p}$. Thus when constructing $\left(A^{\daleth}_{\alpha}\right)^{p}$ we are never required to add elements to the left of 0.

In case (iii), this is either by construction, or by the fact that case (ii) holds for $\tsadi$, which we have just covered. □

Proof of Claim 4.21.2. Mostly this follows by examining the construction. For example, (a) follows from construction in case (ii) and in case (iii) because case (ii) holds for $\tsadi$. For (d), as we observed, the assumptions imply that case (i) holds and we leave $\left(A^{\daleth}_{\alpha}\right)^{p}$ empty. □

Proof of Claim 4.21.3. (a) and (b) are by construction. For the rest, suppose that $\xi > \alpha$ and that $\left(A^{\daleth}_{\alpha}\right)^{p}$ and $\left(A^{\daleth}_{\xi}\right)^{p}$ are both nonempty. Then case (ii) holds for $(\alpha, \daleth)$ and $\beta = \beta^{\daleth}_{\alpha}$ is defined; $\beta \leqslant \xi$.

For (c), By applying this item at level β, it suffices to observe that by construction, $\left(f^{\daleth}_{\alpha,\beta}\right)^{p}$ is an order-preserving map from $\left(A^{\daleth}_{\alpha}\right)^{p}$ onto $\left(A^{\daleth}_{\beta}\right)^{p}$.

(d) is by construction and induction: we define $\left(f^{\daleth}_{\alpha,\xi}\right)^{p} = \left(f^{\daleth}_{\beta,\xi}\right)^{p} \circ \left(f^{\daleth}_{\alpha,\beta}\right)^{p}$ and $\left(f^{\daleth}_{\alpha,\varepsilon}\right)^{p} = \left(f^{\daleth}_{\beta,\varepsilon}\right)^{p} \circ \left(f^{\daleth}_{\alpha,\beta}\right)^{p}$; so we use $\left(f^{\daleth}_{\beta,\varepsilon}\right)^{p} = \left(f^{\daleth}_{\xi,\varepsilon}\right)^{p} \circ \left(f^{\daleth}_{\beta,\xi}\right)^{p}$.

(e) is by construction; by induction, it suffices to show for $\xi = \beta$. □

Proof of Claim 4.21.4. (a): we assume of course that $\left(A^{\daleth}_{\alpha}\right)^{o}$ is nonempty; so case (i) does not hold. In case (ii) for $(\alpha, \daleth)$, by construction we define $\left(A^{\daleth}_{\alpha}\right)^{p}$ to extend $\left(A^{\daleth}_{\alpha}\right)^{o}$. Suppose that case (iii) holds.

Suppose that $\mathtt{instr}_t(\alpha, \daleth)$ is finite. Since o is admissible for stage t, $|\left(A^{\daleth}_{\alpha}\right)^{o}| = \mathtt{instr}^{A}_{t}(\alpha, \daleth)$ (Definition 4.19(5)), and so we set $\left(A^{\daleth}_{\alpha}\right)^{o} = \left(A^{\daleth}_{\alpha}\right)^{p}$.

Suppose that $\mathtt{instr}_t(\alpha, \daleth) = \tsadi$. Since o is admissible for stage t, $\left(A^{\daleth}_{\alpha}\right)^{o} = \left(A^{\tsadi}_{\alpha}\right)^{o}$ (Definition 4.19(6)) and by case (ii), $\left(A^{\tsadi}_{\alpha}\right)^{o} \subseteq \left(A^{\tsadi}_{\alpha}\right)^{p}$; since we set $\left(A^{\daleth}_{\alpha}\right)^{p} = \left(A^{\tsadi}_{\alpha}\right)^{p}$, we get the desired extension $\left(A^{\daleth}_{\alpha}\right)^{o} \subseteq \left(A^{\daleth}_{\alpha}\right)^{p}$.

(b): By induction, and since we are taking compositions, it suffices to show that if $\gamma = \gamma^{\daleth}_{\alpha}$ is defined then $\left(f^{\daleth}_{\alpha,\gamma}\right)^{o} \subseteq \left(f^{\daleth}_{\alpha,\gamma}\right)^{p}$. Case (ii) holds. By construction, $\left(f^{\daleth}_{\alpha,\beta}\right)^{p}$ extends $h^{\daleth}_{\alpha}$. Also, $\left(f^{\daleth}_{\alpha,\gamma}\right)^{p} = \left(f^{\daleth}_{\beta,\gamma}\right)^{p} \circ \left(f^{\daleth}_{\alpha,\beta}\right)^{p}$. Then Claim 4.21.7(ii) gives the desired extension. □

This concludes the proofs of the four claims made before the construction. We now complete the verifications that p is an object, that all conditions of Definition 4.12 except possibly for item 10 holds for p, and that $o \leqslant_{\delta_*} p$.

We have observed that only finitely many linear orderings $\left(A^{\daleth}_{\alpha}\right)^{p}$ are nonempty; the finiteness requirements (1) and (2) of Definition 4.6 are the same as for o. All other items have already been checked, and so p is an object. In fact it is not difficult to see that all items of Definition 4.12 except for (10) have been checked, or follow immediately from our instructions.

We check that $o \leqslant_{\delta_*} p$. (4) of Definition 4.10 follows from Definition 4.19(3) and the construction (case (iii)). For other items, we use the following. For brevity, we fix some $\daleth \in \omega^3$, and for $\alpha < \zeta$ we let:

- $f_{\alpha,\zeta} = \left(f^{\daleth}_{\alpha,\zeta}\right)^{p}$;

- For all $z \in \left(A^{\urcorner}_{\zeta}\right)^p$,

$$C_{\alpha,\zeta}(z) = \{x \in \left(A^{\urcorner}_{\alpha}\right)^o \,:\, f_{\alpha,\zeta}(x) = z\};$$
$$D_{\alpha,\zeta}(z) = \{x \in \left(A^{\urcorner}_{\alpha}\right)^p \,:\, f_{\alpha,\zeta}(x) = z\}.$$

The following claim will be used below, but also verifies Definition 4.10(7).

Claim 4.21.8. Let $\alpha < \zeta$. Suppose that $\left(A^{\urcorner}_{\alpha}\right)^o$ and $\left(A^{\urcorner}_{\zeta}\right)^p$ are nonempty, and further suppose that for all $\xi \in (\alpha, \zeta)$, $\left(A^{\urcorner}_{\xi}\right)^o$ is empty.

(a) If $|\left(A^{\urcorner}_{\zeta}\right)^p| \geqslant 3$ and $\left(A^{\urcorner}_{\zeta}\right)^o$ is empty, then $f_{\alpha,\zeta} \restriction \left(A^{\urcorner}_{\alpha}\right)^o$ is injective.

Let $z \in \operatorname{range} f_{\alpha,\zeta} \restriction \left(A^{\urcorner}_{\alpha}\right)^o$.

(b) If either $z \in \left(A^{\urcorner}_{\zeta}\right)^o$ or $|\left(A^{\urcorner}_{\zeta}\right)^p| \geqslant 3$ then $\min C_{\alpha,\zeta}(z) = \min D_{\alpha,\zeta}(z)$.
(c) If $\zeta = \alpha+1$ and $z \in \left(A^{\urcorner}_{\zeta}\right)^o$ then $C_{\alpha,\alpha+1}(z)$ is an initial segment of $D_{\alpha,\alpha+1}(z)$.

Proof. We know that $\beta = \beta^{\urcorner}_{\alpha}$ is defined, and $\zeta \geqslant \beta$.

(a): $\left(A^{\urcorner}_{\zeta}\right)^o$ being empty implies that either $\gamma = \gamma^{\urcorner}_{\alpha}$ is undefined, or $\gamma > \zeta$. In either case, by Claim 4.21.7(iii), $\left(f^{\urcorner}_{\beta,\zeta}\right)^p \circ h^{\urcorner}_{\alpha}$ is injective; by construction, this map is precisely $f_{\alpha,\zeta} \restriction \left(A^{\urcorner}_{\alpha}\right)^o$.

For the other items, let $w = f_{\alpha,\beta}(y) = h^{\urcorner}_{\alpha}(y)$. We show that $w = \min D_{\beta,\zeta}(z)$. $\left(A^{\urcorner}_{\zeta}\right)^o$ is nonempty. Then $\zeta = \gamma$, and since $f_{\alpha,\gamma}$ extends $\left(f^{\urcorner}_{\alpha,\gamma}\right)^o$, we have $z \in \left(A^{\urcorner}_{\gamma}\right)^o$. In this case Claim 4.21.7(iv) says that $w = \min D_{\beta,\zeta}(z)$.

If, on the other hand, $\left(A^{\urcorner}_{\zeta}\right)^o$ is empty, so $|\left(A^{\urcorner}_{\zeta}\right)^p| \geqslant 3$, then Claim 4.21.7(iii) says that $w = \min D_{\beta,\zeta}(z)$.

In either case, by construction, $C_{\alpha,\beta}(w)$ is an initial segment of $D_{\alpha,\beta}(w)$. In particular, $y = \min D_{\alpha,\beta}(w)$; it follows that $y = \min D_{\alpha,\zeta}(z)$ as well. And if $\zeta = \alpha+1$ and $\left(A^{\urcorner}_{\alpha+1}\right)^o$ is nonempty then $\gamma = \beta = \alpha+1$, and $h^{\urcorner}_{\alpha} = \left(f^{\urcorner}_{\alpha,\alpha+1}\right)^o$, so $w = z$, so as observed, by construction, $C_{\alpha,\alpha+1}(z)$ is an initial segment of $D_{\alpha,\alpha+1}(z)$. □

The following claim verifies (6) and (9).

Claim 4.21.9. Suppose that $\alpha < \zeta$, that $\left(A^{\urcorner}_{\alpha}\right)^o$ and $\left(A^{\urcorner}_{\zeta}\right)^p$ are nonempty, and let $z \in \operatorname{range} f_{\alpha,\zeta} \restriction \left(A^{\urcorner}_{\alpha}\right)^o$. If either $z \in \left(A^{\urcorner}_{\zeta}\right)^o$ or $|\left(A^{\urcorner}_{\zeta}\right)^p| \geqslant 3$ then $\min C_{\alpha,\zeta}(z) = \min D_{\alpha,\zeta}(z)$.

Proof. We prove the claim by reverse induction on α. Let $y = \min C_{\alpha,\zeta}(z)$. If for all $\xi \in (\alpha, \zeta)$, $\left(A^{\urcorner}_{\xi}\right)^o$ is empty, then we appeal to Claim 4.21.8(b). Otherwise, let $\xi \in (\alpha, \zeta)$ be least with $\left(A^{\urcorner}_{\xi}\right)^o$ nonempty. Let $w = f_{\alpha,\xi}(y)$ which note is the same as $\left(f^{\urcorner}_{\alpha,\xi}\right)^o(y)$; $w \in \left(A^{\urcorner}_{\xi}\right)^o$. Since $\left(f^{\urcorner}_{\alpha,\xi}\right)^o$ is onto an initial segment of $\left(A^{\urcorner}_{\xi}\right)^o$, $w = \min C_{\xi,\zeta}(z)$. By induction, $w = \min D_{\xi,\zeta}(z)$. By Claim 4.21.8(b), $y = \min C_{\alpha,\xi}(w) = \min D_{\alpha,\xi}(w)$. It follows that $y = \min D_{\alpha,\zeta}(z)$ as well. □

The following claim verifies Definition 4.10(8), and therefore completes the verification:

Claim 4.21.10. Suppose that $\alpha < \zeta$, that $x, y \in \left(A^{\urcorner}_{\alpha}\right)^o$, that $(x \sim_{\zeta} y)^p$, and that for no $\xi \in [\alpha, \zeta]$ do we have $1 \leqslant |\left(A^{\urcorner}_{\xi}\right)^p| \leqslant 2$. Then $(x \sim y)^o$.

Proof. We prove the claim by reverse induction on α. First note that by shrinking ζ, we may assume that $\left(A^{\daleth}_{\zeta}\right)^{p}$ is nonempty, that is, that $f_{\alpha,\zeta}(x) = f_{\alpha,\zeta}(y)$. We may also assume that x and y are distinct, so $\zeta > \alpha$.

By Claim 4.21.8(a), there is some $\xi \in (\alpha, \zeta]$, such that $\left(A^{\daleth}_{\xi}\right)^{o}$ is nonempty. Let $x' = \left(f^{\daleth}_{\alpha,\xi}\right)^{o}(x)$ and $y' = \left(f^{\daleth}_{\alpha,\xi}\right)^{o}(y)$. Then $(x' \sim_{\zeta} y')^{p}$. By induction, $(x' \sim_{\zeta} y')^{p}$, and so $(x \sim_{\zeta} y)^{p}$ as well. □

Construction of a t-object q such that $q \geqslant_{\delta_*} p$. To meet item 10 of Definition 4.12, we add linear orderings at "large" successor levels below limit levels. We define an object q as follows. Of course $(G_{\alpha})^{p} = (G_{\alpha})^{q}$ and $\left(r^{\daleth}_{\alpha}\right)^{p} = \left(r^{\daleth}_{\alpha}\right)^{q}$ as is required of t-objects. If level α of p is nonempty, then the α^{th} level of q is identical to the α^{th} level of p. Suppose that $\lambda \leqslant \delta_*$ is a limit ordinal, that $(\lambda, \daleth)$ requires attention at stage t, and that $\daleth \notin V_t(\lambda)$. By Definition 4.12(7), $|\left(A^{\daleth}_{\lambda}\right)^{p}| \geqslant 3$. Note that for all $\beta < \lambda$, $\daleth \notin V_t(\beta)$.

Find some successor $\beta < \lambda$ such that all levels ζ for $\beta - 1 \leqslant \zeta < \lambda$ of p are empty. Since $\daleth \notin V_t(\beta)$, $(\daleth, \beta)$ does not require attention at stage t. We let $\left(A^{\daleth}_{\beta}\right)^{q} = \left(A^{\daleth}_{\lambda}\right)^{p}$; we let $\left(f^{\daleth}_{\beta,\lambda}\right)^{q} = \left(f^{\daleth}_{\beta,\beta}\right)^{q}$ be the identity on $\left(A^{\daleth}_{\lambda}\right)^{p}$. For $\alpha < \beta$ we let $\left(f^{\daleth}_{\alpha,\beta}\right)^{q} = \left(f^{\daleth}_{\alpha,\lambda}\right)^{p}$; For $\alpha > \lambda$ we let $\left(f^{\daleth}_{\beta,\alpha}\right)^{q} = \left(f^{\daleth}_{\lambda,\alpha}\right)^{p}$. We define B and g's similarly. There are only finitely many pairs $(\lambda, \daleth)$ for which this action is required, so q is finite.

An examination shows that q is a t-object and $q \geqslant_{\delta_*} p$. The items to check are (4) and (5) of Definition 4.12. For the former, we use the fact that $\daleth \notin V_t(\beta)$. The latter follows from $|\left(A^{\daleth}_{\lambda}\right)^{p}| \geqslant 3$. The rest of the items of Definition 4.12 do not apply as $(\beta, \daleth)$ does not require attention at stage t.

This concludes the proof of Lemma 4.21, and so of Proposition 4.15, and so of Proposition 4.14, and so of Proposition 4.1, and so of Theorem 1.1.

We observe that our techniques can also be used to show Fokina et al.'s result that isomorphism of computable structures is computably complete for Σ^1_1 equivalence relations. A Σ^1_1 equivalence relation is an ω_1^{CK}-intersection of ever finer hyperarithmetic equivalence relations: $E = \bigcap_{\alpha < \omega_1^{CK}} E(\alpha)$. Construct linear orders $A^{k,m,i}$ for $k, m, i \in \mathbb{N}$ so that if $mE(\alpha)i$ but $\neg mE(\alpha+1)i$, then $A^{k,m,i}$ will have order-type $\omega^{\alpha+1}$ for each k with $mE(\alpha+1)k$, and $A^{k,m,i}$ will have order-type $\omega^{\alpha+1} \cdot 2$ for each k with $\neg mE(\alpha+1)k$. Then the structure $\mathcal{M}_k$ will consist of all $A^{k,m,i}$, for $m, i \in \omega$, each labeled with (m, i) in some fashion.

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