TWO MORE CHARACTERIZATIONS OF K-TRIVIALITY

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Abstract. We give two new characterizations of K-triviality. We show that if for all Y such that Ω is Y-random, Ω is (Y ⊕ A)-random, then A is K-trivial. The other direction was proved by Stephan and Yu, giving us the first titular characterization of K-triviality and answering a question of Yu. We also prove that if A is K-trivial, then for all Y such that Ω is Y-random, (Y ⊕ A) ≡ LR Y. This answers a question of Merkle and Yu. The other direction is immediate, so we have the second characterization of K-triviality.

The proof of the first characterization uses a new cupping result. We prove that if A ≰ LR B, then for every set X there is a B-random set Y such that X is computable from Y ⊕ A.

1. Preliminaries

We assume that the reader is familiar with basic notions from computability theory and effective randomness. For more information on these topics, we recommend either Nies [12] or Downey and Hirschfeldt [4].

The K-trivial sets have played an important role in the development of effective randomness. A set A ∈ 2ω is K-trivial if K(A↾n) ≤+ K(n), where K denotes prefix-free Kolmogorov complexity. Chaitin [1] proved that such sets are always Δ02, while Solovay [10] constructed a noncomputable K-trivial set. Although these results date back to the 1970s, the importance of K-triviality did not become apparent until the 2000s, when several nontrivial characterizations were discovered. In particular:

Theorem 1.1 (Nies [11]; Hirschfeldt, Nies, and Stephan [6]). The following are equivalent for a set A ∈ 2ω:

(a) A is K-trivial,
(b) A is low for K: K^A(n) ≥+ K(n),
(c) A is low for randomness: every random set is A-random,
(d) A is a base for randomness: there is an A-random set X ≥T A.

Nies [11] generalized (c) to LR-reducibility: we write A ≤LR B to mean that every B-random set is A-random. In particular, A ≤LR ∅ means that A is low for randomness (hence K-trivial).

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1Throughout this paper, we consistently use random to mean Martin-Löf random.
Much more has been proved about the $K$-trivial sets, including many other characterizations. We give two more. Our results relate to a weakening of lowness for randomness. If $X$ is random, then we say that $Y$ is low for $X$ if $X$ is $Y$-random. This notion was introduced in [6], where it is shown that a set is $K$-trivial if and only if it is $\Delta^0_2$ and low for Chaitin’s $\Omega$. However, many other sets are low for $\Omega$; for example, every 2-random set.

The following recent result regarding $K$-triviality and lowness for $\Omega$ was used by Stephan and Yu to prove one direction of our first characterization (see the discussion before Proposition 3.2). We will need it in the proof of Lemma 3.4.

**Theorem 1.2** (Simpson and Stephan [15, Theorem 3.11]). If $S$ has PA degree and is low for $\Omega$, then $S$ computes every $K$-trivial.

In addition to these facts about the $K$-trivial sets, we will use several fairly well-known theorems from effective randomness. Van Lambalgen’s theorem [17] says that $X \oplus Y$ is random if and only if $X$ is random and $Y$ is $X$-random. Two applications allow us to show that if $X$ is random and $Y$ is $X$-random, then $X$ is $Y$-random. Every set is computable from some random set. Relativizing this to $X$:

**Theorem 1.3** (Kučera [9]; Gács [5]). For any sets $X$ and $C$, there is an $X$-random set $Y$ such that $C \leq_T Y \oplus X$.

Any random set Turing below a $Z$-random set is also $Z$-random. Relativizing this to $Y$:

**Theorem 1.4** (Miller and Yu [10, Theorem 4.3]). Assume that $X \leq_T W \oplus Y$, $X$ is $Y$-random, and $W$ is $Z \oplus Y$-random. Then $X$ is $Z \oplus Y$-random.

Finally, we will use the relativized form of the “randomness preservation” basis theorem:

**Theorem 1.5** (Downey, Hirschfeldt, Miller, Nies [3]; Reimann and Slaman [14]). If $W$ is $Y$-random and $P$ is a nonempty $\Pi^0_1[Y]$ class, then there is a set $S \in P$ that is low for $W$.

2. Cupping with $B$-random sets

As promised in the abstract, we prove the following cupping result.

**Theorem 2.1.** Assume that $A \not\leq_{LR} B$. Then for any set $X$, there is a $B$-random set $Y$ such that $X \leq_T Y \oplus A$ (in fact, we make $Y$ weakly 2-random relative to $B$).

This theorem should be compared to the work of Day and Miller [2]. They proved that a set $A$ is not $K$-trivial if and only if there is a random set $Y \not\leq_T \emptyset'$ such that $\emptyset' \leq_T Y \oplus A$. Note that one direction of this follows from Theorem 2.1 by taking $B = \emptyset$ and $X = \emptyset'$. This is because $A$ is not $K$-trivial if and only if $A \not\leq_{LR} \emptyset$, and if $Y$ is weakly 2-random, then $Y \not\leq_T \emptyset'$. Day and Miller generalized this basic cupping result by adding requirements to control the degrees of $Y'$ and $Y \oplus A$. Theorem 2.1 offers a different generalization.

Our proof uses a result of Kjos-Hanssen. We state it here in a slightly stronger form than he stated it, though without adding any essential content.

**Theorem 2.2** (Kjos-Hanssen [8]). $A \not\leq_{LR} B$ if and only if there is a $\Sigma^0_1[A]$ class $U$ of measure less than one that intersects every positive measure $\Pi^0_1[B]$ class. Furthermore, for any $\varepsilon > 0$, we can ensure that $\lambda(U) < \varepsilon$. 


Kjos-Hanssen showed that \( A \leq_{LR} B \) if and only if each \( \Pi^0_1[A] \) class of positive measure has a \( \Pi^0_1[B] \) subclass of positive measure\(^2\). Taking the contrapositive: \( A \nleq_{LR} B \) if and only if there is a \( \Pi^0_1[A] \) class \( T \) of positive measure that does not have a positive measure \( \Pi^0_1[B] \) subclass. So \( U = 2^\omega \setminus T \) would be a \( \Sigma^0_1[A] \) class of measure less than one that intersects every positive measure \( \Pi^0_1[B] \) class.

The fact that \( U \) can be taken to have arbitrarily small measure also follows from the work in [3]. We use this fact below, so for completeness, we sketch the argument. Assume that \( A \nleq_{LR} B \). So there is a \( B \)-random set \( X \) that is not \( A \)-random. Let \( U \) be a \( \Sigma^0_1[A] \) class containing every non-\( A \)-random set. We may assume, of course, that the measure of \( U \) is as small as we like. Let \( P \) be a positive measure \( \Pi^0_1[B] \) class. Relativizing a result of Kučera [9], every \( B \)-random set has a tail in \( P \), so there is a tail \( Y \) of \( X \) in \( P \). But \( Y \) is not \( A \)-random, so \( Y \in U \).

We need some basic notation for the proof of Theorem 2.1. If \( A \subseteq 2^\omega \) is measurable and \( \sigma \in 2^{<\omega} \), let \( \lambda(P \mid \sigma) \) denote the relative measure of \( P \) in \( \sigma \), i.e., \( \lambda(P \cap [\sigma]) / \lambda([\sigma]) \). If \( \sigma \in 2^{<\omega} \) and \( W \subseteq 2^{<\omega} \), let \( \sigma W = \{ \sigma \tau : \tau \in W \} \).

**Proof of Theorem 2.1.** Suppose that \( A \nleq_{LR} B \). By Theorem 2.2 there is a \( \Sigma^0_1[A] \) class \( U \) such that \( \lambda(U) < 0.1 \) and \( U \) intersects every positive measure \( \Pi^0_1[B] \) class. Let \( W \) be an \( A \)-c.e. prefix-free set of strings such that \( U = [W]^{\omega} \).

Let \( X \) be any set. We will construct \( Y = X(1)\sigma_0 X(2)\sigma_2 \cdots \) such that each \( \sigma_i \in W \). In this way, it is clear that \( X \leq_T Y \oplus A \). To ensure that \( Y \) is weakly \( 2 \)-random relative to \( B \), we build it inside a nested sequence of \( \Pi^0_1[B] \) classes \( P_n \) of positive measure such that \( \bigcap_{n \in \omega} P_n \) is a subset of every \( \Sigma^0_1[B] \) class of measure one. The following claim will let us hit \( W \) and code the next bit of \( X \) while staying inside the current \( \Pi^0_1[B] \) class.

**Claim.** For any string \( \sigma \in 2^{<\omega} \) and any \( \Pi^0_1[B] \) class \( P \) such that \( \lambda(P \mid \sigma) > 0.1 \), there is a \( \tau \supseteq \sigma \) such that \( \tau \in \sigma W \) and \( \lambda(P \mid \tau) \geq 0.8 \).

**Proof.** We first extend \( \sigma \) to a string \( \rho \) that has no prefix in \( \sigma W \) and such that \( \lambda(P \mid \rho) > 0.9 \). Let \( Q = 2^\omega \setminus [\sigma W]^{\omega} \). As \( \lambda(Q \mid \sigma) > 0.9 \) and \( \lambda(P \mid \sigma) > 0.1 \), we have \( \lambda(Q \cap P \mid \sigma) > 0 \). By the Lebesgue density theorem, there is a \( \rho \supseteq \sigma \) such that \( \lambda(Q \cap P \mid \rho) > 0.9 \). In particular, \( \lambda(P \mid \rho) > 0.9 \) and \( \lambda(Q \mid \rho) > 0.9 \); the latter implies that \( \rho \) cannot have a prefix in \( \sigma W \).

We now extend \( \rho \) to a string \( \tau \) satisfying the claim: \( \tau \in \sigma W \) and \( \lambda(P \mid \tau) \geq 0.8 \). Consider the \( \Pi^0_1(B) \) class \( \bar{P} = \{ X \in P \cap [\rho] : (\forall n \geq |\rho|) \lambda(P \mid X \cap n) \geq 0.8 \} \). In other words, \( \bar{P} \) is the subclass of \( P \cap [\rho] \) in which we remove every basic neighborhood inside \( [\rho] \) where the relative measure of \( P \) drops below 0.8. It is not hard to show that we remove at most 0.8 from the relative measure of \( P \cap [\rho] \) inside \( [\rho] \) (consider the antichain of maximal basic neighborhoods that are removed). But \( \lambda(P \mid \rho) > 0.9 \), so \( \lambda(\bar{P} \mid \rho) > 0.1 \). In particular, \( \bar{P} \) is a positive measure subclass of \( [\sigma] \), so by the choice of \( U = [W]^{\omega} \), it must be the case that \( [\sigma W]^{\omega} \) intersects \( \bar{P} \). Take \( \tau \in \sigma W \) such that \( \bar{P} \cap [\tau] \neq \emptyset \). By the definition of \( \bar{P} \), we have \( \lambda(P \mid \tau) \geq 0.8 \).

We are ready to construct \( Y \). We will construct it as the limit of a sequence \( \tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \cdots \) of strings, while staying inside a decreasing sequence \( P_0 \supseteq \]
We start stage \( n \), where \( \lambda \) is enough that \( Y \) is weakly 2-random relative to \( B \). (Note that this is true at stage 0.) First, we want to make progress towards \( \oplus \) being 2-random relative to \( B \). Let \( X \) be random. A set \( A \) is low for \( X \) preserving if for all \( Y \),

\[
Y \text{ is low for } X \implies Y \oplus A \text{ is low for } X.
\]

This notion was recently introduced by Yu Liang, who called it absolutely low for \( X \). Stephan and Yu proved that every K-trivial is low for \( \Omega \) preserving (see [7, Fact 1.8]). Yu asked if the converse is true: if a set is low for \( \Omega \) preserving, is it K-trivial? We show that this holds.

**Proposition 3.2.** If \( X \) is random, then low for \( X \) preserving implies K-triviality.

**Proof.** Assume that \( A \) is low for \( X \) preserving.

First, we claim that \( A \leq_{LR} X \). If not, then Theorem 2.1 gives us an \( X \)-random set \( Y \) such that \( X \leq_T Y \oplus A \). By Van Lambalgen’s theorem, \( X \) is \( Y \)-random. But \( X \leq_T Y \oplus A \) implies that \( X \) is not \((Y \oplus A)\)-random. This contradicts the assumption that \( A \) is low for \( X \) preserving. Therefore, \( A \leq_{LR} X \).

By Theorem 1.3 there is an \( X \)-random set \( Y \) such that \( A \leq_T Y \oplus X \). By Van Lambalgen’s theorem, \( X \) is \( Y \)-random and because \( A \) is low for \( X \) preserving, we have that \( X \) is \((Y \oplus A)\)-random. Furthermore, because \( Y \) is \( X \)-random and \( A \leq_{LR} X \), we know that \( Y \) is \( A \)-random. Therefore, by Van Lambalgen’s theorem relative to \( A \), \( Y \oplus X \) is \( A \)-random. But \( Y \oplus X \) computes \( A \), so \( A \) is a base for randomness. Therefore, it is K-trivial (see Theorem 1.4).

Together with the result of Stephan and Yu, we get a new characterization of K-triviality.

**Theorem 3.3.** A set \( A \) is K-trivial if and only if it is low for \( \Omega \) preserving.

Our next lemma can be viewed as a slight generalization of Stephan and Yu’s result. Assume that \( A \) is K-trivial and \( Y \) is low for \( \Omega \). Stephan and Yu showed that \( Y \oplus A \) is also low for \( \Omega \). Merkle and Yu [7, Question 1.11] asked if, in fact, \( Y \oplus A \) has exactly the same derandomizing power as \( Y \). This is the case:

**Lemma 3.4.** If \( A \) is K-trivial and \( Y \) is low for \( \Omega \), then \( Y \equiv_{LR} (Y \oplus A) \).
Proof. Let $A$ be $K$-trivial and $Y$ be low for $\Omega$. Let $X$ be any $Y$-random. By Theorem 1.3, there is a $Y$-random set $W$ such that both $\Omega$ and $X$ are computable from $W \oplus Y$. There is a nonempty $\Pi^0_1[Y]$ class containing only members with PA degree relative to $Y$. So by Theorem 1.5, there is a low for $W$ set $S$ with PA degree relative to $Y$. Thus $W$ is $S$-random and $\Omega \leq_T S$. By Theorem 1.4, both $X$ and $\Omega$ are also $S$-random. Since $S$ has PA degree and is low for $\Omega$, by Theorem 1.2, $S$ computes every $K$-trivial. In particular, $A \leq_T S$. Because $Y \oplus A \leq_T S$ and $X$ is $S$-random, $X$ is $\Omega \oplus A$-random. But $X$ was any $Y$-random set, so $Y \equiv_{LR} Y \oplus A$. □

The converse to Lemma 3.4 is easy, giving us our second characterization of $K$-triviality.

**Theorem 3.5.** A set $A$ is $K$-trivial if and only if for all $Y$:

$Y$ is low for $\Omega \implies Y \equiv_{LR} (Y \oplus A)$.

**Proof.** One direction is Lemma 3.4. For the other direction, assume that $A$ has the given property. Note $\Omega$ is $\emptyset$-random, so $\emptyset \equiv_{LR} \emptyset \oplus A \equiv_{LR} A$. In other words, $A$ is low for randomness, hence $K$-trivial (see Theorem 1.1). □

It is natural to ask if low for $X$ preserving is equivalent to $K$-triviality for all random $X$. As we shall see, this is not the case, though it is true for some $X$.

**Proposition 3.6.** If $\Omega \leq_T X$ and $X$ is random, then low for $X$ preserving is equivalent to $K$-triviality.

**Proof.** One direction is given by Proposition 3.2. For the other direction, let $A$ be $K$-trivial and take any such that $X = Y$-random. By (the unrelativized form of) Theorem 1.4, $\Omega$ is also $Y$-random. By Lemma 3.4, $Y \equiv_{LR} (Y \oplus A)$. Therefore, $X$ is $(Y \oplus A)$-random. □

For certain other $X$, low for $X$ preserving is equivalent to being computable.

**Proposition 3.7.** If $X$ is Schnorr$[\emptyset']$ random but not $2$-random, then only the computable sets are low for $X$ preserving.

**Proof.** We prove the contrapositive. Assume that $A$ is not computable. If $A$ is not $\Delta^0_3$, then it is not $K$-trivial, hence by Proposition 3.2, it is not low for $X$ preserving. So assume that $A$ is $\Delta^0_3$. By Posner–Robinson [13], there is a low set $Y$ such that $Y \oplus A \equiv_T \emptyset'$. Because $X$ is Schnorr$[\emptyset']$ random, it is random relative to any low set, so it is $Y$-random. But $X$ is not $2$-random, so it is not $(Y \oplus A)$-random. Therefore, $A$ is not low for $X$ preserving. □

**References**


4In fact, this property characterizes Schnorr$[\emptyset']$ randomness: Yu [18] showed that $X$ is Schnorr$[\emptyset']$ random if and only if $X$ is $Z$-random for every low set $Z$. 


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