# Invariance and Noninvariance in the Lattice of $\Pi^0_1$ Classes

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- Joint work with Peter Cholak

## $\Pi^0_1$ Classes

- A (computably bounded) $\Pi^0_1$ class $C$ can be defined as the set $[T]$ of infinite paths through a computable tree $T \subseteq 2^{<\omega}$.
- Examples abound: See Cenzer-Jockusch-Remmel articles.
- $L = L(2^\omega)$ is the lattice of $\Pi^0_1$ classes ordered by inclusion.

## Facts about $L$

- $L$ is distributive We like to think of $\Pi^0_1$-classes as subsets of the Cantor space $2^\omega$ where the sets $I(\sigma) = \{x \mid \sigma \prec x\}$ form a basis of open intervals.
- Then any clopen subset of $2^\omega$ is just a finite union of intervals.
- In $L$, complemented = clopen.
- $L$ can be viewed under duality as the lattice of c.e. ideals in a computable copy of the free boolean algebra.
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## Degrees in $\Pi^0_1$ classes

- There are several degrees one can associate with a class $C$.
- The degrees of *members of the class.* (Extensively studied)
- The degrees of set of initial segments of members of $C$. (A c.e. degrees). This is the same as the degree of $I(C)$, the corresponding ideal.
- etc.

## Autos of $L$

- (Remmel) $L$ has $2^{\aleph_0}$ automorphisms.
- Remmel's proof is like Lachlan's that the lattice of c.e. sets mod finite sets has uncountably many autos.

## This classes

- A $\Pi^0_1$ class $P$ is called *thin* if $P$ is infinite and for every $\Pi^0_1$ class $Q \subseteq P$ there is a clopen set $F \subset 2^\omega$ such that $Q = P \cap F$.
- $P$ is called *minimal* if for all $Q \subseteq P$, either $Q$ or $P - Q$ is finite.
- This classes are first formulated in Downey's Thesis, after Rick Smith, but were (under duality) first constructed by Martin and Pour-El.
- (Duality) A c.e. ideal $I$ is call maximal if it is nonprime and for all c.e. ideals $F \supseteq I$, $F = \langle I \cup \{\theta\} \rangle$, for some $\theta$.

- $T$ is thin iff $I(T)$ is maximal.

# Thin classes in general

- There is a long paper with many results of degrees of and degrees of members of thin classes, due to Cenzer, Downey, Jockusch and Shore, in the APAL. For instance:
- There is a c.e. degree $\boldsymbol{a}$ containing no (member of) a thin class.
- The c.e. degrees containing (members of) thin classes are dense.
- assorted results on members of thin classes of specified rank. For instance if $P$ is a member of a thin class then $P' \leq_T P \oplus \emptyset''$. If $P$ is a member of a

computable minimal class the $P$ is low.

## Perfect Thin Classes

- a class is *perfect* if it has no isolated points. They must be uncountable.
- (Martin Pour-El) Perfect thin classes exist.
- A c.e. degree $\boldsymbol{a}$ is called anc iff for all $f \leq_{wtt} \emptyset'$, there is a $g \leq_T \boldsymbol{a}$ such that $\exists^\infty s(g(s) > f(s)$.
- (Downey, Jockusch, Stob, 1989) If $\boldsymbol{a}$ is an anc degree then $\boldsymbol{a}$ contains a thin class.

# An analog of Martin's thm

- (Martin) Maximal sets are invariant for the high degrees.
- (Cholak, Coles, Downey, Herrmann, TAMS, 2001) $\boldsymbol{a}$ is anc iff $\boldsymbol{a}$ contains a thin class, and thin classes always have anc degree.

# Analog of Soare's Theorem

- (Soare) Given two maximal c.e. sets $A$, $B$ there is an auto of $\mathcal{E}$ taking $A$ to $B$.

- (Cholak, Coles, Downey, Herrmann, TAMS, 2001) Given two perfect thin $\Pi^0_1$ classes $A$, $B$there is an auto of $L$ taking $A$ to $B$.

## Possible extensions

- (Cholak, Coles, Downey, Herrmann, TAMS, 2001) $A$ is a thin class iff $L(A, \downarrow) =_{\text{def}} \{B : B \subseteq A\}$ is a ($\Delta^0_2$-) boolean algebra.
- (Question) Is there any other boolean algebra except the free BA where $L(A, \downarrow)$ determines a single orbit?
- Question (Cenzer-Jockusch-Remmel) what about minimal classes?

## The Main result

- (Cholak and Downey) There exist two minimal classes $A$ and $B$ which are not automorphic. (Nies' theorems show that they are not even elementary equivalent)

## The Proof

- We say that a minimal class $S$ is *cohesively minimal* if it obeys the following property $\mathcal{P}$ below. (All quantifiers range over that lattice of $\Pi^0_1$ classes.)

  For all $F \supseteq S$ one of the following holds.

  (i) There exists a finite $G \subseteq S$ and a clopen $C \subseteq F$, such that, for $\widehat{S} = S - G$, $\widehat{S} \subseteq C$.

  (ii) There is a finite $G \subseteq S$, such that, for $\widehat{S} = S - G$, and for all isolated $I \subseteq \widehat{S}$, there is no clopen $C$ such that

$$(I \subseteq C) \wedge (C \cap F = C).$$

- There is a cohesively minimal $\Pi^0_1$ class.
- There is a minimal $\Pi^0_1$ class that is not cohesively minimal.

- Define a rank one class $[T]$ to be *tame* if its rank one point has degree computable from $\emptyset'$ and there is a $\Delta^0_2$ function $g$ such that

  (i) $g(e) = \langle \sigma_{m(e)}, \varphi_{m(e)} \rangle$ where $\varphi_m$ is the characteristic function of an isolated point $I_m$ of $[T]$, with $\sigma_m$ an initial segment of $I_m$.

  (ii) $\{I_{m(e)} : e \in \omega\}$ list all the isolated points of $[T]$, and

  (iii) $\{\sigma_{m(e)} : e \in \omega\}$ form an antichain of strings.

- For instance, any rank one class generated by $[T]$, where $T$ has no dead ends is tame

- Suppose that we have $\nu$ as an initial segment of the rank one point, and the tree splits at $\nu$. Then simply count which of $\nu\widehat{\ }0$ or $\nu\widehat{\ }1$ has the most split extensions to determine which way the rank one point goes. The $\sigma_i$ are the shortest strings which are initial segments of the isolated points, with the property that they are not initial segments of any other isolated points. You can order them, first by length, and then lexicographically.

# Lemma 1

- Suppose that $S$ is a tame minimal class Then $S$ does not satisfy $\mathcal{P}$.

## Tame Autos

- (essentially Cenzer and Remmel) Suppose that $S$ and $T$ are tame minimal classes. Then $S$ and $T$ are automorphic
- (i) Any two computable minimal classes are automorphic.
  (ii) Being computable is not definable in the lattice of $\Pi_1^0$ classes.

- The proof uses the *Extension Lemma of CCDH*
- (CCDH, Theorem 8.1, special case) Suppose that there is a computable permutation $p$ and computably enumerable bases $\gamma_i^j$, such that additionally

$$\forall k \exists s \forall i > s((\sigma_k^1 \prec \gamma_i^1 \text{ iff } \sigma_k^2 \prec \gamma_{p(i)}^2)$$

$$\wedge(\tau_k^1 \prec \gamma_i^1 \text{ iff } \tau_k^2 \prec \gamma_{p(i)}^2)).$$

Then $[T_1]$ is automorphic to $[T_2]$.

# Requirements

$\mathcal{M}_e$ : $[U_e] \subseteq S \to \exists$

clopen $D(D \cap S = [U_e]$.

$\mathcal{R}_e$ : $S \subseteq [V_e] \to (i)$ or $(ii)$ below holds

$(i)\exists$ finite $G \subset S(\exists$ clopen $C \subseteq [V_e]$

$(S - G \subseteq C \wedge [V_e] \cap C = C)$.

$(ii)\exists$ finite $G \subset S(\forall$ isolated

$I \subset S - G$ $\forall$ clopen $C$

$(I \subseteq C \to (C \cap [V_e] \neq C))$.

- $\mathcal{M}_e$, thinness requirements.
- Suppose $[U_e] \subset S$.
- Then at some stage of the construction, for some string $\sigma$ of length $s$ in $T_s$, we will have $\sigma \notin U_{e,s}$. (That is, there can be no extension of $\sigma$ in $[U_e]$.) Then we can win the requirement $\mathcal{M}_e$ by simply making sure that the rank one point in $[T]$ extends $\sigma$, and hence almost all members of $S$ will extend $\sigma$.

- We will need to get the cohesiveness on the isolated paths of $S$. We do this by trying to verify (ii) of the definition of property $P$, on almost all paths.
- diagram.

## A Conjecture

- The only Boolean algebras $B$, which have the property that any two thin classes with $B$ as their lattices of subalgebras are automorphic, are ones with a finite number of atoms.

- Thank you.