Presenting Reals

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- Some Computability-Theoretical Aspects of Reals and Randomness, to appear in a volume from the *Lecture Notes in Logic* series published by the Association for Symbolic Logic.

- Presentations of computably enumerable reals, (with G. LaForte) *Theoretical Computer Science* Vol. 284 (2002), 539-555.

- Computably Enumerable Reals and Uniformly Presentable Ideals, (with S. Terwijn) *Archive for Mathematical Logic* Vol. 48 (2002), 29-40.

Notation

- "real" is taken without comment as member of 2^ω, Cantor space, with basic clopen sets $[\sigma] = \{\sigma\tau : \tau \in 2^\omega\}$.
- Their measure is $\mu([\sigma]) = 2^{-|\sigma|}$.
- for uniformity, a real is always nonrational.

- Concerned with "representing" reals. And the interaction of this with computability.
- Classically three methods are as dyadic expansions, Cauchy sequences, or cuts.
- We could define a computable real as one whose dyadic expansion $\alpha = .A$ had A a computable set.
- (Turing) α is a computable real iff it is the limit of a computable sequence of rationals $q_i : i \in B$ and there is a computable algorithm M so that for all n,

$$|\alpha - q_{M(n)}| < 2^{-n}.$$

C.e. Reals

- A computably enumerable real (or left computable) is α such that $L(\alpha) = \{q \in \mathbb{Q} : q < \alpha\}$ is a c.e. set.
- Probably first studied explicitly by Soare in the 60's although there was work by Rice and other earlier. In particular Soare constructed a c.e. real α not of the form $.A$ for a c.e. set A.

- These are the correct notions for effective measure theory as c.e. reals are the measures of the domains of *prefix-free* Turing machines. Viz: α is a c.e. real iff there is a prefix free M such that $\alpha = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|}$.
- To see this note that $\alpha = \lim_s \alpha_s$ where $\alpha_s = \sum_{M_s(\sigma)\downarrow} 2^{-|\sigma|}$.

- The following was certainly implicit in earlier work.
- (Calude, Khoussainov, Hertling, Wang) TFAE (i) α is the limit of a computable enumerable monotone increasing (in the real ordering) sequence of rationals.
  (ii) α is computably enumerable.
  (iii) There is an infinite computably enumerable prefix free set A with $\alpha = \Sigma_{n \in A} 2^{-|n|}$.
  (iv) There is a computable prefix free set A such that $\alpha = \Sigma_{n \in A} 2^{-|n|}$.

(v)There is a computable function $f(x, y)$ of two variables, such that

(a) If, for some k, s we have $f(k, s+1) = 0$ yet $f(k, s) = 1$ then there is some $k' < k$ such that $f(k', s) = 0$ and $f(k', s+1) = 1$.

(b) $\alpha = \cdot a_1 a_2 ...$ is a dyadic expansion of α with $a_i = \lim_s f(i, s)$.

(vi) There is a computable increasing sequence of rationals with limit α.

Representing Reals

- We say that a c.e. sequence of rationals $\{q_i : i \in \omega\}$ with monotonic limit α *represents* α.
- Representations were first effectively analyzed by Calude, Coles, Hertling, Khoussainov.

- If a real is computable then every representation must be computable (exercise). Suppose that a c.e. real is noncomputable. What else can be said about its representations? For instance, the natural degree of a c.e. real is the degree of its left cut: $\deg L(\alpha)$. Does α always have a representation of degree $\deg L(\alpha)$? of other degrees?

Degrees

- (Calude et. al.) α has a representation of degree $\deg L(\alpha)$.
- (Soare) If $B = \{q_i\}$ is a representation of α then $B \leq_T L(\alpha)$ and in fact $B \leq_{wtt} L(\alpha)$ where wtt denotes weak truth table reducibility

- (Calude et. al.) Every representation of α is half of a c.e. splitting of $L(\alpha)$. (Proof: Clearly, if A represents α then A must be an infinite c.e. subset of $L(\alpha)$. The thing to note is that $L(\alpha) - A$ is also c.e.$\dot{\text{G}}$iven rational q, if q occurs in $L(\alpha)$, we need only wait till either q occurs in A or some rational bigger than q does.)

- We would *like* to prove that if A is half of a splitting of $L(\alpha)$ then A represents α.

- But it is not difficult to prove that this is not true. We know that if A represents α then there needs to be a computable function g with range A so that, as reals, $g(i) < g(i+1)$. Use this for diagonalization.

- Sometimes true: (Calude et. al.)
  Let A be a representation of α. For subsets B of A, the following are equivalent.

  (a) B represents α.

  (b) B is half of a splitting of A.

- The answer lies in m-degrees:
- Theorem (Downey) The following are equivalent
  (a) $\boldsymbol{b}$ is the m-degree of a splitting of $L(x)$.
  (b) $\boldsymbol{b}$ is the m-degree of a representation of x.

Corollaries

- There exist computably enumerable reals a_i such that the collection of T-degrees of representations $R(a_i)$ have the following properties.

  (i) $R(a_1)$ consists of every c.e. (m-) degree $\leq Deg(a_1)$.

  (ii) $R(a_2)$ forms an atomless boolean algebra, which is nowhere dense in the c.e. degrees.

  – If a c.e. real has representations of every degree below it, then the c.e. real is low_2 or Turing complete.

Presentations

- (Downey) Return to the characterizations of c.e. reals. Say a c.e. prefix-free set A *presents* α if $\alpha = \mu(A)$, equivalently, $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}$.
- Questions What can be said about the spectrum of degree that present α.

- Calude et. al. observed that there is always a *computable* presentation of any given c.e. α.
- Theorem (Downey and Laforte) There is a noncomputable α such that, if A presents α, the A is computable.
- Let's call such a real *nearly computable.*

The Proof

- Requirements $R_e : W_e$ presents α, implies W_e computable.
- $P_e : \alpha$ is not computable via φ_e.
- $\alpha_s = \cdot a_{0,s} \ldots$.
- Meeting the P_e is to pick some position i, keep $a_i = 0[s]$ and then wait till $\varphi_e(i) = 0$, then change $a_i = [t]$ some $t > s$.
- Meeting R_k. We must build a computable f so that after some stage $f(d)$ no string of length $\leq d$ can enter W_k *after stage* $f(d)$.

Length of Agreement

- R_k will be a current "length of agreement". This is a number $m(k, s)$ such that $\alpha_s - \sum_{n \in W_{k,s}} 2^{-|n|} < 2^{-m(k,s)}$.
- We can assume that $\alpha_s \geq \sum_{n \in W_{k,s}} 2^{-|n|}$, for otherwise, we can win by withholding numbers from α.

The Main Idea

- Now the idea is that when we see some P_e require attention via some position i for e bigger than k, if i is smaller than $m(k, s)$, the interesting case, then we wish to put a relatively big number into α, by changing position i for the sake of P_e, yet we wish to not allow numbers of low length to enter W_k.

- The idea is to slowly work backwards. So first we will make position $m(k,s)+1 = 1$ by adding something of length $2^{-m(k,s)+1}$ into A_{s+1}.

- We then do nothing until W_k responds by giving us a stage $t > s$ with $\alpha_t - \sum_{n \in W_{k,t}} 2^{-|n|} < 2^{-m(k,s)+1}$.

- Note that W_k can only change on strings of long length, since we only changed A slightly. Think of someone wanting to dump bad shares on the market. they "drip feed" the shares, so that the marker won't notice.

- Now there are two outcomes. Either at some stage, we don't get recovery, so that W_k does not present A, or W_k responds at each stage and we get a change only on long strings. This means that we can compute W_k.

- Harder with two requirements: Now you must coordinate $m(k_1, s)$, and $m(k_2, s)$. If $k_1 < k_2$ then need to allow $m(k_1, s)$ to increase whilst waiting for k_2 recovery. Like the tower of Hanoi.
- Here we would make sure that $m(k_1, s) >> m(k_2, s)$.
- When we see a k_2 recovers, then we will *delay* $m(k_1, s)$ increases *enough* stages that we enumerate quanta of $2^{-m(k_2,s)}$ enter in lots of k_1-verified $2^{-m(k_1,s)}$.

- Π^0_2 argument on a tree of strategies, on full implementation. Similar to embedding 1-3-1 in the c.e. Turing degrees...connection?
- Diagram

- Downey and LaForte such α can have high c.e. degree and hence be "far" from computable.
- Cannot be improved to tt-degrees.
- (Guohua Wu) There exist c.e. noncomputable degrees $\boldsymbol{a}$ such that if α is any c.e. real of degree $\leq \boldsymbol{a}$, which is noncomputable, then α has a noncomputable presentation. (This uses a gap-co-gap $\mathbf{0}'''$ argument.)

What can be said

- Downey and Laforte:
- If α is of "promptly simple" degree then α has a noncomputable presentation. (e.g. if α is Turing complete)
- If α is Turing complete it has a Turing complete presentation.
- Can also have promptly simple α such that each presentation is below a given low degree, etc.
- Suppose that A presents α and $B \leq_{wtt} A$ is c.e. Then α has a

presentation of wtt-degree B. (This uses Kraft Chaitin.)

- If A and B present α so does $0A \cup 1B$ and hence the wtt degrees of presentations of α forms a Σ^0_3 *ideal.*

Realizing Ideals

- Downey and Terwijn: The answer is "anything that is not explicitly ruled out can be realized." Let $I(\alpha) = \{W_e : \exists A$ a presentation of $\alpha with W_e \equiv_{wtt} A\}$
- Suppose that I is Σ_3^0 and that $I = \{W_e : e \in I\}$ forms an ideal in the c.e. wtt-degrees. Then there is a computably enumerable real α such that $I(\alpha) = I$.
- The proof is kind a combinations of the drip feed strategy with a coding strategy.

- Downey and Terwijn also prove a theorem in the spirit of Rice's Theorem concerning the complexity of I.
- Theorem: $I(\alpha)$ is either $\{W_e : e \in \omega\}$ or Σ^0_3-complete (as an index set).

Other work

- One can look beyond c.e. reals.
- Here's a small sample. I am not aware of too much work on the relevant degrees.
- Ho proved that α is a limit of a computable sequence of rationals (not montone) iff α is computable from $\mathbf{o}'$.
- Ambos-Spies, Weihrauch and Zheng looked at reals of the form $\alpha - \beta$ for α, β c.e. They showed, rather surpisingly that this set forms a field. (division is not obvious) and they can be characterized by the behavour of the oscillations in representations.

- Zheng has shown that there are α, β c.e. with $\alpha - \beta$ not a ω-c.e. real.

- Thank You