

# *Recent Advances in Multivariate Algorithmics*

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# THE GOALS

- ▶ “Understand computation”
- ▶ What does it mean to be a **computable process**?
- ▶ What does it mean to be an **efficiently computable process**?
- ▶ **How** to find a best algorithm for a process?
- ▶ **How** to **know** you cannot do better?
- ▶ **How** to **cope** with apparently hard problems?
- ▶ **What** has this got to do with computation on **real data**.

# TURING AND FRIENDS, 1936:

- ▶ Turing, Kleene, Church and other **clarified** step by step computation; and using analysis arguably real computation.
- ▶ **Turing Machine** Machine is a **Box** with a finite number of **Internal States** (i.e. mental states)
- ▶ Reads/writes on a two way potentially infinite tape.
- ▶ Action : can move **Left, Right**, or **(over-)Print a symbol**,
- ▶ Depending on (state, symbol)

# WHY TURING?

- ▶ The arguments by Church for the acceptance of  $\lambda$ -definability as capturing were (i) by example (i) confluence (ii) step by step arguments echoing logical proof systems and (iii) failure of diagonalization.
- ▶ These were **not accepted** at the time. See e.g. Davis, Gandy 1995, Soare 2012, Kleene 1995.
- ▶ First and foremost Turing has a **conceptual analysis** giving what many regard as a **proof** of the thesis that TM's capture what is computable.
- ▶ This **analysis** is the **fundamental contribution** of Turing's paper.
- ▶ See “The Universal Turing Machine: A Half Century Survey” R. Herken (ed) Springer 1995 (2nd Ed).

# TURING'S ANALYSIS

- ▶ He considers an **abstract human computer** (a modern term)
- ▶ By limitations of sensory and mental apparatus we have
  - (i) fixed bound for the symbols.
  - (ii) fixed bound for number of squares
  - (iii) fixed bound to the number of actions at each step
  - (iv) fixed bound on the movement.
  - (v) fixed bound on the number of states.
- ▶ This justifies TM's
- ▶ Also examples like Bessel functions.
- ▶ Gandy, Soare (and others) argue that Turing **proves** any function calculable by an **abstract human** is computable by a TM.
- ▶ For instance, Turing's analysis was the only thing that convinced Gödel.

► Gandy (1995):

*What Turing did, by his analysis of the processes and limitations of calculations of human beings, was to clear away, with a single stroke of his broom, this dependency on contemporary experience, and produce a characterization-within clearly perceived limits- which will stand for all time..... What Turing also did was to show that **calculation can be broken down into the iteration (controlled by a “program”) of extremely simple concrete operations**; so concrete that they can easily be described in terms of **(physical) mechanisms**. (The operations of  $\lambda$ -calculus are much more abstract.)*

# ALGORITHMS AND COMPLEXITY

- ▶ Basic idea **count the steps** as a function of the **size** of the input. (Hartmanis, Sterns, Edmonds)
- ▶ Classical idealization **Efficient**= P = **polynomial time**= those problems which have algorithms running in  $O(n^c)$  where  $n$  is the size of input.
- ▶ **Controversial** in the 1960's.
- ▶ How to prove something is not e.g. in P?
- ▶ Kind of a holy grail in complexity theory.

- ▶ Russians (Levin) **perabor**, Cook identified **nondeterministic polynomial time**.
- ▶ These are defined as problems  $L$  such that there is a polynomial time checking process  $R(x, y)$  such that

$$x \in L \text{ iff } \exists y (|y| \leq |x|^c \wedge R(x, y)).$$

- ▶ Examples: SATISFIABILITY, VERTEX COVER (vertices cover edges), DOMINATING SET (vertices cover vertices), HAMILTON CYCLE. (Karp)
- ▶ NP completeness is **everywhere**.
- ▶ We cannot prove  $P \neq NP$ , etc.

# WHAT WE ARE ABLE TO PROVE ABOUT LOWER BOUNDS FOR NP PROBLEMS

## BUT WHAT'S THE REAL COST

- ▶ In the early 1990's a reasonably successful systematic coping strategy was developed.
- ▶ Building on early work of Fellows and Langston, Mike Fellows and I suggested that focusing on the parameters of the input would yield insight into feasibility.
- ▶ **When** is the **only** thing you know about an input is its size?
- ▶ Answer **Never, except in cryptography by design.**
- ▶ We know things like planarity, engineering bounds, topology, depth, inductive designs.

# TWO BASIC EXAMPLES

- ▶ VERTEX COVER

**Input:** A Graph  $G$ .

**Parameter :** A positive integer  $k$ .

**Question:** Does  $G$  have a size  $k$  vertex cover? (Vertices cover edges.)

- ▶ DOMINATING SET

**Input:** A Graph  $G$ .

**Parameter :** A positive integer  $k$ .

**Question:** Does  $G$  have a size  $k$  dominating set? (Vertices cover vertices.)

- ▶ VERTEX COVER is solvable by an algorithm  $\mathcal{O}$  in time  $f(k)|G|$ , a behaviour we call **fixed parameter tractability**, (Specifically  $1.28^k k^2 + c|G|$ , with  $c$  a small absolute constant independent of  $k$ .)
- ▶ Whereas the only known algorithm for DOMINATING SET is complete search of the possible  $k$ -subsets, which takes time  $\Omega(|G|^k)$ .

- ▶ In the below I will mostly talk for convenience about graphs.
- ▶ I could just as easily be talking about many other areas.
- ▶ In the Computer Journal alone, there is biological, artificial intelligence, constraint satisfaction, geometric problems, scheduling, cognitive science, voting, combinatorial optimization, phylogeny. Model checking is the basis of Flum-Grohe.

# BASIC DEFINITION(S)

- ▶ Setting : Languages  $L \subseteq \Sigma^* \times \Sigma^*$ .
- ▶ Example (Graph, Parameter).
- ▶ We say that a language  $L$  is **fixed parameter tractable** if there is a algorithm  $M$ , a constant  $C$  and a function  $f$  such that for all  $x, k$ ,

$$(x, k) \in L \text{ iff } M(x) = 1 \text{ and}$$

the running time of  $M(x)$  is  $f(k)|x|^C$ .

- ▶ Without even going into details, think of all the graphs you have given names to and each has a relevant parameter: planar, bounded genus, bounded cutwidth, pathwidth, treewidth, degree, interval, etc, etc.
- ▶ Also **nature** is kind in that for many practical problems the input (often designed by **us**) is nicely ordered.

# POSITIVE TECHNIQUES

- ▶ Elementary ones
- ▶ Almost elementary
- ▶ Logical metatheorems
- ▶ Limits

- ▶ I believe that the most important practical technique is called **kernelization**.
- ▶ pre-processing, or reducing For parameterized complexity  $(I, k) \mapsto (I', k')$  and  $|I'| \leq g(k)$ .

▶ TRAIN COVERING BY STATIONS

**Instance:** A bipartite graph  $G = (V_S \cup V_T, E)$ , where the set of vertices  $V_S$  represents railway stations and the set of vertices  $V_T$  represents trains.  $E$  contains an edge  $(s, t)$ ,  $s \in V_S, t \in V_T$ , iff the train  $t$  stops at the station  $s$ .

**Problem:** Find a minimum set  $V' \subseteq V_S$  such that  $V'$  covers  $V_T$ , that is, for every vertex  $t \in V_T$ , there is some  $s \in V'$  such that  $(s, t) \in E$ .

▶ REDUCTION RULE TCS1:

Let  $N(t)$  denote the neighbours of  $t$  in  $V_S$ . If  $N(t) \subseteq N(t')$  then remove  $t'$  and all adjacent edges of  $t'$  from  $G$ . If there is a station that covers  $t$ , then this station also covers  $t'$ .

▶ REDUCTION RULE TCS2:

Let  $N(s)$  denote the neighbours of  $s$  in  $V_T$ . If  $N(s) \subseteq N(s')$  then remove  $s$  and all adjacent edges of  $s$  from  $G$ . If there is a train covered by  $s$ , then this train is also covered by  $s'$ .

- ▶ European train schedule, gave a graph consisting of around  $1.6 \cdot 10^5$  vertices and  $1.6 \cdot 10^6$  edges.
- ▶ Solved in minutes.
- ▶ This has also been applied in practice as a subroutine in **practical heuristical** algorithms.

- ▶ Reduce the parameterized problem to a **kernel** whose size depends **solely on the parameter**
- ▶ As compared to the classical case where this process is a central heuristic we get a **provable performance guarantee**.
- ▶ We remark that often the performance is **much better** than we should expect **especially when elementary methods are used**.

# VERTEX COVER

- ▶ REDUCTION RULE VC1:  
Remove all isolated vertices.
- ▶ REDUCTION RULE VC2:  
For any degree one vertex  $v$ , add its single neighbour  $u$  to the solution set and remove  $u$  and all of its incident edges from the graph.
- ▶ Note  $(G, k) \rightarrow (G', k - 1)$ .
- ▶ (S. Buss) REDUCTION RULE VC3:  
If there is a vertex  $v$  of degree at least  $k + 1$ , add  $v$  to the solution set and remove  $v$  and all of its incident edges from the graph.
- ▶ The result is a graph with no vertices of degree  $> k$  and this can have a VC of size  $k$  only if it has  $< k^2$  many edges.

# STRATEGIES FOR IMPROVING I: BOUNDED SEARCH TREES

- ▶ Buss's algorithm gives crudely a  $2n + k^{k^2}$  algorithm for  $k$ -VC.
- ▶ Here is another algorithm: (DF) Take any edge  $e = v_1 v_2$ . **either  $v_1$  or  $v_2$  is in any VC.** Begin a tree  $T$  with first children  $v_1$  and  $v_2$ . At each child delete all edges covered by the  $v_j$ .
- ▶ repeat to depth  $k$ .
- ▶ Gives a  $O(2^k \cdot n)$  algorithm.
- ▶ Now combine the two: Gives a  $2n + 2^k k^2$  algorithm.

- ▶ It is worth remarking that there are problems notably FPT by bounded search tree (type checking in ML) that are not known to have polynomial size kernels, and some “provably” don’t.
- ▶ Another easy example for bounded search trees is PLANAR INDEPENDENT SET. (Start with a degree 5 vertex, branching rule of size 6)

# PRUNING TREES AND CLEVER REDUCTION RULES

- ▶ If  $G$  has paths of degree 2, then there are simple reduction rules to deal with them first. Thus we consider that  $G$  is of min degree 3.

## BRANCHING RULE VC2:

If there is a degree two vertex  $v$  in  $G$ , with neighbours  $w_1$  and  $w_2$ , then either both  $w_1$  and  $w_2$  are in a minimum size cover, or  $v$  together with **all other neighbours** of  $w_1$  and  $w_2$  are in a minimum size cover.

- ▶ Now when considering the kernel, for each vertex considered **either**  $v$  is included or **all** of its neighbours (at least)  $\{p, q\}$  are included.
- ▶ Now the tree looks different. The first child nodes are labeled  $v$  or  $\{p, q\}$ , and on the right branch the parameter drops by 2 instead of 1. or similarly with the  $w_i$  case.

- ▶ Now the size of the search tree and hence the time complexity is determined by some recurrence relation.
- ▶ many, many versions of this idea with increasingly sophisticated reduction rules.
- ▶ This method has a 2005 (Fomin, Grandoni, Kratsch) incarnation called **measure and conquer** where the branching rules are given *rational valued* weights, and decisions as to what to do are figured out by optimization.
- ▶ For example the best exact algorithm for SET COVER and DOMINATING SET use this. (van Rooij-Bodlaender point out that this can be used for algorithm design as well.)
- ▶ Jianer Chen and others use this in many FPT algorithms such as the state of the art for FEEDBACK VERTEX SET and VERTEX COVER.

## THEOREM (NEMHAUSER AND TROTTER (1975))

*There is a  $2k$  kernel.*

- ▶ Proof use LP  $\{x_i \mid i \in V\}$ . Solve

$$\sum_{i \in V} x_i \leq k, \text{ subject to}$$

$$x_i + x_j \geq 1, \text{ for } ij \in E(G),$$

$$x_i \in \{0, \frac{1}{2}, 1\}.$$

- ▶ The current champion using this approach is a  $O^*(1.286^k)$  (Chen01) The best is  $O^*(1.2745^k)$  (Chen10 using this, iterative compression, struction, measure and conquer, and other methods).
- ▶ Here the useful  $O^*$  notation only looks at the **exponential** part of the algorithm.

# INTERACTIONS

- ▶ Now we can ask lots of questions. How small can the kernel be?
- ▶ Notice that applying the kernelization to the unbounded problem yields a approximation algorithm.
- ▶ Using the PCP theorem we know that no kernel can be smaller than  $1.36k$  unless  $P=NP$  (Dinur and Safra) as no better approximation is possible. Is this tight?
- ▶ Assuming the “Unique Games Conjecture” the  $2k$  kernel is tight (Khot etc).
- ▶ Actually we know that no  $O^*(1 + \epsilon)^k$  algorithm is possible unless ETH fails.
- ▶ ETH  $n$ -variable 3SAT is not in  $DTIME(2^{o(n)})$ . **That is, not only does  $P \neq NP$  but you can't beat complete search meaningfully.**

- ▶ (Niedermeier and Rossmanith, 2000) showed that iteratively combining kernelization and bounded search trees often performs much better than either one alone or one followed by the other.
- ▶ Begin a search tree, and apply kernelization, then continue etc. Analysing the combinatorics shows a significant reduction in time complexity, which is very effective in practice.

- ▶ Reed, Smith and Vetta 2004. For the problem of “within  $k$  of being bipartite” (by deletion of edges).

## DEFINITION (COMPRESSION ROUTINE)

A **compression routine** is an algorithm that, given a problem instance  $I$  and a solution of size  $k$ , either calculates a smaller solution or proves that the given solution is of minimum size.

## AN EXAMPLE, VC AGAIN!

- ▶  $(G = (V, E), k)$ , start with  $V' = \emptyset$ , and (solution)  $C = \emptyset$ .
- ▶ Add a new vertex  $v$  to both  $V'$  and  $C$ ,  
 $V' \leftarrow V' \cup \{v\}$ ,  $C \leftarrow C \cup \{v\}$ .
- ▶ Now call the compression routine on the pair  $(G[V'], C)$ , where  $G[V']$  is the subgraph induced by  $V'$  in  $G$ , to obtain a new solution  $C'$ . If  $|C'| > k$  then we output NO, otherwise we set  $C \leftarrow C'$ .
- ▶ If we successfully complete the  $n$ th step where  $V' = V$ , we output  $C$  with  $|C| \leq k$ . Note that  $C$  will be an optimal solution for  $G$ . (Algo runs in time  $O(2^k mn)$ .)

- ▶ This was first successfully applied by Reed, Smith, Vetta to GRAPH BIPARTITIZATION. The algorithm is similar, building a minimal bipartitization at each step and using what we can call acceptable partitions for the search step.
- ▶ The best now is  $O^*(3.83^k)$ , and it works better with algorithm engineering (Gray Codes, tree pruning) with (e.g.) biological data Hüffner 2004.
- ▶ It is a crucial step for the best two algorithms for VERTEX COVER (Chen, Kanj, Xia 2010,  $O^*(1.2745^k)$ ) and FEEDBACK VERTEX SET (Can I remove  $k$  vertices and get an acyclic graph?) (Cao, Chen, Liu, 2009).

- ▶ I remark that **in practice** these methods work **much better** than we might expect.
- ▶ Langston's work with irradiated mice, ETH group in Zurich, Karsten Weihe
- ▶ See **The Computer Journal** especially articles by Langston et al.

# LESS PRACTICAL ALGORITHMS

- ▶ In what follows we look at algorithms that in general seem less practical but can sometimes work in practice.

- ▶ K-SUBGRAPH ISOMORPHISM

**Instance:**  $G = (V, E)$  and a graph  $H = (V^H, E^H)$  with  $|V^H| = k$ .

**Parameter:** A positive integer  $k$  (or  $V^H$ ).

**Question:** Is  $H$  isomorphic to a subgraph in  $G$ ?

- ▶ e.g.  $k$ -PATH. Is  $H$  is a path of length  $k$ .
- ▶ Idea: Randomly colour the vertices of  $G$  with  $k$  colours and expect that there is a **colourful** solution; all the vertices of  $V'$  have different colours.
- ▶  $G$  uniformly at random with  $k$  colors, a set of  $k$  distinct vertices will obtain different colours with probability  $(k!)/k^k$ . This probability is lower-bounded by  $e^{-k}$ , so we need to repeat the process  $e^k$  times to have high probability of obtaining the required colouring.

- ▶  $k$ -PATH has a  $2^{O(k)} \cdot \log |V|$  colourings, and  $k!$  orderings.  $k$ -path in time  $O(k \cdot |V|^2)$ , using derandomization techniques.
- ▶ Recent improvements using multilinear detection and finite fields. (Kanj, Lokshtanov and Saurabh)

# BOUNDED WIDTH METRICS

- ▶ Graphs constructed inductively. Treewidth, Pathwidth, Branchwidth, Cliquewidth mixed width etc.  $k$ -Inductive graphs, plus old favourites such as planarity etc, which can be viewed as **local width**. e.g.:

## DEFINITION

Let  $G = (V, E)$  be a graph. A **tree decomposition**,  $TD$ , of  $G$  is a pair  $(T, \mathcal{X})$  where

1.  $T = (I, F)$  is a tree, and
2.  $\mathcal{X} = \{X_i \mid i \in I\}$  is a family of subsets of  $V$ , one for each node of  $T$ , such that

(i)  $\bigcup_{i \in I} X_i = V$ ,

(ii) for every edge  $\{v, w\} \in E$ , there is an  $i \in I$  with  $v \in X_i$  and  $w \in X_i$ , and

(iii) for all  $i, j, k \in I$ , if  $j$  is on the path from  $i$  to  $k$  in  $T$ , then  $X_i \cap X_k \subseteq X_j$ .

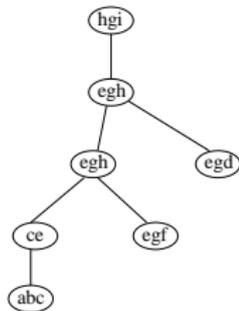
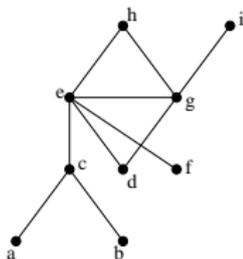
- ▶ This gives the following well-known definition.

### DEFINITION

The **width** of a tree decomposition  $((I, F), \{X_i \mid i \in I\})$  is  $\max_{i \in I} |X_i| - 1$ . The treewidth of a graph  $G$ , denoted by  $tw(G)$ , is the minimum width over all possible tree decompositions of  $G$ .

- ▶ The following refers to any of these inductively defined graphs families. Notes that many commercial constructions of, for example chips are inductively defined.
  1. Find a bounded-width tree (path) decomposition of the input graph that exhibits the underlying tree (path) structure.
  2. Perform dynamic programming on this decomposition to solve the problem.

# AN EXAMPLE FOR INDEPENDENT SET



$\emptyset$	a	b	c	ab	ac	bc	abc
0	1	1	1	2	-	-	-

# BODLAENDER'S THEOREM

- ▶ The following theorem shows that treewidth is FPT. Improves many earlier results showing this. The constant is about  $2^{35k^3}$ .

## THEOREM (BODLAENDER)

*k*-TREEWIDTH is linear time FPT

- ▶ **Not** practical because of large hidden  $O$  term.
- ▶ Unknown if there is a practical FPT treewidth algorithm
- ▶ Nevertheless approximation and algorithms specific to known decomps run well at least sometimes.

# MONADIC SECOND ORDER LOGIC

- ▶ Two sorted structure with variables for sets of objects.
- ▶ 1. **Additional atomic formulas:** For all set variables  $X$  and individual variables  $y$ ,  $Xy$  is an MSO-formula.
- ▶ 2. **Set quantification:** If  $\phi$  is an MSO-formula and  $X$  is a set variable, then  $\exists X \phi$  is an MSO -formula, and  $\forall X \phi$  is an MSO-formula.
- ▶ Eg  $k$ -col

$$\exists X_1, \dots, \exists X_k \left( \forall x \bigvee_{i=1}^k X_i x \wedge \forall x \forall y \left( E(x, y) \rightarrow \bigwedge_{i=1}^k \neg (X_i x \wedge X_i y) \right) \right)$$

- ▶ **Instance:** A structure  $\mathcal{A} \in \mathcal{D}$ , and a sentence (no free variables)  $\phi \in \Phi$ .  
**Question:** Does  $\mathcal{A}$  satisfy  $\phi$ ?
- ▶ PSPACE-complete for FO and MSO.

# COURCELLE'S AND SEESE'S THEOREMS

## THEOREM (COURCELLE 1990)

*The model-checking problem for MSO restricted to graphs of bounded treewidth is linear-time fixed-parameter tractable.*

Detlef Seese has proved a converse to Courcelle's theorem.

## THEOREM (SEESE 1991)

*Suppose that  $\mathcal{F}$  is any family of graphs for which the model-checking problem for MSO is decidable, then there is a number  $n$  such that, for all  $G \in \mathcal{F}$ , the treewidth of  $G$  is less than  $n$ .*

# THE FRICK GROHE THEOREM

- ▶ considers the treewidth growth rate for **families** of graphs
- ▶ Examples Bounded degree, bounded treewidth, bounded genus, excluding a minor

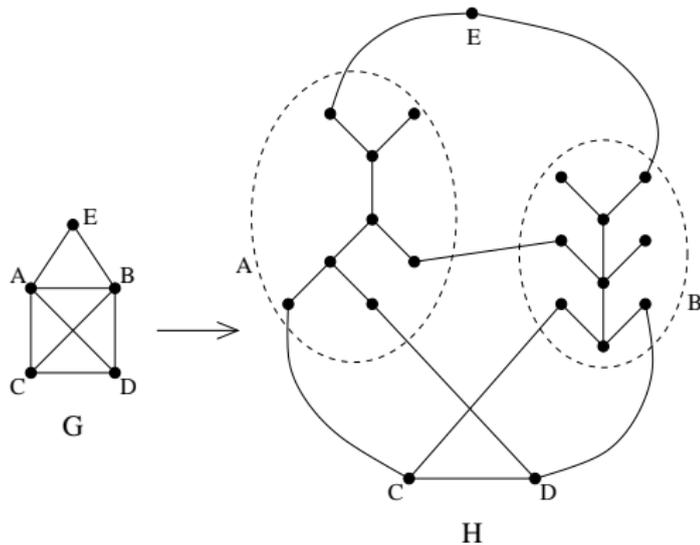
## THEOREM (FRICK AND GROHE 1999)

*Parameterized problems that can be described as model-checking problems for FO are fixed-parameter tractable on classes of graphs of bounded local treewidth.*

- ▶ For example DOMINATING SET, INDEPENDENT SET, or SUBGRAPH ISOMORPHISM are FPT on planar graphs, or on graphs of bounded degree

# MORE EXOTIC METHODS

- ▶ minor ordering



- ▶ Robertson-Seymour Finite graphs are WQO's under minor ordering.  $H \leq_{\text{minor}} G$  is  $O(|G|^3)$  FPT for a fixed  $H$ .

▶ THEOREM (MINOR-CLOSED MEMBERSHIP)

*If  $\mathcal{F}$  is a minor-closed class of graphs then membership of a graph  $G$  in  $\mathcal{F}$  can be determined in time  $O(f(k) \cdot |G|^3)$ , where  $k$  is the collective size of the graphs in the obstruction set for  $\mathcal{F}$ .*

- ▶ Likely I won't have time to discuss what this means but see DF for more details.

- ▶ Natural basic hardness class:  $W[1]$ . Does not matter what it is, save to say that the analog of Cook's Theorem is SHORT NONDETERMINISTIC TURING MACHINE ACCEPTANCE

**Instance:** A nondeterministic Turing Machine  $M$  and a positive integer  $k$ .

**Parameter:**  $k$ .

**Question:** Does  $M$  have a computation path accepting the empty string in at most  $k$  steps?

- ▶ If one believes the philosophical argument that Cook's Theorem provides compelling evidence that SAT is intractable, then one surely must believe the same for the parametric intractability of SHORT NONDETERMINISTIC TURING MACHINE ACCEPTANCE.
- ▶ Moreover, recent work has shown that if SHORT NTM is fpt then  $n$ -variable 3SAT is in  $\text{DTIME}(2^{o(n)})$

- ▶ Given two parameterized languages  $L, \widehat{L} \subseteq \Sigma^* \times \Sigma^*$ , say  $L \leq_{FPT} \widehat{L}$  iff there are (computable)  $f, x \mapsto x', k \mapsto k'$  and a constant  $c$ , such that for all  $x$ ,

$$(x, k) \in L \text{ iff } (x', k') \in \widehat{L},$$

in time  $f(k)|x|^c$ .

- ▶ Lots of technical question still open here.

# ANALOG OF THE COOK-LEVIN THEOREM

## THEOREM (DOWNEY, FELLOWS, CAI, CHEN)

WEIGHTED 3SAT  $\equiv_{FTP}$  SHORT NTM ACCEPTANCE.

WEIGHTED 3SAT

*Input: A 3 CNF formula  $\phi$*

*Parameter:  $k$*

*Question: Does  $\phi$  has a satisfying assignment of Hamming weight  $k$ , meaning exactly  $k$  literals made true.*

- ▶ Many hardness results, like e.g. DOMINATING SET, CLIQUE, INDEPENDENT SET, etc

- ▶ Think about the usual poly reduction from SAT to 3SAT. It takes a clause of size  $p$ , and turns it into many clauses of size 3. **But** the weight control goes awry. A weight 4 assignment could go to anything.
- ▶ We **don't think**  $\text{WEIGHTED CNF SAT} \leq_{ftp} \text{WEIGHTED 3 SAT}$ .
- ▶ Gives rise to a hierarchy:

$$W[1] \subseteq W[2] \subseteq W[3] \dots W[\text{SAT}] \subseteq W[P] \subseteq XP.$$

- ▶  $XP$  is quite important, it is the languages which are in  $\text{DTIME}(n^f(k))$  with various levels of uniformity, depending on the choice of reductions.

- ▶ Notice that there are at least two ways to parameterize:  
Parameterize the part of the problem you want to look at  
and to parameterize the problem itself.
- ▶ This point of view makes this sometime a promise  
problem. Input something, promise it is parameterized, and  
ask questions about it.
- ▶ **Two interpretations** one with certificate one only with a  
promise. e.g. CLIQUEWIDTH, PATHWIDTH.
- ▶ Some recent work “lowers the hardness barrier”; perhaps  
giving better inapproximability results.

- ▶ Recall the exponential time hypothesis is (ETH)  $n$ -variable 3-SATISFIABILITY is not solvable in  $\text{DTIME}(2^{o(n)})$ . (Impagliazzo Paturi and Zane.) Recall, this refinement of  $P \neq NP$  says **not only does  $P \neq NP$  but you can't beat complete search meaningfully.**
- ▶ This is seen an important refinement of  $P \neq NP$  that is widely held to be true.
- ▶ It is related to FPT as we now see.

# THE MINIMOB

- ▶ INPUT A parametrically miniature problem  
QUESTION Is it a yes instance?  
e.g. INPUT a graph  $G$  of size  $k \log n$  with  $n, k$  in unary.  
Does it have a vertex cover of size  $d$ ?
- ▶ Get mini Vertex cover, mini Dominating set, Minisat etc.
- ▶ Core problem: minicircuitsat.

THEOREM (CHOR, FELLOWS AND JUEDES ; DOWNEY ET. AL. )

*The  $M[1]$  complete problems such as MIN-3SAT are in FPT iff the exponential time hypothesis **fails**.*

- ▶ That is, more or less, EPT is the “same” as  $M[1] \neq FPT$ .
- ▶ **And now we have a method of demonstrating no good subexponential algorithm; Show  $M[1]$  hardness.**
- ▶ Chen-Grohe established an isomorphism between the complexity degree structures.

- ▶ This new programme regards the classes like  $W[1]$  as artifacts of the basic problem of proving hardness under reasonable assumptions, and strikes at membership of XP, and **realizes the dream of upper and lower bounds matching**.
- ▶ Eg INDEPENDENT SET and DOMINATING SET which certainly are in XP. But what's the best exponent we can hope for for slice  $k$ ? They are clearly solvable in time  $O(n^{k+1})$ .

## THEOREM (CHEN ET. AL 05)

*The following hold:*

- INDEPENDENT SET *cannot be solved in time  $n^{o(k)}$  unless  $FPT=M[1]$ .*
- DOMINATING SET *cannot be solved in time  $n^{o(k)}$  unless  $FPT=M[2]$ .*

- ▶ The proofs recycle and miniaturize various NP and W[1] completeness results.
- ▶ **Many** recent results of similar ilk based on ETH or SETH, such as results on treewidth etc.
- ▶ For example, **Most known algorithms for e.g. treewidth problems are optimal.**

# WHERE ELSE?

- ▶ Another area is approximation. Here we ask for an algorithm which either says “no solution of size  $k$ ” or here is one of size  $2k$  (say).
- ▶ For example BIN PACKING is has to  $(k, 2k)$ -approx, but  $k$ -INDEPENDENT DOMINATING SET has not approx of the form  $(k, F(k))$  for any computable  $F$  unless  $FPT = W[1]$ . (DFMccartin)
- ▶ Flum Grohe show that all natural  $W[P]$  complete problems don't have approx of the form  $(k, F(k))$  for any computable  $F$  unless  $FPT = W[P]$ .

# REMEMBER KERNELIZATION?

- ▶ When can we show that a FPT problem likely has no polynomial size kernel?
- ▶ Notice that if  $P=NP$  then all have constant size kernel, so some reasonable assumption is needed.

## DEFINITION (BODLAENDER, DOWNEY, FELLOWS, HERMELIN)

An **OR-distillation algorithm** for a classical problem  $L \subseteq \Sigma^*$  (like SAT is an algorithm that

- ▶ receives as input a sequence  $(x_1, \dots, x_t)$ , with  $x_i \in \Sigma^*$  for each  $1 \leq i \leq t$ ,
- ▶ uses time polynomial in  $\sum_{i=1}^t |x_i|$ ,
- ▶ and outputs a string  $y \in \Sigma^*$  with
  1.  $y \in L \iff x_i \in L$  for some  $1 \leq i \leq t$ .
  2.  $|y|$  is polynomial in  $\max_{1 \leq i \leq t} |x_i|$ .
- ▶ Similarly AND-distillation.

# THE FORTNOW-SANTHANAM LEMMA

## LEMMA (FORTNOW AND SANTHANAM 2007)

*If any NP complete problem has a distillation algorithm then  $PH = \Sigma_3^P$ . That is, the polynomial time hierarchy collapses to three or fewer levels*

- ▶ Here  $\Sigma_3^P$  is  $NP^{NP^{NP}}$ .
- ▶ Strictly speaking the prove that  $co - NP \subseteq NP \setminus poly$ .

# HOW DOES THIS RELATE TO KERNELIZATION?

- ▶ Bodlaender, Downey, Fellows, Hermelin identified a property called **composition** and proved the following.

**LEMMA (BODLAENDER, DOWNEY, FELLOWS, HERMELIN)**

*Let  $L$  be a compositional parameterized problem whose derived classical problem  $L_c$  is NP-complete. If  $L$  has a polynomial kernel, then  $L_c$  is also distillable.*

- ▶ One guarantee that a problem is compositional if the disjoint union of two instances is a yes iff at least one is a yes with the same parameter. (BDFH)

- ▶  $k$ -PATH,  $k$ -CYCLE,  $k$ -CHEAP TOUR,  $k$ -EXACT CYCLE, and  $k$ -BOUNDED TREEWIDTH SUBGRAPH
- ▶  $k, \sigma$ -SHORT NONDETERMINISTIC TURING MACHINE COMPUTATION (Needs work)
- ▶ **Many** recent examples, Bodlaender, Kratch, Lokshantov, Saurabh etc. Also using (poly,poly)-reductions, co-nondeterminism, small interactive protocols, etc.
- ▶ Now a big cottage industry.

# AND-COMPOSITION AND DISTILLATION

- ▶ A just graduated PhD student Andrew Drucker from MIT who has shown this also implies collapse. This implies all the below don't have poly kernels, unless.... The proof is remarkable.
- ▶ Applications: Graph width metrics:
- ▶ CUTWIDTH, TREewidth, PROBLEMS WITH TREewidth PROMISES, EG.. COLOURING

## OTHER RESULTS

- ▶ Burhmann and Hitchcock: There are no subexponential size hard sets for NP unless PH collapses. (le **many** hard instances)
- ▶ Using transformations, Bodlaender, Thomasse and Yeo show that DISJOINT CYCLES, HAMILTON CIRCUIT PARAMETERIZED BY TREewidth etc don't have poly kernels unless collapse.
- ▶ Also the important DISJOINT PATHS, famously FPT by Robertson and Seymour.
- ▶ Similarly using Dell-Mecklebeek, Kratz showed the non-poly-kernelizability of  $k$ -RAMSEY.
- ▶ Fernau et. al. have shown that there are problems with **Poly Turing Kernels** but **no** poly kernels unless collapse.(!), and these are natural related to spanning trees (Namely DIRECTED  $k$  LEAF SPANNING TREE).

# THE BIGGEST CHALLENGE

- ▶ **Explain** why heuristics deliver. e.g.
- ▶ Parosh-Abdullah infinite state verification,
- ▶ Simplex method
- ▶ Sat solvers (Remember: Sat was supposed to be toxic)
- ▶ Huge linear programmes.

## SOME REFERENCES

- ▶ Parameterized Complexity, springer 1999 DF
- ▶ Invitation to Parameterized Algorithms, 2006 Niedermeier, OUP
- ▶ Parameterized Complexity Theory, 2006, Springer Flum and Grohe
- ▶ Theory of Computing Systems, Vol. 41, October 2007
- ▶ Parameterized Complexity for the Skeptic, D, proceedings CCC, Aarhus, (see my homepage)
- ▶ The Computer Journal, (ed Downey, Fellows, Langston)
- ▶ Confronting intractability via parameters, Downey Thilikos, Computing Reviews
- ▶ Fundamentals of Parameterized Complexity, Downey-Fellows, this year.

# WHAT SHOULD YOU DO?

- ▶ You should buy that **new** wonderful book...(and its friends)
- ▶ **Thanks!**