Generalized Collatz Functions and Computability

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Collatz Conjecture

- Define **Collatz function** as
$$g(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ 3n+1 & \text{if } n \text{ odd} \end{cases}.$$
- Consider $g(n), g(g(n)), g(g(g(n))), \ldots$. Lothar Collatz (1932) conjectures that $\forall n\ \exists k$ $g^{(k)}(n) = 1$. i.e., that $g^{(t)}(n)$ is a power of 2, for some t.
- Least k with $g^{(k)}(27) = 1$, is 111, and there is some $j \leq 111$ with $g^{(j)}(27) = 9232$.
- Erdos "Mathematics is not ready for such questions."

- Is $\{n : \exists k[g^{(k)}(n) = 1]\}$ a computable set of numbers. That is, is the question "On input n decide whether $\exists k[g^{(k)}(n) = 1]$" an alkgorithmically decidable question.

- *Known* $\{n : \exists k[g^{(k)}(n) = 1]\}$ has density 1.

Conway's Generalization

- Given $d \in \mathbb{N}$, and $a_i, b_i \in \mathbb{Q}^{\geq 0}$, for $0 \leq i < d$, define

  $$g(n) = a_i n + b_i \text{ if } n \equiv i(\mod d)$$

- We assume $g(n) \in \mathbb{N}$. Henceforth, $g = g_{\langle a1,\ldots,a_d,b_1,\ldots,b_d \rangle}$.
- Conways question:
  What, if anything, can be said of problems with input $\langle a1, \ldots, a_d, b_1, \ldots, b_d \rangle$?
- For instance, can we decide whether $\forall n \; \exists k \; g^{(k)}(n)$ is a power of 2?
- What about if $b_i = 0$ for all i so that $\frac{g(n)}{n}$ is periodic?

Conway's answer

- Nothing!

- well nothing algorithmic anyway!
- The rest of this lecture is devoted to proving nothing.

Computability Theory

- "Computable" functions $f : \mathbb{N} \mapsto \mathbb{N}$ means that we have an algorithm for computing $f(n)$ from n.
- Things like strings, rationals etc are coded as integers, so no loss of generality using $\mathbb{N}$.
- A set B is called *computably enumerable* c.e. if it is the range of a computable function.
  $B = \{f(0), f(1), \dots\}$.

- Equivalently B is the *domain* of a computable *partial* function meaning that it is a function whose domain is a *subset* of $\mathbb{N}$. (Put n into the range of f once it is in the domain of $\varphi_s = \{x \leq s : \varphi(x) \downarrow [s]$ (that is, halts at stage s.)$\}$)
- Think of these sets as those for which we have an algorithm that "in theory" would list all the members, but not *in order*. We don't seem to be able to know if the set is *computable* in th sense that we can also list the complement.
- The most famous c.e. set : The halting problem $Halt = \{\langle x, y \rangle :$ the x-program halts on input $y\}$.

- This set is c.e. but non-computable by a simple diagonal argument. (Gödel, Turing, Post, etc.)

- For suppose that H was computable. Then considerthe function $d(x) = \varphi_x(x) + 1$ if $\varphi_x(x) \downarrow$ ($\downarrow$=halts), and $d(x) = 0$ otherwise. Then d would itself be computable, and hence $d = \varphi_z$ for some z. But d halts everywhere, and hence $d(z) = \varphi_z(z) + 1 = \varphi_z(z)$ and so $0 = 1$.

Conway's Theorem

- (Conway) If φ is partial computable, then there is a g, as defined above such that

  (i). $\forall n \in \operatorname{dom} \varphi$
  $2^{\varphi(n)}$ is the first power of 2 in: $g(2^n), g^{(2)}(2^n), \ldots$

  (ii). $\forall n \notin \operatorname{dom} \varphi$
  there is no power of 2 in: $g(2^n), g^{(2)}(2^n), \ldots$

- (Corollary) There is no algorithm to decide $\forall n \ \exists k$ such that $g^{(k)}(n)$ is a power of 2.

- The proof works by a series of *reductions*.
- We will say that A *reduces* to B would mean, that the ability to decide B would enable us to decide A.
- In fact, To prove Conway's theorem, we prove a series of *equivalences*:

$$\begin{array}{c}\text{Minsky machine}\\ \downarrow \\ \text{Vector games}\\ \downarrow \\ \text{Rational games}\\ \downarrow \\ \text{Generalised Collatz}\end{array}$$

Minsky machines

- A **Minsky Machine** *or register machine*) is a finite set of registers $\langle R_1, \ldots, R_n \rangle$ with a finite list of numbered instructions, each being one of the following

  – R_i^+, go to ⟨line number⟩

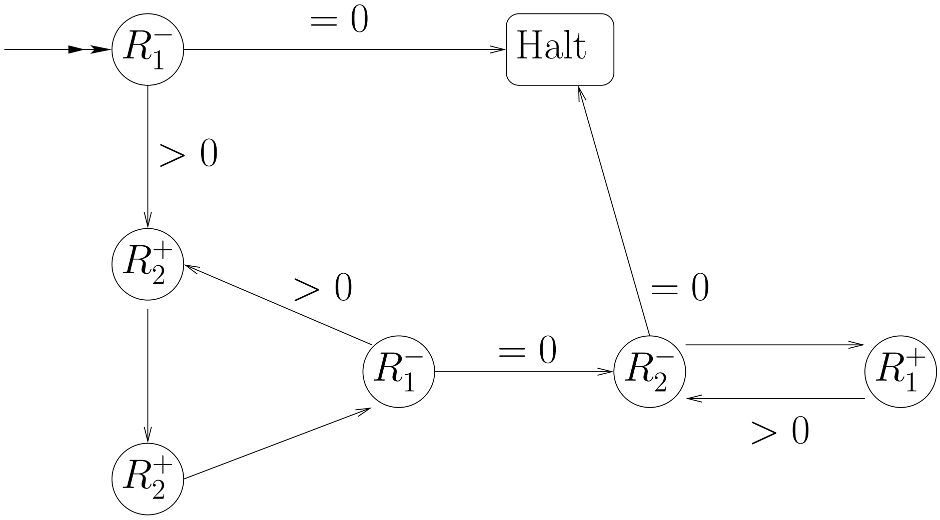
  – R_i^-, if $= 0$ go to ⟨line number⟩
    if > 0 go to ⟨line number⟩

  – Halt

- A register machine *simulates* a partial function $f : \mathbb{N} \to \mathbb{N}$ if when started on line 0 with $\langle x, 0, 0, \ldots, 0 \rangle$ the machine produces $\langle f(x), 0, 0, \ldots, 0 \rangle$.

- An example: $f(x) = 2x$

0. R_1^- if $= 0$ go to 6
   if > 0 go to 1
1. R_2^+ go to 2
2. R_2^+ go to 3
3. R_1^- if $= 0$ go to 4
   if > 0 go to 1
4. R_2^- if $= 0$ go to 6
   if > 0 go to 5
5. R_1^+ go to 4
6. Halt

- 

- φ is partial computable iff φ can be simulated on a Minsky machine.

Vector Games

- A **vector game** is a finite ordered list of vectors $L = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^n$, with $v_i \in \mathbb{R}^+ \cup \{0\}$.
- To play the game:
  Take as input some vector $v \in \mathbb{R}^+$
  Define $g_L(v) = v + v_i$, where i is least so that $v + v_i$ is non-negative
- Compute $g_L(v), g_L^{(2)}(v), \ldots,$ etc.

- A game g_L simulates f if, on input $\langle k, 0, \ldots, 0\rangle$

(i). $\forall k \in \operatorname{dom} f$

$\langle f(k), 0, \ldots, 0\rangle$ is the first vector of form $\langle k', 0, \ldots, 0\rangle$

(ii). $\forall k \notin \operatorname{dom} f$

there never appears a vector of form $\langle k', 0, \ldots, 0\rangle$

- An example

$$
\begin{aligned}
L : & \langle 0,0,0|1,-1\rangle = v_1 \\
& \langle -1,0,0|-3,1\rangle = v_2 \\
& \langle 0,0,0|-3,0\rangle = v_3 \\
& \langle 0,0,1|-2,3\rangle = v_4 \\
& \langle 0,1,0|-1,2\rangle = v_5 \\
& \langle -1,0,0|0,1\rangle = v_6
\end{aligned}
$$

- So on input $x = \langle 3,0,0|0,0\rangle$, we get the sequence

$$g(x) = \langle 2,0,0|0,1\rangle \qquad (v_6)$$

$$g^{(2)}(x) = \langle 2,0,0|1,0\rangle \qquad (v_1)$$

$$g^{(3)}(x) = \langle 2,1,0|0,2\rangle \qquad (v_5)$$

$$g^{(4)}(x) = \langle 2,1,0|2,0\rangle \qquad (v_1)$$

$$g^{(5)}(x) = \langle 2,1,1|0,3\rangle \qquad (v_4)$$

$$g^{(6)}(x) = \langle 2,1,1|3,0\rangle \qquad (v_1)$$

$\cdots$ etc

First reduction

- (Conway) For any partial computable φ there is an L such that

$$g_L = \varphi$$

- Given any partial computable φ, there is a Minsky Machine to compute it. We can then transform this MM into a vector game as follows:

- Suppose we have a MM with n registers, then we construct vectors of arity $n + 2$.

- For $1 \leq i \leq n$, the i^{th} position in each vector corresponds to addition/subtraction on register R_i.

- The $n+1^{\mathrm{th}}$ position corresponds to the current MM instruction and
- the $n+2^{\mathrm{th}}$ position corresponds to the 'next' MM instruction

- To construct L, put $v = \langle 0, \ldots, 0 | 1, -1 \rangle$ at the top.

- then derive the other v_i from the MM, ordering them in decreasing order of k.

- If

$$\text{K} \longrightarrow R_i^+ \longrightarrow \text{t}$$

let

$$v_j = \langle 0, 0, \ldots, \underset{\text{position } i}{1}, \ldots, 0 | -k, t \rangle$$

- if

$$\begin{tikzcd} & & t_1 \\ K \arrow[r] & R_i^- \arrow[ur, "> 0"] \arrow[dr, "=0"'] & \\ & & t_2 \end{tikzcd}$$

let

$$v_j = \langle 0, 0, \ldots, \overset{\text{position } i}{-1}, \ldots, 0 | -k, t_1 \rangle$$
$$v_{j+1} = \langle 0, 0, \ldots, 0, \ldots, 0 | -k, t_2 \rangle$$

- Halt is $\langle 0, \ldots, 0 | -h, 0 \rangle$

- We can then prove that each MM is emulated by a Vector game.

- the point is that as we run down then the last two registers will mean that the first thing to be considered is a k where the "$-k$" part remains non-zero. Then if the instruction wa sto add one to register i we'd do this to the i-th position of the vector, and return t in the last vector position. Note that this is the next place the MM would like to go to. Then the top vector would ve invoked transferring the t to the 2nd last position. The negative case is similar.

Second reduction

- A **rational game** is a finite list of ordered rationals $L = \{r_1, \ldots, r_n\}$.
- To play the game:
  Take as input some $n \in \mathbb{N}$
  Define $g_L(n) = r_i n$, where i is least so that $nr_i \in \mathbb{N}$
  Compute $g_L(n), g_L^{(2)}(n), \ldots$ etc.

- A game g_L simulates a partial function φ if, on input 2^n

(i). $\forall n \in \operatorname{dom} \varphi$

$2^{\varphi(n)}$ is the first power of 2 in $g_L(2^n), g_L^{(2)}(2^n), \ldots$

(ii). $\forall n \notin \operatorname{dom} \varphi$

there is no power of 2 in $g_L(2^n), g_L^{(2)}(2^n), \ldots$

- We can emulate vector games by rational games, where $v_i = \langle a, b, c, \ldots \rangle$ is replaced by $r_i = 2^a 3^b 5^c \cdots$
- example: $\langle 0, 0, 0 | 1, -1 \rangle$ becomes $2^0 3^0 5^0 7^1 11^{-1} = \frac{7}{11}$.

- Vector game $\to$ Rational game:
- For a vector game of arity k, let $2, 3, 5, \ldots, p_k$ be the first k prime numbers.
- Then, for all vector $v_i = \langle v_{i1}, v_{i2}, \ldots, v_{ik} \rangle$ in the game, encode as $2^{v_{i1}} 3^{v_{i2}} \cdots p_k^{v_{ik}} = r_i$ and hence preserving order of the vectors in the order of the rationals.
- So the input vector $v = \langle x, 0, \ldots, 0 \rangle$ translate to 2^x, as expected by the rational game.
  And the output vector $v = \langle f(x), 0, \ldots, 0 \rangle$ translates to $2^{f(x)}$

as expected also, for all $x \in \operatorname{dom} f$. Choosing the first vector v_i in order to preserve $\langle v_1, \ldots, v_k \rangle \in \mathbb{R}^+$ corresponds tp choosing the first rational to preserve $2^{v_1} 3^{v_2} \cdots p_k^{v_k} \in \mathbb{N}^+$.

- Find the highest prime needed for representation of a rational in the rational game, say p_k.
  Then make a vector game of arity k by constructing a vector v_i from each r_i as follows:

- Let $r_i = 2^{r_{1i}} 3^{r_{2i}} \cdots p_k^{r_{ki}}$, then $v_i = \langle r_{1i}, r_{2i}, \ldots, r_{ki} \rangle$.
  Then the input rational $r = 2^x$ corresponds to the vector $\langle x, 0, \ldots, 0 \rangle$ as expected by the vector game. And the output rational $r = 2^{f(x)}$ corresponds to the vector $\langle f(x), 0, \ldots, 0 \rangle$ as expected also, for all $x \in \operatorname{dom} f$.

- Choosing the first rational to preserve $2^{r_{1i}}3^{r_{2i}}\cdots p_k^{r_{ki}} \in \mathbb{N}^+$ corresponds to choosing the first vector to preserve $\langle r_{1i}, r_{2i}, \ldots, r_{ki}\rangle \in \mathbb{R}^+$.

Last reduction

- (Lemma) Given $r_1, \ldots, r_n \in \mathbb{Q}^+$, $x \in \mathbb{N}$ and a rational game $f(x) = xr_i$ (where i least such that $xr_i \in \mathbb{N}^+$), then there is a number p and pairwise disjoint sets $D_1, D_2, \ldots, D_n$ with $\bigcup_{i=1}^{n} D_i = \{0, 1, \ldots, p-1\}$ such that $f(x) = r_i x$ where i is uniquely determined by $x \equiv y (\mod p)$ for some $y \in D_i$.
- i.e., Rational game $\equiv$ Generalised Collatz

- From a rational game we can construct a generalised Collatz function as follows

- Let $r_i = \frac{a_i}{b_i}$, $a_i, b_i \in \mathbb{N}^+$, and $p = b_1 b_2 \cdots b_n$.

- Note: $r_i x \in \mathbb{N}$ iff $b_i | a_i x$, i.e., iff $p | (\frac{p}{b_i}) a_i x$

- Construct sets D_i by

$$D_1 = \{x < p : p | (\frac{p}{b_i}) a_i x\}$$

$$D_{j+1} = \{x < p : x \notin \bigcup_{i=1}^{j} D_i$$

$$\text{and } p | \frac{p}{b_{j+1}} a_{j+1} x\}.$$

- Now it is exam time.

- Thank you.