ON THE GEOMETRY OF PLANAR MOTIONS

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[Received 11 June 1991]

1. Motions and Centrodes

THERE are a number of ways of describing the motion of a rigid body in $\mathbb{R}^2$. The most general is to regard the motion as a subset of the total configuration space of the body. If the body is under no constraints then this space can be represented by the Euclidean group $E(2)$ of orientation-preserving isometries of $\mathbb{R}^2$, which is isomorphic to a semi-direct product $SO(2) \times \mathbb{R}^2$ of rotation and translation subgroups. So a motion with one degree of freedom can be represented as a continuous map μ: $M \to E(2)$ where M is a one-dimensional manifold. If the body is subject to constraints then the configuration space may be embedded as an algebraic set in a Euclidean space of an n-dimensional torus (see, for example, [7]). A further description can be given in terms of the *centrodes* of the motion defined below.

Suppose, for some $x_0 \in M$, there is a local parametrisation φ: $I \to M$ where $I \subseteq \mathbb{R}$ is an open interval, $0 \in I$ and $\varphi(0) = x_0$. We may replace μ locally by the map $\mu \circ \varphi$: $I \to E(2)$. For our purposes a *planar motion* shall be a map μ: $I \to E(2)$.

Suppose that for each $t \in I$, $\mu(t) = (A(t), a(t)) \in SO(2) \times \mathbb{R}^2 \cong E(2)$; then $A(t)$ can be lifted to the universal cover $\mathbb{R}$ of $SO(2)$ giving a map θ: $I \to \mathbb{R}$ such that

$$A(t) = \exp \theta(t)J = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{bmatrix} \quad \text{where} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For any $x \in \mathbb{R}^2$, its *trajectory* is given by $\Phi_x(t) = \mu(t) \cdot x = A(t)x + a(t)$.

It is a classical theorem of kinematics that every displacement of a plane rigid body can be achieved either by rotation about some point in the plane or by translation. If it is the case that for all $t \neq 0$ in some neighbourhood of $0 \in I$ the former holds and that the limit as $t \to 0$ of the centres of rotation exists then the limit is called the *instantaneous centre of rotation*. The locus of instantaneous centres for all $t \in I$ is called the *fixed* (respectively, *moving*) *centrode* if it is constructed in the coordinates of the ambient space (respectively, the coordinates of the body). Denote the centrodes by c_f, c_m. Then clearly $c_f(t) = \mu(t) \cdot c_m(t)$.

In the case that a motion has well-defined, non-singular centrodes, the motion can be generated by "rolling" the moving centrode on the fixed centrode. This is a typical way to describe a motion since for many mechanisms, e.g. gears, rollers, the motion is generated by the non-sliding contact of (the profiles of) components.

Given $\mu(t)=(A(t), a(t))$ differentiable, x is an instantaneous centre at $t=t_0$ provided $\dot{A}(t_0)\cdot x+\dot{a}(t_0)=0$. It is clear that a unique x exists if and only if $\det \dot{A}(t_0)\neq 0$ and then $x=-\dot{A}^{-1}\dot{a}$. Since $\dot{A}=\dot{\theta}JA$, this is equivalent to $\dot{\theta}(t_0)\neq 0$. By definition, the trajectory of an instantaneous centre has a singularity; in general, other trajectories are non-singular.

Suppose c_m, c_f are immersed one-dimensional manifolds in $\mathbb{R}^2$. They may be parametrised by arc-length measured from some given point on each. The corresponding motion $(A(s), a(s))$ can then be derived from the equations

$$\begin{aligned} c_f(s)&=A(s)c_m(s)+a(s)\\ \dot{c}_f(s)&=A(s)\dot{c}_m(s). \end{aligned} \tag{1}$$

This really expresses the rolling construction in a formal way.

Since we shall mainly consider motions locally (in time) it will usually be sufficient to consider germs of motions. Let $\mathcal{M}$ denote the set of germs μ: $I, 0\to E(2)$ and $\mathcal{M}_0$ the subset for which $\mu(0)=\text{id}$. Let Diff^0 denote the group of diffeomorphism germs $I, 0\to I, 0$. Two germs μ_1: $I, 0\to E(2)$, ϕ_1 and μ_2: $I, 0\to E(2)$, ϕ_2 are *$\mathcal{I}$-equivalent* if there exist $h\in \text{Diff}^0$ and $\sigma, \tau\in E(2)$ such that the following diagram commutes:

$$\begin{array}{ccc} I, 0 & \xrightarrow{\mu_1} & E(2), \phi_1\\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \rho_{\sigma,\tau}}\\ I, 0 & \xrightarrow{\mu_2} & E(2), \phi_2 \end{array}$$

where $\rho_{\sigma,\tau}$ denotes the map $\phi\mapsto\sigma\phi\tau$. It is clear that any motion-germ is $\mathcal{I}$-equivalent to one through the identity in the isometry group, and that for two such germs the map ρ reduces to conjugacy in the group (i.e. $\tau=\sigma^{-1}$).

$\mathcal{I}$-equivalence is implicitly used in the classical analysis of planar motions—it allows us to assume that, provided the instantaneous centre at $t=0$ is well defined, then $c_m(0)=c_f(0)=0$, with the centrodes tangent to the x-axis (provided they are non-singular) and that the centrodes are parametrised by arc-length. On the other hand, it preserves the essential geometry of the motion.

Our aim is to explore the geometry of the trajectories of a motion and, in particular, to relate this to the nature of the singularity at the instantaneous centre. Much of the geometry can be found in the classical literature [**1**, **9**]. In particular, Veldkamp [**12**] has explored the degeneracies that can arise in planar motions, making use of Bottema, or instantaneous, invariants. We take a new viewpoint by regarding the motion as an unfolding of any trajectory and then asking the natural question: is this unfolding versal? One reason for considering versality is that it frequently turns out to be a generic property and, if so in this case, would determine what geometry we might usually expect to see.

The classical constructions include the inflection circle and cubic of stationary curvature, which are defined in § 2. They are given uniquely, up to $\mathcal{I}$-equivalence, by certain invariants. In § 3 we show that, in order to give an adequate theory, it is desirable to extend the notion of centrode. In § 4, the singularity types of trajectories are classified under $\mathcal{I}$-, right-left ($\mathcal{A}$-) and contact ($\mathcal{K}$-) equivalence. The key idea of unfolding action is introduced in § 5, together with some general theory of unfoldings. Conditions for versality are derived in § 6. The main theorems of the paper are in § 7. Theorems 7.1, 2, 5 and 6 relate the singularity types and versality conditions to the geometry of the motion. Corollary 7.4 and Theorem 7.7 establish genericity for versality under $\mathcal{K}$- and $\mathcal{A}$-equivalence respectively.

2. Local Geometry of Trajectories

Suppose that $\mu \in \mathcal{M}$ as above, that $c_m(0)$ is defined and $\dot{c}_m(0) \neq 0$. Those points $x = (x_1, x_2) \in \mathbb{R}^2$ whose trajectories have inflections at $t = 0$ are given by $\dot{\Phi}_x(0) \times \ddot{\Phi}_x(0) = 0$, which defines a conic in the plane. In fact it is a circle, called the *inflection circle*. If we assume that $c_m(0) = 0$ then the circle has equation $x_1^2 + x_2^2 - \frac{1}{[\dot{\theta}(0)]^2}(x_1\ddot{a}_1(0) + x_2\ddot{a}_2(0)) = 0$ or equivalently

$$x_1^2 + x_2^2 + \frac{1}{\dot{\theta}(0)}[x_1\dot{c}_{m2}(0) - x_2\dot{c}_{m1}(0)] = 0$$

so it is tangent to the moving centrode and has radius $r_1 = \frac{\|\ddot{a}(0)\|}{2[\dot{\theta}(0)]^2} = \frac{\|\dot{c}_m(0)\|}{2\dot{\theta}(0)}$ (cf [**9**, § 5.8]). Clearly r_I is an $\mathcal{I}$-invariant. Of interest here are the curvatures κ_f and κ_m of the fixed and moving centrodes. These are given by the equations (where $u \times v = u_1v_2 - u_2v_1$):

$$\begin{aligned} \|\dot{c}_m\|^3\kappa_m &= \frac{2\|\ddot{a}\|^2}{\dot{\theta}} + \frac{1}{\dot{\theta}^2}\dot{a} \times \ddot{a} \\ \|\dot{c}_f\|^3\kappa_f &= \frac{\|\ddot{a}\|^2}{\dot{\theta}} + \frac{1}{\dot{\theta}^2}\dot{a} \times \ddot{a}. \end{aligned} \tag{2}$$

Rearranging these gives the well known Euler–Savary equation [**9**,

§ 5.3.2]

$$\kappa_m - \kappa_f = \frac{1}{2r_I}. \tag{3}$$

Similarly, we may derive those points whose trajectories have stationary curvature at $t = 0$. These lie on the curve

$$\|\dot{\Phi}_x\|^2 (\dot{\Phi}_x \times \dddot{\Phi}_x) = 3(\dot{\Phi}_x \times \ddot{\Phi}_x)(\dot{\Phi}_x \cdot \ddot{\Phi}_x)$$

(all derivatives at $t = 0$) which reduces to a cubic

$$(x_1^2 + x_2^2)(x_1\ddot{a}_2 - x_2\ddot{a}_1) + \frac{1}{3\dot{\theta}}(x_1^2 + x_2^2)(x_1\dddot{a}_1 + x_2\dddot{a}_2)$$
$$-\frac{\ddot{\theta}}{\dot{\theta}^2}(x_1^2 + x_2^2)(x_1\ddot{a}_1 + x_2\ddot{a}_2) - \frac{1}{\dot{\theta}^2}(x_1\ddot{a}_2 - x_2\ddot{a}_1)(x_1\ddot{a}_1 + x_2\ddot{a}_2) = 0. \tag{4}$$

The cubic of stationary curvature passes twice through the instantaneous centre, once tangent to and once normal to the centrodes. Both these points are intersections with the inflection circle and there is a further intersection, called Ball's point, where the line

$$\ddot{\theta}(x_1\ddot{a}_1 + x_2\ddot{a}_2) = \tfrac{1}{3}\dot{\theta}(x_1\dddot{a}_1 + x_2\dddot{a}_2) \tag{5}$$

meets the circle. The trajectory of a Ball's point therefore has a higher inflection at $t = 0$. The locus of Ball's points is called Ball's curve.

There are two numerical $\mathscr{I}$-invariants which completely determine the cubic. Rotating the coordinate system so that $\ddot{a}_1 = 0$, these can be written as

$$\frac{1}{M} = \frac{3\dot{\theta}^2\ddot{a}_2 + \dot{\theta}\dddot{a}_1}{3\ddot{a}_2^2} \quad \text{and} \quad \frac{1}{N} = \frac{3\ddot{\theta}\ddot{a}_2 + \dot{\theta}\dddot{a}_2}{\ddot{a}_2^2}$$

and the cubic then becomes

$$\left(\frac{x}{M} + \frac{y}{N}\right)(x^2 + y^2) - xy = 0.$$

The slope of the asymptotic line to the cubic is $-N/M$. The invariants can also be expressed as

$$\frac{1}{M} = \frac{1}{3}(2\kappa_m - \kappa_f), \qquad \frac{1}{N} = -\frac{1}{3}\frac{\dot{r}_I}{\|\dot{c}_m\|\, r_I}. \tag{6}$$

(see [**9**, § 5.7]). Moreover $M/2$ and $N/2$ are respectively the radii of curvature of the cubic at 0 tangent to and normal to the centrodes.

3. Extending the Action and Centrodes

When a motion is instantaneously translatory ($\dot{A}=0$), the centrodes are not defined. It is natural to embed $\mathbb{R}^2$ in the real projective plane $\mathbb{P}^2$ and extend the notion of instantaneous centre to include points at infinity, as follows.

The action of $E(2)$ can readily be homogenised by embedding $E(2)$ in $PSL(3)$: if $\mu=(\exp \theta J, a)\in E(2)$ and $p=(x,y,w)\in\mathbb{P}^2$ (in homogeneous coordinates) then $\mu\cdot p=(x\cos\theta+y\sin\theta+a_1w, -x\sin\theta+y\cos\theta+a_2w, w)$. This clearly leaves the line at infinity $\ell_\infty(w=0)$ invariant and restricts to the standard action on $\mathbb{R}^2(w=1)$. Moreover, if $(x,y,0)$ are homogeneous coordinates of a point in ℓ_∞, set $\tan\alpha=\frac{y}{x}$. Then α defines a local coordinate on ℓ_∞ and the action of $E(2)$ in this coordinate is given by $\mu\cdot\alpha=\alpha+\theta$.

In [4], infinitesimal motions $(B,b)\in\mathbf{e}(2)$, the Lie algebra of the Euclidean group, were classified as follows:

$$L^0\colon\ B\neq 0$$
$$H^2\colon\ B=0,\ b\neq 0$$
$$L^2\colon\ B=0,\ b=0,$$

each type having a characteristic Killing vector field. In particular, type L^0 has a unique fixed point. This corresponds to the instantaneous centre of rotation for a motion $\mu(t)=(A(t),a(t))$ as above.

This remains true for the homogeneous action. For type H^2, the Killing field extended to $\mathbb{P}^2$ fixes ℓ_∞, while type L^2 fixes all of $\mathbb{P}^2$. Nevertheless it is still possible in most cases to identify a unique point as "instantaneous centre of rotation". Suppose (B,b) is of type H^2 and $\mu(t)=(\exp\theta(t)J, a(t))$ satisfies $\mu(0)=\text{id}$, $\dot{\mu}(0)=(B,b)$. Provided $\dot{\mu}(t)$ is of type L^0 for t close to zero then the centrode is defined in $\mathbb{P}^2$ for $t\neq 0$. Its limit as $t\to 0$ is $(b_2,-b_1,0)\in\mathbb{P}^2$, and this is independent of choice of μ. These extensions of the centrodes for a motion μ are given by

$$c_m^*(t)=(\dot{a}_1(t)\sin\theta(t)-\dot{a}_2(t)\cos\theta(t),\ \dot{a}_1(t)\cos\theta(t)+\dot{a}_2(t)\sin\theta(t),\ \dot{\theta}(t)) \tag{7}$$

and similarly for c_f^*. These hold provided $\dot{\mu}(t)$ does not have type L^2. Even in this case though, we can proceed in the same way, by taking the limit of the defined centrode, provided the L^2 point is isolated and the limit exists. These maximally defined centrodes will be called the *extended centrodes* and denoted still c_m^*, c_f^*.

There is a very large (open and dense) set in the space of C^∞ motions (with the Whitney topology) for which extended centrodes can be defined; it includes all non-constant analytic motions [3]. Moreover the centrodes so defined are themselves C^∞. (If μ is C^r and $n \leqslant r$ is the least number for which $\mu^{(n)}(t_0) \neq 0$, then the centrode is at least C^k at t_0, where $k = \max\{r - 2n - 1, 0\}$.)

It is still true, as noted above, that points on the extended centrode are singularities on their trajectories. However for H^2 and L^2 points there are other trajectories in $\mathbb{P}^2$ possessing singularities also. In the next section we explore the nature of these singularities.

4. Classifying Trajectories

A starting point is to observe that $\mathscr{I}$-equivalence of motions induces an equivalence between germs of trajectories also. It is necessary to distinguish between points of $\mathbb{R}^2$ and points on the line at infinity, ℓ_∞, since these form distinct orbits of the extended action of $E(2)$. If μ_1, μ_2 are $\mathscr{I}$-equivalent motions and $x \in \mathbb{R}^2$ (respectively ℓ_∞), then

$$\mu_2(t) \cdot x = \sigma\mu_1(h(t))\tau \cdot x$$

for appropriate σ, τ, h. This suggests an equivalence relation between germs of paths in $\mathbb{R}^2$.

To describe this formally, let $\mathscr{E}_n$ denote the ring of germs of smooth functions $\mathbb{R}^n$, $0 \to \mathbb{R}$ and $\mathbf{m}_n$ its maximal ideal (germs $f \in \mathscr{E}_n$ such that $f(0) = 0$); $\mathscr{E}_{n,p}$ is the module of smooth germs $\mathbb{R}^n$, $0 \to \mathbb{R}^p$, over $\mathscr{E}_n$, and $\mathscr{E}^0_{n,p} = \mathbf{m}_n \cdot \mathscr{E}_{n,p}$. Using local coordinates on ℓ_∞, we shall regard $\mathscr{E}_1$ as the ring of germs $\mathbb{R}$, $0 \to \ell_\infty$. Then two germs $f_1, f_2 \in \mathscr{E}_{1,2}$, $\mathscr{E}_1$ are *$\mathscr{I}$-equivalent* if there exist $h \in \mathrm{Diff}^0$ and $\sigma \in E(2)$ such that $f_2(t) = \sigma \cdot f_1(h(t))$. Thus, if μ_1, μ_2 are $\mathscr{I}$-equivalent motions, then the trajectories of x, $\tau \cdot x$ are $\mathscr{I}$-equivalent germs.

Before arriving at normal forms for $\mathscr{I}$-equivalence classes, we describe two coarser equivalence relations frequently used in the study of singularities. These are $\mathscr{A}$ (right-left) equivalence and $\mathscr{K}$ (contact) equivalence. It is easiest to describe these first on $\mathscr{E}^0_{n,p}$. Let Diff^0_k denote the group of germs of C^∞ diffeomorphisms $\mathbb{R}^k$, $0 \to \mathbb{R}^k$, 0. Then $f, g \in \mathscr{E}^0_{n,p}$ are *$\mathscr{A}$-equivalent* if there exist $\varphi \in \mathrm{Diff}^0_n$, $\psi \in \mathrm{Diff}^0_p$ so that $g = \psi \circ f \circ \varphi^{-1}$. In other words, we allow differentiable changes of coordinates in domain and range. Now let $\mathscr{C}$ denote the subgroup of Diff^0_{n+p} of maps of the form (id_n, θ) where $\theta(x, 0) \equiv 0$. Then $f, g \in \mathscr{E}^0_{n,p}$ are *$\mathscr{K}$-equivalent* if there exist $(\mathrm{id}, \theta) \in \mathscr{C}$ and $\varphi \in \mathrm{Diff}^0_n$ so that $g = \theta(\varphi^{-1}, f \circ \varphi^{-1})$. This means that the graphs of f, g have the same degree of contact at 0 with $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^n \times \mathbb{R}^p$. In order to extend the equivalences to $\mathscr{E}_{n,p}$, the groups Diff^p_0 and $\mathscr{C}$ have to be replaced by pseudogroups of local diffeomorphisms so that we can move between different targets in $\mathbb{R}^p$.

It should be clear that the three equivalence relations on $\mathscr{E}_{1,2}$ and $\mathscr{E}_1$

TABLE 1. *Classification of germs in* $\mathscr{E}_{1,2}$

Boardman symbol	$\mathscr{K}$-normal form	$\mathscr{K}$-codimension	Arnold type	$\mathscr{A}$-normal form	$\mathscr{A}$-codimension
Σ^0	$t \mapsto (t, 0)$	1	non-sing	$t \mapsto (t, 0)$	0
$\Sigma^{1,0}$	$t \mapsto (t^2, 0)$	3	$A_{2p} (p \geq 1)$	$t \mapsto (t^2, t^{2p+1})$	p
$\Sigma^{1,1,0}$	$t \mapsto (t^3, 0)$	5	E_6	$t \mapsto (t^3, t^4)$ $t \mapsto (t^3, t^4 + t^5)$	3
			E_8	$t \mapsto (t^3, t^5)$ $t \mapsto (t^3, t^5 \pm t^7)$	4

are connected by the implications $\mathscr{I} \Rightarrow \mathscr{A} \Rightarrow \mathscr{K}$. In $\mathscr{E}_{1,2}$ the $\mathscr{K}$-types are classified by their Boardman symbol [6]. If 1_r denotes a string of r 1's, then a normal form for a $\Sigma^{1_r,0}$ singularly is $t \to (t^{r+1}, 0)$. The $\mathscr{A}$-classification for simple map-germs $\mathbb{C}, 0 \to \mathbb{C}^2, 0$ was obtained in [2] and translates to the real case with only minor modification. The classification is closely connected to that for $\mathscr{E}_2$ under $\mathscr{K}$-equivalence. Normal forms are given in Table 1.

We may now return to $\mathscr{I}$-equivalence and observe that the $\mathscr{K}$-classes are unions of $\mathscr{I}$-orbits.

PROPOSITION 4.1. *If $f \in \mathscr{E}_{1,2}$ and has Boardman type $\Sigma^{1_r,0}$ then it is $\mathscr{I}$-equivalent to*

$$t \mapsto (t^{r+1}, \alpha t^{r+2} + \eta) \tag{8}$$

where $\alpha \in \mathbb{R}$ and $\eta \in \mathbf{m}_1^{r+3}$.

Proof. First translate $f(0)$ to $0 \in \mathbb{R}^2$ then rotate so that only the first coordinate contains the lowest power t^{r+1}, i.e. has the form $at^{r+1} + 0(t^{r+2})$ where $a \neq 0$. Now reparametrise to reduce the first coordinate to t^{r+1}.

In $\mathscr{E}_1$ the $\mathscr{A}$ and $\mathscr{K}$ equivalence classes coincide and are classified by Arnold type A_k with normal form $t \mapsto t^{k+1}$. Since all the hard work is done by change of coordinates in the domain, $\mathscr{I}$-equivalence also coincides.

5. The Unfolding Actions

It is clear from Proposition 4.1 that there are many moduli associated with each $\mathscr{I}$-equivalence class. Whereas some of the geometry is lost under $\mathscr{A}$-equivalence, at least there exist simple singularities so that a reasonable classification can be achieved. (It would be nice to reverse the procedure we used in introducing $\mathscr{I}$-equivalence, by *extending* $\mathscr{A}$-

equivalence from trajectories to motions, but in such a way as to achieve a finite classification. Unfortunately, it is not clear how to do this in a natural way.)

Associated with the germ of a planar motion μ: I, $0 \to E(2)$ are its action on $\mathbb{R}^2$ and extended action in $\mathbb{P}^2$, respectively Φ^0: $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ and Φ^e: $\mathbb{R} \times \mathbb{P}^2 \to \mathbb{P}^2$, given by $(t, x) \to \mu(t) \cdot x$ (strictly, germs of these maps at $0 \times \mathbb{R}^2$, $0 \times \mathbb{P}^2$ respectively). For points $x \in \ell_\infty$, as noted in the previous section, it is natural to consider the restriction of the action to ℓ_∞, that is Φ^∞: $\mathbb{R} \times \ell_\infty \to \ell_\infty$. Associated with each action is a corresponding map $\Psi^\#$ (where $\# = 0, e, \infty$) given by $\Psi^\#(t, x) = (\Phi^\#(t, x), x)$. This is suggested by the idea of unfolding in singularity theory and for this reason we have called them the *finite, extended* and *infinite unfolding actions* respectively.

Since the standard unfolding theory is a local one it will suffice to express everything in local coordinates. Formally, if f: $\mathbb{R}^n, x_0 \to \mathbb{R}^p, y_0$ is a map germ then a *k-parameter unfolding* of f is a germ F: $\mathbb{R}^n \times \mathbb{R}^k, (x_0, u_0) \to \mathbb{R}^p \times \mathbb{R}^k, (y_0, u_0)$ where $F(x, u) = (\hat{F}(x, u), u)$ and $\hat{F}(x, u_0) = f(x)$. The map $\hat{F}$ is called a deformation and can be thought of as a k-parameter family of germs $\hat{F}_u$: $\mathbb{R}^n \to \mathbb{R}^p$. If α: $\mathbb{R}^\ell \to \mathbb{R}^k$ is C^∞ then there is an *induced* unfolding α^*F: $\mathbb{R}^n \times \mathbb{R}^\ell, (x_0, v_0) \to \mathbb{R}^p \times \mathbb{R}^\ell, (y_0, v_0)$ defined by $\alpha^*F(x, v) = (\hat{F}(x, \alpha(v)), v)$. Suppose F_1, F_2 are k-parameter unfoldings of $f_1, f_2 \in \mathscr{E}^0_{n,p}$ respectively and that f_1, f_2 are $\mathscr{A}$-equivalent, i.e. $f_2 = \psi \circ f \circ \varphi^{-1}$. The unfoldings are *$\mathscr{A}$-isomorphic* if $F_2 = \Psi \circ F_1 \circ \Phi^{-1}$, where Φ, Ψ are unfoldings of φ, ψ respectively. They are $\mathscr{A}$-equivalent (as unfoldings) if F_2 is $\mathscr{A}$-isomorphic to an induced unfolding α^*F_1 where α is a diffeomorphism. A map-germ is called *stable* if every unfolding of f is $\mathscr{A}$-isomorphic to a trivial unfolding $(x, v) \to (f(x), v)$.

Unfoldings which capture all the local perturbations of a given germ in a nice way are called versal. Technically, F is a *versal* unfolding of f if every other unfolding of f is a $\mathscr{A}$-isomorphic to an unfolding induced from F. Versality turns out to be analogous to transversality of a map to the orbit of a Lie group acting on a finite-dimensional manifold. This is formalised by introducing "tangent spaces" to the orbits of map-germs under the group actions. A heuristic justification for these can be found in **[6, 8]**.

Define the *Jacobian module* of a germ $f \in \mathscr{E}^0_{n,p}$ to be $J(f) = \mathscr{E}_n \left\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\}$, the $\mathscr{E}_n$-submodule of $\mathscr{E}_{n,p}$ generated by the partial derivatives. If f has components $f_1, \ldots, f_p \in \mathbf{m}_n$ then $I(f)$ denotes the ideal in $\mathscr{E}_n$ generated by them. As well as being an $\mathscr{E}_n$-module, $\mathscr{E}_{n,p}$ can also be regarded as an $\mathscr{E}_p$-module via the induced map f^*: $\mathscr{E}_p \to \mathscr{E}_n$, $\lambda \mapsto \lambda \circ f$. Let $\mathscr{E}_p\{e_1, \ldots, e_p\}$ denote the $\mathscr{E}_p$-submodule of $\mathscr{E}_{n,p}$ generated by the basis vectors $e_1, \ldots, e_p$ in $\mathbb{R}^p$ (thought of as constant maps on $\mathbb{R}^n$).

The tangent spaces at a germ $f \in \mathscr{E}^0_{n,p}$ are defined as follows:

$$\begin{aligned} T\mathscr{A}(f) &= J(f) + \mathscr{E}_p\{e_1, \ldots, e_p\} \subseteq \mathscr{E}_{n,p} \\ T\mathscr{K}(f) &= J(f) + I(f) \cdot \mathscr{E}_{n,p} \subseteq \mathscr{E}_{n,p} \end{aligned} \tag{9}$$

Define the *$\mathscr{G}$-codimension* of $f \in \mathscr{E}^0_{n,p}$ to be $\mathrm{cod}_{\mathscr{G}} f = \dim_{\mathbb{R}} \mathscr{E}_{n,p}/T\mathscr{G}(f)$ where $\mathscr{G} = \mathscr{A}, \mathscr{K}$. These are $\mathscr{G}$-invariants.

Two points should be noted here. First, whereas the contact tangent spaces are $\mathscr{E}_n$-submodules, the right-left tangent spaces are only $\mathscr{E}_p$-submodules via f^*. That makes their calculation harder and generally involves using the Malgrange Preparation Theorem, not just Nakayama's Lemma. Secondly, the constants $e_1, \ldots, e_p$ are always in $\mathscr{E}_p\{e_1, \ldots, e_p\} \subseteq T\mathscr{A}(f)$ whereas this is not the case for $T\mathscr{K}(f)$. This explains why the $\mathscr{K}$-codimensions for the germs in Table 1 exceed the $\mathscr{A}$-codimensions. It is always the case that $\mathrm{cod}_{\mathscr{A}} f \geqslant \mathrm{cod}_{\mathscr{K}} f - p$.

Suppose F: $\mathbb{R}^n \times \mathbb{R}^k, 0 \to \mathbb{R}^p \times \mathbb{R}^k, 0$ is an unfolding of $f \in \mathscr{E}^0_{n,p}$. Define $\dot{F}_i \in \mathscr{E}_{n,p}$ by $\dot{F}_i(x) = \dfrac{\partial \hat{F}}{\partial u_i}(x, 0)$ for $i = 1, \ldots, k$. The following theorem due to Mather completes the analogy between versality and transversality:

THEOREM 5.1. *F is a versal unfolding of f if and only if $\dot{F}_1, \ldots, \dot{F}_k$ span $\mathscr{E}_{n,p}/T\mathscr{G}(f)(\mathscr{G} = \mathscr{A}, \mathscr{K})$.*

A proof can be found in [**13**, Theorem 3.3]. A necessary condition for versality therefore is $\mathrm{cod}_{\mathscr{G}} f \leqslant k$.

In the case $\mathscr{G} = \mathscr{K}$, the failure of the constant germs to be in the tangent space means the unfolding action is never versal at a singularity. What is required is a restricted equivalence called *relative $\mathscr{K}$-equivalence* ($\mathscr{K}_0$-equivalence), though this in turn necessitates amending the definition of the unfolding action. We will not pursue the details here (they can be found in [**3**]), but concentrate on the case $\mathscr{G} = \mathscr{A}$.

6. Versality of the Unfolding Actions

Given the germ of a motion $\mu \in \mathscr{M}$, we would like to determine whether the unfolding actions $\Psi^\#$, $\# = 0, \infty$ are versal as unfoldings of a trajectory $\Phi^\#_\xi$, $\xi \in \mathbb{P}^2$. In order to use Mather's theorem, it is necessary to know both the initial speeds $\dot{\Psi}^\#_i$ and the tangent space $T\mathscr{G}(\Phi_\xi)$.

The first of these is dealt with easily. Suppose $\mu(t) = (A(t), a(t))$. Then for $\xi \in \mathbb{R}^2$ and $\# = 0$, $\dot{\Psi}^0_i$ is the ith column of $A(0)$ for $i = 1, 2$. For $\xi \in \ell_\infty$ and $\# = \infty$, the action of μ is given in local coordinates by $\mu(t)$. $\alpha = \alpha + \theta$. So $\dot{\Psi}^\infty = 1$ (with respect to this coordinate system). This renders the question of versality straightforward since automatically $1 \in T\mathscr{A}(\Phi^\infty_\xi) \subseteq \mathscr{E}_1$. We thus have:

PROPOSITION 6.1. *Ψ^∞ is versal if and only if Φ_ξ^∞ is $\mathscr{A}$-stable, i.e. non-singular or having a non-degenerate (A_1) critical point.*

To tackle the versality at finite points, it is convenient to move from the general trajectory Φ_ξ^0 to a class of *pre-normal forms,* namely the $\mathscr{I}$-equivalence types of Proposition 4.1. We make use of the following general observation. If $f, g \in \mathscr{E}_{n,p}^0$ are $\mathscr{A}$-equivalent, F and G are $\mathscr{A}$-equivalent unfoldings of f, g respectively and if F is versal then G is also versal.

Having determined under what conditions Ψ^0 is a versal unfolding of a pre-normal form it is relatively straightforward to go back and interpret these as conditions on the derivatives of a given motion. To complete the process we show that $\mathscr{I}$-equivalent motions have $\mathscr{A}$-equivalent unfolding actions so by virtue of the above observation we will have determined the necessary and sufficient conditions for versality of an unfolding action.

Observe from Table 1 that the only $\mathscr{A}$-types in $\mathscr{E}_{1,2}$ of codimension ≤ 2, and hence admitting versal unfolding actions, are non-singular, A_2 and A_4.

PROPOSITION 6.2. *Suppose $a \in \mathscr{E}_{1,2}^0$ is given by $t \to (t^2, \alpha t^3 + \beta t^4 + \gamma t^5 + \mu)$ where at least one of $\alpha, \gamma \in \mathbb{R}$ is non-zero and $\mu \in \mathbf{m}_1^6 \cdot \mathscr{E}_{1,2}$. Then a basis for $\mathscr{E}_{1,2}/T\mathscr{A}(a)$ is given by projections into the quotient of the following*

A_2 $(\alpha \neq 0)$:	$(0, t)$
A_4a $(\alpha = 0, \gamma \neq 0$ and $\beta \neq 0)$:	$(0, t), (t, 0)$
A_4b $(\alpha = 0, \gamma \neq 0$ and $\beta = 0)$:	$(0, t), (0, t^3)$

Proof. We follow the method described in **[10]**. This requires the following form of the Malgrange Preparation Theorem:
if $f \in \mathscr{E}_{n,p}^0$ then the following are equivalent:
a) $\mathscr{E}_n = I(f) + \mathbb{R}\{g_1, \ldots, g_k\}$ for some $g_1, \ldots, g_k \in \mathscr{E}_n$;
b) $\mathscr{E}_n = \mathscr{E}_p\{g_1, \ldots g_k\}$.
We will do the calculation for the type A_4a, the other cases being similar. Under the hypotheses, $\dot{a} = (2t, 4\beta t^3 + 5\gamma t^4 + \delta t^5) \bmod \mathbf{m}_1^6 \cdot \mathscr{E}_{1,2}$ for some $\delta \in \mathbb{R}$. We can simplify the problem by using the theory of determinacy of germs. $\mathscr{A}$-equivalence is assumed. A germ is *r-determined* if it is equivalent to any other germ with the same r-jet (i.e. Taylor expansion to order r). Since the given germ is of type A_{2p} $(p \leq 2)$, from Table 1 it is equivalent to (t^2, t^{2p+1}) and hence 5-determined. Employing a standard result of determinancy theory (see, for example, [5]), $\mathbf{m}_1^6 \cdot \mathscr{E}_{1,2} \subseteq T\mathscr{A}(a)$.

Now $I(a) = \mathbf{m}_1^2$ so $\mathscr{E}_1 = I(a) + \mathbb{R}\{1, t\}$ and by the Preparation Theorem, $\mathscr{E}_1 = \mathscr{E}_2\{1, t\}$. Since $\mathscr{E}_{1,2} = \mathscr{E}_1 \oplus \mathscr{E}_1$, generators for $\mathscr{E}_{1,2}$ as an $\mathscr{E}_2$-module are $e_1 = (1, 0)$, $e_2(0, 1)$, $g_1 = (t, 0)$, $g_2(0, t)$. We already know that $e_1, e_2 \in T\mathscr{A}(a)$ automatically. Suppose that x, y generate $\mathbf{m}_2$.

Then

$$\begin{aligned} a^*(x)\cdot g_1 &= t^2\cdot(t,0)\\ &= (t^3,0)\\ &= \tfrac{1}{2}t^2\dot a + a^*\left(\frac{2\beta^2}{\gamma}x^2-\frac{2\beta}{\gamma}y\right)\cdot e_2 \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &\in J(a)+\mathscr{E}_2\{e_1,e_2\}+\mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &= T\mathscr{A}(a);\\ a^*(y)\cdot g_1 &= \beta t^4\cdot(t,0) \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &= a^*\left(\frac{\beta}{\gamma}y-\frac{\beta^2}{\gamma}x^2\right)\cdot e_1 \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &\in T\mathscr{A}(a);\\ a^*(y)\cdot g_2 &= \beta t^4\cdot(0,t) \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &= a^*\left(\frac{\beta}{\gamma}y-\frac{\beta^2}{\gamma}x^2\right)\cdot e_2 \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &\in T\mathscr{A}(a) \end{aligned}$$

and

$$\begin{aligned} a^*(x)\cdot g_2 &= t^2\cdot(0,t)\\ &= \frac{1}{4\beta}\dot a - a^*\left(\frac{5\gamma}{4\beta}x^2\right)\cdot e_2 - \frac{\delta}{4\beta^2}a^*(y)\cdot g_2 - \frac{1}{2\beta}g_1 \bmod \mathbf{m}_1^6\cdot\mathscr{E}_{1,2}\\ &\in T\mathscr{A}(a)+\mathbb{R}\{g_1\}, \end{aligned} \tag{10}$$

since we already showed that $a^*(y)\cdot g_2\in T\mathscr{A}(a)$. It follows that $\mathbf{m}_2\{g_1,g_2\}\subseteq T\mathscr{A}(a)+\mathbb{R}\{g_1,g_2\}$ and hence that $\mathscr{E}_{1,2}/T\mathscr{A}(a)\subseteq\mathbb{R}\{g_1,g_2\}$. On the other hand, neither $g_1, g_2\in T\mathscr{A}(a)$, otherwise $\operatorname{cod}_{\mathscr{A}} a<2$.

It follows from (10) that in general for a of type A_4 as in Proposition 6.2, the germ $(t, 2\beta t^3)\in T\mathscr{A}(a)$. This is important in calculating versality.

The next step is to determine conditions for an unfolding action to versally unfold a singular trajectory in pre-normal form.

PROPOSITION 6.3. *Suppose* $a\in\mathscr{E}_{1,2}^0$ *is as in Proposition* 6.2 *and* $\mu=(\exp\theta J, a)\in\mathscr{M}_0$, *where* $\theta(t)=\sum_{r=1}^{3}\frac{\theta_r}{r!}t^r+0(4)$. *Then the unfolding action* Ψ^0 *is versal at* 0 *if and only if either*

a) *a is non-singular,*

b) *a has type A_2 and $\theta_1\neq 0$ or*

c) *a has type A_4, $\theta_1\neq 0$ and $\theta_2\neq 4\beta$.*

Proof. By Mather's theorem we need to check whether the vectors $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$ span $\mathscr{E}_{1,2}/T\mathscr{A}(a)$. Since this requires $\mathrm{cod}_{\mathscr{A}}\, a \leqslant 2$ it is only necessary to consider these three classes of $\mathscr{A}$-type; the non-singular case is trivial.

We have

$$\dot{\Psi}_1^0 = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = \begin{bmatrix}1-\frac{1}{2}\theta_1^2t^2-\frac{1}{2}\theta_1\theta_2t^3\\ \theta_1t+\frac{1}{2}\theta_2t^2+\frac{1}{6}(\theta_3-\theta_1^3)t^3\end{bmatrix}+0(4) \tag{11}$$

and similarly for $\dot{\Psi}_2^0$. Now searching for the bases in Proposition 6.2, for type A_2 we find $\dot{\Psi}_1^0$ is sufficient, provided $\theta_1 \neq 0$. For type A_4a or b, $\begin{bmatrix}0\\ t\end{bmatrix}$ can only be provided by $\dot{\Psi}_1^0$ when $\theta_1 \neq 0$. Then observe that modulo $T\mathscr{A}(a)$, $\dot{\Psi}_2^0 = -\begin{bmatrix}\theta_1 t\\ \frac{1}{2}\theta_1\theta_2t^3\end{bmatrix}$. Given also that $\begin{bmatrix}t\\ 2\beta t^3\end{bmatrix} \in T\mathscr{A}(a)$ we have

$$\dot{\Psi}_2^0 = \begin{bmatrix}\theta_1\left(\dfrac{\theta_2}{4\beta}-1\right)t\\ 0\end{bmatrix} \bmod T\mathscr{A}(a)$$

provided $\beta \neq 0$ (type A_4a) or $-\begin{bmatrix}0\\ \frac{1}{2}\theta_1\theta_2t^3\end{bmatrix} \bmod T\mathscr{A}(a)$ if $\beta = 0$ (type A_4b). In either case, versality is equivalent to $\theta_1 \neq 0$ and $\theta_2 \neq 4\beta$.

To make use of this result for a general motion we make the following connection between $\mathscr{I}$-equivalence of motions and $\mathscr{A}$-equivalence of unfoldings.

PROPOSITION 6.4. *If $\mu, \mu' \in \mathscr{M}$ are $\mathscr{I}$-equivalent motions with $\mu' = \sigma\cdot(\mu\cdot h^{-1})$ and $x \in \mathbb{R}^2$ then Ψ^0, $\Psi^{0\prime}$ are $\mathscr{A}$-isomorphic as unfoldings of the trajectories of x, $\sigma\cdot x$ respectively.*

Proof. This is entirely formal: set Σ, H to be the trivial unfoldings $\Sigma(y,x) = (\sigma\cdot y, x)$ and $H(t,x) = (h(t), x)$, *then* $\Sigma\circ\Psi^0\circ H^{-1} = \Psi^{0\prime}$ as required.

The next step is to observe that given any motion μ and $x \in \mathbb{R}^2$ there is an $\mathscr{I}$-equivalence taking x to 0 and putting its trajectory in pre-normal form, so that the question of whether Ψ^0 versally unfolds a given trajectory can be reduced to applying Proposition 6.3. Conversely we can deduce the following conditions for versality on μ and the trajectory of x (which for convenience we denote $x(t)$):

THEOREM 6.5. *The unfolding action Ψ^0 of $\mu = (\exp \theta J, a)$ versally unfolds the trajectory of $x \in \mathbb{R}^2$ if and only if either*

a) $\dot{x}(0) \neq 0$,

b) $\dot{x}(0) = 0$, $\ddot{x}(0) \times \dddot{x}(0) \neq 0$ *and* $\dot{\theta}(0) \neq 0$

or

c) $\dot{x}(0) = 0$, $\ddot{x}(0) \times \dddot{x}(0) = 0$, $\ddot{x}(0) \neq 0$, $\dot{\theta}(0) \neq 0$ *and* $\ddot{x}(0) \times x^{(iv)}(0) \neq -[\ddot{x}(0) \cdot \dddot{x}(0)]\dot{\theta}(0) + 3\,\|\ddot{x}(0)\|^2\,\ddot{\theta}(0)$

Proof. This amounts to reinterpreting the conditions of Proposition 6.3 by tracing back the moduli that appear in the pre-normal form under $\mathcal{J}$-equivalence. For example, a necessary condition for a singular trajectory to have type A_4 is that its pre-normal form $(t^2, \alpha t^3 + \eta)$, as in Proposition 4.1, have $\alpha = 0$; but $\alpha = \ddot{x}(0) \times \dddot{x}(0)$, giving the second conditions in (b) and (c). A slightly more intricate analysis involving explicit calculation of σ and h in the $\mathcal{J}$-equivalence gives the final conditions of (c).

This effectively solves the "recognition" problem: when does a given motion versally unfold one of its trajectories?

7. Unfolding Actions and Geometry

We have taken the point of view that a planar motion is given by a path in the configuration space $E(2)$. The most succinct description is that using the semi-direct product structure of $E(2)$, which isolates the trajectory of the origin (the translational part). As noted before, other descriptions are possible, for example, in some cases by the prescription of the fixed and moving centrodes. Using the results of §§ 2 and 6 we can now connect aspects of the local geometry of a motion to the singularity types of its trajectories and the way in which the rotational part of the motion unfolds them in a non-trivial way. We find that while the contact theory requires the centrodes to be non-singular, $\mathcal{A}$-versality is related to the curvature of the centrodes.

Thom-Boardman Theory. As we noted in §§ 4 and 6, the $\mathcal{K}$-classification is closely related to Thom-Boardman theory but $\mathcal{K}$-versality is not the appropriate condition on the unfolding action. In fact, relative $\mathcal{K}$-versality turns out to be equivalent to transversality of the $\mathcal{K}$-jet extensions of Ψ^0 to the Thom-Boardman strata (see [3]). We establish below the geometric interpretation of this transversality. This idea also suggests how we should proceed in the case of $\mathcal{A}$-versality.

THEOREM 7.1. *Let $\mu \in \mathcal{M}$ and Ψ^0 be its finite unfolding action. Suppose $\mu(t) = (A(t), a(t))$ and $A(t) = \exp \theta(t) J$ then*

a) $(0, x) \in \Sigma^1\Psi^0$ *if and only if* $\dot{\Phi}_x(0) = 0$ *and* $j^1\Phi^0 \pitchfork \Sigma^1$ *at* $(0, x)$ *if and only if* $\dot{\theta}(0) \neq 0$.

b) $(0, x) \in \Sigma^{1,1}\Psi^0$ *if and only if* $\dot{c}_m(0) = 0$ *but* $j^2\Psi^0 \not\pitchfork \Sigma^{1,1}$ *at* $(0, x)$.

Proof. (a) The condition $(0,x) \in \Sigma^1\Psi^0$ is determined by $\operatorname{corank} \mathrm{d}\Psi^0(0,x)=1$, but

$$\mathrm{d}\Psi^0(t,x)=\underset{\quad 1 \qquad\qquad 2}{\begin{bmatrix} \dot{A}(t)x+\dot{a}(t) & A(t) \\ 0 & I \end{bmatrix}^{2}_{2}} \quad \text{where} \quad I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{12}$$

which has corank 1 iff $\dot{\Phi}_x=\dot{A}x+\dot{a}=0$. In other words, x is an instantaneously stationary point at $t=0$. (Note that the form of $\mathrm{d}\Psi^0$ ensures that Σ^r points for $r \geqslant 2$ are impossible.)

For transversality, an equivalent condition is $\mathrm{d}\Psi^0 \pitchfork L^1(\mathbb{R}^3,\mathbb{R}^4)$ where $L^1(V,W)$ denotes the subset of corank 1 linear maps $V \to W$ in $L(V,W)$ [**8**]. If

$$q=\mathrm{d}\Psi^0(0,x)=\begin{bmatrix} 0 & A(0) \\ 0 & I \end{bmatrix} \in L^1(\mathbb{R}^3,\mathbb{R}^4)$$

then a complement to $T_qL^1(\mathbb{R}^3,\mathbb{R}^4)$ is $L(\ker q, \operatorname{coker} q)$. From (12) above

$$\mathrm{d}^2\Psi^0(0,x)(s,u)=\underset{\qquad 1 \qquad\qquad\qquad 2}{\begin{bmatrix} s\ddot{\Phi}_x(0)+\dot{A}(0)u & s\dot{A}(0) \\ 0 & 0 \end{bmatrix}^{2}_{2}} \tag{13}$$

so the restriction, D, of the second derivative to $\ker q$ is given by the first column. The transversality condition is satisfied provided the image of D: $\ker q \to \mathbb{R}^4$ projects *onto* $\operatorname{coker} q$ or equivalently $\operatorname{im} D+\operatorname{im} q=\mathbb{R}^4$.

This amounts to

$$\operatorname{rank} \underset{\quad 1 \qquad\quad 2 \qquad\quad 2}{\begin{bmatrix} \ddot{\Phi}_x(0) & \dot{A}(0) & A(0) \\ 0 & 0 & I \end{bmatrix}^{2}_{2}}=4,$$

i.e. $\operatorname{rank}[\ddot{\Phi}_x(0)\colon \dot{A}(0)]=2$. But $\dot{A}(0)=\dot{\theta}J$ so its rank is 0 (if $\dot{\theta}=0$) or 2 (if $\dot{\theta} \neq 0$), so we must in fact have $\det \dot{A}(0)=\dot{\theta}(0)^2 \neq 0$. This means of course that x is the instantaneous centre of rotation and that $\Sigma^1\Psi^0=\{(t,c_m(t)) \mid t \in \mathbb{R}\}$ as germs at $t=0$.

(b) This only makes sense if $j^1\Psi^0 \pitchfork \Sigma^1$ at $(0,x)$. In particular, by (a) the (germs of) centrodes must be well defined through $t=0$. Moreover $\mathrm{d}\Psi^0(0,x)=\begin{bmatrix} 0 & A(0) \\ 0 & I \end{bmatrix}$. Using Boardman's formula (see [**8**]), $\dim \Sigma^1\Psi^0=1$ so $(0,x) \in \Sigma^{1,1}\Psi^0$ if and only if $\operatorname{corank} d(\Psi^0 \mid \Sigma^1\Psi^0)(0,x)=1$ if and only if $\mathrm{d}\Psi^0(0,x)(T_{(0,x)}\Sigma^1\Psi^0) \equiv 0$. From the form of $\mathrm{d}\Psi^0$, it follows that $T_{(0,x)}(\Sigma^1\Psi^0)=\mathbb{R} \times \{0\} \subseteq \mathbb{R} \times \mathbb{R}^2$.

This is clearly equivalent to $\dot{c}_m(0)=0$. However, $\operatorname{codim} \Sigma^{1,1}=4$ so that $j^2\mu$ cannot meet this stratum transversely.

The next result describes the situation for the infinite action.

THEOREM 7.2. *Let μ be as in Theorem 7.1 and Ψ^∞ its infinite unfolding action. Then $(0, x) \in \Sigma^1\Psi^\infty$ if and only if $\dot\theta(0) = 0$ and $j^1\Psi^\infty \pitchfork \Sigma^1$ at $(0, x)$ if and only if also $\ddot\theta(0) \neq 0$. Always $\Sigma^{1,1}\Psi^\infty = \emptyset$.*

Proof. In local coordinates Ψ^∞ is given by $(t, \alpha) \to (\theta(t) + \alpha, \alpha)$ so that

$$d\Psi^\infty(t, \alpha) = \begin{bmatrix} \dot\theta(t) & 1 \\ 0 & 1 \end{bmatrix}$$

It follows that $(0, x) \in \Sigma^1\Psi^\infty$ if and only if $\dot\theta(0) = 0$. On the other hand $\Sigma^2\Psi^\infty = \emptyset$.

For transversality, as in Theorem 7.1 we look for $d\Psi^\infty \pitchfork L^1(\mathbb{R}^2, \mathbb{R}^2)$. Since $L^1(\mathbb{R}^2, \mathbb{R}^2)$ is defined locally by $\det^{-1}(0)$ where $\det: L(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R}$, we have $T_{(0,\alpha)}L^1(\mathbb{R}^2, \mathbb{R}^2) = \ker d(\det)\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. At the same time, $\operatorname{im} d^2\Psi^\infty(0, \alpha)$ is spanned by $\begin{bmatrix} \ddot\theta(0) & 0 \\ 0 & 0 \end{bmatrix}$ so transversality is equivalent to $\ddot\theta(0) \neq 0$.

It follows that when transversality holds, $\Sigma^1\Psi^\infty = \{0\} \times \ell_\infty$ so that $\Psi^\infty \mid \Sigma^1\Psi^\infty$ is given by

$$\alpha \mapsto (\theta(0) + \alpha, \alpha)$$

whence $\Sigma^{1,1}\Psi^\infty$ (which is Σ^1 of the restriction) is empty.

COROLLARY 7.3. *Suppose $\mu \in \mathcal{M}$ is as in Theorem 7.1 and $\dot\theta(0) \neq 0$. If μ has a trajectory $x(t) \in \mathscr{E}_{1,2}$ of $\mathscr{K}$-type $t \mapsto (t^3, 0)$ then $\dot c_m(0) = 0$.*

Proof. If $x(t)$ has this $\mathscr{K}$-type then its Thom-Boardman type is $\Sigma^{1,1,0}$ (as in Table 1). But an unfolding has the same type as the germ it unfolds so by Theorem 7.1, since $\dot\theta(0) \neq 0$, we have $\dot c_m(0) = 0$.

COROLLARY 7.4. *Let M be a compact 1-dimensional manifold and suppose $C^\infty(M, E(2))$ has the Whitney C^∞ topology. Then there is an open dense set of motions μ: $M \to E(2)$ for which c_m^*: $M \to \mathbb{P}^2$ is an immersion transverse to ℓ_∞.*

Proof. Apply the Thom Transversality Theorem (see for example [**8**]) together with the continuity of the maps $\mu \mapsto \Psi^0$, $\mu \mapsto \Psi^\infty$ and the transversality conditions in Theorems 7.1, 2. That c_m^* is non-singular at ℓ_∞ and transverse to it comes from $\ddot\theta \neq 0$.

In fact, the class of motions defined by transversality of Ψ^0, Ψ^∞ to the Thom-Boardman strata coincides exactly with the class of two-generic motions defined in [**4**].

$\mathscr{A}$-versality. We can now look at the geometric conditions implied by $\mathscr{A}$-versality of the unfolding actions. Corollary 7.3 already implies some restrictions—no trajectory $\mathscr{K}$-equivalent to $(t^3, 0)$ can occur $\mathscr{A}$-versally. In fact, if the unfolding action Ψ^0 of a motion $\mu \in \mathscr{M}$ versally unfolds every trajectory then by Theorem 6.5 either $\dot{x}(0) \neq 0$ or $\dot{\theta}(0) \neq 0$. It follows that the extended centrodes are well-defined and so, by Theorem 7.1(b), non-singular if finite. If, in addition, Ψ^∞ also versally unfolds the trajectories in ℓ_∞ then by Theorem 6.1, $\ddot{\theta}(0) \neq 0$. Thus, the Thom-Boardman condition and $\mathscr{A}$-versality coincide for Ψ^∞. So, when $c_m(0) \in \ell_\infty$, the centrodes are transverse to ℓ_∞ and in particular non-singular.

Since no new information is obtained from Ψ^∞, we return to the situation where $c_m(0) \notin \ell_\infty$ (i.e. $\dot{\theta}(0) \neq 0$). In addition, in view of the above, we shall assume that $\dot{c}_m(0) \neq 0$.

We seek geometric interpretations of the conditions of Theorem 6.5, or equivalently, since the properties we are interested in are $\mathscr{I}$-invariants, Proposition 6.3. This we can split into two parts: the condition that the singular trajectory be a higher cusp and the condition of versality of the unfolding action.

In relation to the first of these we have:

THEOREM 7.5. *Given a motion μ, whose centrodes c_m, c_f are non-singular at $t=0$, the following are equivalent:*

i) *for some $x \in \mathbb{R}^2$, its trajectory $x(t)$ has an A_{2p} singularity, $p \geq 2$, at $t=0$;*

ii) *the curvatures of the centrodes satisfy $\kappa_m(0) = 2\kappa_f(0)$;*

iii) *the inflection circle and cubic of stationary curvature have order of contact ≥ 4 at the instantaneous centre;*

iv) *Ball's point coincides with the instantaneous centre.*

Notes. (1) The substance of this Theorem is well known and appears in the literature on synthesis of mechanisms. For example, the implication from (ii) to (i) is Beyer's Theorem 49 [**1**], while (ii) implies (iv) appears as example 5B1 in Hunt [**9**]. The methods used there are geometric so we give below brief analytic proofs. Veldkamp [**12**] gives an extensive analysis of curvature properties of trajectories.

(2) That the instantaneous centre and Ball's point coincide when there is a higher cusp ($A_{2p}, p \geq 2$) is perhaps unsurprising. It was well known (see, for example, Salmon [**11**, Article 58]) that higher singularities of *algebraic* curves could be regarded as equivalent to a coincidence of "simple" singularities and that, in particular, the rhamphoid (A_4) cusp was equivalent to a cusp, a node, bitangent and an *inflection* all coinciding. I am now aware of a similar result in the C^∞-category.

Proof. (i)$\Leftrightarrow$(ii). From the proof of Theorem 6.5, (i) is equivalent to $\dot{x}(0)=0$ and $\ddot{x}(0) \times \dddot{x}(0) = 0$. Choosing coordinates so that $x=0$ and writing $\mu = (\exp \theta J, a)$, we have from equations (2):

$$\|\dot{c}\|^3 \kappa_m = \frac{2\,\|\ddot{a}\|^2}{\dot{\theta}}$$

$$\|\dot{c}_f\|^3 \kappa_f = \frac{\|\ddot{a}\|^2}{\dot{\theta}}$$

Since $\|\dot{c}_m\| = \|\dot{c}_f\|$, the result follows.
(ii)$\Leftrightarrow$(iii). Choose coordinates so that the centrodes are tangent to the x-axis at the origin. Then the inflection circle may be parametrised by $\gamma(s) = (r_I \sin s, r_I(1-\cos s))$.

The cubic of stationary curvature is given by

$$F(x, y) = \left(\frac{x}{M} + \frac{y}{N}\right)(x^2 + y^2) - xy = 0$$

whence

$$F(\gamma(s)) = r_I^2(1-\cos s)\left[\left(\frac{2r_I}{M} - 1\right)\sin s + \frac{2r_I}{N}(1-\cos s)\right].$$

Since $1 - \cos s = \frac{1}{2}s^2 + 0(4)$ and $\sin s = s + 0(3)$, it immediately follows that the curves have order of contact $\geqslant 4$ if and only if $\dfrac{1}{M} = \dfrac{1}{2r_1}$, which, by (6) and the Euler-Savary equation (3), is equivalent to $\kappa_m = 2\kappa_f$.

The equivalence of (iii) and (iv) is merely a matter of the definition of Ball's point, although it should be noted that this can include the degenerate case $\left(\dfrac{1}{N} = 0\right)$ in which a component of the cubic of stationary curvature coincides with the inflection circle.

By Euler-Savary, a subsidiary condition to (ii) is that the inflection circle osculates with the moving centrodes $\left(\text{i.e. } \kappa_m = \dfrac{1}{r_I}\right)$.

In the light of Theorem 7.5, we can now look for an interpretation of the failure of versality of the unfolding action around an A_4 singularity. One usually expects versality to translate into a *transversality* condition in some finite dimensional setting. We can almost establish this explicitly in the case of the curvatures of the centrodes.

THEOREM 7.6. *Let $\mu \in \mathcal{M}$ be such that $c_m(0)$ is finite and $\dot{c}_m(0) \neq 0$. If $\kappa_m(0) = 2\kappa_f(0)$ and $\dot{\kappa}_m(0) = 2\dot{\kappa}_f(0)$ then Ψ^0 does not versally unfold the trajectory of $c_m(0)$.*

Note. The two conditions imply non-transversality of the graphs of the functions κ_m and $2\kappa_f$. However, the converse of the result does not hold. In particular, transversality of the graphs does not necessarily imply versality of Ψ^0. The reason for this is that, as we saw in Theorem 7.5, the graphs may intersect (either transversely or not) at A_{2p} singularities where $p \geqslant 3$ and hence versality automatically fails.

Proof. This is merely a matter of calculation. Suppose μ to be given by $(a(t), \exp \theta(t)J)$ where a is in the pre-normal form of Proposition 6.2. By the hypotheses, $\theta(t) = \theta_1 t + \frac{1}{2}\theta_2 t^2 + 0(3)$ where $\theta_1 \neq 0$ and $a(t) = (t^2, \beta t^4 + 0(5))$. From this we deduce $\kappa_m(0) = -\theta_1^2$ and $\dot{\kappa}_f(0) = \theta_1(6\beta - 3\theta_2)$, $\dot{\kappa}_m(0) = \frac{1}{2}\theta_1(12\beta - 9\theta_2)$.

Hence $\dot{\kappa}_m(0) = 2\dot{\kappa}_f(0)$ implies $\theta_2 = 4\beta$. So either a is of type A_{2p}, $p \geqslant 3$ or a has type A_4 and $\theta_2 = 4\beta$. In each case Ψ^0 is non-versal by Proposition 6.3.

Now envisage a one-parameter family of motions μ^s in which the graphs of κ_m^s and $2\kappa_f^s$ come together, meet tangentially and then intersect transversely in a pair of points, or vice versa, as the parameter s passes through some critical value s_0. We thus see that the non-versality condition corresponds to the creation or annihilation of a pair or rhamphoid (or worse) cusps.

What of Ball's curve, the locus of Ball's points? We know from Theorem 7.5 that this meets the centrode when the singular trajectory has a higher cusp. Presumably, the non-versality condition translates as some property of this intersection. It seems likely that Ball's curve and the centrode (in either fixed or moving coordinates) meet tangentially and that non-versality corresponds to a higher order of contact. However, this remains to be confirmed.

We complete this paper by proving a result for $\mathscr{A}$-equivalence akin to Corollary 7.4. The idea is the same as the standard one used in the proof of Theorem 7.1. Namely, we realise the versality conditions as transversality requirements to submanifolds of jet bundles. The fact that the $\mathscr{A}$-equivalence classes refine $\mathscr{K}$-equivalence classes suggests that we should seek refinements of the Thom-Boardman strata.

THEOREM 7.7. *If M is a compact 1-dimensional manifold then there is an open and dense subset of $C^\infty(M, E(2))$ for which the unfolding action Ψ^0 is versal at every $(t, x) \in M \times \mathbb{R}^2$ and Ψ^∞ is versal at every $(t, x) \in M \times \ell_\infty$.*

Proof. As in Corollary 7.4 we apply the Thom Transversality Theorem and the continuity of the maps $\mu \mapsto \Psi^0$, Ψ^∞. What is needed is to construct the appropriate submanifolds of jet spaces $J^k(M \times \mathbb{R}^2, \mathbb{R}^2 \times \mathbb{R}^2)$ and $J^k(M \times \ell_\infty, \ell_\infty \times \ell_\infty)$. (Remember $\ell_\infty \simeq \mathbb{P}^1 \simeq S^1$.) In fact, as we noted before, $\mathscr{A}$-versality is the same as $\mathscr{K}$-versality for Ψ^∞ so we can concentrate on Ψ^0.

Start with $\Sigma^{1,0,\dots,0}$ and $\Sigma^{1,1}$. These are the manifolds we needed in

Theorem 7.1. Requiring transversality to $\Sigma^{1,1}$ removes all singularities worse than cusps. It remains to construct manifolds representing the hierarchy of cusps (A_{2p}). In Theorem 6.5 we established that for a singular path $x(t)$, the condition for an $A_{\geqslant 4}$ cusp is $\ddot{x} \times \dddot{x} = 0$. Likewise a necessary condition for an $A_{\geqslant 2p}$ cusp $(p \geqslant 3)$ is $\ddot{x} \times x^{(2p-1)} = 0$. Working in local coordinates these can be established as subsets of $J_0^{2p-1}(\mathbb{R}^3, \mathbb{R}^4)$.

The jet spaces have rather high dimension; for example when $p = 2$, $\dim J_0^3(\mathbb{R}^3, \mathbb{R}^4) = 80$. However, most of the dimensions are of no interest to us. The jet space $J_0^k(\mathbb{R}^3, \mathbb{R}^4)$ is isomorphic to the direct sum of 4 copies of the space of polynomials in 3 variables with degree $\leqslant k$. If we call the variables τ, ξ_1, ξ_2, then everything of interest to us occurs in the subspace U_k spanned by the vectors $(\tau^r, 0, 0, 0)$ and $(0, \tau^r, 0, 0)$ for $r = 2, \ldots, k$, and we let $(u_1, v_1, \ldots, u_k, v_k)$ denote coordinates with respect to this basis. The reason for this is the very specific form of the unfolding action: from Theorem 7.1, $j^1\Psi^0(0, x) \in \Sigma^1$ if and only if $\dot{\Phi}_x(0) = 0$. It follows that the Σ^1 condition is characterized in U_k by $u_1 = v_1 = 0$. To focus on the cusp hierarchy, we need to excise from U_k the set $(\pi_2^k)^{-1}(\Sigma^{1,1})$, where π_2^k: $J_0^k(\mathbb{R}^3, \mathbb{R}^4) \to J_0^2(\mathbb{R}^3, \mathbb{R}^4)$ is the canonical projection, as these represent jets of unfoldings of worse singularities. Call the resulting open subset $\hat{U}_k$.

For $p \leqslant \frac{1}{2}(k+1)$, let $\tilde{R}_p$ be the subset of $\hat{U}_k$ defined by $u_1 = v_1 = 0$, $u_2 v_{2i-1} - u_{2i-1} v_2 = 0$ for $i = 2, \ldots, p$. Since we are in $\hat{U}_k$ at least one of u_2, $v_2 \neq 0$. It follows that $\tilde{R}_p$ is a submanifold of codimension $p + 1$. Now let $R_p = \pi^{-1}(\tilde{R}_p)$ where π is the obvious projection $J_0^k(\mathbb{R}^3, \mathbb{R}^4) \to U_k$. Then $j^k\Psi^0(0, x) \in R_p$ if and only if Φ_x has an $A_{\geqslant 2p}$ cusp at $t = 0$.

We now establish conditions for transversality to the R_p. Since transversality is a local condition, it is sufficient to work with $\mu \in \mathcal{M}$ where $\mu = (A(t), a(t))$ and a is pre-normal form, i.e. $a(t) = (t^2, \alpha t^3 + \beta t^4 + \cdots)$. Also assume $A(t) = \exp \theta(t) J$ where $\theta(t) = \theta_1 t + \frac{1}{2!}\theta_2 t^2 + \cdots$. Clearly for $p \geqslant 3$, since $\operatorname{codim} R_p \geqslant 4$, $j^k\Psi^0 \pitchfork R_p$ implies $j^k\Psi^0(0, x) \notin R_p$. This therefore excludes cusps worse than A_4 and leaves the case $p = 2$. We may as well take $k = 3$. If $j^3\Psi^0(0, 0) \in R_3$ then $\alpha = 0$. Moreover, a complement to the tangent space $T_{j^3\Psi^0(0,0)} R_3$ is spanned by $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, \ldots, 0, 1)\} \subseteq U_3 \cong \mathbb{R}^6$. We can conveniently write $j^3\Psi^0(t, x)$ as $\left(\dot{\Phi}_x, \frac{1}{2!}\ddot{\Phi}_x, \frac{1}{3!}\dddot{\Phi}_x\right)$. It follows, again using the most convenient notation and from $\Phi_x(t) = A(t) \cdot x + a(t)$, that

$$\mathrm{d}(j^3\Psi^0)(0, 0) = \left([\dot{a} \quad \dot{A}], \frac{1}{2!}[\ddot{a} \quad \ddot{A}], \frac{1}{3!}[a^{(iv)} \quad \dddot{A}]\right)_{(0,0)}$$

which, in coordinates, is

$$\begin{bmatrix} 2 & 0 & -\theta_1 \\ 0 & \theta_1 & 0 \\ 0 & -\frac{1}{2}\theta_1^2 & -\frac{1}{2}\theta_2 \\ 0 & \frac{1}{2}\theta_2 & -\frac{1}{2}\theta_1^2 \\ 0 & -\frac{1}{2}\theta_1\theta_2 & -\frac{1}{6}(\theta_3-\theta_1^3) \\ 4\beta & \frac{1}{6}(\theta_3-\theta_1^3) & -\frac{1}{2}\theta_1\theta_2 \end{bmatrix}$$

The image of this spans a complement to the tangent space to R_3 if and only if $\theta_1 \neq 0$ and $\frac{4\beta}{2} \neq \frac{\frac{1}{2}\theta_1\theta_2}{\theta_1}$, i.e., $\theta_2 \neq 4\beta$. By Theorem 6.3(c), this is equivalent to versality. The theorem follows.

Acknowledgments

I am indebted to Dr Chris Gibson for pointing out the correct formulation of $\mathcal{J}$-equivalence.

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