Dual Quaternions and their Applications in Robot Kinematics

Thomas Berry

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1 Quaternions

Quaternions are a number system that extend the complex numbers to four–dimensions. A general quaternion q is written as $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the fundamental quaternion units. The space of quaternions is denoted by $\mathbb{H}$ after William Hamilton who first described them in 1843.

Definition 1. A pure quaternion is a quaternion that can be written as $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ for real numbers b, c, and d. The space of pure quaternions is denoted by $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$.

Multiplication of quaternions is described by the multiplication table:

x	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Table 1: Multiplication table for quaternions.

The following operations hold for quaternions $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$.

   i. Addition: $q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$.

   ii. Scalar multiplication: $\lambda q = \lambda a + \lambda b\mathbf{i} + \lambda c\mathbf{j} + \lambda d\mathbf{k}$.

  iii. Conjugation: $q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

  iv. Norm: $|q| = \sqrt{qq^*} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

   v. Inverse: $q^{-1} = \frac{q^*}{|q|^2}$.

Using the multiplication rules given in Table 1 and the operations above, it can be shown that the quaternions form a four–dimensional non-commutative algebra over the real numbers.

1.1 Rotations

Quaternions can be used to generate rotations in three–dimensional space, $\mathbb{R}^3$, and so find uses in many areas in applied mathematics.

Theorem 1. *If $t = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$ is an arbitrary unit quaternion where $u \in \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ is a unit vector, then conjugation by t rotates $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ through an angle θ about the axis defined by u.*

Proof. A proof is given in [1, page 14]. □

Corollary 1.1. *The rotations of $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ form a group.*

Proof. The identity rotation, is given by the rotation through 0 degrees around any axis, namely $1 = \cos 0 + u\sin 0$. The inverse of a rotation through axis u by an angle θ is given by the rotation through axis u by angle $-\theta$. If a rotation r_1 is induced by conjugation by the unit quaternion $t_1 = \cos\frac{\theta_1}{2} + u_1\sin\frac{\theta_1}{2}$ and r_2 is induced by conjugation by $t_1 = \cos\frac{\theta_2}{2} + u_2\sin\frac{\theta_2}{2}$, then $r = r_1 r_2$ is induced by $q \mapsto t_2(t_1 q t_1^{-1})t_2^{-1} = (t_2 t_1)q(t_2 t_1)^{-1}$, which is conjugation by the unit quaternion $t = t_2 t_1 = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$ for some axis u and angle θ. □

The group of rotations that preserve orientation is denoted SO(3).

Remark 1. Representing conjugation by the map $v \mapsto tvt^{-1}$, we can see from Theorem 1 that the quaternions t and $-t$ generate the same rotation of $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. The pair t and $-t$ are called antipodal pairs of quaternions. The unit quaternions t form a 3–sphere $S^3 \cong \mathrm{Spin}(3)$, so there is a 2–to–1 map S^3 to SO(3).

2 Dual Numbers

Definition 2. Dual numbers are numbers of the form $a + \epsilon b$ where ϵ is a nilpotent number with $\epsilon^2 = 0$. The space of dual numbers is denoted by $\mathbb{D} = \mathbb{R} \oplus \epsilon\mathbb{R}$.

Dual numbers were first considered by Clifford in 1873 and further developed by Eduard Study in the early 1900s. They are an extension of the real numbers that, unlike the complex numbers, do not form a ring. We can represent a dual number by a matrix by letting

$$\epsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{so then} \quad a + \epsilon b = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}. \tag{1}$$

Using this matrix representation, addition and multiplication of dual numbers simply reduces to the addition and multiplication of matrices of the form (1), which is associative and commutative (c.f. complex numbers).

The following operation laws for dual numbers show that they form an algebra over the real numbers, but do not form a field.

i. Addition: $(a + \epsilon b) + (c + \epsilon d) = (a + c) + \epsilon(b + d)$

ii. Multiplication: $(a + \epsilon b)(c + \epsilon d) = ac + \epsilon(ad + bc)$

iii. Dual Conjugation: If $z = a + \epsilon b$ then $\overline{z} = a - \epsilon b$

iv. Division:
$$\frac{a + \epsilon b}{c + \epsilon d} = \frac{a}{c} + \epsilon\frac{bc - ad}{c^2}.$$

Thus, pure dual numbers $(c = 0)$ have no inverse, so dual numbers do *not* form a field.

v. Norm: $|a + \epsilon b| = \sqrt{(a + \epsilon b)(a - \epsilon b)} = \sqrt{a^2} = |a|$

3   Dual Quaternions

Definition 3. Dual quaternions are quaternions of the form $q = q_p + \epsilon q_d$ where q_p and q_d are quaternions representing the primal and dual parts of q, and ϵ denotes the dual unit. Equivalently, we can define the dual quaternions as an eight–dimensional vector space over $\mathbb{R}$ with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \epsilon, \epsilon\mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k})$. The algebra of dual quaternions is denoted by $\mathbb{DH} = \mathbb{D} \otimes \mathbb{H}$.

Dual quaternions are useful for problems in robot kinematics as they provide a simple way to represent rotations and translations in 3–dimensional space without requiring lengthy computations. A rigid transformation in $\mathbb{R}^3$ (a rotation and translation) would typically require separate computations for the rotation and the translation, usually using quaternions or matrices for the rotation, and addition of vectors for the translation. Dual quaternions, however, combine the rotation and translation information into a single object.

The following is a list of operations on dual quaternions.

i. Scalar multiplication: $\lambda q = \lambda q_p + \epsilon\lambda q_d$, for $\lambda \in \mathbb{R}$.

ii. Addition: $p + q = p_p + q_p + \epsilon(p_d + q_d)$.

iii. Multiplication: $pq = p_p q_p + \epsilon(p_p q_d + p_d q_p)$.

iv. Quaternion Conjugate: $q^* = q_p^* + \epsilon q_d^*$.

v. Dual Conjugation: $\overline{q} = q_p - \epsilon q_d$

vi. Third type of Conjugate: $\overline{q^*} = q_p^* - \epsilon q_d^*$.

vii. Norm: $|q|^2 = qq^*$.

viii. Inverse: $q^{-1} = \frac{q^*}{|q|^2}$

We can now prove the following.

Theorem 2. *Dual quaternions form a non–commutative ring with multiplicative identity.*

Proof. Using rule ii above and that quaternions form a ring, we see that dual quaternions are closed under addition, addition is abelian, and the additive identity is given by $0 = 0 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. From rule i, we see that the additive inverse of a dual quaternion q is $-q$. Associativity of multiplication follows from the associativity of multiplication of dual numbers. Distributivity of multiplication over addition follow from rules ii and iii. $\square$

3.1   Rotations and Translations

Definition 4. A proper rigid transformation of a Euclidian space is a transformation that preserves orientation, and the Euclidian notion of the distance between any two points. The group of proper rigid transformations of $\mathbb{R}^n$ is called the special Euclidian group and is denoted by $\mathrm{SE}(n)$.

Theorem 3. *Every proper rigid transformation of $\mathbb{R}^n$ can be represented by a rotation R, followed by a transformation $\mathbf{t}$, i.e. $\mathbf{v} \mapsto R\mathbf{v} + \mathbf{t}$.*

In order to generate a rigid transformation of $\mathbb{R}^3$ *without* dual quaternions, we identify with each rotation quaternion $r = \cos\frac{\theta}{2} + \hat{\mathbf{u}}\sin\frac{\theta}{2}$ a rotation matrix R, and represent translations by pure quaternions $\mathbf{t} = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k} \in \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ by $\mathbf{v} \mapsto \mathbf{v} + \mathbf{t}$. Thereby allowing a general rigid transformation in $\mathrm{SE}(3)$, being the composition of a rotation and translation, to be represented by $\mathbf{v} \mapsto R\mathbf{v} + \mathbf{t}$. We can, however, use dual quaternions to represent the general transformation $R\mathbf{v} + \mathbf{t}$ in a compact way.

As unit quaternions represent rotation in $\mathbb{R}^3$ by conjugation with a vector, $\mathbf{v} \mapsto q\mathbf{v}q^{-1}$, unit dual quaternions represent general rigid transformations by conjugation using the third type of conjugate given above. We identify $\mathbb{R}^3$ with $\mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ and represent a vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ by a pure quaternion $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \in \mathbb{H}$, which, in turn, we can represent via a one–to–one correspondence as a dual quaternion $1 + \epsilon\mathbf{v} \in \mathbb{DH}$.

Theorem 4. *Conjugation of $1 + \epsilon\mathbf{v}$ for $\mathbf{v} \in \mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ with the third type of conjugate for dual quaternions by the unit dual quaternion $\sigma = r + \frac{\epsilon}{2}\mathbf{t}r$, where $r = \cos\frac{\theta}{2} + \hat{\mathbf{u}}\sin\frac{\theta}{2}$ and $t = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$ represents the rigid transformation generated by rotation by θ through the axis defined by $\hat{\mathbf{u}}$, followed by translation by $\mathbf{t}$.*

Proof. We identify the rotation matrix R with the rotation $r\mathbf{v}r^{-1}$ and show that the general transformation $\mathbf{v} \mapsto R\mathbf{v} + \mathbf{t}$ in $\mathrm{SE}(3)$ can be represented by $\mathbf{v} \mapsto \sigma(1 + \epsilon\mathbf{v})\overline{\sigma^*}$.

We first note that for a pure quaternion $\mathbf{t}$, we have that $\mathbf{t}^* = -\mathbf{t}$. We now compute

$$
\begin{aligned}
\mathbf{v} \mapsto \sigma(1 + \epsilon\mathbf{v})\overline{\sigma^*} &= (r + \frac{\epsilon}{2}\mathbf{t}r)(1 + \epsilon\mathbf{v})(r^* - \frac{\epsilon}{2}r^*\mathbf{t}^*) \\
&= (r + \frac{\epsilon}{2}\mathbf{t}r + \epsilon r\mathbf{v})(r^* - \frac{\epsilon}{2}r^*\mathbf{t}^*) \qquad\qquad (\epsilon^2 = 0) \\
&= rr^* + \epsilon(\frac{1}{2}\mathbf{t}rr^* + r\mathbf{v}r^* - \frac{1}{2}rr^*\mathbf{t}^*) \\
&= 1 + \epsilon(r\mathbf{v}r^* + \mathbf{t}) \qquad\qquad\qquad\qquad (\mathbf{t}^* = -\mathbf{t}) \qquad (2)
\end{aligned}
$$

which can be identified with the transformation $R\mathbf{v} + \mathbf{t}$. $\qquad\qquad\square$

Remark 2. A dual quaternion $p + \epsilon q$ is a *unit* dual quaternion if and only if $pp^* = 1$ and $pq^* + qp^* = 0$. Thus, the unit dual quaternions form a six–dimensional subset of $\mathbb{DH}$ known as the Study quadric. The Study quadric is explained in more detail in §4.2.1.

Remark 3. For a general unit dual quaternion $p + \epsilon q$, the rotational and translational information is given by $r = p$, and $\mathbf{t} = 2qp^*$. Thus, it can be seen from equation (2) that we can represent a pure rotation by a dual quaternion by setting the dual part to zero, and a pure translation by setting the real part to unity.

If we were instead to first translate a point $\mathbf{v} \in \mathbb{R}^3$ followed by a rotation, that is $\mathbf{v} \mapsto R(\mathbf{v} + \mathbf{t})$, we simply modify the above transformation by conjugating by the unit dual quaternion $\sigma' = r + \frac{\epsilon}{2}r\mathbf{t}$, yielding

$$
\mathbf{v} \mapsto \sigma'(1 + \epsilon\mathbf{v})\overline{\sigma'^*} = 1 + \epsilon r(\mathbf{v} + \mathbf{t})r^* \longleftrightarrow R(\mathbf{v} + \mathbf{t}).
$$

Thus, any rigid transformation in $\mathbb{R}^3$ can be modelled compactly using dual quaternions.

4   The Exponential Map

4.1   Exponential of a Quaternion

We can represent a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ by the matrix

$$
Q = \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix}, \qquad\qquad (3)
$$

thereby reducing the exponential of a quaternion to the problem of the exponential of a 2×2 square matrix. We start with the following definition.

Definition 5. The matrix absolute value of a matrix $A = (a_{ij})$ is defined to be

$$
|A| = \sqrt{\sum_{i,j} |a_{ij}|^2}.
$$

From this definition comes the following Lemma.

Lemma 5 (Submultiplicative property). *For any two square matrices A and B we have* $|AB| \leq |A||B|$.

Proof. A proof is given in [1, Page 84]. □

The submultiplicative property is a useful property of square matrices as it allows us to prove the absolute convergence of the exponential series.

Proposition 6 (Convergence of the exponential series). *If A is a $n \times n$ matrix, then the exponential series*

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots \tag{4}$$

converges absolutely, where I_n is the $n \times n$ identity matrix.

Proof. [1] To prove the absolute convergence of equation (4) we must prove the convergence of the series

$$\sum_{k=0}^{\infty} \frac{|A^k|}{k!} = |\mathbf{1}| + |A| + \frac{|A^2|}{2} + \frac{|A^3|}{3!} + \dots$$

By the submultiplicative property we have

$$\sum_{k=0}^{\infty} \frac{|A^k|}{k!} \leq \sum_{k=0}^{\infty} \frac{|A|^k}{k!}.$$

But the latter series is just the series for the real exponential function $\mathrm{e}^{|A|}$, which is convergent. Hence, our series given in equation (4) is absolutely convergent by the comparison test. □

Corollary 6.1. *If A is a $n \times n$ matrix, then the series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges.*

This leads to the following definitions.

Definition 6 (Matrix exponential). The exponential of an $n \times n$ matrix A is given by the series

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I_n + A + \frac{A}{2} + \frac{A^3}{3!} + \dots$$

Definition 7 (Quaternion exponential). The exponential of a general quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is given by the quaternion representation of the matrix exponential of $Q = \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix}$.

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[1]This proof follows the proof given in [1, Page 85].

Proposition 7. The exponential of a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is equivalent to $\sum_{k=0}^{\infty} \frac{q^k}{k!}$.

In order to prove this, we need the following Lemma.

Lemma 8. *We have $q^n \longleftrightarrow Q^n$ for all $n \in \mathbb{N}$.*

Proof. We show this by induction on n.
Base Case: The cases $n = 0, 1$ are trivial. For $n = 2$ we have

$$q^2 = a^2 - u^2 + 2ab\mathbf{i} + 2ac\mathbf{j} + 2ad\mathbf{k} \longleftrightarrow \begin{pmatrix} a^2 - u^2 + 2\mathrm{i}ab & 2a(c + \mathrm{i}d) \\ 2a(-c + \mathrm{i}d) & a^2 - u^2 - 2\mathrm{i}ab \end{pmatrix} = Q^2,$$

where $u = \sqrt{b^2 + c^2 + d^2}$.
Inductive Step: Assume the hypothesis holds for $n = k$, we now show that it holds for $n = k + 1$. We have $q^{k+1} = q^k q \longleftrightarrow Q^k Q = Q^{k+1}$ by the rules for multiplication of matrices.

So we now have that $q^n \longleftrightarrow Q^n$ for all $n \in \mathbb{N}$. $\qquad\square$

Proof (proposition 7). From Lemma 8 we have $q^n \longleftrightarrow Q^n$ for all $n \in \mathbb{N}$. Thus, we only need to show the series $\sum_{k=0}^{\infty} \frac{q^k}{k!}$ converges. We first note that by an inductive argument on the multiplicative property of the norm of a quaternion we have $|q^k| = |q|^k$ for any quaternion q. Hence, we find

$$\sum_{k=0}^{\infty} \frac{|q^k|}{k!} = \sum_{k=0}^{\infty} \frac{|q|^k}{k!},$$

and the latter series is the convergent real exponential function $\mathrm{e}^{|q|}$. Therefore our series is absolutely convergent by the comparison test. $\qquad\square$

4.2   Application to Robot Kinematics

One of the fundamental results from the theory of Lie groups is that the exponential function gives a mapping from the Lie algebra $\mathfrak{g}$ to its corresponding Lie group G. The mapping is neither injective nor surjective in general, but is a homeomorphism in a neighbourhood of the identity [2, Chapter 4, §4]. The possible rigid motions of a robot joint with one degree of freedom comprise a one–parameter subgroup of SE(3). Those are, the groups of the form e^{tX} where $X \in \mathfrak{se}(3)$ and t is a scalar. Thus, in order to understand the possible motions of a robot joint with one degree of freedom, we need to study the one–parameter subgroups of SE(3).

4.2.1   SE(3) and the Study Quadric

We saw in section 3.1 that elements of SE(3) can be represented by dual quaternions. The Lie algebra elements of SE(3) can be represented by pure dual quaternions [3]. That is, dual quaternions of the form $p + \epsilon q$ where p and q have zero real part. This is useful for robot kinematics as the exponential map can be used to connect a mechanical joint, represented by a pure dual quaternion, with the possible displacement allowed by the joint [3]. For example, a revolute joint is represented by a pure dual quaternion $p + \epsilon q$ where $p \cdot q = 0$ [2]. The exponential of such a dual quaternion is the focus of section 4.2.2.

In section 3.1 we wrote a general element of SE(3) as $r + \frac{\epsilon}{2}\mathbf{t}r$, where $r \in \mathrm{Spin}(3)$ (that is, $|r| = 1$) and $\mathbf{t} \in \mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. We now follow the construction of the Study quadric given in [2, Chapter 9, §3]. Writing our element of SE(3) as $h = h_0 + \epsilon h_1$, the condition that $r \in \mathrm{Spin}(3)$ becomes $hh^* = 1$, giving the equations

$$h_0 h_0^* = 1, \tag{5}$$
$$h_0 h_1^* + h_1 h_0^* = 0. \tag{6}$$

The dual quaternions satisfying conditions (5) and (6) are elements of the double cover $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ of SE(3). To obtain elements of SE(3), we need to identify the dual quaternions h and $-h$. This is achieved by identifying the dual quaternion $h = h_0 + \epsilon h_1 = (a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) + \epsilon(b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$ with a point in $\mathbb{PR}^7$ with homogeneous coordinate $(a_0 : a_1 : a_2 : a_3 : b_0 : b_1 : b_2 : b_3)$. Because we have projectivised $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$, h and $-h$ now correspond to the same point in homogeneous coordinates, as required. As a point in projective space is invariant under scalar multiplication, condition (5) is now redundant, and we are left with the quadratic relation (6). The six–dimensional quadric described by this relation is called the Study quadric and plays a fundamental role in robot kinematics.

4.2.2   Exponential of a Dual Quaternion

We now wish to investigate the one–parameter subgroups generated by one degree of free-dom robot joints. The elements of the one–parameter subgroups are given by $\mathrm{e}^{t(p+\epsilon q)}$ where t is a scalar and p and q are pure dual quaternions. We start with the following definition.

Definition 8. The exponential of a general dual quaternion $p + \epsilon q$ is given by

$$\sum_{k=0}^{\infty} \frac{(p + \epsilon q)^k}{k!}. \tag{7}$$

Proposition 9. The series (7) is convergent.

Proof. We can express a general dual quaternion $p + \epsilon q$ as a quaternion with dual number coefficients as $p + \epsilon q = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} + \epsilon(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) = p_0 + q_0 + (p_1 + \epsilon q_1)\mathbf{i} + (p_2 + \epsilon q_2)\mathbf{j} + (p_3 + \epsilon q_3)\mathbf{k}$. The exponential of a dual quaternion is thus reduced to the exponential of a quaternion, so, by Proposition 7, the series (7) converges. □

This allows us to investigate some of the properties of the one–parameter subgroups of revolute joints.

Theorem 10. *If $X = p + \epsilon q$ represents a revolute joint, i.e. $p \cdot q = 0$ where p and q are pure imaginary quaternions, then the one–parameter subgroup $\mathrm{e}^{t(p+\epsilon q)}$ is given by $\cos|tp| + t\frac{\sin|tp|}{|tp|}(p + \epsilon q)$ which characterises a straight line through the identity in the Study quadric for any $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.*

Before we can prove this, we need the following Lemmas.

Lemma 11. *For a non–zero pure imaginary quaternion $v = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ we have $\mathrm{e}^v = \cos|v| + \frac{v}{|v|}\sin|v|$. If $v = 0$, we define $\mathrm{e}^v = 1$.*

Proof. We first note that for a pure quaternion v we have that $v^2 = -(b^2 + c^2 + d^2) = -|v|^2$. So we can write powers of v as $v^2 = -|v|^2$, $v^3 = -|v|^2 v$, $v^4 = |v|^4, \ldots$ Thereby giving

$$
\begin{aligned}
\mathrm{e}^v &= 1 + v - \frac{1}{2!}|v|^2 - \frac{1}{3!}|v|^2 v + \frac{1}{4!}|v|^4 + \frac{1}{5!}|v|^4 v - \frac{1}{6!}|v|^6 - \ldots \\
&= (1 - \frac{1}{2!}|v|^2 + \frac{1}{4!}|v|^4 - \frac{1}{6!}|v|^6 + \ldots) + \frac{v}{|v|}(|v| - \frac{1}{3!}|v|^3 + \frac{1}{5!}|v|^5 + \ldots) \\
&= \cos|v| + \frac{v}{|v|}\sin|v|.
\end{aligned}
$$

$\square$

Lemma 12. *If $X = p + \epsilon q$ represents a revolute joint, i.e. $p \cdot q = 0$, then*

$$
(p + \epsilon q)^k = \begin{cases} p^k & \text{if } k \text{ is even} \\ p^{k-1}(p + \epsilon q) & \text{if } k \text{ is odd.} \end{cases}
$$

Proof. We prove this by induction on k.
Base case: $(p + \epsilon q)^1 = p + \epsilon q = p^0(p + \epsilon q)$. Note that if $p \cdot q = 0$, then $pq = -p \cdot q + p \times q = p \times q = -q \times p = -qp$. So multiplication of p and q is anticommutative. Thus, for $k = 2$ we get $(p + \epsilon q)^2 = p^2 + \epsilon(pq + qp) = p^2$. Now assume the hypothesis holds for $k = n$.
Case 1: n is even, so $n + 1$ is odd. We have $(p + \epsilon q)^{n+1} = (p + \epsilon q)^n(p + \epsilon q) = p^n(p + \epsilon q) = p^{(n+1)-1}(p + \epsilon q)$, as required.
Case 2: n is odd, so $n + 1$ is even. We have $(p + \epsilon q)^{n+1} = (p + \epsilon q)^n(p + \epsilon q) = p^{n-1}(p + \epsilon q)(p + \epsilon q) = p^{n-1}(p + \epsilon q)^2 = p^{n-1}p^2 = p^{n+1}$, as required. $\square$

We can now prove Theorem 10.

Proof (Theorem 10). From the definition of the exponential of a dual quaternion and Lemma 12 we obtain

$$
\begin{aligned}
\mathrm{e}^{t(p+\epsilon q)} &= \sum_{k=0}^{\infty} \frac{t^k(p+\epsilon q)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(tp)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1}p^{2k}(p+\epsilon q)}{(2k+1)!} \\
&= \cosh(tp) + p^{-1}\sinh(tp)(p+\epsilon q).
\end{aligned}
$$

Using Lemma 11, we can write

$$
\cosh(tp) = \frac{\mathrm{e}^{tp}+\mathrm{e}^{-tp}}{2} = \frac{1}{2}\left(\cos|tp| + \frac{t\sin|tp|}{|tp|}p + \cos|tp| - \frac{t\sin|tp|}{|tp|}p\right) = \cos|tp|,
$$

and similarly

$$
\sinh(tp) = \frac{\mathrm{e}^{tp}-\mathrm{e}^{-tp}}{2} = \frac{t\sin|tp|}{|tp|}p.
$$

Thus,

$$
\mathrm{e}^{t(p+\epsilon q)} = \cos|tp| + t\frac{\sin|tp|}{|tp|}(p+\epsilon q), \tag{8}
$$

as required.

The condition for $\mathrm{e}^{t(p+\epsilon q)}$ to lie in the Study quadric is given by condition (6). Writing $h_0 = \cos|tp| + \frac{t\sin|tp|}{|tp|}p$ and $h_1 = \frac{t\sin|tp|}{|tp|}q$, from equation (8) we obtain

$$
\begin{aligned}
h_0 h_1^* + h_1 h_0^* &= \left(\cos|tp| + \frac{t\sin|tp|}{|tp|}p\right)\left(\frac{t\sin|tp|}{|tp|}q\right)^* + \left(\frac{t\sin|tp|}{|tp|}q\right)\left(\cos|tp| + \frac{t\sin|tp|}{|tp|}p\right)^* \\
&= \frac{-t^2\sin^2|tp|}{|tp|^2}(pq+qp) && (p^* = -p \text{ and } q^* = -q) \\
&= 0, && (pq = -qp)
\end{aligned}
$$

thereby showing $\mathrm{e}^{t(p+\epsilon q)}$ lies in the Study quadric.

It is shown in [2, Chapter 11, §2] that straight lines through the identity in the Study quadric have the form $\alpha + \beta a + \beta\epsilon c$ for $\alpha, \beta \in \mathbb{R}$ and quaternions a and c. Comparing this with equation (8) and identifying $\alpha = \cos|tp|$, $\beta = t\frac{\sin|tp|}{|tp|}$, $a = p$, and $c = q$; we see that $\mathrm{e}^{t(p+\epsilon q)}$ characterises a straight line through the identity in the Study quadric.   □

Remark 4. In [4], revolute joints are represented by dual quaternions $(t-h)$ for some $t \in \mathbb{PR}^1$ (here we are treating $t = (t_0 : t_1)$ in homogeneous coordinates as a parameter $t = \frac{t_0}{t_1} \in \mathbb{R}$) and a pure dual quaternion h which represents rotation by π about the line represented by h. Then the possible relative configurations of a pair of linkages joined

by the revolute joint will trace a curve on the Study quadric in $\mathbb{PR}^7$. It is noted that the one–parameter rotation subgroups can be geometrically characterised as lines on the Study quadric through the identity, which we have independently shown with the one–parameter subgroups generated by elements of $\mathfrak{se}(3)$ in Theorem 10.

5   Bond Theory of Mobile Closed 4R Linkages

A mobile closed 4R linkage is a closed chain of four revolute joints that allows relative motion between the links. It is known that there are exactly three types of mobile 4R linkages: the Bennett, spherical, and planar four–bar linkages. The Bennett linkage is a type of mobile over–constrained mechanism defined by the following conditions.

i. Opposite sides of the mechanism have the same length, denoted by a and b.

ii. The oriented angles of the axes of the joints, denoted by ρ and ξ, are equal on opposite sides but with different sign.

iii. The mechanism must satisfy the relation:

$$\frac{\sin\rho}{b} = \frac{\sin\xi}{a}.$$

This special geometric configuration allows a Bennett linkage to have one degree of freedom, and hence positive mobility. Each of the links of a spherical linkage are constrained to rotate about the same fixed point in space, so that the axes of the joints intersect at a single point. This then ensures that the trajectories of the links lie on concentric spheres, hence the name spherical linkage. A planar linkage simply has the property that all of its links move in parallel planes.

This section will give some background knowledge of bond theory before moving on to computing the bond sets for the 4R linkages given above. We now move away from the Lie group method used to describe revolute linkages in §4, projectivise $\mathbb{DH}$ as an eight–dimensional vector space to obtain $\mathbb{PR}^7$, and work in this setting. This section closely follows the theory laid out in the paper [4].

As in Remark 4, a general chain of n revolute joints is represented by n pure unit dual quaternions $h_1, \ldots, h_n$. The one parameter subgroup parameterised by $(t_i - h_i)$ geometrically represents a revolute joint where $t_i \in \mathbb{PR}^1$ determines the rotation angle between the i-th and $(i+1)$-th link. The position of the last link relative to the first link is given by $(t_1 - h_1)(t_2 - h_2)\ldots(t_n - h_n)$. For a closed nR chain, this position should be the identity (represented by a non–zero real number), so we are left with the closure equation

$$(t_1 - h_1)(t_2 - h_2)\ldots(t_n - h_n) \in \mathbb{R}^*. \tag{9}$$

This gives the following definition.

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Definition 9. For a closed nR chain, the configuration set K (or configuration curve) is defined to be $K = \{(t_1, t_2, \ldots, t_n) \in (\mathbb{P}\mathbb{R}^1)^n \mid (t_1 - h_1)(t_2 - h_2) \ldots (t_n - h_n) \in \mathbb{R}^*\}$. The dimension of the Zariski closure of K is called the mobility of the linkage.

We associate with each revolute joint its axis of rotation. This is represented by the same dual quaternion h_i as the joint up to multiplication by -1. The motion of link j with respect to link i for $i < j$ is called the coupling curve $C_{i,j}$ and is given by $(t_{i+1} - h_{i+1})(t_{i+2} - h_{i+2}) \ldots (t_j - h_j)$.

5.1   Bonds

We now introduce the concept of bonds.

Definition 10 (Bonds)**.** Consider a closed nR linkage $(h_1, h_2, \ldots, h_n)$ with configuration curve K and Zariski closure $K_\mathbb{C}$. We define the bond set to be

$$B = \{(t_1, t_2, \ldots, t_n) \in K_\mathbb{C} \mid (t_1 - h_1)(t_2 - h_2) \ldots (t_n - h_n) = 0\}, \tag{10}$$

and call β a bond if $\beta \in B$.

A more rigorous definition of bonds can be found in [4] that allows for singularities to occur on the configuration curve. This will not be encountered for the 4R linkages we consider, however, so is not necessary here.

Remark 5. It is shown in Theorem 13 below that every bond has pure imaginary entries. Thus, it is important to distinguish the basis element $\mathbf{i} \in \mathbb{D}\mathbb{H}$ from the complex unit $\mathrm{i} \in \mathbb{C}$.

Theorem 13. *For any bond* $\beta = (t_1, t_2, \ldots, t_n)$, *there exist indices* $i, j \in \{1, 2, \ldots, n\}$ *such that* $t_i^2 + 1 = t_j^2 + 1 = 0$.

Proof. [2] Let $\beta = (t_1, t_2, \ldots, t_n)$ be a bond. Then, for any $k \in \{1, \ldots, n\}$, the norm of $(t_k - h_k)$ is $(t_k - h_k)(t_k - h_k)^* = (t_k - h_k)(t_k + h_k) = t_k^2 + 1$ as h_k is a pure unit dual quaternion. As β is a bond, we have

$$\prod_{k=1}^{n}(t_k - h_k) = (t_1 - h_1)(t_2 - h_2) \ldots (t_n - h_n) = 0. \tag{11}$$

---

[2]This proof follows the proof of Theorem 2 in [4].

Taking the norm of both sides of this equation we obtain

$$
\begin{aligned}
0 &= \prod_{k=1}^{n}(t_k - h_k)\left(\prod_{k=1}^{n}(t_k - h_k)\right)^{*} \\
&= \prod_{k=1}^{n}(t_k - h_k)\prod_{k=1}^{n}(t_{n+1-k} - h_{n+1-k})^{*} \quad \text{(conjugation reverses order of multiplication)} \\
&= \prod_{k=1}^{n}(t_k - h_k)\prod_{k=1}^{n}(t_{n+1-k} + h_{n+1-k}) \quad \text{(h_k are pure dual quaternions)} \\
&= \prod_{k=1}^{n}(t_k^2 + 1).
\end{aligned}
$$

Hence, we must have that $t_i^2 + 1 = 0$ for some $i \in \{1, \ldots, n\}$.

Without any loss of generality, we now assume that i is the least such number for which this condition holds. In order to show that there is $j \in (\{1, \ldots, n\} \setminus \{i\})$ such that $t_j^2 + 1 = 0$, we multiply equation (11) by $(t_{i-1} + h_{i-1})\ldots(t_2 + h_2)(t_1 + h_1)$ on the left, and by $(t_n + h_n)(t_{n-1} + h_{n-1})\ldots(t_{i+1} + h_{i+1})$ on the right, yielding

$$
\begin{aligned}
0 &= \prod_{k=1}^{i-1}(t_{i-k} + h_{i-k})\prod_{k=1}^{n}(t_k - h_k)\prod_{k=1}^{n-i}(t_{n+1-k} + h_{n+1-k}) \\
&= \left(\prod_{k=1}^{i-1}(t_k^2 + 1)\right)(t_i - h_i)\left(\prod_{k=i+1}^{n}(t_k^2 + 1)\right).
\end{aligned}
$$

But $t_i \in \mathbb{C}$ and h is a pure dual quaternion, so $(t_i - h_i)$ can never vanish. Thus, we conclude there is some $j \in (\{1, \ldots, n\} \setminus \{i\})$ such that $t_j^2 + 1 = 0$. $\qquad\square$

We can now make the following definition.

Definition 11. A bond β is called *typical* if there are exactly two indices $i, j \in \{1, \ldots, n\}$ such that $t_i^2 + 1 = t_j^2 + 1 = 0$.

5.2   Computing Bond Sets for 4R linkages

In this section we compute the bond sets for the three types of closed 4R linkages using Mathematica. The source code used to compute the following examples is given in Appendix B.

Example 1 (Bennett linkage). Consider the Bennett linkage (h_1, h_2, h_3, h_4) with

$$h_1 = \mathbf{i},$$
$$h_2 = 9\epsilon\mathbf{i} + \mathbf{j} - 9\epsilon\mathbf{k},$$
$$h_3 = -\left(\frac{1}{3} + 4\epsilon\right)\mathbf{i} - \left(\frac{2}{3} - 4\epsilon\right)\mathbf{j} + \left(\frac{2}{3} + 2\epsilon\right)\mathbf{k},$$
$$h_4 = \left(\frac{2}{3} + 5\epsilon\right)\mathbf{i} + \left(\frac{1}{3} + 4\epsilon\right)\mathbf{j} + \left(\frac{2}{3} - 7\epsilon\right)\mathbf{k}.$$

Let $X = (t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4)$, then, from the closure condition $X \in \mathbb{R}^*$, we can extract eight polynomial equations in t_1, t_2, t_3 and t_4 as the real–valued coefficients of the eight basis elements of the vector space $\mathbb{DH}$.

In order to try and solve this set of polynomial equations, we first compute a Gröbner basis (explained in Appendix A), and then solve this Gröbner basis to obtain a parameterised representation of the configuration curve. For the Bennett linkage above we obtain:

$$t_1 = t - 1, \quad t_2 = t, \quad t_3 = t - 1, \quad t_4 = -t, \tag{12}$$

where $t \in \mathbb{PR}^1$. We then substitute this back into X and find

$$(t - 1 - h_1)(t - h_2)(t - 1 - h_3)(-t - h_4) = -(t^2 + 1)(t^2 - 2t + 2). \tag{13}$$

The bonds are then found by finding the roots of equation (13). They are found to be $t = \pm \mathrm{i}$ and $t = 1 \pm \mathrm{i}$. Substituting these values of t back into equation (12) we obtain the bond set:

$$B = \{(\pm\mathrm{i}, 1 \pm \mathrm{i}, \pm\mathrm{i}, -1 \pm \mathrm{i}), \ (-1 \pm \mathrm{i}, \pm\mathrm{i}, -1 \pm \mathrm{i}, \mp\mathrm{i})\}.$$

As it can be seen, each bond in B is a typical bond.

$\square$

We now follow the same procedure for the spherical and planar linkages.

Example 2 (Spherical linkage). Consider the spherical linkage (h_1, h_2, h_3, h_4) given by

$$h_1 = \mathbf{i}, \quad h_2 = \mathbf{j}, \quad h_3 = \mathbf{k}, \quad h_4 = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

As for the Bennett linkage, we solve the Gröbner basis obtained from the eight polynomial equations derived from $X = (t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \in \mathbb{R}^*$. The configuration curve is found to be parameterised by

$$t_1 = \frac{-5t^2 + 5 + w}{6t}, \quad t_2 = \frac{-5t^2 - 5 + w}{8t}, \quad t_3 = \frac{25t^2 - 7 - 5w}{24},$$

where $w = \pm\sqrt{25t^4 - 14t^2 + 25}$. Substituting this parameterisation into X we obtain

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) = \frac{5(t^2 + 1)(125 + 5t^2(25t^2 - 14 \pm 5w) \pm 7w)}{288t},$$

which has roots $\pm i$, $\frac{-4}{5} \pm \frac{3}{5}i$, and $\frac{4}{5} \pm \frac{3}{5}i$. The bond set is then computed to be

$$B = \left\{ \left(\pm\frac{i}{3}, \pm i, \frac{1}{3}, \pm i\right), \left(\mp i, -1, \pm i, \frac{4}{5} \pm \frac{3}{5}i\right), \left(\mp i, 1, \pm i, \frac{-4}{5} \pm \frac{3}{5}i\right), (\pm 3i, \pm i, -3, \mp i) \right\}.$$

As for the Bennett linkage, each bond in B is typical.

$\square$

Example 3 (Planar linkage). We now consider the planar linkage given by

$$h_1 = \epsilon\mathbf{i} + \mathbf{k}, \quad h_2 = \epsilon\mathbf{j} + \mathbf{k}, \quad h_3 = \mathbf{k}, \quad h_4 = \epsilon\mathbf{i} + 2\epsilon\mathbf{j} + \mathbf{k}.$$

The configuration curve can be parameterised by

$$t_1 = \frac{5 - t(t-2) + w}{2(t+3)}, \quad t_2 = \frac{t^2 + 1 - w}{4(2-t)}, \quad t_3 = \frac{t(t-4) + 1 - w}{4(t-1)}, \quad t_4 = t,$$

where $w = \pm\sqrt{t^4 - 8t^3 + 2t^2 + 56t - 47}$. This then yields

$$(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) = \frac{(t^2 + 1)(t^5 - 9t^4 + t^3(18 \mp w) + t^2(38 \pm 5w) - 5t(19 \pm w) \mp 7w + 31)}{8(t^3 - 7t + 6)},$$

which has roots $\pm i$ and $4 \pm i$. The bond set is

$$B = \{(\pm i, -2 \pm i, \mp i, 4 \mp i), (\pm i, \mp i, \pm i, \mp i), (2 \pm i, \mp i, -1 \pm 2i, \pm i)\}.$$

As it can be seen, the planar linkage has two non–typical bonds $(\pm i, \mp i, \pm i, \mp i)$.

$\square$

We have demonstrated the existence and computation of bonds. Their significance lies in the fact that bonds only exist for mechanisms with positive mobility. Thus, they provide an avenue for understanding and clarifying the so–called paradoxical mechanisms – those that are unexpectedly mobile, such as the three examples discussed in this section.

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[4] Gábor Hegedüs, Joseph Schicho, and Hans-Peter Schröcker. The theory of bonds: A new method for the analysis of linkages. *Mechanism and Machine Theory*, 70:407–424, 2013.

[5] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms.* Undergraduate Texts in Mathematics. Springer Science+Business Media, second edition, 1997.

Appendix A   Gröbner Bases

Intuitively, a Gröbner basis for a set of polynomials is a set of polynomials that have the same common solutions as the original set of polynomials, but are better–suited for computation. That is, in general, it is much easier to find solutions to the Gröbner basis than to the original set of polynomials. In order to define a Gröbner basis, we first need the following definitions.

Definition 12 (Ideals of polynomials). Let k be a field and let $k[x_1, \ldots, x_n]$ be the ring of polynomials in $x_1, \ldots, x_n$ with coefficients in k. We say a subset $I \subseteq k[x_1, \ldots, x_n]$ is an ideal of polynomials if it satisfies

   i. $0 \in I$.

   ii. If $f, g \in I$, then $f + g \in I$.

   iii. If $f \in I$ and $h \in k[x_1, \ldots, x_n]$, then $hf \in I$.

Definition 13. Let $f_1, \ldots, f_k$ be a finite set of polynomials in $k[x_1, \ldots, x_n]$. We define the ideal I generated by $f_1, \ldots, f_k$ to be

$$I = \langle f_1, \ldots, f_k \rangle = \left\{ \sum_{i=1}^{k} h_i f_i \mid h_1, \ldots, h_k \in k[x_1, \ldots, x_n] \right\},$$

and say that $f_1, \ldots, f_k$ is a basis for I.

Definition 14 (Monomial ordering). Let $n \geq 0$. Then, a monomial ordering on $k[x_1, \ldots, x_n]$ is a relation on the set of monomials x^α, $\alpha \in \mathbb{Z}^n$, satisfying:

   i. $>$ is a total ordering on $\mathbb{Z}^n$.

   ii. if $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n$, then $\alpha + \gamma > \beta + \gamma$.

   iii. Every nonempty subset of $\mathbb{Z}^n$ has a smallest element under $>$.

Definition 15. Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $k[x_1, \ldots, x_n]$ and let $>$ be a monomial order. Then

   i. The multidegree of f is $\max(\alpha \in \mathbb{Z}^n \mid a_\alpha \neq 0)$.

   ii. The leading coefficient of f is $\mathrm{LC}(f) = a_{\mathrm{multideg}(f)} \in k$.

   iii. The leading monomial of f is $\mathrm{LT}(f) = x^{\mathrm{multideg}(f)}$.

   iv. The leading term of f is $\mathrm{LT}(f) = \mathrm{LC}(f) \cdot \mathrm{LM}(f)$.

We can now define a Gröbner basis.

Definition 16 (Gröbner basis)**.** Fix a monomial order. Then, a finite subset $G = \{g_1, \ldots, g_k\}$ of an ideal I is called a Gröbner basis if

$$\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_k) \rangle = \langle \mathrm{LT}(I) \rangle,$$

where $\mathrm{LT}(I)$ is the set of leading terms of elements of I.

From this definition it is not clear whether we can always guarantee that a Gröbner basis can be computed for a set of polynomials. However, we have the following powerful theorem.

Theorem 14 (Hilbert basis theorem)**.** *For every ideal $I \subseteq k[x_1, \ldots, x_n]$ there is some $g_1, \ldots, g_k \in I$ such that $I = \langle g_1, \ldots, g_k \rangle$.*

Proof. A proof is given in [5, Page 74]. □

Corollary 14.1. *Fix a monomial ordering. Then every ideal I other than $\{0\}$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal I is indeed a basis for I.*

This is extremely useful as we can, in theory, compute a Gröbner basis for any non trivial ideal. We have not yet given any examples of the type of monomial orderings that a Gröbner basis would use. Here we present the most common type of monomial ordering used for computer algebra systems: lexicographic ordering.

Definition 17 (Lexicographic order)**.** Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ for some $n \geq 0$. We say $\alpha >_{lex} \beta$ if the left most nonzero entry in vector notation of $\alpha - \beta$ is positive. We write $x^\alpha >_{lex} x^\beta$ if $\alpha >_{lex} \beta$.

Proposition 15. The lex ordering on $\mathbb{Z}^n$ is a monomial ordering.

Proof. A proof is given in [5, Page 55]. □

The Mathematica function GroebnerBasis used in Appendix B uses a lexicographic order to compute a Gröbner basis from a set of polynomials using Buchberger's algorithm. A detailed explanation of Buchberger's algorithm is given in [5, Chapter 2, §7].

Appendix B   Mathematica Code for 4R Linkages

B.1   Bennet Linkage

```
In[1]:=    (* Defining the dual unit *)
           ε/:Power[ε,n_]:=0/; n≥2

           (* Loading the Quaternion package *)
           Needs["Quaternions`"]

           (* Shorthand for "Quaternion" *)
           Q[a_,b_,c_,d_]:=Quaternion[a,b,c,d]

           (* Defining the Bennett dual quaternions *)
           h1=Q[0,1,0,0];
           h2=Q[0,9ε,1,-9ε];
           h3=Q[0,-1/3-4ε,-2/3+ 4ε,2/3+2ε ];
           h4=Q[0,2/3+5ε,1/3+4 ε,2/3-7ε ];

           (* Transforming scalars t1, t2, t3, and t4 into quaternions *)
           t1Q=Q[t1,0,0,0];
           t2Q=Q[t2,0,0,0];
           t3Q=Q[t3,0,0,0];
           t4Q=Q[t4,0,0,0];

           (* Functions for extracting the 8 real coefficients of a dual quaternion *)
           ScalarPart[Q[a_,b_,c_,d_]]:=Simplify[a]

           DualPart[x_]:=Expand[
             Simplify[
              Coefficient[
               Simplify[x],ε,1]
              ]
            ]

           RealPart[x_]:=Expand[
             Simplify[
              Simplify[x]-Coefficient[Simplify[x],ε,1]* ε
             ]
            ]
```

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```
QtoList[Q[a_,b_,c_,d_]]:={a,b,c,d}

RealVec[x_]:=QtoList[RealPart/@x]
DualVec[x_]:=QtoList[DualPart/@x]

(* Quaternion obtained from the closure equation *)
X=Simplify[(t1Q-h1)**(t2Q-h2)**(t3Q-h3)**(t4Q-h4)];

(* Groebner basis for the polynomials obtained from X *)
GBasis=GroebnerBasis[
  Flatten[
   {u*RealVec[X][[1]]-1,RealVec[X][[2;;4]],DualVec[X]}
   ],
   {u,t1,t2,t3,t4}
 ];

(* Function that extracts the polynomials that do not contain u from the
Groebner basis *)
uDelete[x_]:=If[
  ContainsAny[{Coefficient[x,u,1]},{0}]==True,x,Nothing
 ]

SetAttributes[uDelete,Listable]

(* Set of polynomials that do not contain u *)
PolySet=uDelete[GBasis];

"Parameterisation of configuration curve:"
tvals=Solve[
  Table[
   PolySet[[n]]==0,{n,1,Length[PolySet]}
  ],
  {t1,t2,t3,t4},   MaxExtraConditions→1
 ]//Flatten

(* Parameterised representation of the configuration curve *)
ConfigCurve=ScalarPart[
   Simplify[X/.tvals]/.t4→-t
  ];
```

```
           "Configuration curve:"
           Factor[ConfigCurve]

           (* Complex roots of the configuration curve *)
           "Roots of Configuration Curve:"
           Bonds=Solve[ConfigCurve==0]

           (* The bond set *)
           "Bond set:"
           TableForm[
            BondSet={t1,t2,t3,t4}/.tvals/.t4→-t/.Bonds,
            TableDepth→1]
```

Out[1]=   Parameterisation of configuration curve:

Out[2]= $\{$t1→-1-t4,t2→-t4,t3→-1-t4$\}$

Out[3]=   Configuration curve:

Out[4]=   -(1+t^2) (2-2 t+t^2)

Out[5]=   Roots of Configuration Curve:

Out[6]= $\{\{$t→-i$\},\{$t→i$\},\{$t→1-i$\},\{$t→1+i$\}\}$

Out[7]=   Bond set:

Out[8]= $\{$-1-i,-i,-1-i,i$\}$
          $\{$-1+i,i,-1+i,-i$\}$
          $\{$-i,1-i,-i,-1+i$\}$
          $\{$i,1+i,i,-1-i$\}$

B.2   Spherical Linkage

In[1]:= ϵ/:Power[ϵ,n_]:= 0/;n≥2

```
           Needs["Quaternions`"]

           Q[a_,b_,c_,d_]:=Quaternion[a,b,c,d]

           t1Q=Q[t1,0,0,0];
           t2Q=Q[t2,0,0,0];
```

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```
t3Q=Q[t3,0,0,0];
t4Q=Q[t4,0,0,0];

h1=Q[0,1,0,0];
h2=Q[0,0,1,0];
h3=Q[0,0,0,1];
h4=Q[0,3/5,4/5,0];

ScalarPart[Q[a_,b_,c_,d_]]:=Simplify[a]
SetAttributes[ScalarPart,Listable]

DualPart[x_]:=Expand[
  Simplify[
   Coefficient[Simplify[x],ϵ,1]
  ]
 ]

RealPart[x_]:=Expand[
  Simplify[
   Simplify[x]-Coefficient[Simplify[x],ϵ,1]*ϵ
  ]
 ]

QtoList[Q[a_,b_,c_,d_]]:={a,b,c,d}

RealVec[x_]:=QtoList[RealPart/@x]
DualVec[x_]:=QtoList[DualPart/@x]

X=Simplify[(t1Q-h1)**(t2Q-h2)**(t3Q-h3)**(t4Q-h4)];

GBasis=GroebnerBasis[
   Flatten[
    {u*RealVec[X][[1]]-1,RealVec[X][[2;;4]],DualVec[X]}
   ],
   {u,t1,t2,t3,t4}
  ];

uDelete[x_]:=If[
  ContainsAny[
    {Coefficient[x,u,1]},{0}]==True,
  x,Nothing]
```

```mathematica
SetAttributes[uDelete,Listable]

PolySet=uDelete[GBasis];

tvals=ToRules[
    Reduce[
      Table[
        PolySet[[n]]==0,{n,1,Length[PolySet]}],
      Backsubstitution→True]/.t4→t
    ]//List//FullSimplify;

"Parameterisation of configuration curve:"
tvals/.Sqrt[25t^4-14t^2+25]→w

ConfigCurves=ScalarPart[
    FullSimplify[
      X/.tvals]
    ]/.t4→t;

"Configuration curves:"
ConfigCurves/.Sqrt[25t^4-14t^2+25]→w

"Bonds:"
Bonds=Flatten[DeleteDuplicates[Table[
    Solve[ConfigCurves[[i]]==0],
    {i,1,Length[ConfigCurves]}]],1]

"Bond set:"
BondSet=TableForm[
  DeleteDuplicates[Flatten[{t1,t2,t3,t4}/.tvals/. t4→t/.Bonds,1]//Sort],
  TableDepth→1]
```

Out[1]=   Parameterisation of configuration curve:

$$
\text{Out[2]=} \quad \left\{\left\{t3 \to \frac{1}{24}\,(-7+25\ t^2-5\ w), t2 \to \frac{-5-5\ t^2+w}{8\ t}, t1 \to \frac{5-5\ t^2+w}{6\ t}\right\},\right.
$$

$$
\left.\left\{t3 \to \frac{1}{24}(-7+25\ t^2+5\ w), t2 \to -\frac{5+5\ t^2+w}{8t}, t1 \to -\frac{-5+5\ t^2+w}{6\ t}\right\}\right\}
$$

Out[3]=   Configuration curves:

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Out[4]= $\{\dfrac{5\ (1+t^2)\ (125+125\ t^4+7\ w-5\ t^2\ (14+5\ w))}{288\ t}$,

$\dfrac{5\ (1+t^2)\ (125-7\ w+5\ t^2(-14+25\ t^2+5\ w))}{288\ t}\}$

Out[5]=   Bonds:

Out[6]= $\{\{t\rightarrow-\dfrac{4}{5}-\dfrac{3i}{5}\},\ \{t\rightarrow-\dfrac{4}{5}+\dfrac{3i}{5}\},\ \{t\rightarrow-i\},\{t\rightarrow i\},\ \{t\rightarrow\dfrac{4}{5}-\dfrac{3i}{5}\},\ \{t\rightarrow\dfrac{4}{5}+\dfrac{3\ i}{5}\}\}$

Out[7]=   Bond Set:

Out[8]= $\{-\dfrac{i}{3},\ i,\ \dfrac{1}{3},\ i\},\ \{\dfrac{i}{3},-i,\dfrac{1}{3},-i\}$

$\{-i,-1,i,\dfrac{4}{5}+\dfrac{3\ i}{5}\},\ \{-i,1,-i,-\dfrac{4}{5}+\dfrac{3\ i}{5}\}$

$\{i,-1,-i,\dfrac{4}{5}-\dfrac{3\ i}{5}\},\ \{i,1,i,-\dfrac{4}{5}-\dfrac{3\ i}{5}\}$

$\{-3\ i,-i,-3,i\},\ \{3\ i,i,-3,-i\}$

B.3   Planar Linkage

```mathematica
In[1]:=   ϵ/:Power[ϵ,n_]:= 0/;n≥2

          Needs["Quaternions‘"]

          Q[a_,b_,c_,d_]:=Quaternion[a,b,c,d]

          t1Q=Q[t1,0,0,0];
          t2Q=Q[t2,0,0,0];
          t3Q=Q[t3,0,0,0];
          t4Q=Q[t4,0,0,0];

          h1=Q[0,ϵ,0,1];
          h2=Q[0,0,ϵ,1];
          h3=Q[0,0,0,1];
          h4=Q[0,ϵ,2ϵ,1];

          QtoList[Q[a_,b_,c_,d_]]:={a,b,c,d}

          ScalarPart[Q[a_,b_,c_,d_]]:=Simplify[a]
          SetAttributes[ScalarPart,Listable]
```

```
DualPart[x_]:=Expand[
  Simplify[
   Coefficient[Simplify[x],ϵ,1]
  ]
 ]

RealPart[x_]:=Expand[
  Simplify[
   Simplify[x]-Coefficient[Simplify[x],ϵ,1]*ϵ
  ]
 ]

RealVec[x_]:=QtoList[RealPart/@x]
DualVec[x_]:=QtoList[DualPart/@x]

X=Simplify[(t1Q-h1)**(t2Q-h2)**(t3Q-h3)**(t4Q-h4)];

GBasis=GroebnerBasis[
   Flatten[
    {u*RealVec[X][[1]]-1,RealVec[X][[2;;4]],DualVec[X]}
    ],
    {u,t1,t2,t3,t4}
  ];

uDelete[x_]:=If[
  ContainsAny[
    {Coefficient[x,u,1]},{0}]==True,
  x,Nothing]

SetAttributes[uDelete,Listable]

PolySet=uDelete[GBasis];

tvals=ToRules[
     Reduce[
       Table[
        PolySet[[n]]==0,{n,1,Length[PolySet]}],
       Backsubstitution→True]/.t4→t
    ]//List//Simplify;
tvals[[1;;2]]=Nothing;
```

```
"Parameterisation of configuration curve:"
FullSimplify[tvals]/.Sqrt[-47+t (56+t (2+(-8+t) t))]→w

ConfigCurves=ScalarPart[
    FullSimplify[
     X/.tvals]
   ]/.t4→t;

"Parameterisation of closure equation:"
ConfigCurves/.Sqrt[-47+t (56+t (2+(-8+t) t))]→w

"Solutions to closure equation:"
Solutions=DeleteDuplicates[
  Flatten[
   Table[
    Solve[ConfigCurves[[i]]==0],
    {i,1,Length[ConfigCurves]}],
   1]
  ]

"Bond set:"
BondSet=
 TableForm[
  Flatten[
   {t1,t2,t3,t4}/.tvals/.t4→t/.Solutions,
   1]//Sort,
  TableDepth→1]
 ]
```

Out[1]=   Parameterisation of configuration curve:

Out[2]= $\{\{t3 \to \dfrac{1+(-4+t)t-w}{4(-1+t)},\ t2 \to \dfrac{-1-t^2+w}{4(-2+t)},\ t1 \to \dfrac{5-(-2+t)t+w}{2(3+t)}\},$
          $\{t3 \to \dfrac{1+(-4+t)t+w}{4(-1+t)},\ t2 \to \dfrac{1+t^2+w}{8-4t},\ t1\to -\dfrac{-5+(-2+t)t+w}{2\ (3+t)}\}\}$

Out[3]=   Parameterisation of closure equation:

Out[4]= $\{\dfrac{(1+t^2)(31-7w+t(-5(19+w)+t(38+t(18+(-9+t)t-w)+5w)))}{8(6-7t+t^3)},$
          $\dfrac{(1+t^2)(31+7w+t(5\ (-19+w)+t(38-5w+t(18+(-9+t)t+w))))}{8\ 6-7t+t^3)}\}$

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Out[5]=   Solutions to closure equation:

Out[6]= $\{\{t\rightarrow-i\},\{t\rightarrow i\},\{t\rightarrow4-i\},\{t\rightarrow4+i\}\}$

Out[7]=   Bond set:

Out[8]= $\{-\dfrac{2}{5}-\dfrac{i}{5},-2-i,\dfrac{1}{5}+\dfrac{2i}{5},4-i\}$, $\{-\dfrac{2}{5}+\dfrac{i}{5},-2+i,\dfrac{1}{5}-\dfrac{2i}{5},4+i\}$
$\{-i,-2-i,i,4+i\}$, $\{-i,i,-i,i\}$
$\{i,-2+i,-i,4-i\}$, $\{i,-i,i,-i\}$
$\{2-i,i,-1-2\ i,-i\}$, $\{2+i,-i,-1+2\ i,i\}$