Natural Definability in the Nether Regions of the Computably Enumerable Degrees

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GOALS

- Understand the structure of the c.e. degrees by finding naturally definable classes of degrees.
- Understand the dynamic nature of constructions of c.e. degrees.
- Understanding the lower regions of the c.e. degrees.

An example of a naturally definable class

Theorem (Downey and Lempp)

A c.e. degree is contiguous iff it is locally distributive.

Theorem (Ambos-Spies and Fejer)

A c.e. degree is contiguous iff it is not the top of a copy (in the c.e. dgerees) of the non-modular, non-distributive 5 element lattice N_5.

A SWEEPING RESULT

Nies, Shore and Slaman used coding of the standard model of arithmetic in the c.e. degrees to get bi-interpretability up to the second jump:

THEOREM

A relation on the c.e. degrees which is invariant under the double jump is definable in the c.e. degrees iff it is definable in arithmetic.

As a result, all jump-classes, except perhaps for the low degrees, are definable.

Two issues that limit our usage of the Nies-Shore-Slaman bi-interpretability:

- It cannot give definable subclasses of the low_2 degrees.
- The definitions are not natural.

Jump classes and permitting

Various notions of permitting:

- Simple permitting (Yates) – finite injury constructions, such as Friedberg-Muchink.
- Prompt permitting (Ambos-Spies et. al.) – minimal pair.
- High permitting (Martin) – almost every construction.
- Non-low$_2$ and array-non-computable permitting (Downey, Jockusch, Stob) – 1 generic degrees, cupping, tops of lattices in the degrees.

Permitting and definability

Theorem (Ambos-Spies, Jockusch, Shore, Soare)

A c.e. degree is promptly simple iff it is not cappable.

Permitting and domination

Notions of permitting are sometimes closely related to the partial ordering of Baire space under domination.

Facts (Martin; Downey-Jockush-Stob)

1. A degree is high iff it computes a function that dominates all computable functions.
2. A degree **a** is non-low$_2$ iff there is no function computable in $\mathbf{0}'$ which dominates every function computable from **a**.
3. A degree **a** is array-non-computable iff there is no function which is truth-table reducible to $\mathbf{0}'$ and which dominates every function computable from **a**.

Lattice embeddings in initial segments

Every c.e. degree bounds N_5 and distributive lattices. So we need to examine modular, non-distributive lattices. The smallest one is the 1-3-1 (aka M_5).

Theorem (Downey and Shore)

Every non-low$_2$ degree bounds a copy of the 1-3-1 lattice.

Theorem (Downey, Weinstein)

There is a c.e. degree that does not bound a copy of the 1-3-1.

Fact (Walk)

Such a degree can be made array-non-recursive.

Critical triples

Incomparable elements $\mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{b}$ in an upper semi-lattice form a critical triple if $\mathbf{a}_0$ and $\mathbf{a}_1$ are equivalent modulo $\mathbf{b}$, but anything that lies below both $\mathbf{a}_0$ and $\mathbf{a}_1$ is also below $\mathbf{b}$.

For example, the middle elements of the 1-3-1 form a critical triple. A lattice contains a critical triple iff it contains a copy of the 1-3-1.

Downey's and Weinstein's proofs actually show that there is a degree below which there is no critical triple.

The Ershov Hierarchy

This is an absolute hierarchy of Δ^0_2 sets and functions, indexed by the computable ordinals.

A function f is α-c.e. if changes in our guess for $f(x)$ (in a computable approximation for f) have to be accompanied by a descending sequence from α.

Understanding the lower levels of the hierarchy

If $g(x, s)$ is a computable approximation for a Δ^0_2 function f, define the mind changing function of g:

$$m_g(n) = \#\{s : g(x, s) \neq g(x, s+1)\}$$

Fact

A function is ω-c.e. iff it has a computable approximation g such that m_g is dominated by some computable function.

Fact

A function is ω^{n+1}-c.e. iff it has a computable approximation g such that m_g is dominated by some function which is ω^n-c.e.

Ershov and Turing

Ershov's notion of complexity is not related to Turing's. For example, the c.e. sets lie in the bottom of Ershov's hierarchy but can be Turing complete.

How do we use Ershov's hierarchy for the study of the c.e. degrees? By brute force.

Definition

A c.e. degree is totally α-c.e. if every function which it computes is α-c.e.

The hierarchy of totally α-c.e. degrees.

Theorem

There is a degree which is totally α-c.e. and not totally β-c.e. for any $\beta < \alpha$ iff $\alpha = \omega^\gamma$ for some γ.

Facts

- Every array-computable degree is totally ω-c.e.
- Every totally α-c.e. degree is low_2.

Theorem

There are maximal totally α-c.e. degrees, but if $\gamma < \delta$ then no totally ω^γ-c.e. degree is maximal in the class of totally ω^δ-c.e. degrees.

Totally ω-c.e. degrees

Theorem (D, G and Weber)

A c.e. degree is totally ω-c.e. iff it does not bound a critical triple.

Corollary

The low degrees and the superlow degrees are not elementarily equivalent.

Problem

Find a definition of the totally ω^2-c.e. degrees.

A refinement of the hierarchy

Definition

A c.e. degree **a** is totally $< \alpha$-c.e. if every function *f* computable from **a** is β-c.e. for some $\beta < \alpha$.

Theorem

There is a degree which is totally $< \alpha$-c.e. but not totally β-c.e. for any $\beta < \alpha$ iff α is of the form ω^γ for some limit ordinal γ.

TOTALLY $< \omega^\omega$-C.E. DEGREES

THEOREM

A c.e. degree is totally $< \omega^\omega$-c.e. iff it does not bound a copy of the 1-3-1.

Other constructions

Other constructions share the dynamics of the critical triple / 1-3-1 constructions.

- *m*-topped degrees.
- Presentations of left-c.e. reals.
- Degrees which contain wtt-minimal pairs.
- Completely mitotic degrees.

There are related results.

THEOREM (G)

Let $\alpha > \omega$ be an admissible ordinal, and let **a** *be an incomplete α-c.e. degree. Then* **a** *bounds a copy of the 1-3-1 iff* **a** *bounds a critical triple iff* **a** *computes a counting of α.*

COROLLARY

The sentence "there is an incomplete degree which bounds a critical triple but not the 1-3-1" holds in the classical c.e. degrees but not in the α-c.e. degrees (for any admissible $\alpha > \omega$.)