Superatomic Boolean algebras and ATR_0

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Lemma 1 (Hirst). *Let $<$ be a linear ordering of some $A \subset \omega$. The following are equivalent over* RCA_0*:*

1. *Every subset of A has a $<$-least element.*

2. *There are no infinite $<$-decreasing chains.*

Such $<$ is called an *ordinal*.

Simpson writes:

> "... ATR_0 is the weakest set of axioms which permits the development of a decent theory of countable ordinals."

Evidence, for example:

Theorem 2 (Friedman, Friedman-Hirst). *The following are equivalent over* RCA_0*:*

1. *If α, β are ordinals, then either α embeds into β or β embeds into α.*

2. *If α, β are ordinals, then either α is an initial segment of β or β is an initial segment of α.*

3. ATR_0.

Proposition 3 (RCA_0). *The following is equivalent to Π^1_1-CA_0: Given a sequence $\langle X_n \rangle_{n<\omega}$ of linear orderings, the set $\{n : X_n$ is well-founded$\}$ exists.*

Definition 4. An Abelian p-group is *reduced* if it has no divisible subgroup.

Lemma 5 (Friedman-Simpson-Smith). *The following is equivalent to Π^1_1-CA_0 over RCA_0: every Abelian p-group is the direct sum of a reduced group and a divisible group.*

Definition 6. An *Ulm resolution* of a reduced Abelian p-group G is a sequence $\langle G_\beta \rangle_{\beta \leqslant \alpha}$ where α is an ordinal, $G_0 = G$, $G_\alpha = 0$, $G_{\beta+1} = pG_\beta$ and for limit β, $G_\gamma = \cap_{\beta<\gamma} G_\beta$.

Theorem 7 (Friedman-Simpson-Smith). *The following are equivalent over* RCA_0*:*

1. *Every reduced Abelian p-group has an Ulm resolution.*

2. *For any two reduced Abelian p-groups G and H, either G is a direct summand of H or H is a direct summand of G.*

3. ATR_0.

A structure A for a first-order computable language is identified with its atomic diagram $D(A)$. We allow A to be a proper subset of ω.

We work with a class $\mathcal{A}$ of structures (for the same language), closed under isomorphism. Together with $\mathcal{A}$ we are given a notion of embeddability $\preccurlyeq$.

COMP($\mathcal{A}$) is the statement: for every $A, B \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq B$ or $B \preccurlyeq A$.

EQU=ISO($\mathcal{A}$) is the statement: for every $A, B \in \mathcal{A}$, if both $A \preccurlyeq B$ and $B \preccurlyeq A$ then $A \cong \mathcal{B}$.

WQO($\mathcal{A}$) is the statement: if $\langle A_n \rangle_{n<\omega}$ is a sequence of members of $\mathcal{A}$ then there are some $n < m$ such that $A_n \preccurlyeq A_m$.

∃-ISO$(\mathcal{A})$ is the statement: if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{A}$, then the set $\{(n, m) : A_n \cong A_m\}$ exists.

∃-EMB$(\mathcal{A})$ is the statement: if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of structures in $\mathcal{A}$, then the set $\{(n, m) : A_n \preccurlyeq A_m\}$ exists.

We are also given a notion of rank on structures in $\mathcal{A}$.

RK$(\mathcal{A})$ is the statement: every structure in $\mathcal{A}$ is ranked.

Let $\mathcal{O}n$ be the class of ordinals.

Theorem 8. *The following are all equivalent to* ATR_0 *over* RCA_0*:*

1. COMP($\mathcal{O}n$) *(Friedman - Hirst);*

2. EQU=ISO($\mathcal{O}n$) *(Friedman - Hirst);*

3. WQO($\mathcal{O}n$) *(Shore);*

4. $\exists$-ISO($\mathcal{O}n$)*;*

5. $\exists$-EMB($\mathcal{O}n$)*.*

A *tree* is a subset of $\omega^{<\omega}$, closed under initial segments. A tree is *well-founded* is it has no infinite path. Let $\mathcal{WFT}$ denote the class of well-founded trees.

A *rank function* on a tree T is a function f from T onto an ordinal α such that for all $\sigma \in T$,

$$f(\sigma) = \mathsf{sup}\{f(\tau) + 1 : \tau \in T, \tau \supsetneq \sigma\}.$$

The main property of rank: A ranked tree T embeds into a ranked tree S iff $\mathsf{rk}(T) \preccurlyeq \mathsf{rk}(S)$.

Theorem 9 (Hirst). *Under* ATR_0, *every well-founded tree is ranked.*

Theorem 10 (ACA_0). *Assume that every well-founded tree is ranked. Then* $\exists$-EMB($\mathcal{WFT}$), COMP($\mathcal{WFT}$) *and* WQO($\mathcal{WFT}$) *all hold.*

Theorem 11 (RCA_0). *The following are all equivalent to* ATR_0*:*

1. RK($\mathcal{WFT}$);

2. $\exists$-ISO($\mathcal{WFT}$);

3. $\exists$-EMB($\mathcal{WFT}$);

4. COMP($\mathcal{WFT}$);

5. WQO($\mathcal{WFT}$).

Lemma 12 (RCA_0). *Let B be a Boolean algebra. The following are equivalent:*

1. *B contains an infinite free set.*
2. *There is an embedding of the full binary tree into B (preserving $\leqslant$ and $\perp$).*
3. *B has an atomless subalgebra.*

Such a Boolean algebra is not called *superatomic*.

Lemma 13 (ACA_0). *A Boolean algebra B is superatomic iff it has no atomless quotient.*

Let $\mathcal{SABA}$ denote the class of superatomic Boolean algebras.

The *atomic ideal* of a Boolean algebra is the ideal generated by its atoms.

A *Cantor-Bendixon resuolution* of a Boolean algebra B is a sequence of ideals $\langle I_\beta \rangle_{\beta \leqslant \alpha}$ such that $I_0 = \{0\}$, $I_\alpha = B$, unions are taken at limit stages and for all $\beta < \alpha$, $I_{\beta+1}$ is the pullback to B of the atomic ideal of B/I_β.

Superatomic Boolean algebras are characterized by the length of their resolution, together with the number of atoms in $B/I_{\alpha-1}$. For invariant pairs (α, n) and (β, m), let $(\alpha, n) \cong (\beta, m)$ if $\alpha \cong \beta$ and $n = m$; and let $(\alpha, n) \preccurlyeq (\beta, m)$ if $\alpha \preccurlyeq \beta$ and $n \leqslant m$. If A, B are ranked, then $A \cong B$ iff $\mathrm{inv}(A) \cong \mathrm{inv}(B)$ and $A \preccurlyeq B$ iff $\mathrm{inv}(A) \preccurlyeq \mathrm{inv}(B)$.

Theorem 14 (ACA_0**).** *Assume that every superatomic Boolean algebra is ranked. Then the following all hold:* $\exists$-ISO($\mathcal{SABA}$), $\exists$-EMB($\mathcal{SABA}$), COMP($\mathcal{SABA}$), EQU=ISO($\mathcal{SABA}$) *and* WQO($\mathcal{SABA}$).

Theorem 15 (RCA_0**).** *The following are all equivalent to* ATR_0*:*

1. RK($\mathcal{SABA}$);
2. $\exists$-ISO($\mathcal{SABA}$);
3. $\exists$-EMB($\mathcal{SABA}$);
4. COMP($\mathcal{SABA}$);
5. EQU=ISO($\mathcal{SABA}$);
6. WQO($\mathcal{SABA}$)+ACA_0.

Definition 16. Let $\mathcal{A}, \mathcal{B}$ be classes of structures. Then $\mathcal{A} \leqslant_{TW} \mathcal{B}$ if there is some computable $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1}\mathcal{B} = \mathcal{A}$.

The following definition was made by Calvert-Cummins-Knight-S.Miller.

Definition 17. Let $\mathcal{A}, \mathcal{B}$ be classes of structures. Then $\mathcal{A} \leqslant_c \mathcal{B}$ if uniformly, for any $A \in \mathcal{A}$, from any enumeration of A one can produce an enumeration of some $\Phi(A) \in \mathcal{B}$, preserving isomorphism and non-isomorphism.

Theorem 18. *The following classes are all TW-and c-equivalent (via the same reductions):*

1. *Ordinals;*

2. *Superatomic Boolean algebras;*

3. *Fat well-founded trees;*

4. *Fat reduced Abelian p-groups.*

Let $\mathcal{A}, \mathcal{B}$ be a pair of these classes. Under the assumption that the structures in $\mathcal{A}$ are ranked, one can usually show in RCA_0 that we have a c-reduction; in ACA_0 we can show that it preserves non-embedding as well. Thus ATR_0 suffices.

However: for $\mathcal{A} = \mathcal{O}n$, ranking comes for free, so we get by with less.

Corollary 19 (RCA_0). *Both* $\exists$-ISO *and* $\exists$-EMB *for reduced Abelian p-groups are equivalent to* ATR_0*.*