A complete Π^1_1 equivalence relation

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The complexity of equivalence relations

Let E and F be equivalence relations on sets X and Y. A function $f : X \to Y$ induces a map from X/E to Y/F if for all $a, b \in X$, if $a \, E \, b$ then $f(a) \, F \, f(b)$. A reduction of E to F is an injection of X/E into Y/F.

Descriptive set theory studies, for example, reductions which are induced by Borel functions. One motivation is understanding when classification problems have good invariants.

In computability

When we throw effectiveness into the mix we can study equivalence relations on the natural numbers. Here we require that the reduction is induced by a computable function.

The study began with Malcev and Ershov (in the guise of the study of numberings). Quite a lot of work recently, for example: Bernardi-Sorbi; Fokina-Friedman; Gao-Gerdes; Coskey-Hamkins-R. Miller; Andrews-Lempp-J. Miller-Ng-San Mauro-Sorbi;Fokina-Friedman-Harizanov-Knight-McCoy-Montalbán.

Σ^1_1 equivalence relations

Theorem
(Fokina,Friedman,Harizanov,Knight,McCoy,Montalbán)
Isomorphism of computable structures is complete among Σ^1_1 *equivalence relations on* ω *with respect to computable reductions.*

Σ^1_1 equivalence relations

How do we work effectively with Σ^1_1 sets? We cannot search the reals.

Theorem (Spector,Gandy)

A set of numbers is Σ^1_1 if and only if it is Π_1 definable over the structure $L_{\omega_1^{ck}}$.

Thus a Σ^1_1 set is co-c.e. if we allow an enumeration procedure to take ω_1^{ck} many steps.

Admissibility

The structure $L_{\omega_1^{ck}}$ is admissible. Technically this says that every function $f : \omega \to \omega_1^{ck}$ which is Δ_1 definable over $L_{\omega_1^{ck}}$ is bounded below ω_1^{ck}.

Here for example is an application:

Lemma
Every Σ_1^1 equivalence relation is the limit of an effective ω_1^{ck}-sequence of finer and finer hyperarithmetic equivalence relations.

Proof.
Let $a, b \in \mathbb{N}$ such that $a \not\sim b$. For all $n \in \mathbb{N}$, either $n \not\sim a$ or $n \not\sim b$.
Admissibility says by some stage $\alpha < \omega_1^{ck}$ we see this for all n.
By admissibility again we can (cofinally) find stages α such that $\sim$, as co-enumerated up to stage α, is in fact an equivalence relation. $\square$

FFHKMM - sketch of proof

Theorem (Fokina,Friedman,Harizanov,Knight,McCoy,Montalbán)
Isomorphism of computable structures is complete among Σ^1_1 equivalence relations on ω with respect to computable reductions.

Let $\sim$ be a Σ^1_1 equivalence relation. Let $\langle \sim_\alpha \rangle_{\alpha < \omega_1^{\mathrm{ck}}}$ be an effective refining sequence of hyperarithmetic equivalence relations with limit $\sim$.

For each $n \in \mathbb{N}$ we define a structure $\mathcal{M}_n$. It consists of disjoint linear orderings $\mathcal{L}_{n,a}$ (tagged by a) such that:

- If $n \not\sim a$ then $\mathcal{L}_{n,a} \cong \omega^\alpha$, where α is least such that $n \not\sim_\alpha a$.
- If $n \sim a$ then $\mathcal{L}_{n,a}$ is Harrison's linear ordering.

And note that $\mathcal{L}_{n,a}$ depends only on the $\sim$-equivalence class of n.

What about Π^1_1

It is natural to ask about the higher analogue of c.e. equivalence relations, namely, Π^1_1 equivalence relations. The Spector-Gandy theorem tells us that the existence of a hyperarithmetic isomorphism between structures is a Π^1_1 fact.

Theorem

Hyperarithmetic isomorphism between computable structures is complete among Π^1_1 equivalence relations with respect to computable reductions.

A rough plan

Let $\sim$ be a Π^1_1 equivalence relation; let $\langle \sim_\alpha \rangle$ be an effective ω_1^{ck}-sequence of hyperarithmetic equivalence relations getting coarser and coarser and whose union is $\sim$.

We build structures $\mathcal{M}_n$. The plan:

- If $m \sim n$ then eventually, all decisions we make for $\mathcal{M}_m$ are identical to those we make for $\mathcal{M}_n$.
- If $m \not\sim n$ then we actively diagonalise against all possible hyperarithmetic isomorphisms.

Diagnoalizing

Let φ_e effectively enumerate all partial Π^1_1 functions.
Each structure $\mathcal{M}_n$ will consist of disjoint tagged components indexed by pairs (e,k). Each component contains two elements $a_{e,k}$ and $b_{e,k}$ which are distinguished from the rest but not from each other. Each one of these is related to a linear ordering $A_{e,k}$ and $B_{e,k}$. Any isomorphism from an $\mathcal{M}_n$ to $\mathcal{M}_k$ must map $a_{e,k}$ and $b_{e,k}$ to themselves, or swap between them.

The plan:

- Suppose that $\varphi_e(a_{e,k})$ converges at stage $\alpha < \omega_1^{\text{ck}}$. Let $A_{e,k}^{\mathcal{M}_n} \cong \omega^\alpha$ and $B_{e,k}^{\mathcal{M}_n} \cong \omega^\alpha \cdot 2$ for all $n \sim_\alpha k$. If $\varphi_e(a_{e,k}) = a_{e,k}$ then let $A_{e,k}^{\mathcal{M}_n} \cong \omega^\alpha \cdot 2$ and $B_{e,k}^{\mathcal{M}_n} \cong \omega^\alpha$ for all $n \not\sim_\alpha k$. If $\varphi_e(a_{e,k}) = b_{e,k}$, swap the latter.
- If $\varphi_e(a_{e,k})\uparrow$ then we let both $A_{e,k}^{\mathcal{M}_n}$ and $B_{e,k}^{\mathcal{M}_n}$ be isomorphic to Harrison's linear ordering, for all $n \in \mathbb{N}$.

Why would this work?

Suppose that $n \sim m$. We want to show that $\mathcal{M}_n \cong \mathcal{M}_m$ via a hyperarithmetic isomorphism.

Suppose that we discover that $n \sim m$ at stage $\beta < \omega_1^{\mathrm{ck}}$. We construct an isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_m$ using (roughly) $\mathbf{0}^{(\beta)}$. Fix a pair (e, k).

- If $\varphi_e(a_{e,k})\downarrow$ by stage β then $\mathbf{0}^{(\beta)}$ knows this fact and can construct the isomorphism between the components which have ordertype ω^α and those which have ordertype $\omega^\alpha \cdot 2$, using the fact that $\alpha < \beta$.
- If $\varphi_e(a_{e,k})\uparrow$ at stage β, then whatever happens later (either $F_e(a_{e,k})\downarrow$ at stage $\alpha > \beta$, or never converges), on the (e, k) component the construction acts the same for $\mathcal{M}_m$ and $\mathcal{M}_n$.

But why are the structures computable?

(You should have asked this about Σ^1_1 as well!)

In the Σ^1_1 case, manipulation of computable trees suffices to build the structures computably.

The Π^1_1 construction appears a bit too complicated for this approach. We need to get our hands dirty.

Getting hands dirty = Using Ash-Knight / Harrington iterated priority arguments

We use a presentation of the technique given by Montalbán.

True stages

Suppose that at stage s, n enters $\varnothing'$. Let $\nabla^1_s = \varnothing'_s \upharpoonright_n$.

- There are infinitely many stages s for which $\nabla^1_s \prec \varnothing'$.

These are the 1-true stages in our approximation of $\varnothing'$.
A stage $t < s$ appears to be 1-true at stage s if by stage s we still don't have a proof that t is not a true stage: $\nabla^1_t \prec \varnothing'_s$.

We repeat the process relative to the 1-true stages. We enumerate $\varnothing''$, at stage s using the oracle ∇^1_s. Capping at the smallest number which just entered, we get ∇^2_s. A stage is 2-true if it is 1-true and further $\nabla^2_s \prec \varnothing''$. Similarly we get the notion of a stage appearing 2-true at a later stage.

Significant work is required to ensure that there are α-true stages for $\alpha \geqslant \omega$.

Iterated priority arguments

Fix a computable ordinal δ. Say we want to build a computable structure $\mathcal{N}$ (for example, one of the $\mathcal{M}_n$'s) but relying on questions asked of $\varnothing^{(\delta)}$. To do so, we rely on our approximations to $\varnothing^{(\delta)}$.

In fact we consider all ordinals $\alpha \leqslant \delta$. Together with $\mathcal{N}$ we approximate the Σ_α-diagram of $\mathcal{N}$. The main instruction is:

- If s is an α-true stage then Σ_α facts listed about $\mathcal{N}_s$ are true of $\mathcal{N}$. Indeed, if s appears α-true at stage $t > s$, then our stage t approximation for the Σ_α diagram (of $\mathcal{N}_t$) agrees with the one at stage s.

Every stage appears to be 0-true at any later stage. So the atomic diagrams of the $\mathcal{N}_s$ all agree, i.e., we are building a computable structure.

Iterated priority arguments

The construction succeeds if we have a strategy for recovering from errors: say $\beta < \alpha$ and s appears β-true but not α-true at stage $t > s$. At stage t we believe that $\mathcal{N}_s$ was wrong about some Σ_α fact, and we want to extend $\mathcal{N}_t$ to fix this. However we must preserve all Σ_β facts while doing so. Explaining how to do this is the combinatorial heart of the construction.

Lifting to ω_1^{ck}

The iterated priority argument machinery is done along a fixed computable ordinal δ: we need the relations "*s* appears α-true at *t*" to be computable, uniformly in $\alpha < \delta$. Our construction of the structures $\mathcal{M}_n$ though goes all the way up to ω_1^{ck}.

Overspill allows us to use pseudo-ordinals. One way to understand these is using a nonstandard model of set theory.

A nonstandard universe

Theorem (Gandy)

Every nonempty Σ^1_1 set contains an element X which preserves ω_1^{ck}: $\omega_1^X = \omega_1^{\mathrm{ck}}$.

Let $\mathcal{A}$ be the collection of binary relations $E \subset \omega^2$ such that (ω, E) is an ω-model of Zermelo-Franekel set theory. The set $\mathcal{A}$ is hyperarithmetic. Find some $H = (\omega, E)$ in $\mathcal{A}$ which preserves ω_1^{ck}.

- H is an ω-model. And so "computable" in the sense of H means computable, arithmetic in the sense of H means arithmetic, etc. In particular, every computable ordinal lies in the well-founded part of H, and every hyperarithmetic set is in H.
- The well-founded part of H cannot contain the ordinal ω_1^{ck}, since every well-founded ordinal in H has an H-computable copy.

A nonstandard universe

Hence the well-founded part of H has height precisely ω_1^{ck}. This well-founded part is not definable in H. In particular ω_1^{ck} cannot be precisely the collection of elements of H which H thinks are computable ordinals. So the computable ordinals “spill over” to the ill-founded part of H. The “pseudo-ordinals” are truly computable linear orderings, but H does not realise that they are ill-founded. They look like Harrison’s linear ordering.

To do our construction we fix a computable pseudo-ordinal $\delta \in H$. We do the Ash-Knight construction along δ.

Does it still work?

We check that going beyond ω_1^{ck} does not spoil the construction.

First we observe that if $n \nsim m$ then $\mathcal{M}_n$ and $\mathcal{M}_m$ are not hyperarithmetically isomorphic.

It is possible that $n \sim_\beta m$ for some ill-founded $\beta < \delta$. Nonetheless, if $\varphi_e \colon \mathcal{M}_n \to \mathcal{M}_m$ is truly hyperarithmetic then $\varphi_e(a_{e,m})$ converges at stage $\alpha < \omega_1^{\mathrm{ck}}$, and so $n \nsim_\alpha m$, so at stage α we diagonalise against φ_e.

Does it still work?

The proof that if $n \sim m$ then $\mathcal{M}_n \cong \mathcal{M}_m$ hyperarithmetically is the same. The proof shows that for all $\beta < \delta$, if $n \sim_\beta m$ then $\mathbf{0}^{(\beta)}$ computes an isomorphism between $\mathcal{M}_n$ and $\mathcal{M}_m$.

- If $\varphi_e(a_{e,k})\downarrow$ by stage β then $\mathbf{0}^{(\beta)}$ knows this fact and can construct the isomorphism between the components which have ordertype ω^α and those which have ordertype $\omega^\alpha \cdot 2$, using the fact that $\alpha < \beta$.
- If $\varphi_e(a_{e,k})\uparrow$ at stage β, then whatever happens later (either $F_e(a_{e,k})\downarrow$ at stage $\alpha > \beta$, or never converges), on the (e,k) component the construction acts the same for $\mathcal{M}_m$ and $\mathcal{M}_n$.

If $n \not\sim m$ but $n \sim_\beta m$ for some ill-founded β, then indeed $\mathbf{0}^{(\beta)}$ (an object in H) computes an isomorphism, but $\mathbf{0}^{(\beta)}$ is not really hyperarithmetic (it computes every hyperarithmetic set).

Thank you