Computing K-trivial sets by incomplete random sets

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Random sets and c.e. sets

What are the possible interactions, in the Turing degrees, between c.e. sets and random sets?

- An incomplete c.e. set cannot compute a random set (follows from Arslanov)
- Chaitin's Ω computes every c.e. set. However, this is atypical: the cone above a non-computable set is null.

Two questions:

1. Which random sets can compute non-computable c.e. sets?
2. Which c.e. sets can be computed from an incomplete random set? [Recall that Stephan says that incomplete randoms are not PA, and in that sense quite far from being complete.]

Computing c.e. sets

Theorem (Kučera)

Every Δ^0_2 random set computes a non-computable c.e. set.

Theorem (Hirschfeldt,Miller)

A random set computes a non-computable c.e. set if and only if it is not an element of any null Π^0_2 class if and only if it forms a minimal pair with $\emptyset'$.

Computability and compressibility

Theorem (Chaitin)

A set A is computable if and only if $C(A \upharpoonright_n) \leqslant^+ C(n)$.

However, Solovay built a non-computable set A such that $K(A \upharpoonright_n) \leqslant^+ K(n)$, and called these sets K-trivial. Such sets are "anti-random". Chaitin's argument shows that such sets are Δ^0_2.

Theorem (Nies)

1. *Every K-trivial set is superlow.*
2. *A set is K-trivial iff it is low for ML-randomness iff it is low for K.*
3. *Every K-trivial set is computable from a c.e., K-trivial set.*

Theorem (Hirschfeldt,Nies,Stephan)

A set A is K-trivial if and only if it is computable from some random^A set.

The covering problem

Corollary (Hirschfeldt,Nies,Stephan)

If Z is random and incomplete, A is c.e. and $A \leqslant_T Z$, then A is K-trivial.

Stephan asked if the converse holds:

Question

Is every K-trivial set computable from an incomplete random set?

Strong variants

Theorem (Kučera,Slaman)

There is a low degree which bounds all K-trivial sets.

Question

Is there a single incomplete random set Turing above all K-trivial sets? Is every K-trivial set computable from a low random set?

Fact

If Y and Z are relatively random and $A \leqslant_T Y, Z$ then A is K-trivial.

Question

Is every K-trivial set computable from both halves of a random?

Cost functions

Cost functions measure how much a computable approximation of a Δ^0_2 set changes. Formalised by Nies, they are a generalisation of the Downey-Hirschfeldt-Nies-Stephan construction of a c.e., K-trivial set.

Notation: $\mathbf{c}_s(x)$ is the cost of changing our guess about $A(x)$ at stage s. For example, if we want A to be K-trivial, putting x into A at stage s would require us to issue new descriptions of $A_{s+1} \upharpoonright_y$ for all $y > x$, and so the cost would be

$$\mathbf{c}_{\mathcal{K}}(x, s) = \sum_{y>x} 2^{-K_s(y)}.$$

Cost functions

We say that a set A obeys a cost function $\mathbf{c}$ if there is some approximation $\langle A_s \rangle$ for A such that the total amount paid

$$\sum_s \mathbf{c}_s(x_s) \; [\![x_s \text{ is least } x \text{ for which } A_{s+1}(x) \neq A_s(x)]\!]$$

is finite.

Nies showed that a set is K-trivial if and only if it obeys the standard cost function $\mathbf{c}_{\mathcal{K}}$.

The Kučera and Hirschfeldt-Miller theorems are also essentially cost-function constructions.

Generalised tests

Recall that a Martin-Löf test is a uniform sequence $\langle \mathcal{U}_n \rangle$ of effectively open sets such that $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. The intersection is not only a null, effectively G_δ set; it is effectively null.

To define stronger notions of randomness we can relax the second condition. For example, weak 2 randomness (mentioned in the Hirschfeldt-Miller theorem) is determined by dropping the second condition completely.

Cost functions can be used to calibrate the rate of convergence of $\lambda(\mathcal{U}_n)$ to 0 between these two extremes. A $\mathbf{c}$-test is one for which $\lambda(\mathcal{U}_n) \leqslant \mathbf{c}(n)$.

Lemma

If a set A obeys a cost function $\mathbf{c}$*, then A is computable from every random set which is captured by a* $\mathbf{c}$*-test.*

Additive cost functions

Nies showed that K-trivial sets obey all cost functions of the form $\mathbf{c}_s(n) = \beta_s - \beta_n$ for some left-c.e. real β. These are called additive cost functions, since they satisfy $\mathbf{c}(a, c) = \mathbf{c}(a, b) + \mathbf{c}(b, c)$.

Corollary

If A is K-trivial, then A is computable from every random set which is captured by some additive-cost-function-test.

Changing test components

Demuth used another way to strengthen Martin-Löf randomness. This time we keep $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$ but relax the requirement that the sequence is uniform. Each $\mathcal{U}_n$ is effectively open, but its index is not necessarily obtained computably, but only approximated. We change our mind finitely many times about what $\mathcal{U}_n$ actually is. The more times we are allowed to change our mind, the more powerful the test notion.

- No restrictions: weak 2 randomness.
- Computable bound: weak Demuth randomness.
- bound $O(2^n)$: balanced randomness (Figueira,Hirschfeldt,Miller,Ng,Nies).

Coherently moving tests

We say that the changes to $\mathcal{U}_n$ are coherent if for all n, if $s < t$ are successive stages at which we change $\mathcal{U}_{n+1}$, then we also change $\mathcal{U}_n$ at stage s or at stage t. Intuition: compared to a balanced test, the opponent cannot “reserve changes” for later use.

Proposition

Additive-cost-function-tests and coherently-moving-tests capture the same reals.

We call the resulting randomness notion OW-randomness.

Corollary

Every K-trivial set is computable from every random set which is not OW-random.

Smart K-trivials

Theorem

There is a K-trivial set which is not computable from any OW-random set.

(New idea: no allocation of capital).

Corollary

If every K-trivial set is computable from some incomplete random set, then there is an incomplete random set which computes every K-trivial set.

Computational strength

Proposition (FHMNN)

If $X \oplus Y$ is random, then either X or Y is not balanced random.

Corollary

There is a K-trivial set which is not computable from both halves of a random set.

Proposition (FHMNN)

If X is random but not balanced random, then X is not superlow.

Proposition

If X is random but not OW random, then X is superhigh. Indeed, every X-random is 2-random.

Corollary

There is a K-trivial set which is not computable from any low, random set.

Analytic concepts

Proposition

If X is OW random and d is a c.e. martingale, then

$$\lim_{n\to\infty} d\left(X\restriction_n\right)$$

exists.

Corollary

If X is OW random and $\mathcal{P}$ is an effectively closed set with $X \in \mathcal{P}$, then

$$\lim_{n\to\infty} \lambda(\mathcal{P}|X\restriction_n) = 1.$$

Notation: $\rho(\mathcal{P}|X) = 1$.

Density and completeness

Theorem (Franklin,Ng;Bienvenu,Hölzl,Miller,Nies)

Let X be random. Then $X \not\geq_T \varnothing'$ if and only if for every effectively closed set $\mathcal{P}$ with $X \in \mathcal{P}$,

$$\underline{\rho}(\mathcal{P}|X) = \liminf_{n\to\infty} \lambda(\mathcal{P}|X\upharpoonright_n) > 0.$$

And we notice: if X is random and $\underline{\rho}(\mathcal{P}|X) < 1$ for some effectively closed set $\mathcal{P}$ containing X, then, since X cannot be OW random, X must compute all K-trivial sets.

The solution

Theorem (Day,Miller)

There is a random set X such that:

- $\underline{\rho}(\mathcal{P}|X) > 0$ *for every effectively closed set $\mathcal{P}$ containing X (and so, is incomplete); and*
- *For some effectively closed set $\mathcal{P}$ containing X, $\underline{\rho}(\mathcal{P}|X) < 1$.*

Corollary (everyone)

There is an incomplete random set X which computes every K-trivial set.

Questions

- Does density characterise OW randomness? (related work by the Madison group)
- How does density and OW randomness relate to LR-hardness?
- How do the K-trivial sets look under the 'reducibility' $A \leqslant B$ if every random above B is also above A? (There is a greatest degree; surely they are not linearly ordered?)
- What is lowness for OW randomness?
- What are the randoms which compute all SJT sets?