# Cupping and jump classes in the c.e. degrees

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# Cupping

The c.e. Turing degrees form an upper semilattice: every two c.e. degrees $\mathbf{a}$ and $\mathbf{b}$ have a least upper bound $\mathbf{a} \vee \mathbf{b}$.

**Definition**

A c.e. degree $\mathbf{a}$ is cuppable if there is some c.e. degree $\mathbf{b} < \mathbf{0}'$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$.

Such a degree $\mathbf{b}$ is called a cupping partner for $\mathbf{a}$. It contains the "missing information" that helps $\mathbf{a}$ decide the halting problem.

# Existence of cuppable degrees

**Theorem (Sacks)**

*There is an incomplete cuppable degree. In other words, $\mathbf{0}'$ is join-reducible.*

In fact, every c.e. degree is join-reducible.

# Noncuppable degrees

**Theorem (Cooper;Yates)**

*Not every c.e. degree is cuppable.*

The noncuppable c.e. degrees form a definable ideal.

# Noncuppability is special to the c.e. degrees

**Theorem (Posner,Robinson)**

*Every $\Delta^0_2$ degree is cuppable to $\mathbf{0}'$ by some incomplete $\Delta^0_2$ degree.*

# Jump classes

The jump operator is used to classify strata of c.e. degrees: those that are closer to being computable, compared to those that are closer to being complete.

**Definition**

- A c.e. degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}'$. It is high if $\mathbf{a}' = \mathbf{0}''$.
- A c.e. degree $\mathbf{a}$ is low$_2$ if $\mathbf{a}'' = \mathbf{0}''$. It is high$_2$ if $\mathbf{a}'' = \mathbf{0}'''$.
- etc.

# Cuppability and jump classes

**Theorem (Sacks)**

*There is a low cuppable degree; in fact a low cuppable degree which has a low cupping partner.*

**Theorem (Harrington)**

*There is a high noncuppable degree.*

# Capping

A notion dual to cupping is that of capping.

**Definition**

A c.e. degree $\mathbf{a}$ is cappable if there is some c.e. degree $\mathbf{b} > \mathbf{0}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

In other words, $\mathbf{a}$ and $\mathbf{b}$ have no “common information”.

A major difference between cupping and capping is that the c.e. degrees do not form a lower semilattice: some pairs of degrees do not have greatest lower bounds.

# Cupping and capping

There is some interaction between cupping and capping.

**Theorem (Harrington)**

*Every c.e. degree is either cappable or cuppable. Some c.e. degrees are both.*

But:

**Theorem (Lachlan)**

*Let* **a** *be a cappable and cuppable c.e. degree. No degree* **b** *can be both a cupping and a capping partner for* **a***.*

# Cupping, capping and lowness

The full picture of the interaction between cupping and capping:

**Theorem (Ambos-Spies,Jockusch,Shore,Soare)**

*A c.e. degree is noncappable if and only if it has a low cupping partner.*

We say that it is low cuppable.

The main concept enabling this characterisation is that of prompt permission; noncappable c.e. sets resemble $\emptyset'$ in that they get enumerated quickly. The theorem gives a definable decomposition of the c.e. degrees into a filter and an ideal.

# Cupping and lowness

A corollary of Harrington’s and the decomposition theorems is the following:

**Corollary**

*Not every cuppable degree is low cuppable.*

Compare:

**Theorem (Shore)**

*Every cappable degree is high cappable.*

# High cupping

**Theorem**
*There is a cuppable degree which only has high cupping partners.*

So non-high cuppability is distinct from cuppability.

# Notes on the proof

The proof is a modification of the construction of a noncuppable degree. In that construction, we enumerate a set $A$, and need to:

- make it nonrecursive – this requires enumerating numbers into $A$; and
- ensure that if $W$ is another c.e. set, and $A \oplus W \geqslant_T \emptyset'$, then $\emptyset' \leqslant_T W$.

To overcome the tension between these requirements, before enumerating numbers into $A$, we agitate $W$ to change by enumerating a small number into $\emptyset'$, and freezing $A$. This gives us clearance to redefine the reduction of $\emptyset'$ to $W$ and so to enumerate a number into $A$ without ruining that reduction.

# Notes on the proof

In the current construction, we need to make $A$ cuppable – which means enumerating more numbers into $A$.

Now if $A \oplus W$ is complete, then we need to merely ensure that $A$ is high, or "complete in the limit": we construct a $W$-computable sequence $\langle X_s \rangle$ such that $\lim_s X_s = \emptyset'$. Every subrequirement is now allowed to be injured finitely often, which enables the extra enumerations into $A$.

# The continuity of cupping

**Theorem (Ambos-Spies,Lachlan,Soare)**

*No cuppable degree has a minimal cupping partner.*

# A simple case of the continuity of cupping

Being array recursive is a strong form of being $\text{low}_2$, yet incomparable with lowness. In the c.e. degrees, it coincides with being c.e. traceable.

**Theorem**

*Let* $\mathbf{b}$ *be a low cupping partner for* $\mathbf{a}$*. Then there is some* $\mathbf{c} \leqslant \mathbf{b}$*, a cupping partner for* $\mathbf{a}$*, which is both low and array recursive.*

# Array recursive cupping

**Theorem (Downey,G,J.Miller,Weber)**

*Every low cuppable degree is AR cuppable, but some AR cuppable degree is not low cuppable.*

Hence not only cuppability in general is distinct from low cuppability, but in fact $\text{low}_2$ cuppability is distinct from low cuppability.

# A certain collapse of the hierarchy

**Theorem**

*Every $\text{low}_2$ cuppable degree is AR cuppable.*

**Question**

*Do some levels of the hierarchy of the $\text{low}_n$ and non-$\text{high}_n$ cuppable degrees collapse?*

# Joining to degrees other than $\mathbf{0}'$

Some degrees are badly not low cuppable:

**Theorem (Cholak,Groszek,Slaman)**

*There is a degree* $\mathbf{a}$ *such that for every low degree* $\mathbf{b}$, $\mathbf{a} \vee \mathbf{b}$ *is low.*

This gives another ideal of the c.e. degrees. However, this result cannot be extended to other jump classes:

**Theorem (Jockusch,Li,Yang)**

*Every c.e. degree can be joined to a high degree by a* $\text{low}_2$ *degree.*

# Refining the jump classes using strong reducibilities

**Definition (Bickford,Mills;Mohrherr)**

A set $A$ is superlow if $A' \leqslant_{\mathrm{wtt}} \emptyset'$.

This notion is invariant in the Turing degrees.

# Superlowness and cupping

**Theorem (Diamondstone)**

*Not every low cuppable degree is superlow cuppable.*

**Theorem (Ng)**

*There is a c.e. degree* $\mathbf{a}$ *such that for every superlow c.e. degree* $\mathbf{b}$*,* $\mathbf{a} \vee \mathbf{b}$ *is superlow.*

# A common specialisation

**Theorem (G,Nies)**

*There is a low cuppable degree* $\mathbf{a}$ *such that for every superlow c.e. degree* $\mathbf{b}$*,* $\mathbf{a} \vee \mathbf{b}$ *is superlow.*

In fact, every strongly jump-traceable c.e. degree has this property of being “almost superdeep”.

# $12^{th}$ Asian Logic Conference

- $15^{th}$-$20^{th}$ December 2011, in Wellington.
- Additional workshop for students on the $14^{th}$.
- There is some NSF funding for US-based people to attend. Talk to Antonio or write to Rod.