Hyperarithmeticity through an Algebraic lens

Noam Greenberg, Antonio Montalbán, Ted Slaman

13th October 2010

Degree spectra

Recall that for a countable structure $\mathcal{M}$ in a computable language $\mathcal{L}$, the Turing degree of $\mathcal{M}$ is the degree of the uniform join of the interpretation of the nonlogical symbols of $\mathcal{L}$ in $\mathcal{M}$. The Turing degree of $\mathcal{M}$ is also the degree of the atomic (or quantifier-free) diagram of $\mathcal{M}$.

Definition

The degree spectrum of a countable structure $\mathcal{M}$ is the collection of Turing degrees of isomorphic copies of $\mathcal{M}$.

We write $\mathrm{Spec}(\mathcal{M})$.

A research programme

> *Characterise the sets of degrees that are degree spectra of structures.*

Motivating idea: classes of degrees that are not cones cannot be captured entirely by sets of natural numbers. But a countable structure nevertheless “captures” its spectrum.

A structure theorem

Theorem (Knight)

Let $\mathcal{M}$ be a structure. Either

1. $\mathrm{Spec}(\mathcal{M}) = \{\mathbf{0}\}$*; or*
2. $\mathrm{Spec}(\mathcal{M})$ *is closed upwards in* $\mathcal{D}$*.*

Examples and nonexamples

- Every upper cone is a degree spectrum.
- But a finite or countable union of incomparable cones is not a degree spectrum.

A subprogramme

Which complements of ideals of $\mathcal{D}$ *are degree spectra?*

Naturally we focus on countable ideals. In particular:

For which degrees $\mathbf{d}$ *is* $\mathcal{D}(\not\leq \mathbf{d})$ *a degree spectrum?*

The simplest ideals

Theorem (Slaman;Wehner)

$\mathcal{D} \setminus \{\mathbf{0}\}$ *is a degree spectrum.*

Theorem (Kalimullin)

$\mathcal{D} \setminus \Delta^0_2$ *is a degree spectrum.*

Large spectra

Every degree spectrum is $\boldsymbol{\Sigma}^1_1$, and so is measurable. Since it is a set of degrees, it is either null or co-null.

The complements of countable ideals are co-null.

The bounding theorem

Theorem (GMS;Nies,Kalimullin)

If Spec($\mathcal{M}$) *is co-null, then* $\mathcal{O} \in$ Spec($\mathcal{M}$)*.*

In fact, every Π^1_1-random set is in Spec($\mathcal{M}$); note that $\mathcal{O}$ computes Π^1_1-random sets.

Corollary

There are only countably many structures $\mathcal{M}$ *such that* Spec($\mathcal{M}$) *is co-null.*

Corollary

There are only countably many countable ideals $\mathcal{I}$ *of* $\mathcal{D}$ *such that* $\mathcal{D} \setminus \mathcal{I}$ *are degree spectra. There are only countably many degrees* $\mathbf{d}$ *such that* $\mathcal{D}(\not\leq \mathbf{d})$ *is a degree spectrum.*

Proof of the bounding theorem

Suppose that $\mathrm{Spec}(\mathcal{M})$ is co-null. There is a Turing functional Φ such that

$$\lambda \{X \in 2^\omega : \Phi(X) \cong \mathcal{M}\} > 1/2.$$

Let

$$B = \{(X, Y) : \Phi(X) \cong \Phi(Y)\}.$$

Then B is Σ^1_1. Then

$$C = \{X : \lambda B_X > 1/2\}$$

has positive measure, is Σ^1_1, and is contained in $\mathrm{Spec}(\mathcal{M})$.

Is the $\mathcal{O}$-bound sharp?

We cannot improve the bound $\mathcal{O}$ in the bounding theorem to a hyperarithmetic degree.

Theorem

The collection of nonhyperarithmetic degrees is a degree spectrum.

Construction of a universally nonhyperarithmetic structure

Relativising the Slaman-Wehner theorem, we get, for any computable ordinal α, a structure $\mathcal{M}_\alpha$ such that

$$\mathrm{Spec}(\mathcal{M}_\alpha) \cap \mathcal{D}(\geqslant \mathbf{0}^{(\alpha)}) = \mathcal{D}(> \mathbf{0}^{\alpha}).$$

Inverting the α-jump (Ash), we get a structure $\mathcal{N}_\alpha$ whose degree spectrum is the collection of non-low$_\alpha$ degrees:

$$\mathrm{Spec}(\mathcal{N}_\alpha) = \left\{ \mathbf{d} \, : \, \mathbf{d}^{(\alpha)} > \mathbf{0}^{(\alpha)} \right\}.$$

Observation: A degree is hyperarithmetic if and only if it is low$_\alpha$ for some α.
Hence a "stringing" of all the structures $\mathcal{N}_\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$ should work.
However, this stringing cannot be done computably, as $\mathcal{O}$ is Π^1_1.
Solution: work with a non-standard extension of $\mathcal{O}$ (overspill). For nonstandard α, the "no" and "yes" fibers of $\mathcal{N}_\alpha$ are isomorphic, and so $\mathcal{N}_\alpha$ is computable.

Capturing randomness precisely

Theorem (J. Miller)
If $\mathrm{Spec}(\mathcal{M})$ *is co-null, then it contains a non-random set.*

So an algebraic structure cannot capture a notion of randomness

Capturing genericity

Unlike randomness, notions of genericity can be algebraically captured. The following theorem follows from an careful examination of a theorem of Kumabe and Slaman's.

Theorem

The collection of array nonrecursive degrees is a degree spectrum.

(Recall that the array nonrecursive degrees are those that compute pb-generic sets.)

Separating randomness and genericity

Note that the collection of array nonrecursive degrees is null (every 2-random degree is array recursive).

We can also separate randomness and genericity in the other direction:

Theorem

There is a degree spectrum which is meagre and co-null.

Weaker reducibilities

The fact that the collection of nonhyperarithmetic degrees is a degree spectrum, implies that the analogue of the Slaman-Wehner theorem holds in the hyperdegrees.

Going further up fails.

Theorem

The Slaman-Wehner theorem fails for the degrees of constructibility. That is, if for every nonconstructible real X, $L[X]$ contains a copy of $\mathcal{M}$, then $\mathcal{M}$ has a constructible copy.

Constructible structures

Suppose that for every nonconstructible $X \in 2^\omega$, there is a copy of $\mathcal{M}$ in $L[X]$.

For every X, fix a bijection j_X from $\omega_1^{L[X]}$ to $2^\omega \cap L[X]$. Since ω_1 is inaccessible from reals, for almost all X, $\omega_1^{L[X]} = \omega_1^L$.

The relation $Y = j_X(\alpha)$ is $\Delta^1_1(R)$ in any real code R for α. Hence there is some $\alpha < \omega_1^L$ such that the collection of X such that $j_X(\alpha) \cong \mathcal{M}$ is non-null.

An argument as above now shows that there is a copy of $\mathcal{M}$ constructible from $\mathcal{O}^R$, where R codes α. We can find such R in L.

A finer analysis

A more delicate programme is to classify the collections of degrees which are degree spectra of structures in a given class. For example, one asks what are the degree spectra of linear orderings. There are some restrictions:

Theorem (Richter)

The only cone which is the degree spectrum of a linear ordering is $\mathcal{D}$.

Question

Is there a linear ordering $\mathcal{L}$ *such that* $\text{Spec}(\mathcal{L}) = \mathcal{D} \setminus \{0\}$?

Structures that capture nonhyperarithmeticity

Theorem

There is a linear ordering $\mathcal{L}$ whose degree spectrum is the collection of nonhyperarithmetic degrees.

Theorem

There is no structure whose theory is uncountably categorical, whose degree spectrum is the collection of nonhyperarithmetic degrees.

Question

Is there a stable one?

The complements of cones

Theorem (Kalimullin)

For any r.e. degree $\mathbf{d}$, $\mathcal{D}(\not\leq \mathbf{d})$ *is a degree spectrum. There is a degree* $\mathbf{d} < \mathbf{0}''$ *such that* $\mathcal{D}(\not\leq \mathbf{d})$ *is not a degree spectrum.*

Nothing else is known.

Further questions

Question

Is $\mathcal{D}(\not\leq \mathbf{0}'')$ a degree spectrum?

Question

Is the collection of nonarithmetic degrees a degree spectrum?

Question

If $\mathcal{D}(\not\leq \mathbf{d})$ is a degree spectrum, is $\mathbf{d}$ hyperarithmetic?