More on strongly jump-traceable reals

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Traceability

- Originated in work of Raisonnier on rapid filters.
- Imported into computability by Terwijn and Zambella for characterising lowness for Schnorr randmoness. (recursive traceability)
- Used by Ishmukhametov for constructing strong minimal covers. (c.e. traceability)
- Relates to array computability.

Definitions

A trace for a partial function $p\colon \omega \to \omega$ is a uniformly c.e. sequence of finite sets $\langle T_x \rangle$ such that for all $x \in \operatorname{dom} p$, $p(x) \in T_x$.

An order is a non-decreasing and unbounded recursive function.

A trace $\langle T_x \rangle$ obeys an order h if for all x, $|T_x| \leqslant h(x)$.

Jump-traceability

Let h be an order. A Turing degree $\mathbf{a}$ is called *h-jump-traceable* if every $\mathbf{a}$-partial recursive function p has a trace which obeys h.

A Turing degree is jump-traceable if it is h-jump-traceable for some order h.

STRONG TRACEABILITY

Once a uniform bound for traces of total functions is given, one can slow it down. This is not so for partial functions.

DEFINITION (NIES, FIGUEIRA, STEPHAN)

A Turing degree **a** is strongly jump-traceable if it is h-jump-traceable for every order h.

They also proved they exist.

There is a difference

Compare the following:

- [Nies] There is a perfect Π^0_1 class of jump-traceable reals.
- There are only countably many strongly jump-traceable reals.

and:

- [Nies] A c.e. degree is jump-traceable iff it is superlow; every K-trivial degree is jump-traceable.
- The c.e., strongly jump-traceable degrees are strictly contained in the K-trivial degrees.

Are they essentially c.e.?

Compare the following, both due to Nies.

- Every K-trivial real is bounded by a recursively enumerable one.
- The c.e. jump-traceable degrees are the same as the c.e. superlow degrees. However, no inclusion holds in the ω-c.e. degrees.

How does strong jump-traceability behave?

PARTIAL ANSWERS

Define strong superlowness in an analogous way.

THEOREM (FIGUEIRA, NIES, STEPHAN)

1. *On the c.e. degrees, strong superlowness and strong jump-traceability coincide.*
2. *Every strongly superlow degree is strongly jump-traceable.*

The latter uses a characterisation of strong jump-traceability as "almost low for C".

THEOREM

Every strongly jump-traceable set is Δ^0_2.

ENUMERABILITY CONJECTURE

The following conjecture implies that studying the c.e., strongly jump-traceable degrees is all that is necessary.

CONJECTURE

Every strongly jump-traceable set is bounded by a c.e. one.

Possible weaker variations would also be useful.

One corollary would be the coincidence of strong superlowness and strong jump-traceability.

A structure theorem

Theorem

The c.e., strongly jump-traceable degrees form an ideal.

The proof uses the independent version of the box-promotion method. Its logical structure is simpler than the general method, but the combinatorial details have daunted some.

Cost functions

Say we want to enumerate a c.e. set A which doesn't change too often. One way to quantify this is using a cost function.

Definition

A cost function is a recursive function $c_s(x)\colon \mathbb{N}^2 \to \mathbb{Q}^+$ which is non-decreasing in s.

Usually we also expect that for each x, the limit $c(x) = c_s(x)$ exists and that $\lim_x c(x) = 0$. Often $c_s(x)$ is non-increasing in x.

Obeying cost functions

A computable approximation $\langle A_s \rangle$ of a Δ^0_2 set A obeys a cost function c if the sum

$$\sum_s c_s(x)[\![x \text{ is least such that } A_{s+1}(x) \neq A_s(x)]\!]$$

is finite.

We say that a Δ^0_2 set obeys a cost function c if there is some computable approximation for A which obeys c. In the c.e. world, we restrict ourselves to computable enumerations.

Theorem (Downey, Hirschfeldt, Nies, Stephan; Kummer)

If c is a cost function (which satisfies the desirable properties) then there is a promptly simple c.e. set A which obeys c.

The K-trivial cost function

The best-known cost function is $c_{\mathcal{K}}$, the cost function which characterises K-triviality, defined by

$$c_{\mathcal{K},s}(x) = \sum_{y \geq x} 2^{-K_s(y)}.$$

Theorem (Nies)

A set A is K-trivial iff it obeys $c_{\mathcal{K}}$.

Benign cost functions

Nice cost functions don't surprise us by amassing cost repeatedly.

Definition

A cost function c is benign if there is a computable function $b\colon \mathbb{Q}^+ \to \mathbb{N}$ such that for every rational $\epsilon > 0$, every collection $\mathcal{I}$ of pairwise disjoint intervals of the form $[n, s)$ such that for all $[n, s) \in \mathcal{I}$,

$$c_s(n) \geq \epsilon$$

contains at most $b(\epsilon)$ many such intervals.

For example

For example, $c_{\mathcal{K}}$ is benign, because if $n < s \leqslant m < t$ and

$$c_{\mathcal{K},s}(n), c_{\mathcal{K},t}(m) > \epsilon$$

Then the descriptions in the universal prefix-free machine which induce these costs are disjoint. So the witness for $c_{\mathcal{K}}$ is $b(\epsilon) = 1/\epsilon$.

A characterisation

Theorem

A c.e. set A is strongly jump-traceable iff it obeys every benign cost function.

However, no single benign cost function is sufficient for characterising the strongly jump-traceable c.e. sets.

Corollary

The c.e., strongly jump-traceable degrees are strictly contained in the c.e. K-trivial degrees.

Diamonds

Let $\mathcal{C} \subseteq \mathbb{R}$. We let $\mathcal{C}^\diamond$ be the ideal of all c.e. degrees **a** such that for all $X \in \text{MLR} \cap \mathcal{C}$, $\mathbf{a} \leqslant_T X$.

Theorem

1. *[Miller, Hirschfeldt] If $\mathcal{C}$ is a null Σ^0_3 class, then $\mathcal{C}^\diamond$ contains a promptly simple c.e. degree.*
2. *[Nies, Stephan] If $\mathcal{C}$ contains an incomplete random real, then $\mathcal{C}^\diamond$ is contained in the the class of K-trivial degrees.*

The relevance here is that sometimes there is a benign cost function, obedience to which ensures membership in $\mathcal{C}^\diamond$.

$\mathcal{C}$ = LR-COMPLETE

One example is the class of LR-complete (or almost everywhere dominating) degrees.

THEOREM

Every c.e., strongly jump-traceable degree is in (LR-complete)$^\diamond$. As a result, every c.e., strongly jump-traceable degree is ML-coverable and not ML-cuppable.

$\mathcal{C} = \omega$-R.E.

Theorem (following Kučera)

If X is random, then there is a cost function c such that every c.e. set obeying c is X-computable.

If X is also ω-r.e., then the cost function c is benign.

Corollary

Every c.e., strongly jump-traceable degree is in $(\omega\text{-r.e.})^\diamond$.

Question: do we get equality?

Superlow cupping

Theorem (Nies)

For all $B \in \mathbb{R}$ there is a random X such that $(X \oplus B)' \leqslant_{tt} B'$.

Corollary

Every $\mathbf{a} \in (\omega\text{-r.e.})^\diamond$ is almost superdeep.

This extends results of Diamondstone and Ng.

Other topics

- The hierarchy of h-jump-traceable degrees, and K-triviality. [Barmpalias,Downey,G; Ng]
- Stronger notions: relativising sjt. [Ng]
- The corresponding highness properties. [Ng]

QUESTIONS

- The enumerability conjecture (hopefully, my next project).
- A direct box-promotion proof that every c.e. sjt is almost superdeep.
- Does c.e., sjt = $(\omega\text{-r.e.})^\diamond$? Other natural ideals between sjt and K-trivial?
- Questions relating to the highness notions (relates to general questions about pseudo-jump inversion).
- Are these classes definable?