# *Lowness Properties of Computably Enumerable Degrees*

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# Lowness

R.W. Robinson's low-guessing trick led to the belief that all low c.e. degrees (those degrees whose Turing jump is as low as possible, i.e. computable from $\mathbf{0}'$, the halting problem) are "nearly computable" and hence not very interesting.

Classes such as array computable (and contiguous) degrees have challenged this view. Only recently, with the discovery of the $K$-trivial degrees (Downey, Hirschfeldt, Nies), did the full richness of the low degrees manifest itself.

# C.E. SETS AS ORACLES

A theme: understand a class of degrees by understanding how useful these degrees are as oracles for computations.

Suppose that a given c.e. set $D$ is given. We want to construct a set $A$ so that $A \leqslant_T D$.

We do an effective construction, based on an enumeration $D_s$ of $D$. We build a Turing functional $\Gamma$ (a partial, effectively continuous map from $2^\omega$ to $2^\omega$) and approximate $A$, and ensure that $\Gamma(D) = A$.
To do this, the constraint is: if at stage $s$, an initial segment $\sigma$ of $D_s$ of length $n$ is mapped by $\Gamma$ to an initial $\tau$ of $A_s$, then we cannot change $A$ below $|\tau|$ unless we are given a change in $D$ below $|\sigma|$.

# Simple permitting

This gives rise to the notion of permitting. The idea is that the more often a c.e. set changes, the more it can compute.

Simple permitting (Yates) is granted by all non-computable c.e. degrees. Here, repeated requests for change are eventually granted. This allows finite-injury constructions to run successfully. For example, every non-computable c.e. degree bounds an incomparable pair of c.e. degrees.

# High permitting

This is due to Martin. To make an infinite-injury argument work, we need that almost all requests (from an infinite stream of requests) for change will be granted. This holds if the degree $\mathbf{d}$ is high: $\mathbf{d}' \geq \mathbf{0}''$.

This is due to a domination property: Martin showed that a degree is high iff it computes a function $f$ which dominates all computable functions.

# Prompt permitting

This is determined by how quickly permission is granted, rather than the number of permissions.
For example, every promptly permitting degree bounds a minimal pair (a pair of degrees which have greatest lower bound $\mathbf{0}$).

# Array computability

Together with non-low$_2$ permitting, this is a middle ground between simple and high permitting: rather than grant almost all requests, determine in advance how many (finitely many) permissions each requirement needs.

This too is related to domination: a degree is array non-computable iff it computes some function which is not dominated by any function which is $\omega$-c.e.

**LEMMA (SHOENFIELD)**

*A function $f : \omega \to \omega$ is $\Delta_2^0$ (i.e. computable from $\mathbf{0}'$) iff it has a computable approximation: a computable function $g(x, s)$ whose pointwise limit (in the discrete topology on $\omega$) is $f$.*

Associated with any computable approximation $g(x, s)$ is the mind-change function:

$$h(x) = \#\{s : g(x, s+1) \neq g(x, s)\}.$$

A function is $\omega$-c.e. if it has some computable approximation whose mind-change function is bounded by some computable function.

# From domination to approximation

### Fact

*A c.e. degree is array computable iff every function computable from it has a computable approximation whose mind-change function is bounded by the identity function.*

This leads us to think of approximation properties of functions computable in a c.e. set as a key to permitting.

# Totally $\omega$-c.e. degrees

### Definition

A degree $\mathbf{d}$ is totally $\omega$-c.e. if every $f \leqslant_T \mathbf{d}$ is $\omega$-c.e.

The totally $\omega$-c.e. degrees properly contain the array computable degrees and are properly contained in the low$_2$ degrees.

### Theorem

*There are maximal totally $\omega$-c.e. degrees.*

This is analogous to the contiguous degrees.

# Unification

The class of totally $\omega$-c.e. degrees is interesting because it captures the dynamic properties of quite a number of different constructions. For example:

**Theorem (D,G, Weber)**

*A c.e. degree $\mathbf{d}$ is not totally $\omega$-c.e. iff it bounds a critical triple: degrees $\mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{b}$ such that $\mathbf{a}_0 \equiv_{\mathbf{b}} \mathbf{a}_1$ and if $\mathbf{e} \leqslant \mathbf{a}_0, \mathbf{a}_1$ then $\mathbf{e} \leqslant \mathbf{b}$.*

Other constructions involve the wtt-structure of a Turing degree, presentation of left-c.e. reals, and splittings of c.e. sets.
It turns out that this notion sheds light on the dual question: what kind of sets can compute the given set $D$?

**Theorem (Chisholm et. al.; D,G)**

*A c.e. degree is totally $\omega$-c.e. iff every c.e. set $D \in \mathbf{d}$ is wtt-reducible to a ranked set.*

# Definability

The result regarding critical triples shows that the totally $\omega$-c.e. degrees are definable in the c.e. degrees in a natural way.

**Theorem (Nies, Shore, Slaman)**

*A relation on the c.e. degrees which is invariant under the double jump is definable in the c.e. degrees iff it is definable in arithmetic.*

# NATURALLY DEFINABLE CLASSES

Not many examples!

THEOREM (DOWNEY, LEMPP)

*A c.e. degree is contiguous iff it is locally distributive.*

THEOREM (AMBOS-SPIES, FEJER)

*A c.e. degree is contiguous iff it is not the top of a copy (in the c.e. dgerees) of the non-modular, non-distributive 5 element lattice $N_5$.*

THEOREM (AMBOS-SPIES ET. AL.)

*A degree permits promptly iff it is not the half of a minimal pair.*

# Beyond $\omega$

The Ershov hierararchy for $\Delta_2^0$ functions allows us to consider more complicated sets and functions. A function is $\alpha$-c.e. if it has a computable approximation $g$ such that for every $x$, the sequence of mind-changes for $g(x, s)$ is accompanied with an effective, decreasing sequence from $\alpha$.

We can thus define the totally $\alpha$-c.e. degrees and the totally $< \alpha$-c.e. degrees. All such degrees are low$_2$.

For the lower levels we have a concrete characterization: a function is $\omega^{n+1}$-c.e. if it has a computable approximation whose mind-change function is bounded by some $\omega^n$-c.e. function.

# A PROPER HIERARCHY

**THEOREM**

*There is a totally $\alpha$-c.e. degree which is not totally $< \alpha$-c.e. iff $\alpha = \omega^\gamma$ for some $\gamma$.*

**THEOREM**

*There is a totally $< \alpha$-c.e. degree which is not totally $\beta$-c.e. for any $\beta < \alpha$ iff $\alpha = \omega^\gamma$ for some limit ordinal $\gamma$.*

There are maximal degrees in all levels of the hierarchy, but no degree at any level is maximal for a higher level.

# Another naturally definable level

**Theorem**

*A c.e. degree is not totally $< \omega^\omega$-c.e. iff it bounds a copy of the 1-3-1 lattice.*

This level unifies quite a number of constructions as well. For example, a c.e. degree is not totally $< \omega^\omega$-c.e. iff it contains a pair of c.e. sets $A_0$ and $A_1$ whose wtt-degrees have an infimum which is strictly Turing below that degree.

No *m*-topped degree can be totally $< \omega^\omega$-c.e.; on the other hand, there is an *m*-topped degree which is totally $\omega^\omega$-c.e.

# An application to higher computability theory

### Theorem (G)

*If $\kappa > \omega$ is an admissible ordinal and* **a** *is an incomplete $\kappa$-c.e. degree, then* **a** *bounds a 1-3-1 iff it bounds a critical triple.*

### Corollary

*There is a single, natural, elementary statement which holds in the classical c.e. degrees but not in the $\kappa$-c.e. degrees for an admissible $\kappa > \omega$ (including $\omega_1^{\mathrm{CK}}$).*

# $K$-TRIVIALITY

THEOREM (DOWNEY, HIRSCHFELDT, NIES)

*The following are equivalent for any $A \in 2^\omega$:*

1. *A is K-trivial: the sequence $\langle K(A \upharpoonright n) - K(n) \rangle$ is bounded.*
2. *A is low for ML-randomness: every ML-random real is ML-random over A.*
3. *A is low for K: The sequence $\langle K(n) - K^A(n) \rangle$ is bounded.*
4. *A is a base for ML-randomness: there is some $R \geq_T A$, ML-random over A.*
5. *A has an approximation which obeys a cost-function condition.*

Crucial here is the decanter method. It allows us to show that the collection of $K$-trivial degrees is an ideal, properly contained in the superlow degrees, which is generated by its c.e. elements.

# Smaller classes?

The following classes of reals $A$ are contained in the $K$-trivials:

- ML-non-cuppable: there is no incomplete ML-random $R$ such that $R \oplus A \geq_T \mathbf{0}'$.
- ML-coverable: there is some incomplete ML-random $R \geq_T A$.
- $A$ computable from every ML-random $R$ such that $\mathbf{0}'$ is $K$-trivial relative to $R$.

The cost functions involved in showing these classes are non-empty are more stringent that the standard one. So perhaps these classes are properly contained in the $K$-trivials?

[However, the low for strong 1-randomness, at first considered such a class, were shown (Downey, Miller, Nies, Weber, Yu) to be the same as the $K$-trivials.]

### Question

Is there a natural, proper sub-ideal of the $K$-trivials?

# Traces

### Definition (Terwijn, Zambella)

A trace for a (partial) function $f\colon \omega \to \omega$ is a sequence of finite sets $\langle F_x \rangle$ such that for all $x \in \operatorname{dom} f$,

$$f(x) \in F_x.$$

A trace is computable if the sequence of (canonical indexes for the) finite sets is computable. A trace is c.e. if the sequence of finite sets is uniformly c.e.

# ORDERS

### DEFINITION

An order is a computable, non-decreasing, and unbounded function $h\colon \omega \to \omega$.

A trace $\langle F_x \rangle$ for a function $f$ respects an order $h$ if for all $x$,

$$|F_x| \leqslant h(x).$$

# Computable Traceability

### Definition

A Turing degree **a** is computably traceable if there is some order $h$ such that every (total) $f \leqslant_T \mathbf{a}$ has a computable trace which respect $h$.

### Theorem (Terwijn, Zambella, Kjos-Hanssen)

*A degree* **a** *is computably traceable iff it is low for Schnorr randomness.*

There are $2^{\aleph_0}$ many computably traceable degrees. They are all hymperimmune-free (or **0**-dominated) and so none are $\Delta^0_2$.

# C.E. TRACEABILITY

### DEFINITION

A degree is c.e. traceable if there is some order $h$ such that every (total) $f \leqslant_T \mathbf{a}$ has a c.e. trace which respects $h$.

### THEOREM (ISHMUKHAMETOV)

*A c.e. degree is c.e. traceable iff it is array computable. As a result, a c.e. degree has a strong minimal cover iff it is array computable.*

### THEOREM (STEPHAN)

*A degree is computably traceable iff it is both c.e. traceable and hyperimmune-free.*

# Strong traceability

Let $\Gamma \in \{\text{c.e.}, \text{computably}\}$.

**Definition**

A degree $\mathbf{a}$ is strongly $\Gamma$-traceable if for *every* order $h$, every $f \leqslant_T \mathbf{a}$ has a $\Gamma$-trace which respects $h$.

**Theorem (Terwijn, Zembella)**

*A degree is $\Gamma$-traceable iff it is strongly $\Gamma$-traceable.*

# Jump-traceability

### Definition (Nies)

A degree **a** is jump-traceable if there is an order $h$ such that every function which is *partial* computable in **a** has a c.e. trace which respects $h$.

### Theorem (Nies)

1. *There are $2^{\aleph_0}$ many jump-traceable degrees.*
2. *Every $K$-trivial degree is jump-traceable.*
3. *On the c.e. degrees, superlowness coincides with jump-traceability. They differ on the $\omega$-c.e. degrees.*

# Strong jump-traceability

### Definition (Figueira, Nies, Stephan)

A degree **a** is strongly jump-traceable if for all orders $h$, every function which is partial computable in **a** has a c.e. trace which respects $h$.

Figueira, Nies and Stephan showed that not every jump-traceable degree is strongly jump-traceable, but that the latter exist.

### Theorem (Figueira, Nies, Stephan)

*A set $A$ has strongly jump-traceable degree iff it is "almost low for $C$" in the sense that for every order $h$, for almost all $x$,*

$$C(x) - C^A(x) \leqslant h(C^A(x)).$$

# Strong jump-traceability in the c.e. degrees

**Theorem (Cholak, D, G)**

*In the c.e. degrees, the strongly jump-traceable degrees form an ideal which is strictly contained in the $K$-trivial degrees.*

**Theorem (Cholak, D,G)**

*Every c.e., strongly jump-traceable degree does not ML-cup.*

# What about non-c.e. sets?

**Theorem**

*Every strongly jump-traceable set is $\Delta^0_2$.*

**Question**

Is every strongly jump-traceable set bounded by a c.e. one? Is every strongly jump-traceable set $K$-trivial?

## A similar class

Figueira, Nies and Stephan also define the strongly superlow sets, those sets such that $A'$ is $\omega$-c.e. via arbitrarily slow approximations.

The fact that on the c.e. degrees, superlowness and jump-traceability coincide, yields also FNS's conclusion that on the c.e. degrees, strong jump-traceability and strong superlowness coincide.

**Theorem (Figuiera, Nies, Stephan)**

*Every strongly superlow set is strongly jump-traceable.*

**Theorem**

*Every strongly superlow set is $K$-trivial.*

**Question**

Do strong superlowness and strong jump-traceability coincide for all sets?

# A MIRROR IMAGE: THE HIGH DEGREES

The pseudo-jump inversion technique of Jockusch and Shore allows us to relect the picture up to the high degrees.

### QUESTION

Is there a non-computable degree which is computable from all c.e. degrees $\mathbf{a}$ such that $\mathbf{0}'$ is strongly jump-traceable relative to $\mathbf{a}$? $K$-trivial relative to $\mathbf{a}$?

This is motivated by the facts that there is a low$_2$ c.e. degree which bounds all $K$-trivials (Nies) and a low PA-degree which bounds all $K$-trivials (Kućera-Slaman).

# Goals

- Understand the structure of the c.e. degrees by finding naturally definable classes of degrees.
- Find classes of Turing degrees which are determined by their c.e. elements.
- Understand the dynamic nature of constructions of c.e. degrees.
- Understanding the lower regions of the c.e. degrees.
- Develop new proof techniques (anti-permitting arguments, little boxes).