Notre Dame Journal of Formal Logic

Embedding and Coding Below a 1-Generic Degree

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Abstract We show that the theory of $\mathcal{D}(\leqslant \mathbf{g})$, where $\mathbf{g}$ is a 2-generic or a 1-generic degree below $\mathbf{0}'$, interprets true first order arithmetic. To this end we show that 1-genericity is sufficient to find the parameters needed to code a set of degrees using Slaman and Woodin's method of coding in Turing degrees. We also prove that any recursive lattice can be embedded below a 1-generic degree preserving top and bottom.

1 Introduction

The complexity of the theory of degree structures (as partial orderings) has been for a long time a focus of attention of researchers. Among the noted results we can mention are that the theory of all Turing degrees $\mathrm{Th}(\mathcal{D})$ is undecidable (Lachlan [13]); the theory of $\mathcal{D}(\leqslant \mathbf{0}')$ is undecidable (Lerman [14]); the theory of the recursively enumerable degrees $\mathrm{Th}(\mathcal{R})$ is undecidable (Harrington and Shelah [4]).

A particular method for proving undecidability is embedding models of arithmetic in the degree structure with parameters. If one finds a first order condition on the parameters which ensures that the coded model is the standard one, then the theory of the structure interprets first-order true arithmetic. For structures which are interpretable in arithmetic this shows that the theory is as complicated as possible. Such results were obtained for $\mathcal{D}(\leqslant \mathbf{0}')$ (Shore [21], where the result is extended to $\mathcal{D}(\leqslant \mathbf{a})$ for many other arithmetic degrees $\mathbf{a}$); and for $\mathcal{R}$ (Harrington and Slaman, and also Slaman and Woodin; see [18]). Another important similar result is that $\mathrm{Th}(\mathcal{D})$ is recursively isomorphic to true second-order arithmetic (Simpson [23]). We show in this paper that if $\mathbf{g}$ is 2-generic, or if it is a 1-generic degree below $\mathbf{0}'$, then this method can be employed in $\mathcal{D}(\leqslant \mathbf{g})$ and so we get the same result.

We code models of arithmetic below a 1-generic degree in a direct way, using coding schemes defined in [18]. Further, this coding, together with

Printed May 15, 2004

the technique of comparison maps (again from [18]), shows that if the 1-generic degrees are downward dense in the structure $\mathcal{D}(\leqslant \mathbf{g})$ then the standard models can be isolated. We then quote results of Chong and Jockusch ([2]) and Jockusch ([6]) which show that this condition holds if $\mathbf{g}$ is a 1-generic degree below $\mathbf{0}'$ or if $\mathbf{g}$ is 2-generic. (In fact, Haught, in [5], showed that every nonzero degree below a 1-generic degree below $\mathbf{0}'$ is 1-generic.) We note that this technique cannot be extended to all 1-generic degrees; Both Kumabe ([10]) and Chong and Downey ([1]) show that there is a 1-generic degree which bounds a minimal degree.

The coding tool we use is the coding introduced by Slaman and Woodin ([25]). One of the questions connected with this coding is where can one find the parameters needed for the coding, relative to the structure coded. It follows from the proof of [25, Prop. 2.5], Slaman and Woodin show that a 2-generic suffices. Their claim that parameters can be found below the jump of the coded structure was covered in detail in Odifreddi and Shore [19]. In order to code models below 1-generic degrees, we show here that a 1-generic filter suffices.

The requirement that standard models can be identified in a first order way is quite stringent. If we drop this requirement we get structures in which a class of models satisfying some finite part of arithmetic T is interpreted; this class contains the standard model. Then the theory of the structure can effectively separate the theorems of T and their negations; for sufficiently complicated T this shows that the structure is undecidable. Using our results concerning the coding parameters, we show that if $\mathbf{a}$ bounds a 1-generic degree then $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ is undecidable.

This result, though, can be deduced from earlier work. Jockusch ([6]) showed that every 1-generic degree is recursively enumerable in a strictly lower degree. Relativizing, one can apply the undecidability results of Shore ([21]) which use techniques related to r.e. degrees, to get the aforementioned result. We mention our proof because it is straightforward in its use of genericity and does not appeal to recursive enumerability.

Embeddings of algebraic objects into degree structures have a close connection with undecidability results; indeed all the early undecidability results are established by coding some class of algebraic objects (such as linear orderings, partial orderings and graphs) into the degree structure. A striking example is Lerman's work, [14; 15], which showed that every countable upper semi-lattice can be embedded in $\mathcal{D}$ as an initial segment; so the question about the theories of initial segments involves the theories of such semi-lattices. Further, before Slaman and Woodin introduced their coding, Lerman's results were used in undecidability proofs by using lattices to code models of arithmetic (Nerode and Shore [16; 17; 21]). Later, Shore ([22]) found a simpler method of embedding lattices below any r.e. degree (not as initial segments though). He applied Jockusch's result mentioned earlier to embedding techniques below r.e. degrees and showed that every recursive lattice can be embedded below any 1-generic degree (the power of the technique lies in embedding non-distributive lattices; the result for distributive lattices follows from the fact that the countable atomless Boolean algebra is embeddable below a 1-generic degree, even preserving **0** and **1**). As we did before, we give a direct proof; we are, however, able to improve it to show that embeddings can be found which preserve **0** and **1**.

We remark that Downey, Jockusch and Stob ([3]) showed that every recursive lattice with least and greatest element can be embedded into $\mathcal{D}(\leqslant \mathbf{a})$ preserving **0** and **1**, where **a** is any array nonrecursive degree. These are the degrees which bound pb-generic degrees; this is a notion of genericity which is intermediate between 1 and 2-genericity. Unlike the 1-generics, the array nonrecursive degrees are upward closed. Our theorem cannot be improved in this direction; there is a degree **a** which is a strong minimal cover of a 1-generic degree (Kumabe, [11]). Hence, for example, the diamond lattice cannot be embedded in $\mathcal{D}(\leqslant \mathbf{a})$ preserving **1**.

1.1 Notation Given $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \frown \tau$ for the string π of length $\max\{|\sigma|, |\tau|\}$ such that for all $i < |\pi|$

$$\pi(i) = \begin{cases} \sigma(i) & \text{if } i < |\sigma| \\ \tau(i) & \text{if } |\sigma| \leqslant i < |\tau|. \end{cases}$$

If $\sigma, \tau \in 2^{<\omega}$ and $E \subseteq \omega$, we say that σ and τ are *E-equivalent*, and we write $\sigma \equiv_E \tau$, if for all $x \in E \cap \operatorname{dom}\sigma \cap \operatorname{dom}\tau$, $(\sigma(x) = \tau(x))$.

We assume we have a fixed recursive bijection between ω and V_ω. In particular we identify finite sequences of natural numbers with the number coding the sequence. For $A \subseteq \omega$ and $n < \omega$ we let the n^{th} *column* of A be

$$A^{[n]} \;=\; \{x \in \omega \,:\, \langle n, x\rangle \in A\}.$$

If $F \subseteq \omega$ and for every $i \in F$ we have a set $A_i \subseteq \omega$ then we let

$$\bigoplus_{i \in F} A_i \;=\; \bigcup_{i \in F} \{i\} \times A_i.$$

Thus if $i \in F$ then the i^{th} column of $\bigoplus_{i \in F} A_i$ is again A_i.

If $A \subseteq \omega$ then we denote its Turing degree $\deg_T A$ by **a**. $\mathcal{D}$ is the collection of all Turing degrees. A nonempty set of degrees $\mathcal{J}$ is an *ideal* if it is closed downwards and with respect to the join operation. For example, if **a** is a Turing degree then

$$\mathcal{D}(\leqslant \mathbf{a}) \;=\; (\mathbf{a}) \;=\; \{\mathbf{b} \in \mathcal{D} \,:\, \mathbf{b} \leqslant \mathbf{a}\}$$

is an ideal.

If $\varphi(\bar{x})$ is a formula in the language of upper semi-lattices, then we say that φ is *absolute for ideals* if for every ideal $\mathcal{J}$ and every tuple $\bar{\mathbf{a}} \in \mathcal{J}$,

$$(\mathcal{J}, \leqslant_T) \models \varphi(\bar{\mathbf{a}}) \;\;\Leftrightarrow\;\; (\mathcal{D}, \leqslant_T) \models \varphi(\bar{\mathbf{a}}).$$

A formula φ in the language of upper semi-lattices is *bounded* if all quantifiers appearing in φ are bounded, i.e. of the form $\exists x \leqslant t$, $\forall x \leqslant t$, where t is a term not containing x. Every bounded formula is absolute for ideals.

1.2 1-Genericity We consider the notion of 1-generic filters with regards to various forcing notions.

Definition 1.1 Let $\mathbb{P}$ be a partial ordering on ω (we regard $\mathbb{P}$ as a *forcing notion*). Let $C \subset \omega$. A filter $G \subset \mathbb{P}$ is *C-1-generic* if for every $W \subset \mathbb{P}$ which is recursively enumerable in C, either $G \cap W \neq 0$ or there is some $p \in G$ such that for all $q \leqslant_{\mathbb{P}} p$, $q \notin W$.

A *1-generic degree* is a Turing degree which contains a filter which is 1-generic for *set Cohen forcing* ($2^{<\omega}$, ordered by reverse inclusion).

Let $\mathbb{P}, \mathbb{Q}$ be partial orderings on ω. We say that an injection $i : \mathbb{Q} \to \mathbb{P}$ is a *dense embedding* if i preserves $\leqslant, \perp$ and for every $p \in \mathbb{P}$ there is a $q \in \mathbb{Q}$ such that $i(q) \leqslant_{\mathbb{P}} p$ (see [12, VII.7]).

The following is a recursive analogue of a familiar theorem of set theory.

Proposition 1.2 *Let $i : \mathbb{Q} \to \mathbb{P}$ be a dense embedding. Let $C \geqslant_T i \oplus \mathbb{P} \oplus \mathbb{Q}$.*

1. *Suppose that $G \subset \mathbb{Q}$ is 1-generic over C. Let H be the upward closure of i“G in $\mathbb{P}$. Then $H \subset \mathbb{P}$ is a C-1-generic filter, and $H \leqslant_T G \oplus C$.*
2. *Suppose that $H \subset \mathbb{P}$ is 1-generic over C. Let $G = i^{-1}H$. Then $G \subset \mathbb{Q}$ is C-1-generic filter, and $G \leqslant_T H \oplus C$.*

Proof (1) Let $W \subset \mathbb{P}$ be recursively enumerable in C. Without loss of generality, assume that W is closed downwards (i.e. open). $i^{-1}W$ is also recursively enumerable in C and is open in $\mathbb{Q}$. The fact that i is dense implies that the upward closure of i“$i^{-1}W$ in $\mathbb{P}$ is W.

If $G \cap i^{-1}W \neq 0$ then $H \cap W \neq 0$. Otherwise, there is some $p \in G$ such that no extension of p in $\mathbb{Q}$ is in $i^{-1}W$; so $p \perp_{\mathbb{Q}} r$ for all $r \in i^{-1}W$. This implies that $i(p) \perp_{\mathbb{P}} s$ for all $s \in i$“$i^{-1}W$, so $i(p)$ has no extension in W. It is immediate to check that H is a filter, so H is indeed C-1-generic.

Given any $p \in \mathbb{P}$, by genericity we can find some $q \in G$ such that either $i(q) \leqslant_{\mathbb{P}} p$ or $i(q) \perp_{\mathbb{P}} p$, and this decides whether $p \in H$.

(2) is easier. □

It is well-known that Cohen forcing is universal for all countable forcings: every (nontrivial) countable notion of forcing embeds densely into Cohen forcing (see [9, Prop. 10.20]). Further, for each forcing $\mathbb{P}$ there is a dense embedding $i : \mathbb{P} \to 2^{<\omega}$ which is recursive in $\mathbb{P}$; this is shown, for example, in [24]. For completeness, we show that *function Cohen forcing* ($\omega^{<\omega}$, ordered by reverse inclusion) embeds into set Cohen forcing. We will later (2.9) see another example of this universality.

Proposition 1.3 *There is a recursive, dense embedding of function Cohen forcing into set Cohen forcing.*

Proof For $\sigma \in \omega^n$, let $i(\sigma) = 0^{\sigma(0)}10^{\sigma(1)}1 \cdots 10^{\sigma(n-1)}1$. i is dense because for every $\tau \in 2^{<\omega}$, $\tau^\frown 1 \in \mathrm{range}(i)$. It is clear that i preserves $\subset$ and $\perp$. □

Thus a degree is 1-generic iff it contains some $G \subset \omega^{<\omega}$ which is 1-generic.

2 Slaman and Woodin Coding

Let $\mathcal{J} = \{\mathbf{c}_i : i \in I\}$ be an antichain of Turing degrees. (I could be either ω or some finite set.) We want to find degrees $\mathbf{c}$, $\mathbf{g}_0$ and $\mathbf{g}_1$ such that the elements of $\mathcal{J}$ are the minimal solutions below $\mathbf{c}$ of the following inequality in $\mathbf{x}$

$$(\mathbf{g}_0 \vee \mathbf{x}) \cap (\mathbf{g}_1 \vee \mathbf{x}) \neq (\mathbf{x}). \tag{2.1}$$

For each $i \in I$, let $\hat{C}_i$ be an element of $\mathbf{c}_i$ and let

$$C_i = \{\alpha \in 2^{<\omega} : \alpha \subset \hat{C}_i\}.$$

Let $C = \bigoplus_{i \in I} C_i$ and let $\mathbf{c} = \deg_T C$. Given $F \subseteq I$, we let $C_F = \bigoplus_{i \in F} C_i$.

Let $\mathbb{P}$ be Slaman and Woodin's notion of forcing for their coding. The elements of $\mathbb{P}$ are triples $p = \langle p_0, p_1, F_p \rangle$ where $p_0, p_1 \in 2^{<\omega}$, F_p is a finite subset of I, and $|p_0| = |p_1|$. We call $|p_0|$ the *length* of p and write $|p|$ instead of $|p_0|$. The partial ordering of $\mathbb{P}$ is defined as follows: $q \leqslant_{\mathbb{P}} p$ if:

- $p_0 \subseteq q_0$ and $p_1 \subseteq q_1$;
- $F_p \subseteq F_q$; and
- for all $y = \langle i, x \rangle$ such that $i \in F_p$, $x \in C_i$ and $|p| \leqslant y < |q|$, we have that $q_0(y) = q_1(y)$. In other words, it is required that q_0 and q_1 are $C_{F_p} \smallsetminus |p|$-equivalent.

Note that $\langle \mathbb{P}, \leqslant_{\mathbb{P}}, \perp_{\mathbb{P}} \rangle \leqslant_T C$.

Given a filter $G \subseteq \mathbb{P}$, let $G_0 = \bigcup\{p_0 : p \in G\}$ and $G_1 = \bigcup\{p_1 : p \in G\}$; let $\mathbf{g}_0 = \deg_T G_0$ and $\mathbf{g}_1 = \deg_T G_1$ be their degrees. Recall that a filter $G \subseteq \mathbb{P}$ is C-1-generic if for every set $W \subseteq \mathbb{P}$ which is recursively enumerable in C, either $G \cap W \neq \varnothing$, or there is a $p \in G$ such that $\forall q \leqslant_{\mathbb{P}} p (q \notin W)$. Observe that for every C-1-generic G, $G_0, G_1 \in 2^\omega$ and $\forall k \exists p \in G (k \in F_p)$.

Theorem 2.1 *Let G be a C-1-generic filter on $\mathbb{P}$, and let $\mathbf{g}_0$ and $\mathbf{g}_1$ be defined from G as above. Then $\mathcal{J}$ is the collection of minimal solutions of equation (2.1) below $\mathbf{c}$.*

In order to prove Theorem 2.1, it is sufficient to show that the following requirements are satisfied. Here k varies over I, Φ varies over all Turing functionals, and X varies over all sets which are recursive in C.

- P_k: $C_k \not\equiv_T (G_0 \oplus C_k) \wedge (G_1 \oplus C_k)$ (if the latter exists).
- $M_{X,\Phi}$: If $\Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X} = D$ are total and equal, and if $D \not\leqslant_T X$, then for some k, $C_k \leqslant_T X$.

The P_k requirements ensure that the C_ks are solutions to (2.1) and the $M_{X,\Phi}$ requirements ensure that the C_ks are minimal solutions, and that no other minimal solutions exist below C.

Lemma 2.2 *For every k, P_k is met. Therefore all the sets C_k satisfy equation (2.1).*

This is exactly as in the proof of [25, Prop. 2.5], but for completeness, we present the proof.

Proof Let $E_k = C_k \cap G_0^{[k]}$. It is immediate that $E_k \leqslant_T G_0 \oplus C_k$. However, we also have $E_k \leqslant_T G_1 \oplus C_k$. In fact

$$G_0^{[k]} \cap C_k =^* G_1^{[k]} \cap C_k,$$

because there is some $p \in G$ such that $k \in F_p$; for all $\langle k, x\rangle > |p|$ with $x \in C_k$, we have $G_0^{[k]}(x) = G_1^{[k]}(x)$.

It remains to show that $E_k \nleqslant_T C_k$. Consider a Turing functional Φ and let

$$S_{k,\Phi} = \{q \in \mathbb{P} : \exists x \in C_k (q_0(k,x)\downarrow \neq \Phi^{C_k}(x)\downarrow)\}.$$

Since $S_{k,\Phi}$ is C-r.e., there has to be some $p \in G$ such that either $p \in S_{k,\Phi}$, or $\forall q \leqslant_{\mathbb{P}} p (q \notin S_{k,\Phi})$. In the former case we have $\Phi^{C_k} \neq E_k$. In the latter case, we claim that $\Phi^{C_k}(x)\uparrow$ for all $x \in C_k$ such that $\langle k, x\rangle \geqslant |p|$; for if $\Phi^{C_k}(x)\downarrow$ for some such x then one can easily extend p to a condition in $S_{k,\Phi}$.

Therefore, we have that for all Φ, $\Phi^{C_k} \neq E_k$, and hence $E_k \nleqslant_T C_k$. □

Minimality Requirements Now fix $X \leqslant_T C$ and Φ such that

$$D = \Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X}$$

and such that $D \nleqslant_T X$. We want to show that for some k, $C_k \leqslant_T X$. The general idea of the proof (as done by Slaman and Woodin) is as follows. A *split* of a condition $p \in \mathbb{P}$ is a pair of strings $\sigma, \tau \supseteq p_0$ such that $\Phi^{\sigma \oplus X}$ and $\Phi^{\tau \oplus X}$ are contradictory. Clearly no such split can be a condition in the generic, so by genericity there is some condition $\bar{p}$ which is not extended by splits. Now, every condition has some split, as D is not recursive. So the reason that such a split is not an extension of $\bar{p}$ is that σ and τ contain some contradictory information about $x \in C_k$ for some x and k such that $k \in \bar{F} = F_{\bar{p}}$. The idea is to read off information about C_k by searching for such splits.

Now the way we go about fulfilling this strategy is the new part of the proof so we describe it more closely. As discussed, we will find (in Lemma 2.5) $\bar{F}$ and $\bar{p}$ as above such that for every split (σ, τ) of $\bar{p}$ there is some $k \in \bar{F}$ and some $\gamma \in C_k$ such that $\sigma(\langle k, \gamma\rangle) \neq \tau(\langle k, \gamma\rangle)$. Further, we will look for "special" splits (σ, τ) of $\bar{p}$, which means that for some $k \in \bar{F}$ and $\alpha \in 2^{<\omega}$, if σ and τ differ on some $\langle i, \gamma\rangle$ with $i \in \bar{F}$, then necessarily $i = k$ and $\gamma \supseteq \alpha$. As we are guaranteed such a difference for *some* i and γ, we have $\gamma \in C_k$; as C_k is the set of initial segments of the set $\hat{C}_k$, we must have $\alpha \in C_k$. We will show that recursively in X, for some k, one can enumerate infinitely many such special splits with α arbitrarily long, and thus is able to enumerate infinitely many elements of C_k. As C_k is recursive in any of its infinite subsets, this gives us a method of calculating C_k from X.

Definition 2.3 We call a condition $q \in \mathbb{P}$ *contradictory* if for some x,

$$\Phi^{q_0 \oplus X}(x)\downarrow \neq \Phi^{q_1 \oplus X}(x)\downarrow.$$

Being contradictory is a C-r.e. condition, so, by C-1-genericity, there is some $p \in G$ such that either p is contradictory or no extension of it is contradictory. The former case cannot hold because $\Phi^{G_0 \oplus X} = \Phi^{G_1 \oplus X}$, so the latter is the case.

Definition 2.4 Given $p \in \mathbb{P}$ and a set E, an E-split of p is a pair $\langle \sigma, \tau\rangle$ such that

- $\sigma \supseteq p_0$ and $\tau \supseteq p_0$;
- $\Phi^{\sigma \oplus X}(m)\downarrow \neq \Phi^{\tau \oplus X}(m)\downarrow$ for some m.
- $|\sigma| = |\tau|$.
- $\sigma \equiv_E \tau$.

If $\langle \sigma, \tau \rangle$ is a split, we let $m(\sigma, \tau)$ be the least m such that

$$\Phi^{\sigma \oplus X}(m)\downarrow \neq \Phi^{\tau \oplus X}(m)\downarrow.$$

Lemma 2.5 *There is a finite $F \subseteq I$ and a condition $p \in G$ which has no C_F-split.*

Proof Let $\bar{p}$ be a condition in G which has no contradictory extensions and let $F = F_{\bar{p}}$. Consider the set

$$S = \{q \leqslant_{\mathbb{P}} \bar{p} : \exists \sigma \in 2^{|q|}(\langle \sigma, q_0 \rangle \text{ is a } C_F\text{-split of } \bar{p}\}$$

Since S is C-recursive, by C-1-genericity, there is some $p \in G$ such that either p is in S or no extension of p is in S. Observe that if $p \leqslant_{\mathbb{P}} \bar{p}$ has any C_F split, then we can easily construct some extension of p in S, so it suffices to show that $G \cap S = \varnothing$.

Suppose that $p \in S \cap G$ and let σ be a string such that $\langle \sigma, p_0 \rangle$ is a C_F-split of $\bar{p}$. Let $m = m(\sigma, p_0)$. By our assumptions on X and Φ, there is some extension q of p such that $\Phi^{q_1 \oplus X}(m)\downarrow$. $q_0 \supseteq p_0$ and so $\Phi^{q_0 \oplus X}(m)\downarrow = \Phi^{p_0 \oplus X}(m)\downarrow$. Also, q is not contradictory. To sum it up, we have

$$\Phi^{q_1 \oplus X}(m)\downarrow = \Phi^{q_0 \oplus X}(m) = \Phi^{p_0 \oplus X}(m) \neq \Phi^{\sigma \oplus X}(m)\downarrow.$$

Let $\bar{\sigma} = \sigma \frown q_0$. Then $\langle \bar{\sigma}, q_1, F \rangle$ is a contradictory extension of $\bar{p}$ contradicting our choice of $\bar{p}$. □

Lemma 2.6 *Let E_0, E_1 be recursive sets. Suppose that every $p \in G$ has a $(E_0 \cap E_1)$-split. Then either every $p \in G$ has a E_0-split or every $p \in G$ has a E_1-split.*

Proof Suppose, toward a contradiction, that there is some condition in G which has no E_0-split and some condition in G which has no E_1-split. Then, by taking a lower bound, we find some $\bar{p} \in G$ which has neither any E_0-split nor any E_1-split. We can also assume that $\bar{p}$ has no contradictory extensions. Consider

$$S = \{q \leqslant_{\mathbb{P}} \bar{p} : \exists \sigma, \tau \in 2^{|q|}(\sigma \equiv_{E_0} q_0 \equiv_{E_1} \tau \;\&\; \langle \sigma, \tau \rangle \text{ is a } E_0 \cap E_1\text{-split of } \bar{p}\}$$

Since S is C-recursive, there is some $p \in G$ such that either p is in S or no extension of p is in S. We note that every $p \in G$ has an extension in S: Take any $p \in G$; without loss of generality $p \leqslant \bar{p}$. Let $\langle \sigma, \tau \rangle$ be a B-split of p, and let q_0 be defined as follows:

$$q_0(x) = \begin{cases} \sigma(x) & \text{if } x \in E_0 \\ \tau(x) & \text{if } x \in E_1 \\ p_0(x) & \text{otherwise.} \end{cases}$$

This definition makes sense because $\sigma \equiv_{E_0 \cap E_1} \tau$. We have $\sigma \equiv_{E_0} q_0$ and $\tau \equiv_{E_1} q_0$. Then $\langle q_0, p_1 \frown q_0, F_p \rangle$ extends p and is in S.

Thus we have some $p \in S \cap G$. Let σ and τ witness that $p \in S$ and let $m = m(\sigma, \tau)$. There is some extension q of p such that $\Phi^{q_0 \oplus X}(m)\downarrow$. Let

$\bar{\sigma} = \sigma \frown q_0$ and $\bar{\tau} = \tau \frown q_0$. Then, either $\langle \bar{\sigma}, q_0 \rangle$ is an E_0-split of $\bar{p}$, or $\langle \bar{\tau}, q_0 \rangle$ is an E_1-split of $\bar{p}$ (according to the value of $\Phi^{q_0 \oplus X}(m)$), contradicting the definition of $\bar{p}$. □

Lemma 2.7 *Let E_0,..., E_{n-1} be recursive sets. Suppose that every $p \in G$ has a $(E_0 \cap E_1 \cap \cdots \cap E_{n-1})$-split. Then, for some $i < n$, every $p \in G$ has a E_i-split.*

Proof The magic word is 'induction'. □

Lemma 2.8 *For every finite set $S \subset \omega$, every $p \in G$ has an S-split.*

Proof If $\max S < |p|$, then the notions of an S-split of p and of a $\varnothing$-split of p coincide. Since we can make $p \in G$ large, if the lemma fails then there is some $p \in G$ with no $\varnothing$-splits. We show that this assumption implies that $D \leqslant_T X$, which contradicts our previous assumptions.

Pick some $p \in G$ which has no $\varnothing$-splits. To compute $D(x)$ recursively in X, one looks for some $\sigma \supseteq p_0$ such that $\Phi^{\sigma \oplus X}(x)\downarrow$. Since p has no $\varnothing$-splits, necessarily $\Phi^{\sigma \oplus X}(x) = \Phi^{G_0 \oplus X}(x) = D(x)$. □

Now we show that for some k, $C_k \leqslant_T X$. By Lemma 2.5, fix a finite $\bar{F}$ and $\bar{p} \in G$ such that $\bar{p}$ has no $C_{\bar{F}}$-splits. Given $\alpha \in 2^{<\omega}$ and $k \in \bar{F}$ let

$$\begin{aligned} E_{k,\alpha} &= \{\langle i, \beta \rangle : i \in \bar{F} \ \& \ (i \neq k \vee \beta \not\supseteq \alpha)\} \\ &= (\bar{F} \times \omega) \smallsetminus \{\langle k, \beta \rangle : \beta \supseteq \alpha\}. \end{aligned}$$

First observe that if there is an $E_{k,\alpha}$-split, $\langle \sigma, \tau \rangle$ of $\bar{p}$, then $\alpha \in C_k$. This is because, since $\bar{p}$ has no $C_{\bar{F}}$-split, σ and τ differ on some $\langle i, \gamma \rangle \in C_{\bar{F}} \smallsetminus E_{k,\alpha}$, and hence $i = k$, $\gamma \supseteq \alpha$ and $\gamma \in C_k$. Therefore $\alpha \in C_k$. So

$$Y := \{\langle k, \alpha \rangle : \text{there is a } E_{k,\alpha}\text{-split of } \bar{p}\}$$

is subset of $C_{\bar{F}}$.

Now, fix $n \in \omega$, and observe that

$$E_n := \bigcap_{k \in \bar{F}, \alpha \in 2^n} E_{k,\alpha} = \{\langle i, \beta \rangle : i \in \bar{F} \ \& \ |\beta| < n\}$$

is finite. So, by Lemma 2.8 every $p \in G$ has a E_n-split. Then, by Lemma 2.7, for some $k \in \bar{F}$ and $\alpha \in 2^n$ there is a $E_{k,\alpha}$-split $\langle \sigma, \tau \rangle$ of $\bar{p}$. Hence, Y is infinite. Then, for some $k \in \bar{F}$,

$$Y_k := \{\alpha : \langle k, \alpha \rangle \in Y\}$$

is an infinite subset of C_k. Note that Y_k is r.e. in X, and therefore $C_k \leqslant_T X$.

2.1 Coding Countable Sets To find the parameters for coding countable sets, we first need to relate genericity for $\mathbb{P}$ with genericity for Cohen forcing. Let $\mathbb{Q} = \omega^{<\omega}$ be function Cohen forcing.

Proposition 2.9 *There is a dense embedding $i : \mathbb{Q} \to \mathbb{P}$ which is recursive in C.*

Proof Let $\{p_i\}$ be a recursive enumeration of the elements of $\mathbb{P}$. We say that a condition $p \in \mathbb{P}$ decides G up to p_n if for all $i \leqslant n$, $p \leqslant_{\mathbb{P}} p_i$ or $p \perp_{\mathbb{P}} p_i$. For every n, the collection of conditions which decide G up to p_n is dense in $\mathbb{P}$; we denote this collection by Ψ_n.

We claim that there is a process, uniformly recursive in C, which, given $p \in \mathbb{P}$ and $n < \omega$, enumerates an infinite maximal antichain below p, recursive in C, of conditions which decide G up to p_n. First, we find an infinite maximal antichain below p. For each $k < \omega$, let $p^k = (p_0^\frown 0^k 1, p_1^\frown 0^k 1, F_p)$; note that $p^k \leqslant_{\mathbb{P}} p$. Now define A_p by inductively deciding whether $p_i \in A_p$: $p_i \leqslant_{\mathbb{P}} p$ is added to A_p if it is one of the p^ks, or if it is incompatible with all of the p^ks and with all elements previously decided to be in A_p. $q \leqslant_{\mathbb{P}} p$ is incompatible with all p^ks iff $\min\{l \geq |p| : q_0(l) \neq 0\} \neq \min\{l \geq |p| : q_1(l) \neq 0\}$; this shows that A_p is recursive in C, and it is immediate that A_p is an infinite, maximal antichain below p.

For every $q \in A_p$ we find a maximal antichain B_q below q contained in Ψ_n in much the same manner; we don't mind if B_q is finite, so we simply apply the inductive process, restricted to elements of Ψ_n. Note that we indeed get a maximal antichain below q, because Ψ_n is dense open below q. Now

$$B_{p,n} := \bigcup_{q \in A_p} B_q$$

is an infinite, maximal antichain below p, is contained in Ψ_n, and can be enumerated recursively in C (uniformly in p and n), by enumerating A_p and B_q for q enumerated in A_p, dovetailing of course.

We can now easily define $i(\sigma)$ by induction on σ; $i(\langle\rangle)$ is the empty condition of $\mathbb{P}$. If $i(\sigma)$ is defined, then $i(\sigma^\frown\{n\})$ is the n^{th} element enumerated in $B_{i(\sigma),|\sigma|}$. Clearly i is recursive in C, and i is an embedding of $\mathbb{Q}$ into $\mathbb{P}$ preserving $\perp$. To see that i is dense, take any p_n. i"ω^n is a maximal antichain in $\mathbb{P}$, so for some $\sigma \in \omega^n$, $i(\sigma)$ is compatible with p_n. Since $i(\sigma)$ decides G up to p_n, we must have $i(\sigma) \leqslant_{\mathbb{P}} p_n$. □

The following is well known.

Proposition 2.10 *Suppose that G is 1-generic over B, that $A_0, A_1 \leqslant_T B$ and that $n, m < \omega$. Then*

$$A_0 \oplus G^{[n]} \leqslant_T A_1 \oplus G^{[m]}$$

iff $A_0 \leqslant_T A_1$ and $n = m$. □

We finally show how to code countable sets. This follows [25, Prop. 2.15].

Theorem 2.11 *There is a bounded formula $\psi(x, \bar{y})$ in the language of upper semi-lattices such that whenever we have a sequence of reals $\langle C_i \rangle$, a real $C \geqslant_T \bigoplus_i C_i$ and some G which is 1-generic over C, then there is a tuple $\bar{\mathbf{a}}$ of degrees below $\mathbf{g} \vee \mathbf{c}$ such that*

$$\mathbf{x} \in \{\mathbf{c}_i : i < \omega\} \Leftrightarrow \mathcal{D} \models \psi(\mathbf{x}, \bar{\mathbf{a}}).$$

Of course, $\mathbf{c}_i = \deg_T C_i$, $\mathbf{c} = \deg_T C$ and $\mathbf{g} = \deg_T G$.

Proof By Theorem 2.1 and Proposition 1.2, there is a bounded formula $\varphi(x, y, z_0, z_1)$ such that for every countable antichain of degrees $\mathcal{C} = \{\mathbf{c}_i\}$ and every G which is 1-generic over $C = \bigoplus C_i$, there are $G_0, G_1 \leqslant_T C \oplus G$ such that $\mathcal{C}$ is definable by the formula $\varphi(x, \mathbf{c}, \mathbf{g}_0, \mathbf{g}_1)$.

Let $\mathcal{C} = \{\mathbf{c}_i : i < \omega\}$. Let $G_i = (G^{[0]})^{[i]}$, and let $\mathcal{G} = \{\mathbf{g}_i : i < \omega\}$; $\mathcal{G}$ is an antichain, as $G^{[0]}$ is 1-generic over C. Let $\mathcal{I} = \{\mathbf{c}_i \vee \mathbf{g}_i : i < \omega\}$; $\mathcal{I}$ is an antichain. Note that $\bigoplus G_i$ and $\bigoplus(C_i \oplus G_i)$ are both recursive in $C \oplus G^{[0]}$. As $G^{[1]}$ is 1-generic over $C \oplus G^{[0]}$, there are parameters below $G \oplus C$ coding $\mathcal{I}$ and $\mathcal{G}$ as above.

Now $\mathcal{C}$ is definable from the above parameters and $\mathbf{c}$ by the bounded formula

$$x < \mathbf{c} \,\&\, \exists(g \in \mathcal{G}, z \in \mathcal{I})(z = g \vee x) \qquad \square$$

Porism 2.12 The function taking $\mathbf{c}_i$ to $\mathbf{g}_i$ is definable with the same parameters by the formula

$$x \in \mathcal{C} \;\&\; y \in \mathcal{G} \;\&\; (x \vee y) \in \mathcal{I}.$$

We now code countable functions.

Theorem 2.13 *Suppose that $\mathcal{B} = \{\mathbf{b}_i : i < \omega\}$ and $\mathcal{C} = \{\mathbf{c}_i : i < \omega\}$ are sets of degrees. Let $B_i \in \mathbf{b}_i$, $C_i \in \mathbf{c}_i$, $B = \bigoplus B_i$ and $C = \bigoplus C_i$. Suppose that G is 1-generic over $B \oplus C$. Then the function taking $\mathbf{b}_i$ to $\mathbf{c}_i$ is definable with parameters found below $B \oplus C \oplus G$.*

As for sets, the coding is done uniformly by a bounded formula.

Proof Let $E = B \oplus C \oplus G$. We can find parameters below $\mathbf{e}$ coding the sets $\mathcal{B}$ and $\mathcal{C}$. Again split G: let $G_n = \left(G^{[0]}\right)^{[n]}$. As $G^{[1]}$ is 1-generic over $C \oplus D \oplus G^{[0]}$, we saw in porism 2.12 that the relations $\{(\mathbf{b}_i, \mathbf{g}_i) : i < \omega\}$ and $\{(\mathbf{g}_i, \mathbf{c}_i) : i < \omega\}$ are both definable with parameters below E. Now composition gives the desired function. $\square$

Remark 2.14 Both theorems 2.11 and 2.13 hold if the sets of degrees are finite.

3 Interpreting Arithmetic

To get undecidability results, we code models of arithmetic into $\mathcal{D}(\leqslant \mathbf{g})$. Let T be a finitely axiomatizable theory in the language of arithmetic which is hereditarily undecidable and ensures that every model of T has a standard part and of course, which holds in the standard model. We can pick T to be Robinson arithmetic, Shoenfield's theory N ([20, Ch. 6]), or PA^-.

We use the terminology of [18] concerning coding schemes. In particular, we use their scheme for coding models of arithmetic in partial orders. Rather than repeat the definitions, we review the needed properties. We have formulas $\varphi_{\mathrm{dom}}, \varphi_0, \varphi_S, \varphi_+$ and $\varphi_\times$ in the language of partial orderings. If $\mathcal{L} = (L; \leqslant_{\mathcal{L}})$ is a partial ordering, then the interpretation of arithmetic in $\mathcal{L}$ is the structure

$$N_{\mathcal{L}} \;=\; (\varphi_{\mathrm{dom}}(\mathcal{L}); \varphi_0(\mathcal{L}), \varphi_S(\mathcal{L}), \varphi_+(\mathcal{L}), \varphi_\times(\mathcal{L}))$$

for the language of arithmetic. Moreover, the scheme (the defining formulas) can be chosen such that there is a recursive partial ordering $\mathcal{L}^*$ such that $N_{\mathcal{L}^*}$ is isomorphic to the standard model of arithmetic.

This scheme can be transformed into a scheme of coding arithmetic in a degree structure such as $\mathcal{D}(\leqslant \mathbf{r})$ via the coding of countable sets; namely, given a tuple of parameters $\bar{\mathbf{a}}$ for ψ (of Theorem 2.11) we let $L_{\bar{\mathbf{a}}}$ be the set coded (defined) by $\psi(x, \bar{\mathbf{a}})$ (we saw that the parameters code the same set or relation in any local degree structure $\mathcal{D}(\leqslant \mathbf{r})$ which contains the parameters) and let $\mathcal{L}_{\bar{\mathbf{a}}}$ be the model $(L_{\bar{\mathbf{a}}}; \leqslant_T)$. Having found a partial ordering we can use the scheme above to interpret arithmetic: we let $M_{\bar{\mathbf{a}}} = N_{\mathcal{L}_{\bar{a}}}$. The correctness condition $\chi(\bar{\mathbf{a}})$ states that $M_{\bar{\mathbf{a}}} \models T$. All formulas involved are bounded, and so $M_{\bar{\mathbf{a}}}$ (and the correctness of $\bar{\mathbf{a}}$) is well-defined and doesn't depend on the ideal in which we're working.

Proposition 3.1 *Suppose that* $\mathbf{g}$ *is 1-generic. Then there are* $\bar{\mathbf{a}} \in D(\leqslant \mathbf{g})$ *such that* $M_{\bar{\mathbf{a}}}$ *is isomorphic to the standard model of arithmetic. (In particular,* $\bar{\mathbf{a}}$ *satisfy the correctness condition.)*

Proof Let $G \in \mathbf{g}$ be a 1-generic set. We know if H is 1-generic and $\mathcal{L} = (\{p_i\}_{i<\omega}, <_{\mathcal{L}})$ is a recursive partial ordering, then there are sets $\{P_i\}_{i<\omega}$ such that $\bigoplus_n P_n \leqslant_T H$ and $p_i \to P_i$ is an embedding of $\mathcal{L}$ into the degrees. Thus the recursive ordering $\mathcal{L}^*$ which was discussed above can be embedded below $G^{[0]}$ in such a uniform way. Theorem 2.11 shows that there is some tuple $\bar{\mathbf{a}}$ below $G^{[0]} \oplus G^{[1]}$ which codes (via ψ) the copy of $\mathcal{L}^*$ embedded below $G^{[0]}$. Then $M_{\bar{\mathbf{a}}} \cong N_{\mathcal{L}^*}$ is isomorphic to the standard model. □

This gives a direct proof of the following corollary, which, as mentioned in the introduction, can be deduced from work of Shore ([21]) and Jockusch ([6]).

Corollary 3.2 *Suppose that* $\mathbf{c}$ *is a degree which bounds a 1-generic degree. Then* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{c}))$ *is undecidable.*

We now employ the technique of comparison maps from [18]. Let φ be the formula coding binary relations. Let $\theta(\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1, \bar{\mathbf{c}})$ be a correctness condition stating that $\varphi(x, y; \bar{\mathbf{c}})$ codes an injective function $h_{\bar{\mathbf{c}}}$ from an initial segment of $M_0 = M_{\bar{\mathbf{a}}_0}$ to an initial segment of $M_1 = M_{\bar{\mathbf{a}}_1}$ which preserves the arithmetical structure. Let $\xi(x, y; \bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1)$ say that there is some $\bar{\mathbf{c}}$ such that $\theta(\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1, \bar{\mathbf{c}})$ holds and $h_{\bar{\mathbf{c}}}(x) = y$. If both tuples $\bar{\mathbf{a}}_i$ satisfy the correctness condition χ, then ξ defines a relation $R_{\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1}$ between M_0 and M_1, which restricted to the standard part of M_0 is a partial isomorphism, defined on a not necessarily proper initial segment of this standard part. Note that R depends heavily on the ideal $\mathcal{J}$ in which we are working, as the quantification of $\bar{\mathbf{c}}$ is unbounded. Given a large enough ideal, $R_{\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1}$ will be total on the standard part of M_0; what we need is that all finite partial isomorphisms of initial segments of M_0 to initial segments of M_1 can be coded by parameters $\bar{\mathbf{c}}$ in $\mathcal{J}$.

If $\bar{\mathbf{a}}_0, \bar{\mathbf{a}}_1 \leqslant \mathbf{b}$ and $\mathbf{g}$ is 1-generic over $\mathbf{b}$, then Theorem 2.13 shows that in fact $\bar{\mathbf{c}}$ can be found below $\mathbf{b} \vee \mathbf{g}$.

Proposition 3.3 *Suppose that* $\mathcal{J}$ *is an ideal and suppose that the 1-generic degrees are downward dense in* $\mathcal{J}$ *(that is, every nonzero* $\mathbf{a} \in \mathcal{J}$ *bounds a 1-generic degree). Then there is a correctness condition* χ^* *such that* $\chi^*(\mathcal{J})$ *is non-empty, and for all* $\bar{\mathbf{a}} \in \mathcal{J}$ *such that* $\mathcal{J} \models \chi^*(\bar{\mathbf{a}})$, $M_{\bar{\mathbf{a}}}$ *is isomorphic to the standard model of arithmetic.*

It follows that $\mathcal{J}$ interprets the standard model (without parameters) and so that first order true arithmetic is reducible to $\mathrm{Th}(\mathcal{J}, \leqslant_T)$.

Proof Let $\chi^*(\bar{\mathbf{a}})$ say that the correctness condition $\chi(\bar{\mathbf{a}})$ holds, and that there is some nonzero $\mathbf{b}$ such that whenever $\bar{\mathbf{a}}' \leqslant \mathbf{b}$ is a tuple such that $\chi(\bar{\mathbf{a}}')$ holds, then $\operatorname{dom} R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'} = M_{\bar{\mathbf{a}}}$ (i.e. $R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'}$ is total).

If $\chi^*(\bar{\mathbf{a}})$ holds in $\mathcal{J}$, let $\mathbf{b}$ witness this fact. Since there is a 1-generic degree below $\mathbf{b}$, there is a standard model $M_{\bar{\mathbf{a}}'}$ with $\bar{\mathbf{a}}' \leqslant \mathbf{b}$. Totality of $R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'}$ implies that $M_{\bar{\mathbf{a}}}$ must be standard.

Now we show existence. Let $\mathbf{g} \in \mathcal{J}$ be 1-generic. Let $G_i = G^{[i]}$. If $\bar{\mathbf{a}} \leqslant \mathbf{g}_0$ codes a standard model then $\chi^*(\bar{\mathbf{a}})$ holds, with witness $\mathbf{b} = \mathbf{g}_1$. This is because parameters $\bar{\mathbf{c}}$ coding the finite comparison maps from $M_{\bar{\mathbf{a}}}$ to any models coded below $\bar{\mathbf{g}}_1$ can be found in $\mathcal{J}$, as $\mathbf{g}_2$ is 1-generic over $\mathbf{g}_0 \vee \mathbf{g}_1$. □

This establishes our main theorems:

Theorem 3.4 *If $\mathbf{g} < \mathbf{0}'$ is 1-generic, then $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is recursively isomorphic to true arithmetic.*

Proof Chong and Jockusch [2] show that the 1-generic degrees are downward dense in $\mathcal{D}(\leqslant \mathbf{g})$ whenever $\mathbf{g}$ is 1-generic and below $\mathbf{0}'$. □

Theorem 3.5 *If $\mathbf{g}$ is 2-generic, then true arithmetic is 1-reducible to $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$.*

Proof Martin (see [6, Thm. 4.1]) showed that the 2-generic degrees are downward dense in $\mathcal{D}(\leqslant \mathbf{g})$ whenever $\mathbf{g}$ is 2-generic. □

Remark 3.6 We remark that this shows that the set of reals A for which $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ computes $\mathbf{0}^{(\omega)}$ is comeager. We also remark that if $\mathbf{g}$ is arithmetic, then $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ can be interpreted in first order true arithmetic; thus for every arithmetic 2-generic degree $\mathbf{g}$, $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is recursively isomorphic to true arithmetic.

Remark 3.7 $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ is constant for arithmetically generic $\mathbf{g}$ (see [15, Ex. IV 2.13]). In fact, the n-quantifier part of $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ can be uniformly decided by $\mathbf{0}^{(\omega)}$. It follows that the theory of $\mathcal{D}(\leqslant \mathbf{g})$ for arithmetically generic degrees $\mathbf{g}$ is recursively isomorphic to true arithmetic.

We get a little more. A degree $\mathbf{a}$ is 1-*REA* if it is recursively enumerable. A degree $\mathbf{a}$ is $n+1$-*REA* if it is r.e. in some $\mathbf{b} \leqslant \mathbf{a}$. We remark that every n-r.e. degree is n-REA ([7]).

Lemma 3.8 *If $\mathbf{a}$ is n-REA for some $n < \omega$, then for all $\mathbf{b} < \mathbf{a}$ there is some $\mathbf{g} \in (\mathbf{b}, \mathbf{a})$ which is 1-generic over $\mathbf{b}$.*

Proof We show that if $\mathbf{a}$ is n-REA then for all $\mathbf{b} < \mathbf{a}$ there is some $\mathbf{c} \in [\mathbf{b}, \mathbf{a})$ such that $\mathbf{a}$ is r.e. in $\mathbf{c}$. The lemma follows by relativizing to $\mathbf{b}$ the fact that every r.e. degree bounds a 1-generic degree (see [26, Ex. VI 3.9]).

Let $\mathbf{a}$ be n-REA; let this be witnessed by $\mathbf{0} = \mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots < \mathbf{a}_n = \mathbf{a}$ (i.e. $\mathbf{a}_{i+1}$ is r.e. in $\mathbf{a}_i$). Let $\mathbf{b} < \mathbf{a}$. Let $i < n$ be the least such that $\mathbf{b} \vee \mathbf{a}_{i+1} = \mathbf{a}$. Since $\mathbf{a}_0 = \mathbf{0} < \mathbf{a}$, we have $\mathbf{b} \vee \mathbf{a}_i < \mathbf{a}$. We claim that $\mathbf{a}$ is r.e. in $\mathbf{b} \vee \mathbf{a}_i$. Since $\mathbf{a} = \mathbf{b} \vee \mathbf{a}_{i+1}$, it is sufficient to show that $\mathbf{b} \vee \mathbf{a}_i$ can enumerate $\mathbf{a}_{i+1}$. But $\mathbf{a}_{i+1}$ is r.e. in $\mathbf{a}_i$. □

Theorem 3.9 *If* $\mathbf{c}$ *is* n*-REA then* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{c}))$ *is recursively isomorphic to true arithmetic.*

Proof The correctness condition $\chi^*(\bar{\mathbf{a}})$ will say that $\chi(\bar{\mathbf{a}})$ holds, that $\vee\bar{\mathbf{a}} < \mathbf{c}$ and that for every $\bar{\mathbf{a}}'$ such that $(\vee\bar{\mathbf{a}}') \vee (\vee\bar{\mathbf{a}}) < \mathbf{c}$ (and such that $\chi(\bar{\mathbf{a}}')$ holds), $R_{\bar{\mathbf{a}},\bar{\mathbf{a}}'}$ is total.

If $\chi^*(\bar{\mathbf{a}})$ holds then there is some 1-generic $\mathbf{g} \in (\vee\bar{\mathbf{a}}, \mathbf{a})$ and so some standard model $M_{\bar{\mathbf{a}}'}$ coded below $\mathbf{g}$; it follows that $M_{\bar{\mathbf{a}}}$ must be standard.

$\chi^*(\mathcal{D}(\leqslant \mathbf{c}))$ is not empty. Let $\mathbf{g}_0 < \mathbf{c}$ be some 1-generic degree, and let $\mathbf{a} < \mathbf{g}_0$ code a standard model. For every $\mathbf{b} \in (\mathbf{g}_0, \mathbf{c})$ which bounds some $\bar{\mathbf{a}}'$ which code a model $M_{\bar{\mathbf{a}}'}$, and for every final initial segment of $M_{\bar{\mathbf{a}}}$, there are $\bar{c}$ which code the isomorphism between this initial segment and its copy in $M_{\bar{\mathbf{a}}'}$; this is because there is some $\mathbf{g}_1 < \mathbf{c}$ which is 1-generic over $\mathbf{b}$. It follows that $\chi^*(\bar{\mathbf{a}})$ holds. □

We are left with a couple of questions for which we do not yet know an answer.

Question 3.10 *Is there a 1-generic degree* $\mathbf{g}$ *such that* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ *does not interpret true arithmetic? Is there one such that* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{g}))$ *is more complicated than true arithmetic?*

Question 3.11 *Suppose that* $\mathbf{a}$ *bounds a 1-generic degree. Does* $\mathrm{Th}(\mathcal{D}(\leqslant \mathbf{a}))$ *interpret true arithmetic?*

4 Lattice Embeddings

In this section we show how to embed lattices into $\mathcal{D}(\leqslant \mathbf{g})$, where $\mathbf{g}$ is 1-generic, preserving $\mathbf{0}$ and $\mathbf{1}$ (we consider only lattices with $\mathbf{0}$ and $\mathbf{1}$, where $\mathbf{0} \neq \mathbf{1}$).

We start by defining lattice tables. Whitman [27] observed that every lattice can be embedded in a lattice of equivalence relations. Then, Jónsson showed how to construct a lattice table that also satisfies (3) below in [8] (he maintained the notation of equivalence relations). Shore, in [22] pointed out that, using Jónsson's construction, for every recursive lattice we can get a uniformly recursive lattice table, also satisfying (3). We modify Shore's proof a bit to get a recursive lattice table that also satisfies (4).

Definition 4.1 Let L be a lattice and T be a set of functions from L to ω. Given $\alpha, \beta \in T$ and $p \in L$, we write $\alpha \sim_p \beta$ if $\alpha(p) = \beta(p)$ (observe that $\sim_p$ is an equivalence relation and that one can think of $\alpha(p)$ as a name for the equivalence class of α.) We say that T is a *lattice table* for L if for all $p, q, r \in L$

1. $p \leqslant q \iff \forall \alpha, \beta \in T(\alpha \sim_q \beta \implies \alpha \sim_p \beta)$
2. $p \vee q = r \implies \forall \alpha, \beta \in T(\alpha \sim_p \beta \;\&\; \alpha \sim_q \beta \implies \alpha \sim_r \beta)$
3. $p \wedge q = r \;\&\; \alpha \sim_r \beta \implies \exists \gamma_1, \gamma_2, \gamma_3 \in T \;\; (\alpha \sim_p \gamma_1 \sim_q \gamma_2 \sim_p \gamma_3 \sim_q \beta)$.
4. $\forall \alpha\, \beta \in T, (\alpha \neq \beta \implies \alpha \sim_{\mathbf{0}} \beta \;\&\; \alpha \not\sim_{\mathbf{1}} \beta)$.

The definition follows [22, Thm. 7] but adds condition (4).

We say that a lattice table T is recursive if there is some numbering of the elements of T which makes them uniformly recursive.

Proposition 4.2 *Every recursive lattice L has a recursive lattice table.*

Shore [22, Thm. 7] constructs a lattice table satisfying (1)-(3). We show how to modify Shore's construction to add condition (4).

Sketch of Proof One first defines a set of functions $T_0 = \{\beta_{p,i} : p \in L, i < 2\}$ by letting

$$\beta_{p,0}(q) = \begin{cases} \langle p, 0\rangle & \text{if } q \neq \mathbf{0} \\ 0 & \text{if } q = \mathbf{0} \end{cases} \qquad \beta_{p,1}(q) = \begin{cases} \beta_{p,0}(q) & \text{if } q \leqslant p \\ \langle p, 1\rangle. & \text{if } q \nleqslant p \end{cases}$$

Note that $\beta_{\mathbf{1},0} = \beta_{\mathbf{1},1}$. It is easy to check that (1), (2) and (4) are satisfied for T_0. Now suppose that a set of functions T_i which satisfies (1), (2) and (4), and such that $\bigcup_{\alpha \in T_i} \alpha``L$ is coinfinite, is given. Suppose that $p, q, r \in L$ and $\alpha, \beta \in T_i$ are such that $p \wedge q = r$ and $\alpha \sim_r \beta$, and such that (3) fails in this situation. Then we enlarge T_i to T_{i+1} by adding three functions $\gamma_0, \gamma_1, \gamma_2$ defined as follows. Let w, x, y and z be new numbers not in the range of any of the functions in T_i.

$$\begin{aligned}
\gamma_0(s) &= \begin{cases} \alpha(s) & \text{if } s \leqslant p \\ w & \text{if } s \nleqslant p \end{cases} \\
\gamma_1(s) &= \begin{cases} \gamma_0(s) & \text{if } s \leqslant q \\ x & \text{if } s \leqslant p \ \&\ s \nleqslant q \\ y & \text{otherwise} \end{cases} \\
\gamma_3(s) &= \begin{cases} \beta(s) & \text{if } s \leqslant q \\ x & \text{if } s \leqslant p \ \&\ s \nleqslant q \\ z & \text{otherwise.} \end{cases}
\end{aligned}$$

Then T_{i+1} satisfies the induction hypothesis, and also (3) for p, q, r, α, β. (To check (4) one has to note that neither $p = \mathbf{1}$ nor $q = \mathbf{1}$.) Also note that a recursive index for T_{i+1} can be uniformly obtained from a recursive index for T_i. By bookkeeping we get a uniformly recursive lattice table as desired. $\square$

Suppose that T is a lattice table for a lattice L. For $p \in L$ and $\sigma \in T^{\leqslant\omega}$ we define $h_p^\sigma \in \omega^{|\sigma|}$ by letting $h_p^\sigma(i) = (\sigma(i))(p)$ (in the language of equivalence relations, h_p^σ gives the sequence of $\sim_p$-equivalence classes of the elements of σ). If $\sigma, \tau \in T^{\leqslant\omega}$, write $\sigma \sim_p \tau$ if $\forall i < \min\{|\sigma|, |\tau|\}\,(\sigma(i) \sim_p \tau(i))$. Observe that if $|\sigma| = |\tau|$, then $\sigma \sim_p \tau$ iff $h_p^\sigma = h_p^\tau$.

In [22, Thm. 8], Shore constructs, given a recursive lattice table T for a lattice L, a function $g \leqslant 0'$ such that $p \to \deg_T(h_p^g)$ is an embedding of L into $\mathcal{D}(\leqslant 0')$. Moreover, given an r.e. degree $\mathbf{a}$, he constructs $g \leqslant \mathbf{a}$ such that $p \to \deg_T(h_p^g)$ is an embedding of L into $\mathcal{D}(\leqslant \mathbf{a})$. These embeddings preserve neither $\mathbf{0}$ nor $\mathbf{1}$.

Here we prove the following:

Proposition 4.3 *If T is a recursive lattice table for a lattice L, and if $g \in T^\omega$ is 1-generic, then the map $p \mapsto h_p^g$ is a lattice embedding of L into $\mathcal{D}(\leqslant_T \mathbf{g})$ preserving $\mathbf{0}$ and $\mathbf{1}$.*

This last proposition implies the following theorem.

Theorem 4.4 *If* $\mathbf{g}$ *is 1-generic, then every recursive lattice can be embedded in* $\mathcal{D}(\leqslant \mathbf{g})$ *preserving* $\mathbf{0}$ *and* $\mathbf{1}$.

By 1-generic we mean a 1-generic filter for the forcing $T^{<\omega}$ which can be identified with function Cohen forcing by the numbering of T which makes T recursive. Lemmas 4.5 and 4.6 below follow from Shore's proof. Lemma 4.7 is new.

Let $g \in T^\omega$ be 1-generic. We start by proving the facts about h^g which do not need genericity.

Lemma 4.5

1. $h^g_{\mathbf{1}} \equiv_T g$.
2. $h^g_{\mathbf{0}}$ *is a constant function, and hence recursive.*
3. *if* $p \leqslant q$ *then* $h^g_p \leqslant_T h^g_q$.
4. *if* $p \vee q = r$, *then* $h^g_p \oplus h^g_q \equiv_T h^g_r$

Proof (1) and (2) follow from 4.1(4); $g(i)$ is the unique $\alpha \in T$ such that $\alpha(\mathbf{1}) = h^g_{\mathbf{1}}(i)$. For part (3) consider $p \leqslant q$. Take $i \in \omega$; we want to compute $h^g_p(i)$ using h^g_q. Find $\alpha \in T$ such that $\alpha(q) = h^g_q(i)$. Since $\alpha \sim_q g(i)$, $\alpha \sim_p g(a)$, so $h^g_p(i) = \alpha(p)$.

For part (4), we already have from (3) that $h^g_p \oplus h^g_q \leqslant_T h^g_r$. Take $i \in \omega$; we want to compute $h^g_r(i)$ using h^g_p and h^g_q. Find $\alpha \in T$ such that $\alpha(p) = h^g_p(i)$ and $\alpha(q) = h^g_q(i)$. Then, since $\alpha \sim_p g(i)$ and $\alpha \sim_q g(i)$, we have $\alpha \sim_r g(i)$, so $h^g_r(i) = \alpha(r)$. □

Now we show that h^g is a poset embedding.

Lemma 4.6 *If* $p \not\leqslant q$, *then* $h^g_p \not\leqslant_T h^g_q$.

Proof Consider a Turing functional Φ and suppose that $\Phi^{h^g_q}$ is total. We want to show that $h^g_p \neq \Phi^{h^g_q}$. Let

$$S = \{\tau \in T^{<\omega} : \exists x \ \ (h^\tau_p(x) \neq \Phi^{h^\tau_q}(x) \downarrow)\}.$$

By 1-genericity, there is a $\tau_0 \subset g$ such that either $\tau_0 \in S$ or $\forall \sigma \supseteq \tau_0 (\sigma \notin S)$. The former case clearly implies that $h^g_p \neq \Phi^{h^g_q}$. We show that the latter case is impossible. Assume, toward a contradiction, that $\tau_0 \subset g$ and $\forall \sigma \supseteq \tau_0(\sigma \notin S)$. Let α and β be such that $\alpha \sim_q \beta$ but $\alpha \not\sim_p \beta$. By 1-genericity there is some $\tau_1 \subset g$, such that for some $x \geq |\tau_0|$, $\tau_1(x) = \alpha$. Since $\Phi^{h^g_q}$ is total, there is a $\tau_2 \subset g$ extending τ_1 such that $\Phi^{h^{\tau_2}_q}(x)\downarrow$. Let σ be obtained from τ_2 just by changing the value at x to β. Then $\tau_2 \sim_q \sigma$, so $\Phi^{h^\sigma_q}(x)\downarrow = \Phi^{h^{\tau_2}_q}(x)$ but $h^\sigma_p(x) = \beta(p) \neq \alpha(p) = h^{\tau_2}_p(x)$. So either σ or τ_2 is in S and both extend τ, contradicting our assumption. □

Finally we prove that h^g preserves meet.

Lemma 4.7 *If* $p \wedge q = r$, *then* $h^g_p \wedge h^g_q \equiv_T h^g_r$.

Proof From Lemma 4.5 we have that $h^g_p, h^g_q \geqslant_T h^g_r$. We may assume that $p \neq q$ and use Posner's trick. Suppose that $D = \Phi^{h^g_p} = \Phi^{h^g_q}$. We want to show that $D \leqslant_T h_r^g$. First consider

$$S_0 = \{\tau : \exists x(\Phi^{h_p^\tau}(x)\downarrow \neq \Phi^{h_q^\tau}(x)\downarrow)\}.$$

Clearly g does not meet S_0, so there is a $\tau_0 \subset g$ such that $\forall \sigma \supseteq \tau_0$ $(\sigma \notin S_0)$. Now consider

$$S_1 = \{\tau \supseteq \tau_0 : \exists \sigma_0, \sigma_1, \sigma_2, \sigma_3 \in O_{\tau_0,|\tau|}, \exists x \in \omega \\ (\Phi^{h_p^{\sigma_0}}(x)\downarrow \neq \Phi^{h_q^{\sigma_3}}(x) \;\&\; \sigma_0 \sim_p \sigma_1 \sim_q \tau \sim_p \sigma_2 \sim_q \sigma_3)\},$$

where $O_{\tau,n} = \{\sigma \in T^n : \sigma \supseteq \tau\}$. We claim that no $\tau \subset g$ is in S_1. Suppose $\tau \subset g$ is in S_1 and σ_0, σ_1, σ_2, σ_3 and x witness it. Extend τ to $\bar\tau$ such that $\Phi^{h_p^{\bar\tau}}(x)\downarrow = \Phi^{h_q^{\bar\tau}}(x)\downarrow$. For $i = 0,1,2,3$, let $\bar\sigma_i = \sigma_i \frown \bar\tau$. Either $\Phi^{h_p^{\bar\sigma_0}}(x) \neq \Phi^{h_q^{\bar\tau}}(x)$ or $\Phi^{h_p^{\bar\tau}}(x) \neq \Phi^{h_q^{\bar\sigma_3}}(x)$. Suppose $\Phi^{h_p^{\bar\sigma_0}}(x) \neq \Phi^{h_q^{\bar\tau}}(x)$. Since $\bar\sigma_0 \sim_p \bar\sigma_1$, $\Phi^{h_p^{\bar\sigma_1}}(x)\downarrow = \Phi^{h_p^{\bar\sigma_0}}(x)$, and since $\bar\tau \sim_q \bar\sigma_1$, $\Phi^{h_q^{\bar\sigma_1}}(x)\downarrow = \Phi^{h_q^{\bar\tau}}(x)$. Therefore $\bar\sigma_1 \in S_0$ and extends τ_0. This contradicts the definition of τ_0 and proves our claim. So there is some $\tau_1 \subset g$ such that $\forall \sigma \supseteq \tau_1$ $(\sigma \notin S_1)$.

Now we claim that for all $\sigma \supseteq \tau_1$ such that $\sigma \sim_r g$ and for all x such that $\Phi^{h_p^\sigma}(x)\downarrow$, we have $\Phi^{h_p^\sigma}(x) = D(x)$. Otherwise, find some $\sigma \sim_r g$ which extends τ_1 and find an x such that $\Phi^{h_p^\sigma}(x)\downarrow \neq D(x) = \Phi^{h_q^g}(x)$. Let $\sigma_3 \subset g$ be such that $\Phi^{h_q^{\sigma_3}}(x)\downarrow$ and $|\sigma_3| \geq \sigma$. Let $\sigma_0 = \sigma \frown \sigma_3$. Since $\sigma_0 \sim_r \sigma_3$ and both extend τ_1, by definition 4.1.(3), there exist σ_1, σ_2 and τ, extending τ_1, such that

$$\sigma_0 \sim_p \sigma_1 \sim_q \tau \sim_p \sigma_2 \sim_q \sigma_3.$$

But then τ is an extension of τ_1 in S_1. This contradiction proves our second claim.

Finally we show that $D \leqslant_T h_r^g$. Take $x \in \omega$. To compute $D(x)$ recursively in h_r^g look for $\sigma \supseteq \tau_1$ such that $\Phi^{h_p^\sigma}(x)\downarrow$ and $\forall i < |\sigma|$ $(\sigma(i)(r) = h_r^g(y))$. (Notice that $\forall i < |\sigma|(\sigma(i)(r) = h_r^g(y))$ is equivalent to $\sigma \sim_r g$.) Some initial segment of g serves as such a σ, so the search will end. Then $D(x) = \Phi^{h_p^\sigma}(x)$. □

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Acknowledgments

We would like to thank our advisor Richard Shore for his guidance and useful comments. We would also like to thank the referee for his/her detailed report. Both authors were partially supported by NSF Grant DMS-0100035. The first author was also supported by a fellowship from the Lady Davis Trust.

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