COMPUTABLE STRUCTURE THEORY USING ADMISSIBLE RECURSION THEORY ON ω_1

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ABSTRACT. We use the theory of recursion on admissible ordinals to develop an analogue of classical computable model theory and effective algebra for structures of size $\aleph_1$, which, under our assumptions, is equal to the continuum. We discuss both general concepts, such as computable categoricity, and particular classes of examples, such as fields and linear orderings.

1. INTRODUCTION

Our aim is to develop computable structure theory for uncountable structures. In this paper we focus on structures of size $\aleph_1$. The fundamental decision to be made, when trying to formulate such a theory, is the choice of computability tools that we intend to use. To discover which structures are computable, we need to first describe which subsets of the domain are computable, and which functions are computable. In this paper, we use admissible recursion theory (also known as α-recursion theory) over the domain ω_1. We believe that this choice yields an interesting computable structure theory. It also illuminates the concepts and techniques of classical computable structure theory by observing similarities and differences between the countable and uncountable settings. In particular, it seems that as is the case for degree theory and for the study of the lattice of c.e. sets, the difference between true finiteness and its analogue in the generalised case, namely countability in our case, is fundamental to some constructions and reveals a deep gap between classical computability and attempts to generalise it to the realm of the uncountable.

In this paper, we make a sweeping simplifying assumption about our set-theoretic universe. We suppose Every real is constructible; in other words, L_{ω_1} is the collection of all hereditarily countable sets. The reason for this assumption is to make every subset of ω_1 amenable for L_{ω_1}. Recall that L_{ω_1} is the domain for our model of computation; it will be important for us that for all $A \subseteq L_{\omega_1}$, L_{ω_1} is closed under the function $x \mapsto A \cap x$. Admissible computability would be far more complicated if we had to deal with non-amenable sets. In the non-amenable setting, we would have to make more decisions about what it means for an uncountable structure to be computable. This may be a subject of future research.

We choose to focus on the least uncountable cardinal $\aleph_1$, rather than work with arbitrary uncountable cardinals κ, mostly for simplicity. Most of the theory carries over without changes to any *successor* cardinal. The limit case, in particular the singular case, is more difficult. We remark, however, that some results regarding

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linear orderings do not seem to easily generalise from ω_1 to higher successor cardinals. Another reason to concentrate on ω_1 is that under our assumptions (which imply the continuum hypothesis), it coincides with the size of the continuum, which has natural mathematical interest.

2. Admissible computability on ω_1

In this section we develop admissible computability over ω_1 from the very beginning. Since ω_1 is a regular cardinal, this development is significantly easier than the general treatment over an arbitrary admissible ordinal. For the general theory and historical notes, see [30].

We remark here that various older and more recent results show the robustness of the notion of computability that we are about to define. This notion can be defined not only using definitions by existential formulas, as we do below, but also using inductive schemes such as E-recursion or ω_1-calculability (again see Sacks [30]) or other schemes (Takeuti, Kripke, Machover, and others, see [31]), and generalised Turing machines that have a tape of length ω_1 and run for countably many steps (see [20]).

2.1. Computable sets and functions.

As mentioned above, the "universal domain" for our theory of computability is L_{ω_1}. Computability in this setting is the result of investigating the early layers of definability on this domain.

We work in the language of set theory, which contains equality and one binary relation symbol $\in$. A formula in this language is *bounded*, or Δ_0, if it is built from the atomic formulas using Boolean combinations and the bounded quantifiers $\exists\, x \in y$ and $\forall\, x \in y$. A formula is Σ_1, if it is of the form $\exists\, x_1, \dots, x_n\, \varphi$, where φ is bounded. We remark that in this section, unlike later ones, all of the formulas are finitary.

We sometimes enrich our language by adding one constant for each element of L_{ω_1}. As is often done in model theory, we identify the elements of L_{ω_1} with the constants that they name. We use $\Delta_0(L_{\omega_1})$ to denote the collection of bounded formulas of this enriched language. Of course a formula of this enriched language is bounded if and only if it is of the form $\varphi(\bar{a}, \bar{x})$, where $\varphi(\bar{y}, \bar{x})$ is a bounded formula in the language of pure set theory (with no added constants), and $\bar{a}$ is a tuple of (constants naming) elements of L_{ω_1}. Similarly, we define the collection $\Sigma_1(L_{\omega_1})$ of Σ_1 formulas of the augmented language.

The syntactic notions have semantic counterparts on the intended structure $(L_{\omega_1}, \in)$. An n-ary relation $R \subseteq (L_{\omega_1})^n$ is $\Sigma_1(L_{\omega_1})$ if it is defined by some $\Sigma_1(L_{\omega_1})$ formula. These relations have a familiar name.

Definition 2.1.

- A relation $R \subseteq (L_{\omega_1})^n$ is *computably enumerable*, or *c.e.*, if it is $\Sigma_1(L_{\omega_1})$.
- A relation $R \subseteq (L_{\omega_1})^n$ is *computable* if both R and $(L_{\omega_1})^n \setminus R$ are computably enumerable.
- A partial function $f\colon (L_{\omega_1})^n \to L_{\omega_1}$ is *partial computable* if its graph
$$\{(\bar{a}, f(\bar{a})) \,:\, \bar{a} \in \operatorname{dom} f\}$$
is a computably enumerable relation.
- A *computable* function $f\colon (L_{\omega_1})^n \to L_{\omega_1}$ is a partial computable function whose domain is computable.

Traditionally, the terminology for the notions above is "ω_1-recursively enumerable", "ω_1-recursive", etc. We drop the prefix ω_1 right from the start (and shift to "computable" rather than "recursive"). When we wish to refer to classical computability, we say that we are in the "setting of ω", or the "standard" or "classical" setting.

In the definition above, we referred to relations of any finite arity. This could lead to confusion when we recall that L_{ω_1} is closed under taking ordered n-tuples. There is no confusion because the "tupling" functions are computable: for example, the function $x, y \mapsto \langle x, y\rangle$ and the functions $\langle x, y\rangle \mapsto x$, $\langle x, y\rangle \mapsto y$ are all computable (the set of ordered pairs is computable). Hence a relation $R \subseteq (L_{\omega_1})^2$ is computable (or computably enumerable) if and only if the set

$$\{\langle a, b\rangle \,:\, R(a, b)\}$$

is a computable (computably enumerable) subset of L_{ω_1}. Henceforth, we ignore such subtleties.

Indeed, this can also be applied to countable arities, which we shall discuss later.

Proposition 2.2. *A set $A \subseteq L_{\omega_1}$ is computable iff its characteristic function χ_A is computable.*

Proof. If χ_A is computable, then A is $\Sigma_1(L_{\omega_1})$-definable by the formula $\chi_A(x) = 1$; the complement of A is definable by a similar formula.

For the other direction, suppose that $\psi(x)$ defines A and that $\theta(x)$ defines $L_{\omega_1} \setminus A$, and that both formulas are $\Sigma_1(L_{\omega_1})$. Then $\chi_A(x) = y$ is $\Sigma_1(L_{\omega_1})$-definable by the formula

$$[y = 1 \;\&\; \psi(x)] \vee [y = 0 \;\&\; \theta(x)]\,. \qquad \square$$

Lemma 2.3.

(1) *If $R(\bar{y})$ is a computably enumerable relation, then R is definable by a formula of the form $\exists x\, \varphi(x, \bar{y})$, where φ is $\Delta_0(L_{\omega_1})$.*

(2) *If $R(x, \bar{y})$ is a computably enumerable relation, and Q is a quantifier ($\exists$ or $\forall$), then the relation $Q\, x \in z\, R(x, \bar{y})$ is also computably enumerable.*

Proof. (1): We replace $\exists\, x_1, \ldots, x_n\, \psi(\bar{x}, \bar{y})$ by

$$\exists t\; [\exists x_1, \ldots, x_n \in t\, \psi(\bar{x}, \bar{y})]\,,$$

using the fact that L_{ω_1} is closed under taking finite subsets.

(2): This is immediate for the existential quantifier. For the universal quantifier, $\forall\, x \in z\, \exists w\, \psi(x, w, \bar{y})$ is equivalent to $\exists t \exists w\, \forall x \in z\, \exists w \in t\, \psi(x, w, \bar{y})$, using the fact that each $z \in L_{\omega_1}$ is countable, so picking witnesses for any relation can be done within a countable set. $\square$

The main method of constructing computable functions is by recursion. This is possible by the following.

Proposition 2.4. *Let $I\colon L_{\omega_1} \to L_{\omega_1}$ be a computable function. Then there is a unique computable function $f\colon \omega_1 \to L_{\omega_1}$ such that for all $\alpha < \omega_1$,*

$$f(\alpha) = I(f\restriction_{\alpha}).$$

Proof. Define f according to the inductive formula. The point is that if $\alpha < \omega_1$ and $f\restriction_{\alpha}$ is already defined, then $f\restriction_{\alpha} \in L_{\omega_1}$, and so $f(\alpha) = I(f\restriction_{\alpha})$ is defined. Uniqueness follows by induction as well. The main point is that f is computable: $f(\alpha) = a$ iff there is some $g \in L_{\omega_1}$ whose domain is an ordinal greater than α and that satisfies the inductive formula defining f and such that $g(\alpha) = a$. Unravelling, this is a $\Sigma_1(L_{\omega_1})$ formula that defines the graph of f. □

Defining computable functions by recursion allows us to view them dynamically, as is common in classical computability. Processes of computation, described by $\Delta_0(L_{\omega_1})$ formulas, for instance, and taking only countably many steps, can be used to define computable functions. An intuition for informal definitions of such computable objects develops with experience.

Recall that the constructible universe is globally well-ordered: there is a well-ordering $<_L$ of all constructible sets, such that whenever $\alpha < \beta$, $<_L\restriction_{L_\alpha}$ is an initial segment of $<_L\restriction_{L_\beta}$.

Lemma 2.5.

(1) *The function $\alpha \mapsto L_\alpha$ (for $\alpha < \omega_1$) is computable.*
(2) *The function $\alpha \mapsto <_L\restriction_{L_\alpha}$ (for $\alpha < \omega_1$) is computable.*

Proof. Both are constructed by recursion on α (Proposition 2.4). The corresponding recursive rules I can be written to be Σ_1.

For (1), $I(f)$ is the collection of definable subsets of $f(\alpha)$, if $\operatorname{dom} f = \alpha + 1$ is a successor ordinal; otherwise, $I(f)$ is the union of the range of f. A similar rule defines $<_L\restriction_{L_\alpha}$ given the sequence $\langle <_L\restriction_{L_\beta}\rangle_{\beta<\alpha}$. □

Corollary 2.6.

(1) *$<_L\restriction_{L_{\omega_1}}$ is a computable relation.*
(2) *The map*

$$a \mapsto \{b \in L_{\omega_1} \,:\, b < a\}$$

is computable.
(3) *The map that takes $\alpha < \omega_1$ to the α^{th} element of $<_L$ is computable.*

Proof. (1): Let $a, b \in L_{\omega_1}$. Then $a <_L b$ iff there is some $\alpha < \omega_1$ such that $a, b \in L_\alpha$ and a precedes b in the ordering $<_L\restriction_{L_\alpha}$. This shows that $<_L\restriction_{L_{\omega_1}}$ is a computably enumerable relation. However, it is also a total ordering, so it must be computable. Alternatively, $a <_L b$ iff for every α such that $a, b \in L_\alpha$, a precedes b in the ordering $<_L\restriction_{L_\alpha}$; this shows that $<_L\restriction_{L_{\omega_1}}$ is co-c.e.

(2): x is the set of $<_L$-predecessors of a iff there is (for all) α such that $a \in L_\alpha$ and such that x is the set of $<_L$-predecessors of a which are in L_α. The last part involves only quantification over L_α, i.e., bounded quantification. By Lemma 2.3(2), this defines a Σ_1 relation.

(3): a is the α^{th} element of $<_L$ iff in L_{ω_1}, there is some order-preserving map from α to the set of $<_L$-predecessors of a. □

The map of Corollary 2.6(3) is a computable bijection between ω_1 and L_{ω_1}. Using this map, we may pass without mention between investigating computable and c.e. subsets of ω_1 and subsets of L_{ω_1}. This can be done in general:

Proposition 2.7. *The following are equivalent for a non-empty set $A \subseteq L_{\omega_1}$:*

(1) *A is computably enumerable.*
(2) *A is the range of a computable function $f\colon \omega_1 \to L_{\omega_1}$.*
(3) *A is the domain of a partial computable function.*

Moreover, an uncountable set $A \subseteq L_{\omega_1}$ is c.e. iff it is the range of a total, 1-1 function $f\colon \omega_1 \to L_{\omega_1}$.

Proof. The implications (2)→(1) and (3)→(1) are immediate.

For (1)→(3), let $\exists y\, \varphi(x, y)$ be a $\Sigma_1(L_{\omega_1})$ formula that defines A, where φ is bounded. Let $f(x) = y$ if y is the least, according to $<_L$, such that $\varphi(x, y)$ holds. Then f is partial computable and $\operatorname{dom} f = A$.

For (1)→(2), again let $\exists y\, \varphi(x, y)$ define A, with φ bounded. Suppose first that A is uncountable. By recursion, define a function $f\colon \omega_1 \to L_{\omega_1}$ by letting $f(\alpha) = x$ if there is some y such that:

- $\langle x, y\rangle$ is the $<_L$-least pair such that $\varphi(x, y)$ holds; and
- $x \notin \operatorname{range} f\restriction_\alpha$.

Then f is total and the range of f is A.

If A is countable, but nonempty, we let $g\colon \omega \to A$ be a function whose range is A; $g \in L_{\omega_1}$. Let a be any element of A. We then let $f\colon \omega_1 \to L_{\omega_1}$ be defined as follows:

$$f(\alpha) = \begin{cases} g(\alpha), & \text{if } \alpha < \omega; \\ a, & \text{if } \alpha \geqslant \omega. \end{cases}$$ ☐

2.2. **The universal c.e. set.** In what follows, we regard formulas as mathematical objects, elements of L_{ω_1}.

Lemma 2.8. *The relation*

$$\{(\alpha, \varphi) \,:\, \varphi \text{ is a sentence over } L_\alpha \;\&\; L_\alpha \models \varphi\}$$

is computable.

Proof. This is proved by induction on the complexity of φ. The point is that all quantifiers are bounded, as they range only over L_α. The induction is turned into a computable definition in the style of the proof of Proposition 2.4: $L_\alpha \models \psi(\bar a)$ if there is a sequence of relations on L_α, defined by the subformulas of ψ, and $\bar a$ belongs to the last relation on the list (the one defined by $\psi(\bar x)$). ☐

Corollary 2.9. *The collection of $\Sigma_1(L_{\omega_1})$ sentences that are true in L_{ω_1} is computably enumerable.*

Proof. Let $\varphi(\bar x)$ be a Σ_1 formula and let $\bar a \in L_{\omega_1}$. Then $L_{\omega_1} \models \varphi(\bar a)$ if and only if there is some $\alpha < \omega_1$ such that $\bar a \in L_\alpha$ and $L_\alpha \models \varphi(\bar a)$. The point is that since L_α is transitive, bounded formulas are absolute between L_α and L_{ω_1}. ☐

Corollary 2.9 allows us to define a universal c.e. set. The collection $\mathcal{W}$ of all $\Sigma_1(L_{\omega_1})$ formulas with one free variable is computable. Let $\alpha \mapsto \psi_\alpha$ be a computable bijection between ω_1 and $\mathcal{W}$. For $\alpha < \omega_1$, let W_α be the subset of L_{ω_1} defined by ψ_α. The collection $\{W_\alpha \,:\, \alpha < \omega_1\}$ is the collection of all c.e. sets. Corollary 2.9 ensures that

$$\bigoplus_{\alpha<\omega_1} W_\alpha = \{(\alpha, x) \,:\, x \in W_\alpha\}$$

is a c.e. set. We take this to be the *universal* c.e. set. Of course, it depends on the enumeration $\langle \psi_\alpha \rangle$. Diagonalisation holds:

Proposition 2.10. *The universal c.e. set is not computable.*

Proof. Suppose, for contradiction, that the universal c.e. set $\bigoplus_\alpha W_\alpha$ is computable. Then the diagonal set

$$K = \{\alpha < \omega_1 \,:\, \alpha \in W_\alpha\}$$

is computable, and so its complement $\bar{K}$ is computable as well. Hence, there is some α such that $\bar{K} = W_\alpha$. We get a contradiction by examining whether $\alpha \in K$. □

Similarly, there is an effective enumeration of all partial computable functions, and, hence, there is a universal partial computable function. Let $R(\alpha, x, z)$ be a computable relation such that for all x, $x \in W_\alpha$ iff there is some z such that $R(\alpha, x, z)$. Let $\varphi_\alpha(x) = y$ if there is some z such that $R(\alpha, \langle x, y\rangle, z)$ and such that for no $\langle y', z'\rangle <_L \langle y, z\rangle$ do we have $R(\alpha, \langle x, y'\rangle, z')$. Then $\langle \varphi_\alpha\rangle_{\alpha<\omega_1}$ is an effective enumeration of all partial computable functions: the set $\{(\alpha, x, y) \,:\, \varphi_\alpha(x) = y\}$ is computably enumerable.

From the universal c.e. set, we also obtain a uniform enumeration of all c.e. sets. Let f be a computable, injective function from ω_1 onto the universal c.e. set. For $\alpha, s < \omega_1$, we let $W_{\alpha,s}$ be the collection of $a \in L_{\omega_1}$ such that there is some $t < s$ such that $f(t) = \langle \alpha, a\rangle$. Each $W_{\alpha,s}$ is countable, the function $\alpha, s \mapsto W_{\alpha,s}$ is computable, and for every $\alpha < \omega_1$, the sequence $\langle W_{\alpha,s}\rangle_s$ is increasing and $W_\alpha = \bigcup_{s<\omega_1} W_{\alpha,s}$.

2.3. **Some classical theorems.** The proofs of the following two theorems are again analogous to the classical ones.

Proposition 2.11 (s-m-n theorem)**.** *If $f(x, y)$ is a partial computable function, then there is a (total) computable function g such that for all x, $\varphi_{g(x)} = f(x, -)$.*

Proof. Effectively in x, we find a $\Sigma_1(L_{\omega_1})$ formula θ_x that defines the graph of $f(x, -)$; since $\alpha \mapsto \psi_\alpha$ is a bijection, from θ_x we can effectively find some $\alpha = g(x)$ such that $\theta_x = \psi_\alpha$ and so $\varphi_\alpha = f(x, -)$ □

Theorem 2.12 (Recursion theorem)**.** *If $f\colon \omega_1 \to \omega_1$ is a total computable function, then there is some $\alpha < \omega_1$ such that $\varphi_\alpha = \varphi_{f(\alpha)}$.*

Proof. We solve the equation

$$\varphi_{\varphi_\alpha(\alpha)} = \varphi_{f(\varphi_\alpha(\alpha))}. \tag{1}$$

By Proposition 2.11, there is a computable function g such that for all α,

$$\varphi_{g(\alpha)} = \varphi_{f(\varphi_\alpha(\alpha))}.$$

There is some α^* such that $g = \varphi_{\alpha^*}$. So, α^* solves Equation 1. □

2.4. **Relative computability.** The basic notion of computable enumerability can be relativised to an oracle; equivalently, it can be used to define this relativisation, using functionals.

Definition 2.13. An *enumeration functional* is a c.e. set of pairs (σ, a) where $\sigma \in 2^{<\omega_1}$ and $a \in L_{\omega_1}$.

Here $2^{<\omega_1}$ is the collection of all binary strings of countable length, that is, functions from some countable ordinal to $\{0, 1\}$. As is standard in classical computability, we identify sets with their characteristic functions. We are considering subsets of ω_1. So, for $\sigma \in 2^{<\omega_1}$ and $B \subseteq \omega_1$ we write $\sigma \subset B$ if $\sigma \subset \chi_B$; that is, if for all $\alpha < \operatorname{dom}\sigma$, $x \in B \Leftrightarrow \sigma(x) = 1$.

If Φ is an enumeration functional and $B \subseteq \omega_1$, we let

$$\Phi^B = \{a : \exists \sigma \subset B\ [(\sigma, a) \in \Phi]\}.$$

In another direction, we consider an enrichment of the language of set theory with constants for the elements of L_{ω_1} by one unary predicate symbol. In this language, too, we define the collection of bounded and existential formulas. For a given $B \subseteq \omega_1$, we say that a subset of L_{ω_1} is $\Sigma_1(L_{\omega_1}, B)$ if it is defined over the structure $(L_{\omega_1}, \in, B)$ (again with the constants interpreted by themselves) by a Σ_1 formula.

Proposition 2.14. *Let $B \subseteq \omega_1$. The following are equivalent for $A \subseteq L_{\omega_1}$:*

(1) *There is some enumeration functional Φ such that $A = \Phi^B$.*
(2) *A is $\Sigma_1(L_{\omega_1}, B)$.*

Such a set A is called *c.e. in* (or *relative to*) B.

Proof. The collection of countable initial segments of B is $\Sigma_1(L_{\omega_1}, B)$ (in fact it is $\Delta_0(L_{\omega_1}, B)$), as it is defined by bounded universal quantification on the domain of the binary string. Hence if Φ is an enumeration functional, then the relation

$$\exists \sigma \subset B\ (\sigma, a) \in \Phi$$

is $\Sigma_1(L_{\omega_1}, B)$, which shows that Φ^B is $\Sigma_1(L_{\omega_1}, B)$.

In the other direction, let $\varphi(x, \bar{b})$ be an existential formula in the enriched language with a unary predicate. Let Φ be the collection of pairs (σ, a) such that for $\alpha = \operatorname{dom} \sigma$, $a, \bar{b} \in L_\alpha$ and $(L_\alpha, \in, \sigma) \models \varphi(a, \bar{b})$ (where by σ we mean the subset of α whose characteristic function is σ). Then Φ is an enumeration functional, and Φ^B is the subset of L_{ω_1} defined by $\varphi(x, \bar{b})$ over $(L_{\omega_1}, \in, B)$. $\square$

A set is B-computable if it is both c.e. and co-c.e. relative to B. A partial function is *B-partial computable* if its graph is c.e. in B, and B-computable if also its domain is B-computable. The relativisation of Proposition 2.7 holds: for any set $B \subseteq \omega_1$, a set $A \subseteq L_{\omega_1}$ is c.e. in B if and only if it is the domain of a B-partial computable function if and only if it is empty or the range of a B-computable function.

The results of Subsection 2.2 also relativise: for any $B \subseteq \omega_1$, there is a universal B-c.e. set and universal B-partial computable function. This is because there is an effective enumeration $\langle \Phi_\alpha \rangle$ of enumeration functionals, and letting $W_\alpha^B = \Phi_\alpha^B$, the universal set $\bigoplus_\alpha W_\alpha^B$ is c.e. in B. Equivalently, there is an effective enumeration of all existential formulas in the enriched language with a unary predicate, and the satisfaction relation for these formulas is c.e. in the oracle B. The universal B-c.e. set is often denoted B'.

The universal B-c.e. set also gives us a uniform B-computable enumeration of all B-c.e. sets. This, in fact, can be also done uniformly in B. For a given enumeration functional Φ and a binary string σ, let

$$\Phi^\sigma = \{a : \exists \sigma' \subseteq \sigma\ [(\sigma', a) \in \Phi_{\operatorname{dom} \sigma}]\},$$

where $\langle \Phi_s \rangle_{s<\omega_1}$ is an effective enumeration of Φ. For all σ, Φ^σ is countable, and the function $\sigma \mapsto \Phi^\sigma$ is computable (indeed, the function $\sigma, \alpha \mapsto W_\alpha^\sigma$ is computable). If $\sigma \subset s'$ then $\Phi^\sigma \subseteq \Phi^{\sigma'}$, and for all $B \subseteq \omega_1$, $\Phi^B = \bigcup_{\sigma \subset B} \Phi^\sigma$.

Relative computability can also be given by functionals.

Definition 2.15. A *Turing functional* is a c.e. set Φ of pairs of binary strings of countable length, which is *consistent* in the sense that if $(\sigma,\tau),(\sigma',\tau') \in \Phi$ and $\sigma \subseteq \sigma'$ then τ and τ' are compatible, that is, $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$.

If Φ is a Turing functional and $B \subseteq \omega_1$, then we let

$$\Phi^B = \bigcup \{\tau \,:\, \exists \sigma \subset B \; [(\sigma,\tau) \in \Phi]\}\,.$$

For all B, $\Phi^B \in 2^{\leqslant \omega_1}$; we say that Φ^B is *total* if $\Phi^B \in 2^{\omega_1}$.

Proposition 2.16. *Let $A, B \subseteq \omega_1$. Then A is computable in B if and only if there is a Turing functional Φ such that $A = \Phi^B$.*

Proof. Suppose that $A = \Phi^B$ for some Turing functional Φ. Then $a \in A$ if and only if there is some $\sigma \subset B$ and some τ such that $\tau(a) = 1$ and $(\sigma,\tau) \in \Phi$. This is a $\Sigma_1(L_{\omega_1}, B)$ definition of A. By replacing 1 by 0, we get a definition of $\bar{A}$.

For the other direction, we use the regularity of ω_1. Suppose that A is B-computable. Let φ and ψ be existential formulas that define A and $\bar{A}$, respectively, over $(L_{\omega_1}, \in, B)$. Let Φ be the collection of pairs (σ,τ) of countable binary strings such that for $\alpha = \operatorname{dom}\sigma$, L_α contains the parameters of φ and ψ, $\alpha > \operatorname{dom}\tau$, and for all $a < \operatorname{dom}\tau$,

- If $\tau(a) = 1$, then $(L_\alpha, \in, \sigma) \models \varphi(a)$ & $\neg\psi(a)$;
- If $\tau(a) = 0$, then $(L_\alpha, \in, \sigma) \models \neg\varphi(a)$ & $\psi(a)$.

Then Φ is a Turing functional. The main point is that $\Phi^B = A$ because countably much information about A is decided by a countable level of $(L_{\omega_1}, \in, B)$. □

We write $A \leqslant_{\mathrm{T}} B$ if A is computable from B.

Proposition 2.17. *The relation $\leqslant_{\mathrm{T}}$ is reflexive and transitive.*

Proof. Reflexivity is immediate. Suppose that $A \leqslant_{\mathrm{T}} B \leqslant_{\mathrm{T}} C$; let Φ and Ψ be Turing functionals such that $A = \Phi^B$ and $B = \Psi^C$. Let Θ be the collection of pairs (σ,ρ) of countable binary strings such that there are strings σ', ρ', τ and τ' with $\sigma \subseteq \sigma'$, $\rho' \subseteq \rho$, $\tau' \subseteq \tau$ and $(\rho',\tau) \in \Psi$, $(\tau',\sigma') \in \Phi$. Then Θ is a Turing functional, and $\Theta^C = A$. □

We write $A \equiv_{\mathrm{T}} B$ if $A \leqslant_{\mathrm{T}} B$ and $B \leqslant_{\mathrm{T}} A$. By Proposition 2.17, this is an equivalence relation on the subsets of ω_1. The equivalence classes are called the (ω_1)-Turing degrees. The relation $\leqslant_{\mathrm{T}}$ induces a partial ordering on the Turing degrees. The map $B \mapsto B'$ is degree-invariant and induces an order-preserving, increasing map on the degrees.

The jump function can be iterated, giving an analogue of the arithmetical hierarchy along all countable ordinals. Details can be found in [5].

3. Computable structures

In classical computability, a structure with a computable domain is identified with its atomic diagram. We can use the same definition for ω_1. It turns out that for some of our applications, the correct generalisation of the concept of a (model theoretic) structure should allow countably infinite arities.

3.1. **Definitions.**

Definition 3.1. A *signature* (or *language*) is a collection of function and relation symbols, together with an arity function, which associates with every symbol a countable ordinal.

A constant symbol is a function symbol with arity 0; a proposition is a predicate symbol of arity 0. From now on, we assume that all signatures are computable. This means that the set of symbols is computable, and the function mapping a symbol to its arity is computable. We call a signature *finitary* if every relation and function symbol of the signature has finite arity.

Definition 3.2. A *structure* $\mathcal{M}$ for a signature $\mathcal{L}$ consists of a non-empty set M and an interpretation of the symbols of $\mathcal{L}$:

- for every α-ary relation symbol R, a set $R^{\mathcal{M}} \subseteq M^\alpha$;
- for every α-ary function symbol f, a function $f^{\mathcal{M}} \colon M^\alpha \to M$.

Here M^α is the collection of all sequences of element of M of length α. The key point regarding infinite arities, is that if $M \subseteq L_{\omega_1}$, then for all α, $M^\alpha \subset L_{\omega_1}$, and in fact, if M is computable, then M^α is computable, uniformly in α.

3.1.1. *The degree of a structure.* Let $\mathcal{M}$ be a structure for a computable signature $\mathcal{L}$, and suppose that the universe M of $\mathcal{M}$ is a subset of L_{ω_1}. To give $\mathcal{M}$ a Turing degree, we identify $\mathcal{M}$ with the 'infinite join" of the relations and functions of $\mathcal{M}$. Let $\tilde{\mathcal{M}}$ consist of:

- the pairs $(R, \bar{a})$ where R is an α-ary relation symbol of $\mathcal{L}$ and $\bar{a} \in R^{\mathcal{M}}$;
- the triples $(f, \bar{a}, b)$ where f is an α-ary relation symbol of $\mathcal{L}$, $\bar{a} \in M^\alpha$ and $f^{\mathcal{M}}(\bar{a}) = b$.

Then $\tilde{\mathcal{M}} \subset L_{\omega_1}$, and so has a Turing degree. We often identify $\mathcal{M}$ with $\tilde{\mathcal{M}}$, and so we write $\deg_T(\mathcal{M})$ to denote the Turing degree of $\tilde{\mathcal{M}}$. We remark that $\tilde{\mathcal{M}}$ is Turing equivalent to the atomic diagram $D(\mathcal{M})$ of $\mathcal{M}$, which we define in Section 5. If $\mathcal{L}$ is finitary, then our definition of $D(\mathcal{M})$ agrees with the standard definition of the atomic diagram in elementary first-order logic, and the Turing equivalence of $\tilde{M}$ and $D(\mathcal{M})$ is immediate.

We assume that every language contains the binary relation symbol $=$, which is always interpreted as true equality. It follows that the domain M of a structure $\mathcal{M}$ is computable from $\deg_T(\mathcal{M})$, as $a \in M$ if and only if $(=, (a, a)) \in \tilde{\mathcal{M}}$, if and only if the sentence $a = a$ is in $D(\mathcal{M})$.

3.2. **Examples.**

We give some natural examples of computable structures, all of finitary signature.

Proposition 3.3. *There is a computable copy of the ordered real field* $(\mathbb{R}; +, \cdot, <, 0, 1)$.

Proof. Any standard set-theoretic construction of the real numbers will do. We use, for example, Dedekind cuts. We fix a copy of the rational numbers $(\mathbb{Q}; +, \cdot, <, 0, 1)$ (this is a countable object, hence an element of L_{ω_1}). Recall that a Dedekind cut is a nonempty initial segment of $\mathbb{Q}$ with no greatest element. Writing the definition in the language of set theory, we see that all quantification ranges over $\mathbb{Q}$ and the potential cut, and so is bounded: an element A of L_{ω_1} is a Dedekind cut if

$$\begin{aligned} \forall x \in A\,(x \in \mathbb{Q}) \quad & \& \quad \exists x \in \mathbb{Q}\,(x \in A) \;\; \& \;\; \exists x \in \mathbb{Q}\,(x \notin A) \\ & \& \quad \forall x \in A\, \forall y \in \mathbb{Q}\,(y < x \to y \in A) \\ & \& \quad \forall x \in A\, \exists y \in A\,(x < y). \end{aligned}$$

It follows that the collection of all Dedekind cuts is computable. We let $\mathbb{R}$ be the collection of all Dedekind cuts. The ordering on $\mathbb{R}$ is set containment: $A \leqslant B$ if $\forall x \in A\,(x \in B)$, which is a computable relation. Addition is also computable: $C = A + B$ if

$$\forall x \in \mathbb{Q}\,(x \in C \;\Leftrightarrow\; \exists y \in A\, \exists z \in B\,(x = y + z)).$$

Similarly, multiplication on $\mathbb{R}$ (defined in the standard way on cuts) is computable. □

Note that the standard embedding of the rationals into the reals is also computable.

Proposition 3.4. *There is a computable copy of the complex field* $(\mathbb{C}; +, \cdot, 0, 1)$.

Proof. The algebraic construction works. Let $(\mathbb{R}; +, \cdot, 0, 1)$ be a computable copy of the real field. Let $\mathbb{C} = \mathbb{R}^2$; this is a computable subset of L_{ω_1}. The standard formulas for the operations of addition and multiplication in terms of the real and imaginary parts show that these operations are computable. □

Fix a computable copy $(\mathbb{R}; +, \cdot, <, 0, 1)$ of the ordered real field.

Lemma 3.5. *Every continuous function* $f\colon \mathbb{R} \to \mathbb{R}$ *is computable.*

By cardinality considerations, most functions from $\mathbb{R}$ to $\mathbb{R}$ are not computable.

Proof. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. $f\restriction_{\mathbb{Q}}$ is countable, and so an element of L_{ω_1}; so we can use it as a parameter in the following computable definition of f: for $a, b \in \mathbb{R}$, $f(a) = b$ if and only if

$$\begin{aligned} \forall \epsilon \in \mathbb{Q}\; \exists \delta \in \mathbb{Q}\; \big[\epsilon \leqslant 0 \quad \vee \quad & \big(\delta > 0 \;\&\; \forall x \in \mathbb{Q} \\ & (a - \delta < x < a + \delta) \to (b - \epsilon < f\restriction_{\mathbb{Q}}(x) < b + \epsilon)\big)\big]. \end{aligned}$$

□

It follows that every total analytic function on $\mathbb{R}$ is computable. This fact has a uniform proof.

Lemma 3.6.

(1) *The collection of triples* $(\langle a_0, a_1, \ldots\rangle, b, c)$ *such that* $\sum a_n b^n = c$ *is computable.*
(2) *The set of sequences* $\langle a_0, a_1, \ldots\rangle$ *such that* $\sum_n a_n x^n$ *defines a total analytic function on* $\mathbb{R}$ *is computable.*

Proof. For (1), we have $\sum a_n b^n = c$ if and only if for all rational $\epsilon > 0$ there is some natural number N such that for all natural numbers $m > N$ we have $|c - \sum_{n \leqslant m} a_n b^n| < \epsilon$. As this quantification ranges over the countable sets $\mathbb{N}$ and $\mathbb{Q}$, this is a computable definition.

Similarly, the collection of pairs $(\langle a_0, a_1, \ldots\rangle, b)$ such that $\sum a_n b^n$ converges to a finite value is also computable; we say that $\left\langle \sum_{n \leqslant m} a_n b^n \right\rangle_{m<\omega}$ is a Cauchy sequence.

(2) follows from the root test, or directly from the previous sentence: a power series converges everywhere if and only if it converges on all integers. □

Corollary 3.7. *There is a computable expansion of the real field consisting of all total analytic functions.*

3.3. **A basic result of computable model theory.** Several basic results of classical computable model theory transfer to ω_1, often with similar or simplified proofs. For example, Morley and Millar ([26], [23]) showed that a countable, complete, decidable theory T has a decidable saturated model if and only if there is a computable enumeration of the complete types consistent with T.

In this subsection, we restrict ourselves to elementary (first-order) theories in countable languages. That is, the languages we deal with are finitary, and all formulas are finitary. We call a structure for a finitary language *decidable* if its elementary diagram is computable.

Proposition 3.8. *Every countable elementary theory with infinite models has a decidable saturated model of size* $\aleph_1$.

Proof. There are two main points:

(1) There is a computable function, which, given an elementary theory in a countable language, produces a complete extension of the theory in the same language.
(2) For any countable elementary theory T, there is an effective listing of all complete types consistent with T (uniformly in T).

To see (1), we first note that the function that takes a countable signature $\mathcal{L}$ and produces the collection of all $\mathcal{L}$-sentences is computable; we state the existence of a countable sequence that gives the recursive construction of all $\mathcal{L}$-formulas. Hence the set of pairs $(T, \mathcal{L})$ such that T is a complete $\mathcal{L}$-theory is computable. Now to get a completion for a given theory, we merely output the $<_L$-least complete extension. The argument for (2) is similar.

Let T_0 be a countable elementary theory with infinite models. Without loss of generality, we suppose that T_0 is complete, so every model of T_0 is infinite. Let $\langle c_\alpha\rangle_{\alpha<\omega_1}$ be a computable enumeration of an uncountable collection of new constants. For $\beta \leqslant \omega_1$, let $C_\beta = \{c_\alpha : \alpha < \beta\}$. By effective recursion, we define an increasing sequence $\langle T_\alpha\rangle_{\alpha<\omega_1}$ of countable elementary first order theories, each complete for the language $\mathcal{L} \cup C_\alpha$, where $\mathcal{L}$ is the language of T_0.

At stage $\alpha+1$, we are given T_α. We let $\left\langle p^\alpha_\beta \right\rangle_{\beta<\omega_1}$ be a computable enumeration of all complete 1-types consistent with T_α. Let (β, γ) be the least pair of ordinals $\beta, \gamma \leqslant \alpha$ which has not been dealt with in a previous stage. Since T_γ is the restriction of T_α to its language $\mathcal{L} \cup C_\gamma$, the type p^γ_β is consistent with T_α. Since c_α is not a symbol of the language $\mathcal{L} \cap C_\alpha$ of T_α, the set

$$T_\alpha \cup p^\gamma_\beta(c_\alpha)$$

is consistent. We let $T_{\alpha+1}$ be a completion of $T_\alpha \cup p^\gamma_\beta(c_\alpha)$, in the language $\mathcal{L} \cup C_{\alpha+1}$.

At limit stages α, we let $T_\alpha = \bigcup_{\gamma<\alpha} T_\alpha$.

Now $T_{\omega_1} = \cup_{\alpha<\omega_1} T_\alpha$ is a computable and complete $\mathcal{L} \cup C_{\omega_1}$ theory extending T_0. As in the standard Henkin construction, we derive a structure $\mathcal{M}$ from T_{ω_1}. The universe M of $\mathcal{M}$ is the collection of $<_L$-least elements of the equivalence classes of C_{ω_1} under the equivalence relation $T_{\omega_1} \vdash c = d$; M is computable since T_{ω_1} is computable. The interpretation of the symbols of $\mathcal{L}$ which defines $\mathcal{M}$ is straightforward; for example, for any relation symbol R of $\mathcal{L}$, for any $\bar{c}$ in M, $\bar{c} \in R^{\mathcal{M}}$ if and only if $T_{\omega_1} \vdash R(\bar{c})$. Since T_{ω_1} is computable, so is $\mathcal{M}$.

By construction, for any $\alpha < \omega_1$, for any 1-type p of $\mathcal{L} \cup C_\alpha$ consistent with T_α, there is some $c \in M$ such that $T_{\omega_1} \vdash p(c)$. Let $\varphi(x)$ be an $\mathcal{L} \cup C_{\omega_1}$-formula such that $T_{\omega_1} \vdash \exists x\, \varphi$. Since φ is finite, there is some $\alpha < \omega_1$ such that φ is an $\mathcal{L} \cup C_\alpha$-formula. Since T_α is the restriction of T_{ω_1} to $\mathcal{L} \cup C_\alpha$, we have $T_\alpha \vdash \exists x\, \varphi$. It follows that there is some $\beta < \omega_1$ such that $\varphi(x) \in p^\alpha_\beta$. There is some $\delta > \alpha$ such that $\varphi(c_\delta) \in T_{\delta+1}$, and so there is some $c \in C_{\omega_1}$ such that $\varphi(c) \in T_{\omega_1}$. Thus, T_{ω_1} has the witness property, which implies that for all $\bar{c} \in M$ and all $\mathcal{L}$-formulas φ, $\mathcal{M} \models \varphi(\bar{c})$ if and only if $T_{\omega_1} \vdash \varphi(\bar{c})$. It follows that $\mathcal{M} \models T_0$. Hence, $\mathcal{M}$ is infinite. It also follows that $\mathcal{M}$ is decidable, again because T_{ω_1} is computable.

Now a similar argument shows that $\mathcal{M}$ is $\aleph_1$-saturated (and so that $\mathcal{M}$ is uncountable). Let A be a countable subset of M, and let p be a 1-type over A consistent with the theory of $(\mathcal{M}, A)$. There is some α such that $A \subset C_\alpha$. There is some 1-type q of $\mathcal{L} \cup C_\alpha$, consistent with T_α, extending p. The construction of T_{ω_1} and the argument above show that q, and so p, is realised in $\mathcal{M}$. □

Corollary 3.9. *Let T be a countable, uncountably categorical elementary theory. The unique model of T of size $\aleph_1$ has a computable copy.*

4. Elementary effective algebra

We give examples of analogues of some results from classical effective algebra, some of which require new approaches.

4.1. **Vector spaces.** In the standard setting, Metakides and Nerode [22] showed that there is a computable infinite-dimensional $\mathbb{Q}$-vector space with no infinite c.e. linearly independent set. By contrast, for any finite field F, any computable F-vector space has a computable basis. In the setting of ω_1, we can generalise these results and obtain the following.

Proposition 4.1. *For any countable field F, every computable F-vector space has a computable basis.*

Theorem 4.2. *There is computable $\mathbb{R}$-vector space, of uncountable dimension, with no uncountable c.e. independent set.*

Proof of Proposition 4.1. Let F be a countable field, and let V be a computable F-vector space. The collection of countable subsets of V that are linearly independent is computable: formally, if we write down the property of being independent, we see that we only need quantify over the countable field F, and so we get a (finitary) $\Delta_1(L_{\omega_1})$ property in the language of set theory defining the collection of countable independent subsets of V. Informally, a countable tuple is independent if satisfies a countable conjunction of computable statements saying that the non-trivial linear combinations do not result in 0_V, and the countable conjunction of computable statements is also computable. We can therefore build a computable basis for V by effective recursion. □

In the rest of this Subsection, we sketch a proof of Theorem 4.2. The proof of Metakides and Nerode's can be adjusted to the setting of ω_1. We present a generalisation of Ash's proof of the same result. In the uncountable, the algebraic aspects of the proof are cleaner. This is because the rationals have no finite subfields, but the reals have countable subfields.

To construct a computable $\mathbb{R}$-vector space of uncountable dimension, with no uncountable c.e. independent set, we start with a canonical presentation V of $\mathbb{R}^{\omega_1}$, by specifying a computable uncountable set B to be the basis of V, and letting V be the collection of formal linear combinations of the elements of B over $\mathbb{R}$. By design, the collection of countable, linearly independent subsets of V is computable.

We construct an isomorphic copy $\hat{V}$ of V by "twisting" V to avoid uncountable independent c.e. subsets. We show that this twisting can be performed without affecting atomic statements about $\hat{V}$ to which we are already committed, thus making $\hat{V}$ computable.

We recursively define an increasing sequence $\langle F_s\rangle_{s<\omega_1}$ of countable subfields of $\mathbb{R}$ (at limit stages we take unions) whose union is $\mathbb{R}$. If F is a subfield of a field G, and U is a vector space over G, then we denote by $U\restriction_F$ the reduct of U to an F-vector space. We construct a (not necessarily increasing) sequence $\langle U_s\rangle$ of countable subsets of V such that for all s, U_s is an F_s-vector space, a subspace of $V\restriction_{F_s}$.

To define $\hat{V}$, we define an increasing sequence of countable sets $\left\langle \hat{U}_s \right\rangle$ and bijections $p_s\colon U_s \to \hat{U}_s$. The map p_s endows $\hat{U}_s$ with an F_s-vector space structure, by making p_s an isomorphism between U_s and $\hat{U}_s$. The point now is that even though the sequence $\langle p_s\rangle$ is not increasing, we will ensure that for all $s<t$, U_s is a subspace of $U_t\restriction_{F_s}$. This would ensure that $\hat{V} = \bigcup_s \hat{U}_s$ is a computable $\mathbb{R}$-vector space.

To ensure that $\hat{V}$ does not contain uncountable c.e. independent sets, we meet the following requirements:

R_α**:** If W_α is uncountable, then it is not a linearly independent subset of $\hat{V}$.

That $\hat{V}$'s dimension is $\aleph_1$ is ensured by the fact that it is isomorphic to V by the Δ^0_2 map $\lim_s p_s$. To ensure that this limit exists and is an isomorphism, we meet the following requirements:

N_α**:** There is some stage s such that for all $t \geqslant s$, $B\cap\alpha \subset U_t$, and $p_t\restriction_{B\cap\alpha} = p_s\restriction_{B\cap\alpha}$.

The construction is a finite injury priority argument, where the requirements are ordered in order-type ω_1. The algebraic content of the proof is the following.

Lemma 4.3. *Let F be a proper subfield of $\mathbb{R}$, and let U be a subspace of $V\restriction_F$. Let $u_0, u_1 \in U$ be F-independent. Then there is an F-linear map $f\colon U \to V\restriction_F$ such that $f(u_0)$ and $f(u_1)$ are not $\mathbb{R}$-independent in V.*

Proof. Let $X \subseteq U$ be the F-span of $\{u_0, u_1\}$, and let Y be a linear complement of X in U, so $U = X \oplus Y$. Let $v \in V$ be any vector which is not in Y, and let $Z \subset V$ be the $\mathbb{R}$-span of $\{v\}$. Since F is a proper subfield of $\mathbb{R}$, the F-vector space $Z\restriction_F$ has dimension greater than 1, so we can find $v_0, v_1 \in Z$ which are F-independent;

so the F-span of $\{v_0, v_1\}$ is isomorphic to X. We then let f be the F-linear map determined by $f\restriction_Y = \mathrm{id}_Y$ and $f(u_i) = v_i$ for $i = 0, 1$. $\square$

At stage s of the construction, every requirement R_α is provided with a restraint $p_s(\alpha)$ which is the restriction of p_s to some F_s-linear subspace $U_s(\alpha)$ of U_s. A requirement P_α requires attention at stage s if $W_{\alpha,s} \cap \hat{U}_s$ is F_s-independent, and moreover contains a pair $\{u_0, u_1\}$ which is independent over $\hat{U}_s(\alpha) = \operatorname{range} p_s(\alpha)$. Suppose that R_α receives attention at stage s (which happens if it is the strongest requirement that requires attention at stage s). Since $\{u_0, u_1\}$ is independent over $\hat{U}_s(\alpha)$, there is a complement X of $U_s(\alpha)$ in U_s that contains $p_s^{-1}(u_0)$ and $p_s^{-1}(u_1)$. By Lemma 4.3, we find some Y, a subspace of $V\restriction_{F_s}$, disjoint from $U_s(\alpha)$, and an F_s-isomorphism $f\colon Y \to X$ such that $(f \circ p_s)^{-1}(u_0)$ and $(f \circ p_s)^{-1}(u_1)$ are not $\mathbb{R}$-independent in V. We then let $q = f \oplus \mathrm{id}_{U_s(\alpha)}$; q is an isomorphism from $U_s(\alpha) \oplus Y$ to $\hat{U}_s$ such that $q^{-1}(u_0)$ and $q^{-1}(u_1)$ are not $\mathbb{R}$-independent in V. Let F_{s+1} be a countable subfield of $\mathbb{R}$ containing F_s such that $q^{-1}(u_0)$ and $q^{-1}(u_1)$ are not F_{s+1}-independent. Let U_{s+1} be the F_{s+1}-span of $U_s(\alpha) \oplus Y$ in V, let $\hat{U}_{s+1}$ be a countable set containing $\hat{U}_s$, and let p_{s+1} be a bijection from U_{s+1} to $\hat{U}_{s+1}$ extending q; as described above, $\hat{U}_{s+1}$ is now equipped with an F_{s+1}-vector space structure that makes p_{s+1} an isomorphism. Since q is an F_s-isomorphism, U_s is a subspace of $U_{s+1}\restriction_{F_s}$. Note that $p_s(\alpha) \subset p_{s+1}$, so the restraint on R_α was respected; and that u_0 and u_1 are not independent in $\hat{U}_{s+1}$, so R_α is met.

A requirement N_α requires attention at stage s if $B \cap \alpha$ is not a subset of U_s. If N_α receives attention, then we let $F_{s+1} = F_s$ and let U_{s+1} be the F_s-span of $U_s \cup (B \cap \alpha)$, and let p_{s+1} be an extension of p_s; the requirement N_α then imposes the restraint $p_{s+1} \subset p_t(\beta)$ for all $\beta > \alpha$.

Special care is required at limit stages s. Since the sequences $\langle F_t \rangle$ and $\left\langle \hat{U}_t \right\rangle$ are increasing, we can take unions at s and obtain F_s and $\hat{U}_s$. We then need to define p_s and U_s; we limit the damage to requirements N_α to a minimum. Let A_s be the collection of all ordinals $\alpha < s$ such that there is some $t_\alpha < s$ such that requirement N_α is not injured at any stage $t \in [t_\alpha, s)$. For any $\alpha \in A_s$, for all $t \in [t_\alpha, s)$, we have $p_t\restriction_{B\cap\alpha} = p_{t_\alpha}\restriction_{B\cap\alpha}$, which is a map from $B \cap \alpha$ to an independent subset of $\hat{U}_t$. It follows that the function

$$\bigcup_{\alpha \in A_s} p_{t_\alpha}\restriction_{B\cap\alpha}$$

is a well-defined function from $B \cap (\sup A_s)$ to an independent subset of $\hat{U}_s$. We let p_s be any linear map extending this function. This definition ensures that no N_α for $\alpha \in A_s$ is injured at stage s. This completes our sketch of the proof of Theorem 4.2.

4.2. **Fields.** In the classical setting, any finite extension of a computable field has a computable copy. We have the following analogous result.

Proposition 4.4. *Every countable extension of a computable field has a computable copy.*

Proof. We start by noting that if F is a computable field and $G = F(a)$ is a one-element extension of F, then G has a computable copy; in fact, if the universe of F is co-uncountable, then there is some computable $G' \supset F$ and $a' \in G'$ such that $\mathrm{id}_F \cup \{a' \mapsto a\}$ determines an isomorphism from G' to G (so $G' = F(a')$). This is done in cases: if a is transcendental over F, then we take $G' = F(x)$ be the field of rational functions over F; if a is algebraic over F, with f being its minimal polynomial, then we let $G' = F(x)/(f)$, which has a computable copy into which F canonically embeds.

Now suppose that F is a computable field and that G is a countable extension of F; that is, $G = F(A)$ for some countable set $A \subset G$. Let $\langle a_n \rangle_{n<\omega}$ be an enumeration of A, and for $n < \omega$, let $G_n = F(a_0, a_1, \dots, a_{n-1})$. We may assume that the universe of F is co-uncountable. By recursion on $n < \omega$ we define computable fields G'_n, whose universes are co-uncountable, and isomorphisms $\varphi_n \colon G'_n \to G_n$. We start with $G'_0 = G_0 = F$, and $\varphi_0 = \mathrm{id}_F$. Given G'_n and φ_n, we know that $G_{n+1} = G_n(a_n)$ and so by the previous paragraph, we may find a computable $G'_{n+1} = G'_n(a'_n)$ and an extension φ_{n+1} of φ_n which is an isomorphism from G'_{n+1} to G_{n+1}, determined by mapping a'_n to a_n.

Since $\omega < \omega_1$, not only is each G'_n computable, but in fact they are uniformly computable. It follows that $G' = \bigcup_n G'_n$ is computable. The map $\bigcup_n \varphi_n$ shows that G' is isomorphic to G. □

Van der Waerden [33] gave the idea for producing a computable field without a splitting algorithm; this was fleshed out by Fröhlich and Shepherdson [12]: to the field of rationals we add a $(p_n)^{\text{th}}$ root of unity for $n \in \emptyset'$, where $\langle p_n \rangle$ is an enumeration of all prime numbers. This field has the feature that no computable copy has a splitting algorithm. For more on the early results of effective algebra and computable model theory, see Miller [25]. The idea of van der Waerden, and Fröhlich and Shepherdson, cannot be immediately applied in the setting of ω_1; there are no new prime numbers. Nevertheless, the analogous results hold.

Proposition 4.5. *There is a computable field F with no splitting algorithm: the collection of irreducible polynomials in $F[x]$ is not computable.*

We need the following:

Lemma 4.6. *Let $(\mathbb{C}; +, \cdot, 0, 1)$ be a computable copy of the complex field. There is an uncountable computable subset of $\mathbb{C}$ which is algebraically independent.*

In fact, we can obtain an algebraic basis for $\mathbb{C}$ over $\mathbb{Q}$.

Proof. The relation of algebraic independence in $\mathbb{C}$, namely the collection of pairs (A, b) such that $A \subset \mathbb{C}$ is computable and b belongs to the algebraic closure of A in $\mathbb{C}$, is computable, since it requires quantification over only countably many polynomials over A. A computable algebraic basis for $\mathbb{C}$ over $\mathbb{Q}$ can be then defined recursively, at each stage choosing, for the next element, the $<_L$-least element independent from the elements chosen so far. □

Lemma 4.7. *Let F be a field. The collection of elements $a \in F$ such that F contains a square root of a is computable from the set of irreducible polynomials in $F[x]$.*

Proof. F contains a square root of a if and only if the polynomial $x^2 - a$ is reducible in $F[x]$. □

Proof of Proposition 4.5. By Lemma 4.6, let $\{a_\alpha : \alpha < \omega_1\}$ be a computable, algebraically independent subset of $\mathbb{C}$; let

$$F_0 = \mathbb{Q}(a_\alpha)_{\alpha<\omega_1}.$$

It is not difficult to see that F_0 is computable. Now we let

$$F = F_0(\sqrt{a_\alpha})_{\alpha \in \emptyset'}.$$

It is not hard to show that a_α has a square root in F if and only if $\alpha \in \emptyset'$; by Lemma 4.7, the collection of irreducible polynomials in $F[x]$ computes $\emptyset'$.

The field F is not computable (as a subfield of $\mathbb{C}$); the technique of the proof of Proposition 4.4 shows that F has a computable copy which extends F_0, so that the mapping $\alpha \mapsto a_\alpha$ remains computable as a function from ω_1 to F. □

Unlike the Fröhlich-Shepherdson field, the field F of the previous proof has a computable copy with a computable splitting algorithm. Using more robust coding, we get the full result.

Theorem 4.8 (with Hirschfeldt and Montalbán). *There is a computable field F (of characteristic 0) such that for any computable field K isomorphic to F, the collection of irreducible polynomials in $K[x]$ computes $\emptyset'$.*

The reduction of $\emptyset'$ to the collection of irreducible polynomials in $K[x]$ is uniform in K.

The proof of Theorem 4.8 uses a technique of coding of graphs into fields which is due to Friedman and Stanley [10], extending results of [3]. In this context, a (simple, undirected) *graph* is a structure for a language with a single binary relation symbol which is interpreted by a symmetric, irreflexive relation (rather than being a two-sorted structure containing vertices and edges).

Definition 4.9. A graph $G = (V, E)$ is *c.e.* if V is a c.e. set and E is a c.e. set of pairs.

Theorem 4.8 follows from the following two propositions:

Proposition 4.10. *There is a c.e. graph, every copy of which computes $\emptyset'$.*

Proposition 4.11. *For any c.e. graph G there is a field F with a computable copy such that if K is isomorphic to F, then the collection of irreducible polynomials in $K[x]$ computes a copy of G.*

Proof of Proposition 4.10. By taking a computable bijection between ω_1 and the power set of ω, we may assume that $\emptyset'$ is a set of subsets of ω. The graph G is a disjoint union of "daisy graphs". Fix $A \subseteq \omega$. The graph G_A consists of a vertex c_A; for every $n \in A$, a cycle of length $2n + 4$ starting and ending in c_A; for every $n \notin A$, a cycle of length $2n + 5$ starting and ending in c_A; two extra vertices a_A and b_A, both connected by an edge to c_A; and if $A \in \emptyset'$, we connect a_A and b_A by an edge. The graph G is the disjoint union of G_A for all $A \subseteq \omega$.

It is clear that G has a c.e. copy. Suppose that $H \cong G$; let $f \colon G \to H$ be an isomorphism. Let V_A be the collection of vertices of G_A; it is uniformly computable in A. The map $A \mapsto f\restriction_{V_A}$ is H-computable. To see this, let $\tilde{G}_A$ be G_A, except that there is no edge between a_A and b_A. Thus $\tilde{G}_A$ is uniformly computable in A. For any function $g \in L_{\omega_1}$, $g = f\restriction_{V_A}$ if and only if g is an isomorphism between $\tilde{G}_A$ and the restriction of H to $\operatorname{range} g$, except that we ignore any H-edge between $g(a_A)$ and $g(b_A)$. It follows that the map $A \mapsto \{f(a_A), f(b_A)\}$ is H-computable, and $A \in \emptyset'$ if and only if $f(a_A)$ and $f(b_A)$ are connected by an edge in H, which is an H-computable predicate. □

In the rest of the section, we prove Proposition 4.11. Let $G = (V, E)$ be a c.e. graph. By pulling back by an effective enumeration, we may assume that the set V of vertices of G is computable. By Lemma 4.6, we may in fact assume that V is an algebraically independent subset of $\mathbb{C}$. Let F_0 be the compositum of the algebraic closures of $\mathbb{Q}(v)$ for $v \in V$, and let

$$F = F_0(\sqrt{v+w})_{(v,w)\in E}.$$

As in the proof of Proposition 4.5, F is not computable, but has a computable copy; when we discover that $(v, w) \in E$, we add a new element and declare that its square is $v + w$.

For $a, b \in \mathbb{C}$, let $a \leqslant_{\mathbb{C}} b$ if a is algebraic over $\mathbb{Q}(b)$; since $a \leqslant_{\mathbb{C}} b$ if and only if the index $[\mathbb{Q}(a, b) : \mathbb{Q}(a)]$ is finite, and the field extension index is multiplicative, the relation $\leqslant_{\mathbb{C}}$ is transitive; hence algebraic equivalence $a \sim_{\mathbb{C}} b$, defined by $a \leqslant_{\mathbb{C}} b$ & $b \leqslant_{\mathbb{C}} a$, is an equivalence relation. We have $a \sim_{\mathbb{C}} b$ if and only if the algebraic closure $\operatorname{acl}^{\mathbb{C}}(\mathbb{Q}(a))$ of $\mathbb{Q}(a)$ in $\mathbb{C}$ equals the algebraic closure $\operatorname{acl}^{\mathbb{C}}(\mathbb{Q}(b))$ of $\mathbb{Q}(b)$ in $\mathbb{C}$. We use the following important fact.

Fact 4.12 (Friedman, Stanley [10]).

(1) If $a \in F$ is transcendental and $\operatorname{acl}^{\mathbb{C}}(\mathbb{Q}(a)) \subset F$ then there is some $v \in V$ such that $a \sim_{\mathbb{C}} v$.
(2) If $v, w \in V$, $a, b \in F$, $a \sim_{\mathbb{C}} v$, $b \sim_{\mathbb{C}} w$ and F contains a square root of $a+b$, then $(v, w) \in E$.

Lemma 4.13. *Let K be a field. The collection of countable subfields of K which are algebraically closed is computable from K.*

Proof. $A \subset K$ is a subfield if it is closed under the field operations; this involves quantifying over A. A is algebraically closed if every polynomial over A has a root in A; again this involves only quantifying over A. □

Let $K \cong F$; let $f\colon K \to F$ be an isomorphism. We build a copy $H = (V_H, E_H)$ of G, computable from K and the collection of irreducible polynomials in $K[x]$. We let V_H be the set of countable subfields of K which are algebraically closed and distinct from the algebraic closure of $\mathbb{Q}$ in K. By Lemma 4.13, V_H is K-computable. For $A, B \in V_H$, we let $(A, B) \in E_H$ if there are $a \in A$ and $b \in B$ such that K contains a square root of $a + b$. By Lemma 4.7, E_H is computable from the collection of irreducible polynomials in $K[x]$ (as A and B are countable).

Finally, fact 4.12 implies that $g\colon V \to V_H$ defined by $g(v) = \operatorname{acl}^K(\mathbb{Q}(f(v)))$ is an isomorphism from G to H. This completes the proof of Proposition 4.11.

5. Computable categoricity and intrinsically c.e. relations

In the classical setting, much research has gone into two related notions, computable categoricity and intrinsically c.e. relations. There is also work on relative versions of these notions. We recall the definitions, beginning with intrinsically c.e. and relatively intrinsically c.e. relations.

Definition 5.1. Let $\mathcal{A}$ be a computable structure, and let R be a relation on $\mathcal{A}$.

- R is *intrinsically c.e. on* $\mathcal{A}$ if for all computable $\mathcal{B} \cong \mathcal{A}$, for all isomorphisms $f\colon \mathcal{A} \to \mathcal{B}$, $f[R]$ is c.e.
- R is *relatively intrinsically c.e.* on $\mathcal{A}$ if for all $\mathcal{B} \cong \mathcal{A}$, for all isomorphisms $f\colon \mathcal{A} \to \mathcal{B}$, $f[R]$ is c.e. relative to $\mathcal{B}$.

The notion of an intrinsically c.e. relation seems natural, particularly if computable structures are of primary interest. It turns out, however, that the relative version is better behaved. This is true in the standard setting (see [4] and [6]). It is true also in our setting, as we shall soon see.

In a computable graph, the collection of vertices that are contained in an infinite complete subgraph is relatively intrinsically c.e., as is the collection of vertices that are contained in an infinite, totally disconnected subgraph. In a computable partial ordering, the collection of elements that are contained in an infinite chain is relatively intrinsically c.e., as is the collection of elements that are contained in an infinite antichain. In an Abelian group, the set of divisible elements is relatively intrinsically c.e.

Next, we give the definitions of computable categoricity, and relative computable categoricity.

Definition 5.2. Let $\mathcal{A}$ be a computable structure.

- $\mathcal{A}$ is *computably categorical* if for all computable $\mathcal{B} \cong \mathcal{A}$, there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.
- $\mathcal{A}$ is *relatively computably categorical* if for all $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ computable from $\mathcal{B}$.

For example, the linear ordering $(\mathbb{R}, <)$ is relatively computably categorical, whereas the linear orderings $(\omega_1, <)$ and $\mathbb{Q} \times \omega_1$ and $\mathbb{R} \times \omega_1$ (the long line) are not. Again, it turns out that the relative notion is better behaved, just as in the standard setting ([4], [6]).

5.1. **The appropriate infinitary logic.** What is intrinsic to a structure should be definable. Similarly, we expect that if a structure is computably categorical, then there should be an easy way to describe it which would enable the standard back-and-forth construction to produce the promised computable isomorphism. This is indeed the case for the relative notions.

In the standard setting, we use formulas of $\mathcal{L}_{\omega_1\omega}$, of a special kind, to define the intrinsically c.e. relations and the orbits in the computably categorical structures. In particular, we use "computable Σ_1 formulas", where these are c.e. disjunctions of finitary existential formulas, with a fixed finite tuple of variables. Similar results hold in the uncountable case. Below, we say what, in the setting of ω_1, are the computable infinitary Σ_1 formulas. We start by describing the analogue of $\mathcal{L}_{\omega_1\omega}$ in our setting, which is a generalisation of $\mathcal{L}_{\omega_2\omega_1}$, in that it allows quantification over countably many variables, and conjunctions and disjunctions of families of formulas of size $\aleph_1$.

5.1.1. *The infinitary logic.* Let $\mathcal{L}$ be a computable signature (see Subsection 3.1). The collection of $\mathcal{L}$*-terms* is defined recursively, starting with an uncountable computable collection of variables, by applying function symbols of $\mathcal{L}$ with arity α to α-tuples of terms of lower rank. This recursion is effective, and so the collection of all $\mathcal{L}$-terms is computable.

An *atomic* $\mathcal{L}$*-formula* is an expression of the form $R(\bar{t})$, where R is a relation symbol of $\mathcal{L}$ of arity α, and $\bar{t}$ is a tuple of $\mathcal{L}$-terms of length α.

A *(countable) quantifier-free* $\mathcal{L}$-formula (sometimes called a Σ_0 or a Π_0 formula) is recursively obtained from atomic formulas by applying negation and countable conjunctions and disjunctions. Each countable quantifier-free $\mathcal{L}$-formula is an element of L_{ω_1}, and as with terms, the collection of such formulas is computable. These formulas are the analogue of the elementary ($\mathcal{L}_{\omega\omega}$) quantifier-free formulas.

Semantics are defined as expected. This gives us the notion of the *atomic diagram* $D(\mathcal{M})$ of an $\mathcal{L}$-structure, which is the collection of all atomic sentences, and negations of atomic sentences true in $\mathcal{M}$. Really, the sentences are in the language $\mathcal{L}_{\mathcal{M}}$ that is obtained from $\mathcal{L}$ by adding constants naming the elements of $\mathcal{M}$. As mentioned above, the atomic diagram (which is a subset of L_{ω_1}) is Turing equivalent to $\mathcal{M}$, and to the collection of all countable quantifier-free $\mathcal{L}_{\mathcal{M}}$-sentences tnat hold in $\mathcal{M}$.

Infinitary (uncountable) formulas are in general obtained from countable quantifier-free formulas by allowing uncountable conjunctions and disjunctions, negation, and quantification over countably many variables. We restrict to formulaswith only countably many free variables.

5.1.2. *The effective infinitary formulas.* A (countable) *existential formula* is a formula of the form $\exists \bar{u}\, \varphi(\bar{x}, \bar{u})$, where φ is a countable quantifier-free formula, and $\bar{x}$ and $\bar{u}$ are countable tuples of variables. Again, every countable existential formula is an element of L_{ω_1}, and the collection of all countable existential formulas is a computable subset of L_{ω_1}.

A *computable (infinitary)* Σ_1 *formula* is the disjunction of a c.e. set (possibly uncountable) of existential formulas, with a fixed countable tuple $\bar{x}$ of free variables.

5.1.3. *Defining relatively intrinsically c.e. relations.* In the standard setting, every relatively intrinsically c.e. relation is definable by a computable Σ_1 formula with a finite tuple of parameters (see [4] and [6]). The analogous result holds in our setting.

Theorem 5.3. *Let $\mathcal{A}$ be a computable structure, and let R be a relation on $\mathcal{A}$. The following are equivalent:*

(1) *R is relatively intrinsically c.e. on $\mathcal{A}$.*
(2) *There is an expansion of $\mathcal{A}$ by countably many constants, in which R is definable by a computable Σ_1 formula.*

The proof of Theorem 5.3 is given in [15]. Theorem 5.3 was also extended to the arithmetical hierarchy; see [5].

We remark that in the classical setting, Manasse [21] and Goncharov [13] gave examples of relations that are intrinsically c.e. but not relatively intrinsically c.e. It seems likely that such examples can be given in the setting of ω_1.

5.2. **Continuous Scott sets.** A new ingredient enters the picture when we consider computable categoricity. In the classical setting, a computable structure $\mathcal{A}$ is relatively computably categorical if and only if some expansion $(\mathcal{A}, \bar{c})$ of $\mathcal{A}$ by finitely many constants has a formally Σ^0_1 Scott family, where this is a c.e. set Φ of computable Σ_1 formulas that includes definitions of the orbits of all finite tuples in $\mathcal{A}$ under the action of the group of automorphisms of $(\mathcal{A}, \bar{c})$. The formally Σ^0_1 Scott family enables us to effectively carry out a back-and-forth construction between any two copies of $\mathcal{A}$. In fact, taking the disjuncts of the members of Φ, we obtain a c.e. Scott family consisting of finitary existential formulas.

There are similar results for relative Δ^0_α categoricity where α is an arbitrary computable ordinal, where the isomorphism is Δ^0_α relative to the copy, and the formulas in the c.e. Scott family are "computable Σ_α".

In the setting of ω_1, it is not sufficient to have a c.e. set of computable existential formulas that define the orbits of tuples in our structure. The key point is that the back-and-forth construction now needs to pass through limit stages, and we need to preserve the property of effectively describing the orbits of the tuples that we have already included in the domain and range of our partial isomorphism. Below, we add a condition to take care of this problem.

Definition 5.4. Let A be an uncountable set. Recall that $[A]^{\aleph_0}$ denotes the collection of all countable subsets of A. We say that a set $\mathcal{C} \subseteq [A]^{\aleph_0}$ is *unbounded* if for all countable $b \subset A$ there is some $c \in \mathcal{C}$ such that $b \subseteq c$. We say that $\mathcal{C} \subseteq [A]^{\aleph_0}$ is *closed* if whenever $a_0 \subset a_1 \subset a_2 \subset \dots$ is a sequence of elements of $\mathcal{C}$, $\bigcup_n a_n$ is also an element of $\mathcal{C}$.

Definition 5.5. Let $\mathcal{A}$ be a structure. A *continuous formally c.e. Scott family* for $\mathcal{A}$ consists of a computable closed unbounded subset $\mathcal{C}$ of $[A]^{\aleph_0}$ and a c.e. collection Φ of (c.e. indices for) computable Σ_1 formulas such that:

(1) for every $\bar{a} \in \mathcal{C}$ there is a formula $\varphi_{\bar{a}} \in \Phi$ that defines the orbit of $\bar{a}$ in $\mathcal{A}$; and
(2) if $\bar{a}_0 \subset \bar{a}_1 \subset \bar{a}_2 \subset \dots$ is an increasing sequence of elements of $\mathcal{C}$, then
$$\varphi_{\bigcup_n \bar{a}_n} = \bigwedge_n \varphi_{\bar{a}_n}.$$

Theorem 5.6. *The following are equivalent for a computable structure $\mathcal{A}$:*

(1) *$\mathcal{A}$ is relatively computably categorical.*
(2) *There is an expansion of $\mathcal{A}$ by countably many constants, which has a continuous formally c.e. Scott family.*

The restriction to a computable closed unbounded set is necessary. Theorem 5.6 is proved in [15].

As for intrinsically c.e. relations, in the classical setting there are structures that are computably categorical but not relatively so; again we expect the same to hold for ω_1.

6. Linear orderings

A significant body of work concerns the effective properties of countable linear orderings. In the uncountable case, we get both analogues of classical results that require new proofs, and results that are completely opposite to classical ones.

6.1. A saturated linear ordering.

We may apply Proposition 3.8 to the theory DLO of dense linear orderings without endpoints. The resulting saturated model of DLO of size $\aleph_1$, which we denote by η_1, is the ω_1-dense linear ordering: it is characterised by the property that whenever A and B are countable subsets of η_1 with the property that $A < B$ (which means that for all $a \in A$ and $b \in B$, $a < b$), there is some $c \in \eta_1$ such that $A < c < B$ (which means that for all $a \in A$ and $b \in B$, $a < c < b$). Since DLO has quantifier elimination, η_1 is saturated with respect to simple embeddings: if $\mathcal{L}$ and $\mathcal{L}'$ are countable linear orderings, and $f\colon \mathcal{L} \to \mathcal{L}'$, $g\colon \mathcal{L} \to \eta_1$ are order-preserving functions, then there is an order-preserving function $h\colon \mathcal{L}' \to \eta_1$ such that $g = h \circ f$.

In the classical setting, Nurtazin [27] showed that there is a "computable numbering" of the computable linear orderings; that is, a uniformly computable sequence of linear orderings, that includes a copy of every computable linear ordering. We can transfer Nurtazin's proof to the uncountable case.

Proposition 6.1. *There is a uniformly computable sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha<\omega_1}$ of linear orderings such that for every computable linear ordering $\mathcal{L}$ there is some α such that $\mathcal{L} \cong \mathcal{L}_\alpha$.*

Proof. Let $\mathcal{M} = (M, <^{\mathcal{M}})$ be a computable copy of η_1. Since M is computable, by taking a computable bijection we may suppose that $M = L_{\omega_1}$. First, we note that if $\mathcal{L}$ is a computable linear ordering, then we can recursively embed $\mathcal{L}$ into $\mathcal{M}$ such that the image of the embedding is c.e. Next, we show that each c.e. set W_α, the substructure of $\mathcal{M}$ with universe W_α is isomorphic to a computable ordering. For all $a \in W_\alpha$, let $g_\alpha(a) = (a, s)$, where s is the (ordinal) stage at which a enters W_α. We let $\mathcal{L}_\alpha$ be the linear ordering whose universe is range g_α, where the ordering is defined so that g_α is an isomorphism from $(W_\alpha, <^{\mathcal{M}} \restriction_{W_\alpha})$ to $\mathcal{L}_\alpha$. This is the desired uniformly computable sequence representing all computable order types. □

6.2. **Computable well orderings.** Much of hyperarithmetic theory is based on the theory of computable well-orderings. In the uncountable case, there is no analogue for hyperarithmetic theory, and the situation is distorted by the fact that well-foundedness is now a co-c.e. phenomenon rather than Π^1_1. Nevertheless, it is natural to study the computable well-orderings of ω_1 in their own right.

As in the classical setting, the initial segment of any computable well-ordering has a computable copy. Hence the collection of ordinals that are isomorphic to computable linear orderings is an initial segment of the class of ordinals (of size $\aleph_1$, the number of computable linear orderings). As in the classical case, no ordinal beyond ω_1 with a computable copy can be admissible; if L_α is admissible and $\alpha > \omega_1$, then L_α contains every computable well-ordering of ω_1, and given a computable well-ordering $\mathcal{L}$ of ω_1, by effective recursion, L_α must also contain the order-type of $\mathcal{L}$. [By effective recursion in L_α, define a function f from the ordinals onto initial segments of $\mathcal{L}$. At stage β, if $f\restriction_\beta$ is not onto $\mathcal{L}$, we let $f(\beta)$ be the $\mathcal{L}$-least-upper-bound of the range of $f\restriction_\beta$; this definition is effective since it involves only quantification over the elements of $\mathcal{L}$. If the construction doesn't halt before stage α, we get an effective injective map from α onto an initial segment of $\mathcal{L}$; but since $\mathcal{L}$ is a well-ordering, every initial segment of $\mathcal{L}$ is an element of L_α, for a contradiction.]

The following diverges markedly from the classical case, in which the least non-computable ordinal ω_1^{CK} is admissible. It follows from results of S. Friedman [11]. We include a proof for completeness and for illustration of our methods, that will be used for Proposition 6.3.

Proposition 6.2 (Friedman)**.** *The least ordinal that is not isomorphic to any computable well-ordering of ω_1 is not admissible.*

Proof. Let β be this least ordinal; and let α be the least admissible ordinal greater than ω_1. Every computable linear ordering of ω_1 is an element of L_{ω_1+1}, and the collection $\mathcal{W}$ of computable well-orderings is definable over L_{ω_1+1} (since the countable descending sequences are elements of L_{ω_1}). Then $\mathcal{W}$ is an element of L_α. The function that takes every element of $\mathcal{W}$ to its ordertype is L_α-effective, and so by admissibility, the range β of this function is bounded below α. □

Even though by definition there is no (classically) computable linear ordering isomorphic to ω_1^{CK}, there is a (classically) computable linear ordering with an initial segment isomorphic to ω_1^{CK}. This is Harrison's linear ordering, which is isomorphic to $\omega_1^{\mathrm{CK}}(1+\mathbb{Q})$. It is obtained by taking a non-standard extension of ω_1^{CK}, one with no infinite descending hyperarithmetic sequences (equivalently, it is an initial segment of the Kleene-Brouwer ordering of a computable tree that is ill-founded but has no hyperarithmetical paths). We obtain a similar result in the setting of ω_1 using completely different tools.

Proposition 6.3 (with Shore)**.** *There is a computable ordering with an initial segment of order-type β, where β is the least ordinal that is not isomorphic to any computable well-ordering of ω_1.*

Proof. By Proposition 6.1, let $\langle \mathcal{L}_\alpha \rangle$ be a uniformly computable list of all computable linear orderings. We note that the set

$$\{\alpha < \omega_1 \,:\, \mathcal{L}_\alpha \text{ is not a well-ordering}\}$$

is c.e.

We construct a computable linear ordering $\mathcal{L}$ by recursion with priorities. At stage $s < \omega_1$ we have constructed a countable part of $\mathcal{L}$, and identified our guess A_s for the initial segment A that should be isomorphic to β. As long as $\mathcal{L}_\alpha$ apears to be well-founded, we add a copy of $\mathcal{L}_\alpha$ to A. If at stage s we discover that $\mathcal{L}_\alpha$ is not well-founded, we declare that all the elements that were put in order to copy $\mathcal{L}_\alpha$ lie to the right of A, and we injure $\mathcal{L}_\gamma$ for $\gamma > \alpha$, so that we code such well-founded $\mathcal{L}_\gamma$ in A all over again. □

6.3. **Least degrees and jump-degrees.**

Definition 6.4. The *degree spectrum* $\operatorname{degSpec}(\mathcal{M})$ of a structure $\mathcal{M}$ is the collection of Turing degrees that compute a structure isomorphic to $\mathcal{M}$.

Much work has gone into the study of degree spectra of structures in general and of linear orderings in particular. A fundamental theorem of the second author states that if $\mathbf{b} \in \operatorname{degSpec}(\mathcal{M})$ then (except for a trivial case) there is a copy of $\mathcal{M}$ of degree $\mathbf{b}$. This holds for ω_1 as well.

The degree spectrum $\operatorname{degSpec}(\mathcal{M})$ is of particular interest in case it is the cone above some degree $\mathbf{b}$; in this case, we say that $\mathbf{b}$ is *the* degree of (the isomorphism type of) $\mathcal{M}$. In the countable context, Richter [29] showed that no nonzero degree can be the degree of a linear ordering. While it is possible to code an arbitrary subset of ω in the isomorphism type of a countable linear ordering, it is not possible to do this so that the set is computable relative to all copies of the ordering. Richter showed that for any countable linear ordering $\mathcal{L}$, there is some $\mathcal{L}'$, isomorphic to $\mathcal{L}$, which forms a minimal pair with $\mathcal{L}$. This result strongly uses the true finiteness of the finite cardinals, and cannot be replicated in the case of ω_1. In fact, we have the opposite.

Theorem 6.5. *Every Turing degree is the degree of some linear ordering.*

Proof. We show that for any uncountable $A \subseteq \mathbb{R} \setminus \mathbb{Q}$ there is some linear ordering $\mathcal{L}_A$ such that for any $X \subseteq \omega_1$, X computes a copy of $\mathcal{L}_A$ if and only if A is c.e. relative to X. Then, given any Turing degree $\mathbf{b}$, we fix some $A \subseteq \mathbb{R} \setminus \mathbb{Q}$ of degree $\mathbf{b}$; the degree spectrum of the linear ordering $\mathcal{L}_{A \oplus \bar{A}}$ is then precisely the cone above $\mathbf{b}$.

So, fix some uncountable $A \subseteq \mathbb{R}$; let $\mathcal{L}_A = A \cup \mathbb{Q}$, with the ordering inherited from the real line.

Let $X \subseteq \omega_1$. If A is c.e. relative to X, let $f \colon \omega_1 \to A$ be injective and extend f to a function $g \colon \omega_1 \cup \mathbb{Q} \to A \cup \mathbb{Q}$ by letting $g\restriction_{\mathbb{Q}} = \mathrm{id}_{\mathbb{Q}}$. Let $\mathcal{L}$ be the linear ordering defined on $\omega_1 \cup \mathbb{Q}$ that makes g into an isomorphism from $\mathcal{L}$ to $\mathcal{L}_A$; $\mathcal{L}$ is computable in X.

In the other direction, suppose that $\mathcal{L} \cong \mathcal{L}_A$; we show that A is c.e. in $\mathcal{L}$. Let $f \colon \mathcal{L} \to \mathcal{L}_A$ be an isomorphism. Let $Q = f^{-1}\mathbb{Q}$; the point is that $f\restriction_Q$ is countable. For all a in the domain of $\mathcal{L}$, and which is not in Q, let

$$C_a = \left\{ q \in Q \, : \, q <^{\mathcal{L}} a \right\}$$

and

$$D_a = \left\{ q \in Q \, : \, q >^{\mathcal{L}} a \right\}.$$

The sets C_a and D_a (which are countable) are $\mathcal{L}$-computable given a; since $f\restriction_Q \in L_{\omega_1}$, so are $f[C_a]$ and $f[D_a]$. Now $f(a)$ is the unique irrational real number such that

$$f[C_a] < f(a) < f[D_a],$$

This definition involves quantifying only over $f[C_a]$ and $f[D_a]$; so f is computable from $\mathcal{L}$. Hence A, which is the range of f (minus the rationals), is c.e. relative to $\mathcal{L}$. □

In the absence of a degree for a linear ordering $\mathcal{L}$, one can ask about a jump-degree of $\mathcal{L}$: a least degree for the Turing jumps of all isomorphic copies of $\mathcal{L}$, and similarly the double-jump-degree and so on. In the countable case, the second author [18] showed that $\mathbf{0}'$ is the only jump-degree of a countable linear ordering; but every degree above $\mathbf{0}''$ is the double-jump-degree of a linear ordering ([1], [9]).

In recent work, the first author, with Kach, Lempp and Turetsky, showed that in the uncountable setting, every degree above $\mathbf{0}'$ is the proper jump-degree of a linear ordering.

6.4. **Techniques of Jockusch and Soare.** Theorem 6.5 implies the analogue of a result of the second author for the countable case, that for any nonzero Turing degree $\mathbf{b}$, there is a $\mathbf{b}$-computable linear ordering that has no computable copy; in other words, for all $\mathbf{b} > \mathbf{0}$ there is some linear ordering $\mathcal{L}$ such that $\mathbf{b} \in \mathrm{degSpec}(\mathcal{L})$ but $\mathbf{0} \notin \mathrm{degSpec}(\mathcal{L})$. The proof of the result in the countable setting is quite elaborate, relying on double-jump inversion and arguments of Seetapun's. Seetapun's proofs are in turn generalisations of a technique of Jockusch's and Soare's [17], who first proved the result for c.e. degrees $\mathbf{b}$. These techniques can be simplified in the uncountable setting to get analogues of other classical results.

Proposition 6.6. *There is a computable function f such that for all $e < \omega_1$, if e is an index for an infinite computable linear ordering $\mathcal{L}$, $f(e)$ is a computable index for an infinite linear ordering $\mathcal{L}'$ (of universe ω_1) such that $\mathcal{L} \ncong \mathcal{L}'$.*

Thus the fixed-point theorem fails for infinite linear orderings.

Proof. Suppose that we are given an enumeration $(\mathcal{L}_s)_{s<\omega_1}$ of an infinite computable linear ordering $\mathcal{L}$. We know that $\mathcal{L}$ contains either an infinite ascending chain or an infinite descending chain, so we can wait for a stage $s < \omega_1$ at which we see such a chain. If we see an infinite descending chain in $\mathcal{L}$, we output a computable copy of ω_1; if we see an infinite ascending chain in $\mathcal{L}$, we output a computable copy of ω_1^*. □

In the countable setting, Miller [24] extended the Jockusch-Soare and Seetapun technique and showed that there is a linear ordering $\mathcal{L}$ with no computable copy whose degree spectrum contains every hyperimmune degree, in particular every nonzero Δ^0_2 degree. The same holds in the uncountable case.

Theorem 6.7 (Greenberg, Kach, Lempp, Turetsky)**.** *There is a linear ordering of ω_1 that has no computable copy, but whose degree spectrum contains every nonzero Δ^0_2 degree.*

Sketch of proof. We give an axiomatic approach, which originates from understanding the algebraic aspects of Miller's proof for the countable case. A *computable directed system of linear orderings* is a sequence $\langle C_\beta, h_\beta\rangle_{\beta<\omega_1}$ such that:

- for every $\beta < \omega_1$, C_β is a countable linear ordering, and h_β is an embedding of C_β into $C_{\beta+1}$;
- $\beta \mapsto (C_\beta, h_\beta)$ is a computable function;
- C_0 is empty;
- for limit β, C_β is the direct limit of $\langle C_\alpha, h_\alpha\rangle_{\alpha<\beta}$.

What we require is a pair of computable directed systems $\langle A_\beta, f_\beta\rangle_{\beta<\omega_1}$ and $\langle B_\beta, g_\beta\rangle_{\beta<\omega_1}$ of linear orderings with the following properties:

(1) For all β, A_β does not embed into any proper initial segment of A_β.
(2) If C is a nonempty initial segment of B_{ω_1} (the direct limit of the system $\langle B_\beta, g_\beta\rangle$), then for all $\beta < \omega_1$, $A_{\beta+1}$ embeds into $A_\beta + C$.
(3) For all $\beta < \omega_1$ there is some $\alpha > \beta$ such that $A_\beta + B_\beta$ embeds into A_α, extending the embedding of A_β into A_α induced by $\langle f_\gamma\rangle_{\gamma\in[\beta,\alpha)}$.
(4) If β is a limit ordinal, then no proper initial segment of A_β contains copies of A_γ for all $\gamma < \beta$.

The existence of such systems is exactly what is required for Miller's argument to work, with Δ^0_2 (and in fact hyperimmune) permitting. Miller's argument (with ω_1 replaced by ω) used $A_\omega = \omega$ and $B_\omega = \omega^*$, with $A_n = B_n = n$ with the obvious embeddings; we use $A_\beta = \sum_{\gamma<\beta} \mathbb{Z}^\gamma$ and $B_\beta = A^*_\beta$. □

We note that the Jockusch-Soare techniques fail for classes close to linear orderings, such as Boolean algebras: in the countable context, Downey and Jockusch [8] showed that every low Boolean algebra has a computable copy; this was extended by Thurber [32] to low_2 and by Stob and the second author [19] for low_3 and low_4. So far as we know, no work has been yet done on an analogue of these results in the uncountable setting.

6.5. **Computable categoricity.** Dzgoev and Goncharov [14], and, independently, Remmel [28], showed in the standard setting that a computable linear ordering is computably categorical (equivalently, relatively computably categorical) if and only if it contains only finitely many successor pairs (pairs $a < b$ such that b is the immediate successor of a in the linear ordering).

A naïve attempt to generalise this result would guess that a computable linear ordering of size $\aleph_1$ is computably categorical if and only if it contains only countably many successor pairs. This fails in both directions:

- The linear ordering $2 \cdot \mathbb{R}$ (replacing every real number by a successor pair) contains uncountably many successor pairs, but is computably categorical: after fixing a copy of the double rationals in two computable copies, there is a unique extension to an isomorphism, which is computable.
- The linear ordering $\mathbb{Q} \cdot \mathbb{R}$ does not contain any successor pairs, but is not computably categorical: we can build a "bad" computable copy of $\mathbb{Q} \cdot \mathbb{R}$ and defeat all computable attempts at an isomorphism by waiting for the e^{th} computable function to be defined on the e^{th} copy of $\mathbb{Q}$ in the standard copy, and then add a point to the e^{th} copy of $\mathbb{Q}$ in our copy.

When we try to generalise the result of Dzgoev, Goncharov, and Remmel, we encounter the same difficulty as with Richter's result. The Dzgoev-Goncharov-Remmel theorem relies on the true finiteness of the finite cardinals. The correct generalisation involves an effectiveness condition that does not appear in the countable case. We believe that understanding the correct generalisation to the uncountable case sheds new light on the countable case as well.

To state the correct generalisation, we make use of the following notions. Let $\mathcal{L} = (L, <^{\mathcal{L}})$ be a linear ordering, and let C be a subset of the universe L of $\mathcal{L}$. A *C-cut* is a nonempty initial segment of C (by the inherited ordering from $\mathcal{L}$). If A is a C-cut, then the *$\mathcal{L}$-interval* determined by A, is the set

$$I^{\mathcal{L}}(A) = \left\{ b \in L \setminus C \,:\, \forall c \in C \quad \left[c <^{\mathcal{L}} b \Leftrightarrow c \in A \right] \right\}.$$

Theorem 6.8 (Greenberg, Kach, Lempp, Turetsky)**.** *A computable linear ordering $\mathcal{L}$ is computably categorical if and only if there is a countable subset C of the domain of $\mathcal{L}$ and a uniformly c.e. partition $\langle S_n \rangle_{n \geqslant 1}$ of the collection of all C-cuts that define nonempty $\mathcal{L}$-intervals such that for all $n \geqslant 1$, for all $A \in S_n$, $I^{\mathcal{L}}(A)$ either has size n or is isomorphic to η_1.*

For the proof, and for further results on the degree spectrum of the successor relation on computable linear orderings, see [16]. We remark that the proof makes use of our understanding of countable linear orderings, particularly the scattered / nonscattered dichotomy, and does not seem to immediately generalise to higher cardinality.

Question 6.9. *Which ω_2-computable linear orderings of ω_2 are computably categorical?*

7. Conclusion

The following are desirable features for a model of computability in an uncountable setting.

(1) **Applications**. There should be interesting results about familiar mathematical objects such as the field of real numbers.
(2) **Implementation**. There should be an implementation of the model, or at least a way of thinking about the computations that makes them "feel" effective.
(3) **Comprehensibility**. A working mathematician should be able to understand the model.

(4) **Insight**. Studying the new model should deepen our understanding of the standard model.

We believe that our model meets all criteria except possibly the third one. We have given some examples of computable structures, and some results in computable structure theory. We find it pleasing that our model lets us think of the real numbers, with the analytic functions, as a computable structure. The model also provides insight into standard notions and constructions, as was shown when discussing the generalisations and failure thereof of Richter's result and the Dzgoev-Goncharov-Remmel theorem.

As in the countable setting, after gaining experience with admissible recursion, one develops a solid intuition to what constitutes a computable construction and what does not; it becomes natural to describe such constructions in an informal way, and rely on an analogue of Church's thesis. In Sacks's terminology, one gets a dynamic feeling which lifts beyond the static nature of existential formulas in the language of set theory.

As for comprehensibility, the apparatus of constructible sets and the Levy hierarchy of formulas are certainly not widely known. As indicated earlier, though, there are alternative approaches to defining the notions of c.e. and computable subsets of ω_1. The two main kinds of development rely on either inductive schemes for partial computable functions, or by transfinite runs of Turing machines. These definitions may seem more immediately effective and more easily comprehensible. What they do assume—what cannot be omitted by any development of the subject—are the countable ordinals and ω_1. Mathematicians who are familiar with the application of Zorn's Lemma may find this less of a barrier.

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