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DIAGONALLY NON-RECURSIVE FUNCTIONS AND EFFECTIVE HAUSDORFF DIMENSION

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Abstract. We prove that every sufficiently slow growing DNR function computes a real with effective Hausdorff dimension one. We then show that for any recursive unbounded and nondecreasing function j, there is a DNR function bounded by j that does not compute a Martin-Löf random real. Hence there is a real of effective Hausdorff dimension 1 that does not compute a Martin-Löf random real. This answers a question of Reimann and Terwijn.

1. Introducion

Reimann and Terwijn asked the *dimension extraction problem*: can one effectively increase the information density of a sequence with positive information density? For a formal definition of information density, they used the notion of effective Hausdorff dimension. This effective version of the classical Hausdorff dimension of geometric measure theory was first defined by Lutz [10], using a martingale definition of Hausdorff dimension. Unlike classical dimension, it is possible for singletons to have positive dimension, and so Lutz defined the dimension $\dim(A)$ of a binary sequence $A \in 2^\omega$ to be the effective dimension of the singleton $\{A\}$. Later, Mayordomo [12] (but implicit in Ryabko [15]), gave a characterisation using Kolmogorov complexity: for all $A \in 2^\omega$,

$$\dim(A) = \liminf_{n\to\infty} \frac{K(A \upharpoonright n)}{n} = \liminf_{n\to\infty} \frac{C(A \upharpoonright n)}{n},$$

where C is plain Kolmogorov complexity and K is the prefix-free version.[1]

Given this formal notion, the dimension extraction problem is the following: if $\dim(A) > 0$, is there necessarily a $B \leqslant_{\mathrm{T}} A$ such that $\dim(B) > \dim(A)$?[2] The problem was recently solved by the second author [13], who showed that there is an $A \in 2^\omega$ such that $\dim(A) = 1/2$ and if $B \leqslant_{\mathrm{T}} A$, then $\dim(B) \leqslant 1/2$.

Even while it was still open, the dimension extraction problem spawned variations. The one we consider in the present paper is:

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[1] Kolmogorov complexity of a finite binary string σ is the length of the shortest description of σ; it is the measure of the best effective compression of the information encapsulated in the string. Formally, $C(\sigma)$ is the length of the shortest string τ such that $U(\tau) = \sigma$ where U is a universal partial computable function. Prefix-free complexity $K(\sigma)$ is defined similarly, except that the function U is universal among all functions whose domain is an antichain of strings.

[2] Here the relation $\leqslant_{\mathrm{T}}$ denotes Turing reducibility: the formalisation of relative computability, where $A \leqslant_{\mathrm{T}} B$ means that A is computable if a Turing machine has access to an oracle that knows B.

Question 1.1. *Is there an $A \in 2^\omega$ such that $\dim(A) = 1$ and A computes no Martin-Löf random set?*[3]

One motivation for this, and for the dimension extraction problem, is that the obvious ways to obtain nonrandom sets of positive dimension allow us to extract random sets. For example, we could take a random set X (whose dimension is always 1) and water it down by inserting zeros between the bits of X. This would give us a sequence of dimension 1/2; inserting zeros sparsely would result in a sequence of dimension 1. As long as the insertion positions are computable, we can recover the original random set. As another example, a sequence of positive dimension created by flipping a biased coin also computes a random sequence. The dimension extraction question, and Question 1.1, can be understood as asking if there are sequences of positive dimension, and even dimension 1, that behave differently than these examples.

In this paper, we answer Question 1.1 in the positive. We remark that a weaker version of the answer we give was proved by Downey and Greenberg [4], where they show that there is a set of minimal Turing degree whose effective packing dimension is 1; such a set can compute no Martin-Löf random real. Packing dimension is a variant of Hausdorff dimension that uses inner measure; the complexity characterisation uses $\limsup$ instead of $\liminf$.

To answer Question 1.1, we study the computational power of sufficiently slow growing diagonally non-recursive functions. Recall that the *jump function* $J\colon \omega \to \omega$ takes e to the result of the eth program on input e, if it halts. Formally, $J(e) = \varphi_e(e)$, where $\{\varphi_e\}_{e\in\omega}$ is a standard enumeration of the partial recursive functions. A function $f \in \omega^\omega$ is *diagonally non-recursive* (or DNR) if $J(e) \neq f(e)$ for all $e \in \operatorname{dom} J$. Diagonally non-recursive functions were introduced by Jockusch in [7], where he shows that the Turing degrees of DNR functions are exactly those degrees that contain fixed-point free functions (that is, functions f such that for all e, $\varphi_e \neq \varphi_{f(e)}$).

From a computational standpoint, DNR functions, in general, can be quite feeble. For example, it follows from [1] that there is a DNR function that does not compute any set of positive effective Hausdorff dimension, while Kumabe and Lewis [9] showed that there is a DNR function that has minimal Turing degree. On the other hand, DNR *sets*, that is, DNR functions whose range is $\{0, 1\}$, are quite powerful: their degrees are characterized as those that can compute completions of Peano arithmetic [16, 8] and paths through nonempty recursive binary trees. This difference suggests that, possibly, it is the *rate of growth* of a DNR function that may influence its strength as an oracle.

A finer analysis of the computational power of a class of functions is given by considering the issue of uniformity. For any $a \geqslant 2$, let DNR_a be the class of DNR functions whose range is contained in the set $\{0, 1, \ldots, a-1\}$ (so DNR_2 is the class of DNR sets). Jockusch [7] has shown that the Turing degrees of the functions in DNR_a coincide, that is, if $a < b$ then every function in DNR_b computes a function in DNR_a; but that this cannot be done uniformly, that is, there is no single Turing functional that given a function in DNR_b produces one in DNR_a. In modern parlance, the Mučnik (weak) degrees of the classes DNR_a coincide for all

[3] A set is Martin-Löf random if it avoids all effectively presented, effectively null G_δ sets. Equivalently, its initial segments are incompressible by a prefix-free machine.

$a \geqslant 2$, but the Medvedev (strong) degrees of these classes form a strictly decreasing chain (see [17]).

In this paper, we examine what can be computed from an unbounded DNR function that nevertheless does not grow too quickly. For any non-decreasing function $j: \omega \to \omega \setminus \{0,1\}$, let

$$\mathrm{DNR}_j = \{f \in \mathrm{DNR} : \text{for all } e,\ f(e) < j(e)\}$$

be the class of DNR functions that are bounded by j. A DNR function is called *recursively bounded* if it is bounded by some recursive function j. We shall see (Theorem 4.9) that if j grows slowly, then every DNR_j function computes a set whose effective Hausdorff dimension is 1. This will be a consequence of results on the uniform power of bounded DNR functions, culminating in Proposition 4.8, which says that uniformly in a, functions in DNR_a uniformly compute sets of Hausdorff dimension 1.

We then show that for any recursive, non-decreasing and unbounded function $j: \omega \to \omega \setminus \{0,1\}$, there is some $f \in \mathrm{DNR}_j$ that does not compute any Martin-Löf random real (Theorem 5.11), improving the result from [1] that there is some recursively bounded DNR function that does not compute a Martin-Löf random real. Again, our result is obtained as some kind of "overspill" from the bounded case; we utilize a technique from [5], that was used to show that it is not possible to compute random reals uniformly from DNR_3 functions, even though every DNR_a function can compute a Martin-Löf random real.

Theorems 4.9 and 5.11 together yield the positive answer for Question 1.1. In Theorem 5.15, we show that the function produced by Theorem 5.11 can be Δ^0_2, and so the positive answer to Question 1.1 is given by a Δ^0_2 set (Corollary 5.16).

To prove Theorem 4.9, we use variants of Cantor space that we call *h-spaces*. For a recursive, non-decreasing function h, we work in the space

$$\prod_{m<\omega} \{0,1,\dots,h(m)-1\}$$

which for shorthand, in analogy with 2^ω, we name h^ω. In Section 2 we develop the theory of effective dimension for sequences in these spaces. Then, in Section 3, we show that for slow-growing functions h, the notion of dimension in h^ω coincides with dimension in 2^ω or in the Euclidean interval $[0,1]$. This allows us to carry out the plan of stringing together the uniform power of bounded DNR functions to prove Theorem 4.9. Theorem 5.11 is proved in Section 5.

2. Dimension in h-spaces

We generalise effective Hausdorff dimension to spaces that are very similar to Cantor space. Let $h: \omega \to \omega \setminus \{0,1\}$ be a recursive function. We let

$$h^\omega = \prod_{m<\omega} \{0,1,\dots,h(m)-1\}.$$

For finite $n < \omega$, we let

$$h^n = \prod_{m<n} \{0,1,\dots,h(m)-1\}$$

and we let

$$h^{<\omega} = \bigcup_{n<\omega} h^n.$$

An analogue of Lebesgue measure for h^ω is obtained by dividing mass equitably: for $\sigma \in h^{<\omega}$, let

$$\mu(\sigma) = \frac{1}{|h^{|\sigma|}|} = \frac{1}{h(0)\cdots h(|\sigma|-1)}.$$

Then $\mu = \mu^h$ induces a measure $\mu = \mu^h$ on h^ω such that for all $\sigma \in h^{<\omega}$, $\mu([\sigma]) = \mu(\sigma)$. Here as usual $[\sigma] = \{X \in h^\omega : \sigma \subset X\}$. We extend this notation: for sets of strings $A \subseteq h^{<\omega}$, we let $[A] = \bigcup_{\sigma\in A}[\sigma]$. The collection of sets $[A]$ for $A \subseteq h^{<\omega}$ forms a topology on h^ω which makes it a compact Polish space.

Definition 2.1. For a real $s \geqslant 0$ and a set $A \subseteq h^{<\omega}$, the *s-weight* of A is

$$\mathtt{wt}_s(A) = \sum_{\sigma\in A} \mu(\sigma)^s.$$

The *weak s-weight* of A is

$$\mathtt{wkwt}_s(A) = \sup \mathtt{wt}_s(C)[\![C \subseteq A \text{ is prefix-free}]\!].$$

Definition 2.2. Let $s \geqslant 0$. An *s-ML-test* is a uniformly r.e. sequence $\langle A_k\rangle$ of subsets of $h^{<\omega}$ such that for all $k < \omega$, $\mathtt{wt}_s(A_k) \leqslant 2^{-k}$. A *weak s-ML-test* is a uniformly r.e. sequence $\langle A_k\rangle$ of subsets of $h^{<\omega}$ such that for all $k < \omega$, $\mathtt{wkwt}_s(A_k) \leqslant 2^{-k}$.

Definition 2.3. A ML-test $\langle A_k\rangle$ *covers* $X \in h^\omega$ if $X \in \bigcap_k [A_k]$. A sequence $X \in h^\omega$ is *weakly s-null* if it is covered by a weak s-ML-test. It is *s-null* if it is covered by a s-ML-test.

Remark 2.4. In other common parlance, X is called *s-random* if it is not s-null and *strongly s-random* if it is not weakly s-null.

Proposition 2.5 (Reimann)**.** *If $t > s \geqslant 0$ and X is weakly s-null, then X is t-null.*

Proof. We show that there is a constant $c > 0$ such that for all $A \subseteq h^{<\omega}$, $\mathtt{wt}_t(A) \leqslant c \cdot \mathtt{wkwt}_s(A)$.[4] We can then convert a weak s-ML-test to a t-ML-test simply by taking a tail. Let $A \subseteq h^{<\omega}$. For $n < \omega$, let $\gamma_n = 1/|h^n|$ be the measure of any string of length n.

$$\mathtt{wt}_t(A) = \sum_{\sigma\in A} \mu(\sigma)^t = \sum_{n<\omega}\sum_{\sigma\in A\cap h^n} \gamma_n^t = \sum_{n<\omega} \gamma_n^{t-s} \sum_{\sigma\in A\cap h^n} \gamma_n^s$$
$$\leqslant \mathtt{wkwt}_s(A) \sum_{n<\omega} \gamma_n^{t-s},$$

the last equation following of course from the fact that $A \cap h^n$ is prefix-free. Thus we can let $c = \sum_{n<\omega} \gamma_n^{t-s}$, which is finite because for all m, $h(m) \geqslant 2$ and so $\gamma_n \leqslant 2^{-n}$. $\square$

Thus for $X \in h^\omega$, the infimum of all s for which X is s-null equals the infimum of all s for which X is weakly s-null.

[4] The inequality always holds if $\mathtt{wkwt}_s(A)$ is infinite.

Definition 2.6. Let $s \geqslant 0$. A *Solovay s-test* is an r.e. set $A \subseteq h^{<\omega}$ such that $\mathtt{wt}_s(A)$ is finite. We say that $X \in h^\omega$ is *covered* by a Solovay s-test if there are infinitely many initial segments of X in A.

Proposition 2.7 (Reimann, [14])**.** *If X is s-null then X is covered by a Solovay s-test. If X is covered by a Solovay s-test then for all $t > s$, X is t-null.*

Proof. Suppose first that X is s-null, so it is covered by some s-ML-test $\langle A_k \rangle$. Let $A = \bigcup_k A_k$. Then $\mathtt{wt}_s(A) \leqslant 1$. Since every h^n is finite, for all n there is a k such that every string in A_k has length greater than n. Thus A covers X as a Solovay test.

Suppose that A is a Solovay s-test that covers X. Let $t > s$. For all k we can find a length n_k such that for all $\sigma \in h^{<\omega}$ of length greater than n_k we have $\mu(\sigma)^{t-s} < 2^{-k}/\mathtt{wt}_s(A)$ (again recall that the series $\sum \gamma_n^{s-t}$ from the proof of Proposition 2.5 converges) and so if we let $A_k = \{\sigma \in A : |\sigma| > n_k\}$ then

$$\mathtt{wt}_t(A_k) = \sum_{\sigma \in A_k} \mu(\sigma)^t = \sum_{\sigma \in A_k} \mu(\sigma)^{t-s}\mu(\sigma)^s \leqslant 2^{-k}/\mathtt{wt}_s(A) \sum_{\sigma \in A} \mu(\sigma)^s = 2^{-k}. \quad \square$$

We also need to characterize dimension by martingales.

Definition 2.8. A *supermartingale* (for h) is a function $d\colon h^{<\omega} \to \mathbb{R}^+$ such that for all $\sigma \in h^{<\omega}$,

$$\sum_{i<h(|\sigma|)} d(\sigma i) \leqslant h(|\sigma|)d(\sigma).^{5}$$

A supermartingale d is a *martingale* if inequality is replaced by equality for all σ.

If $s \geqslant 0$ is a real number then we say that a supermartingale d is *s-successful* on $X \in h^\omega$ if the sequence

$$\left\langle d(X \upharpoonright n)\mu(X \upharpoonright n)^{1-s} \right\rangle_{n<\omega}$$

is unbounded.

Proposition 2.9 (Lutz [11], Calude-Staiger-Terwijn [2])**.** *If X is s-null then there is a left-r.e. martingale that s-succeeds on X.*

In fact, the proposition is true even if X is weakly s-null; in light of Proposition 2.5, this does not matter for dimension.

Proof. For any $\sigma \in h^{<\omega}$ there is a recursive martingale d_σ such that $d_\sigma(\sigma) = 1$ and $d_\sigma(\langle\rangle) = \mu(\sigma)$. For any $A \subseteq h^{<\omega}$ let

$$d_A = \sum_{\sigma \in A} \mu(\sigma)^{s-1} d_\sigma.$$

Then

$$d_A(\langle\rangle) = \sum_{\sigma \in A} \mu(\sigma)^{s-1} d_\sigma(\langle\rangle) = \mathtt{wt}_s(A)$$

and so if $\mathtt{wt}_s(A)$ is finite, then d is a martingale (inductively, for all $\tau \in h^{<\omega}$, we get $d_A(\tau) < \infty$). If A is r.e. then d_A is left-r.e.

Thus if $\langle A_k \rangle$ is a s-ML-test that covers X, then we can let

$$d = \sum_{k<\omega} k d_{A_k}.$$

[5] Note that this is equivalent to $\sum d(\tau)\mu(\tau) \leqslant d(\sigma)\mu(\sigma)$, where the sum is taken over all immediate successors τ of σ.

Then $d(\langle\rangle) = \sum k2^{-k} < \infty$ and so d is a martingale. If $\sigma \in A_k$ then

$$d(\sigma)\mu(\sigma)^{1-s} \geqslant \left(k\mu(\sigma)^{s-1}d_\sigma(\sigma)\right)\mu(\sigma)^{1-s} = k.$$

Since there is a $\sigma \subset X$ in A_k we get that d s-succeeds on X. □

Proposition 2.10 (Lutz [11], Calude-Staiger-Terwijn [2])**.** *If there is a left-r.e. supermartingale d that s-succeeds on X, then X is weakly s-null.*

Proof. The key is that Kolmogorov's inequality holds in the space h^ω as well: for any prefix-free set $C \subset h^{<\omega}$ we have

$$\sum_{\sigma\in C} d(\sigma)\mu(\sigma) \leqslant d(\langle\rangle).$$

The easiest way to see this is to think of $d(\sigma)\mu(\sigma)$ as inducing a submeasure on h^ω.

Thus if d s-succeeds on X, we can let, for $k < \omega$,

$$A_k = \{\sigma \in h^{<\omega} : d(\sigma)\mu(\sigma)^{1-s} > 2^k\}$$

(by normalizing we assume that $d(\langle\rangle) \leqslant 1$). Then $\langle A_k\rangle$ is uniformly r.e., covers X, and if $C \subseteq A_k$ is prefix-free then

$$\mathtt{wt}_s(C) = \sum_{\sigma\in C}\mu(\sigma)^s \leqslant 2^{-k}\sum_{\sigma\in C} d(\sigma)\mu(\sigma) \leqslant 2^{-k},$$

thus $\mathtt{wkwt}_s(A_k) \leqslant 2^{-k}$. □

Corollary 2.11. *The following four numbers are equal for $X \in h^\omega$. This number $\dim^h(X)$ is the* effective Hausdorff dimension *of X in the space h^ω.*

(1) $\inf\{s$: *There is a left-r.e. supermartingale d that s-succeeds on X*$\}$.
(2) $\inf\{s$: *X is s-null*$\}$.
(3) $\inf\{s$: *X is weakly s-null*$\}$.
(4) $\inf\{s$: *X is covered by a Solovay s-test*$\}$.

The proof of the final result in this section is standard.

Proposition 2.12. *There is an optimal left-r.e. supermartingale d^h for h. That is, for any left-r.e. supermartingale d for h, there is a constant c such that $d(\sigma) \leqslant cd^h(\sigma)$, for all $\sigma \in h^{<\omega}$.*

3. Translating between spaces

There is a natural measure-preserving surjection of h^ω onto the Euclidean interval $[0,1]$. First we map strings to closed intervals: we let $\pi^h(\langle\rangle) = [0,1]$, and if $\pi^h(\sigma) = I$ is defined for some $\sigma \in h^{<\omega}$ then we divide I into $h(|\sigma|)$ many intervals I_k of equal length and let $\pi^h(\sigma k) = I_k$. Note that indeed $\mu^h(\sigma) = \lambda(\pi^h(\sigma))$, where λ is Lebesgue measure on $[0,1]$. Finally we extend π^h continuously to h^ω by letting $\{\pi^h(X)\} = \bigcap_n \pi^h(X \upharpoonright n)$. The fact that $h(n) \geqslant 2$ ensures that this intersection is indeed a singleton. The mapping π^h is not quite 1-1; it is, if we ignore the (countably many) sequences that are eventually constant.

Note that for all $X \in h^\omega$, $X \equiv_T \pi^h(X)$.[6]

[6] Here we extend Turing reducibility and equivalence from Baire space ω^ω to elements of the real line, by, for example, identifying a real number with its binary expansion; all reasonable ways of treating real numbers as oracles for computation, such as using other bases or using quickly converging Cauchy sequences yield the same relation of relative computation.

Some parts of the theory of effective dimension can also be developed for the space $[0,1]$. We do not have martingales, but we can still, for example, define Solovay tests: for $s \geqslant 0$, a *Solovay s-test* for $[0,1]$ is an r.e. collection A of (rational) closed intervals in $[0,1]$ whose s-weight $\mathtt{wt}_s(A) = \sum_{I\in A} \lambda(I)^s$ is finite. Such a test A *covers* a point $x \in [0,1]$ if for infinitely many $I \in A$ we have $x \in I$. We let $\dim^{[0,1]}(x)$ be the infimum of all s such that there is a Solovay s-test (for $[0,1]$) that covers x.

Proposition 3.1. *For any h and all $X \in h^\omega$, $\dim^{[0,1]}(\pi^h(X)) \leqslant \dim^h(X)$.*

Proof. Let $s > \dim^h(X)$ and let A be a Solovay s-test for h that covers X. Since π^h is measure-preserving, the s-weight of $\pi^h[A]$ (in $[0,1]$) is also finite and covers $\pi^h(X)$, so $\dim^{[0,1]}(\pi^h(X)) \leqslant s$. □

Equality does not hold in general. We show below that it holds if h does not grow too quickly or irregularly. Suppose that A is a Solovay s-test for $[0,1]$ that covers $\pi^h(X)$. We would like to cover X by something like $(\pi^h)^{-1}A$. The problem is that the basic sets (rational closed intervals) in $[0,1]$ are finer than the basic sets in h^ω; not every closed rational interval is in the range of π^h. Thus we need to replace A by a coarser collection: replace every $I \in A$ by finitely many intervals in the range of π^h. We can control the Lebesgue measure of such a collection, but if $s < 1$, then the process of replacing large intervals by a collection of smaller ones may increase the s-weight significantly. We show that if h does not grow too irregularly and we increase the exponent, the weight remains finite.

Let $\mathcal{I}_n = \pi^h[h^n]$ and let $\mathcal{I} = \bigcup_n \mathcal{I}_n = \pi^h[h^{<\omega}]$. Recall that $\gamma_n = 1/|h^n|$ is the μ^h-measure of any string in h^n. Thus $\mathcal{I}_n$ consists of $|h^n|$ many closed intervals, each of length γ_n.

For a closed interval $I \subsetneq [0,1]$, let n_I be the unique n such that $\gamma_n \geqslant \lambda(I) > \gamma_{n+1}$. Let k_I be the greatest integer k such that $k\gamma_{n_I+1} \leqslant \lambda(I)$; so $k_I \leqslant h(n_I)$ and there is a set $\hat{I} \subseteq \mathcal{I}_{n_I+1}$ of size $k_I + 2$ such that $I \subseteq \bigcup \hat{I}$.

For a set A of closed rational subintervals of $[0,1]$, let $\hat{A} = \{\hat{I} : I \in A\}$. Thus $\hat{A} \subseteq \mathcal{I}$, and if $x = \pi^h(X)$ and x is covered by A, then X is covered by $(\pi^h)^{-1}\hat{A}$. If A is r.e. then so is $\hat{A}$ (and $(\pi^h)^{-1}\hat{A}$). Note that the t-weight of $\hat{A}$ in $[0,1]$ equals the t-weight of $(\pi^h)^{-1}\hat{A}$ in h^ω.

We express the regularity of h in terms of the following conditions, where $t > s \geqslant 0$:

$$(*)(t,s) \qquad \sum_{n<\omega} \frac{h(n)^{1-s}}{\big(h(0)\cdots h(n)\big)^{t-s}} < \infty.$$

Lemma 3.2. *Suppose that $(*)(t,s)$ holds for h and that A is a set of closed intervals in $[0,1]$ such that $\mathtt{wt}_s(A)$ is finite. Then $\mathtt{wt}_t(\hat{A})$ is also finite.*

Thus for all $X \in h^\omega$, if $(*)(t,s)$ holds for h and $\dim^{[0,1]}(\pi^h(X)) < s$, then $\dim^h(X) \leqslant t$. Thus if $(*)(t,s)$ holds for all $t > s \geqslant 0$ then for all $X \in h^\omega$, $\dim^{[0,1]}(\pi^h(X)) = \dim^h(X)$.

Proof. Let I be any closed interval in $[0,1]$. Let $n = n_I$ and $k = k_I$. Recalling that $\lambda(I)/\gamma_{n+1} \geqslant k$, we have

$$\begin{aligned}\mathtt{wt}_t(\hat{I}) = (k+2)\gamma_{n+1}^t \leqslant 3k\gamma_{n+1}^{t-s}\gamma_{n+1}^s = 3k\frac{\gamma_{n+1}^s}{\lambda(I)^s}\gamma_{n+1}^{t-s}\lambda(I)^s \\ < 3k^{1-s}\gamma_{n+1}^{t-s}\lambda(I)^s \leqslant \frac{3h(n)^{1-s}}{\left(h(0)\dots h(n)\right)^{t-s}}\lambda(I)^s.\end{aligned}$$

because $k \leqslant h(n)$.

Thus if we let $A_n = \{I \in A : n_I = n\}$ then

$$\begin{aligned}\mathtt{wt}_t(\hat{A}) \leqslant \sum_{n<\omega}\sum_{I\in A_n}\mathtt{wt}_t(\hat{I}) \leqslant \sum_{n<\omega}\sum_{I\in A_n}\frac{3h(n)^{1-s}}{\left(h(0)\dots h(n)\right)^{t-s}}\lambda(I)^s \\ \leqslant 3\mathtt{wt}_s(A)\sum_{n<\omega}\frac{h(n)^{1-s}}{\left(h(0)\dots h(n)\right)^{t-s}},\end{aligned}$$

which by assumption is finite. □

Note, for example, that the condition $(*)(t,s)$ holds (for all $t > s \geqslant 0$) for the constant function $h(n) = 2$. Thus dimension in $[0,1]$ is the same as dimension in Cantor space 2^ω. However, the condition holds for some unbounded functions h as well (for example $h(n) = 2^n$).

The reader may note that we do not appear to use the full strength of $(*)(t,s)$ in the proof of Lemma 3.2. It would be enough to assume that the sequence $\left\langle h(n)^{1-s}\left(h(0)\cdots h(n)\right)^{s-t}\right\rangle_{n<\omega}$ is bounded. On the other hand, if this holds for all $t > s \geqslant 0$, then $(*)(t,s)$ holds as well, so we have no need to make a distinction.

In Lemma 3.3, we show that a certain regularity condition on h implies the desired condition $(*)(t,s)$ for all $t > s \geqslant 0$. We then (Lemma 3.4) show that sufficiently slow-growing functions (those bounded by a simple exponential) all satisfy the regularity condition.

Lemma 3.3. *Suppose that*

$$\lim_{n\to\infty}\frac{\log h(n)}{\sum_{m\leqslant n}\log h(m)} = 0.$$

Then for all $t > s \geqslant 0$, the condition $(*)(t,s)$ holds for h, and so for all $X \in h^\omega$, $\dim^h(X) = \dim^{[0,1]}(\pi^h(X))$.

Proof. Let $f(n) = \log h(n)$. Let $t > s \geqslant 0$ and let $\varepsilon < (t-s)/(1-s)$. There is some $N < \omega$ such that for all $n \geqslant N$, $f(n) < \varepsilon\sum_{m\leqslant n} f(m)$. Let $g(n) = h(0)\cdots h(n)$ and let $\delta = \varepsilon(1-s) - (t-s) < 0$. For $n \geqslant N$, $h(n) < g(n)^\varepsilon$ and so

$$\frac{h(n)^{1-s}}{g(n)^{t-s}} < g(n)^\delta.$$

Thus

$$\sum_{n\geqslant N}\frac{h(n)^{1-s}}{\left(h(0)\dots h(n)\right)^{t-s}} < \sum_{n<\omega}\left(2^\delta\right)^{\log_2 g(n)},$$

which is finite becuase $2^\delta < 1$ and $\log_2 g(n) \geqslant n$ (as $h(n) \geqslant 2$). □

The regularity condition of Lemma 3.3 is not, strictly speaking, a slowness condition, because, for example, $h(n) = 2^{n^2}$ satisfies the condition yet there is a monotone function that is dominated by h and does not satisfy the condition. However, the condition holds for all sufficiently slow monotone functions:

Lemma 3.4. *Suppose that h is non-decreasing and dominated by 2^{kn} (for some constant k). Then*

$$\lim_{n\to\infty} \frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} = 0,$$

and so for all $X \in h^\omega$, $\dim^h(X) = \dim^{[0,1]}(\pi^h(X))$.

Proof. There are two cases. If h is bounded then it is eventually constant, and the condition is easily verified. Assume then that h is unbounded. Fix $c < \omega$. There is an $N_c < \omega$ such that for all $n \geqslant N_c$, $\log h(n) > c$. For $n > N_c$,

$$\frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} \leqslant \frac{kn}{c(n - N_c) + \sum_{m<N_c} \log h(m)} \leqslant \frac{k}{c} + \frac{kN_c}{c(n - N_c)}.$$

As n grows, $kN_c/c(n - N_c)$ tends to 0 and so

$$\lim_{n\to\infty} \frac{\log h(n)}{\sum_{m\leqslant n} \log h(m)} \leqslant \frac{k}{c}.$$

As this is true for all $c < \omega$, we get that the limit is 0. □

4. Using slow DNR functions

Our goal is to prove that if q grows sufficiently slowly, then every DNR_q function computes a set of effective Hausdorff dimension 1. The power of slow growing DNR functions is related to the uniform power of bounded DNR functions, so we begin by investigating what can be computed uniformly from functions in DNR_a (even uniformly in a). Along the way we investigate $\mathcal{P}^c_a$, the class of functions $f \in a^\omega$ such that $f(n)$ avoids $J(n, 0)$ through $J(n, c-1)$, where $J(n, m)$ is defined to be $\varphi_n(m)$. This class was previously investigated by Cenzer and Hinman [3] and Corollary 4.6 follows from their work.

Definition 4.1. For positive natural numbers $a > b$, let $\mathcal{Q}^b_a$ be the collection of functions f such that for all $n < \omega$,

- $f(n)$ is a finite set of natural numbers which is a subset of $\{0, \ldots, a-1\}$;
- $|f(n)| = b$;
- If $n \in \operatorname{dom} J$ then $J(n) \notin f(n)$.

Via standard coding, $\mathcal{Q}^b_a$ can be seen as a recursively bounded Π^0_1 subclass of ω^ω. $\mathcal{Q}^1_a$ is essentially the same as DNR_a.

Recall that a class $\mathcal{P} \subseteq \omega^\omega$ is *strongly reducible* (or *Medvedev reducible*) to another class $\mathcal{Q}$ (we write $\mathcal{P} \leqslant_s \mathcal{Q}$) if there is a Turing functional Ψ such that for all $f \in \mathcal{Q}$, $\Psi(f) \in \mathcal{P}$. If $\mathcal{Q}$ is a Π^0_1 class, then without loss of generality, we can assume that Ψ is total and so we get a truth-table reduction.

Lemma 4.2. *For all $a > b > 0$, $\mathcal{Q}^{b+1}_{a+1} \leqslant_s \mathcal{Q}^b_a$, uniformly in a and b.*

The uniformity means that an index for the reduction functional Ψ can be obtained effectively from a and b.

Proof. First, for all $n < \omega$ and $y < a$ we can compute an input $m_{n,y}$ such that:

- For all $x < a$ such that $x \neq y$, $J(m_{n,y})\downarrow = x \Leftrightarrow J(n)\downarrow = x$;
- $J(m_{n,y})\downarrow = y$ iff either $J(n)\downarrow = y$ or $J(n)\downarrow = a$.

Let $f \in \mathcal{Q}^b_a$ and let $n < \omega$. Note that for all $y < a$, if $n \in \operatorname{dom} J$ then $J(n) \notin f(m_{n,y})$ and so $J(n) \notin g(n) = \bigcup_{y<a} f(m_{n,y})$. Further, if there is some $y < a$ such that $y \in f(m_{n,y})$ then we also know that $J(n) \neq a$ and so $J(n) \notin g(n) \cup \{a\}$. Finally, if $|g(n)| = b$ then $f(m_{n,y})$ is in fact constant for all $y < a$ and so in that case $y \in f(m_{n,y})$ for all $y \in g(n)$. Thus we can define Ψ as follows:

$$\Psi(f,n) = \begin{cases} \text{some subset of } g(n) \text{ of size } b+1, & \text{if } |g(n)| > b; \\ g(n) \cup \{a\}, & \text{if } |g(n)| = b. \end{cases} \qquad \square$$

Corollary 4.3. *For all $a \geqslant 2$ and $b \geqslant 0$, $\mathcal{Q}^{b+1}_{a+b} \leqslant_s \mathrm{DNR}_a$, uniformly in a and b.*

Definition 4.4. For $a \geqslant 2$ and $c > 0$, let $\mathcal{P}^c_a$ be the collection of functions $f \in a^\omega$ such that for all n and all $x < c$, if $(n,x) \in \operatorname{dom} J$ then $J(n,x) \neq f(n)$.

Again $\mathcal{P}^1_a \equiv_s \mathrm{DNR}_a$.

Lemma 4.5. *For all $a > b > 0$ and $c \geqslant 1$, if $c(a-b) < a$ then $\mathcal{P}^c_a \leqslant_s \mathcal{Q}^b_a$, uniformly in a, b and c.*

Proof. Given $f \in \mathcal{Q}^b_a$ and $n < \omega$, for all $x < c$, if $J(n,x)\downarrow < a$ then

$$J(n,x) \in \bigcup_{x<c} \{0,1,\dots,a-1\} \smallsetminus f(n,x).$$

The set on the right has size at most $c(a-b)$ and so if $c(a-b) < a$ we can choose an $x < a$ not in that set and define $\Psi(f,n) = x$. $\square$

Corollary 4.6 (See Cenzer and Hinman [3])**.** *For all $a \geqslant 2$ and $c > 0$, $\mathcal{P}^c_{ca} \leqslant_s \mathrm{DNR}_a$, uniformly in a and c.*

Proof. Let $b = c(a-1) - a + 1$. Then $\mathcal{Q}^{b+1}_{a+b} \leqslant_s \mathrm{DNR}_a$ and

$$c((a+b) - (b+1)) = c(a-1) < a+b$$

so $\mathcal{P}^c_{c(a-1)+1} = \mathcal{P}^c_{a+b} \leqslant_s \mathcal{Q}^{b+1}_{a+b}$. Since $c > 0$ we have $c(a-1)+1 \leqslant ca$ and so $\mathcal{P}^c_{c(a-1)+1} \subseteq \mathcal{P}^c_{ca}$ and so $\mathcal{P}^c_{ca} \leqslant_s \mathcal{P}^c_{c(a-1)+1}$. All reductions are uniform. $\square$

We now use the classes $\mathcal{P}^c_{ca}$ to construct sequences of positive dimension. We begin lightly.

Proposition 4.7. *Let $a \geqslant 2$. For every $\varepsilon > 0$, every $f \in \mathrm{DNR}_a$ computes a real X of dimension $\dim(X) > 1 - \varepsilon$. This is uniform in a and ε.*

That each $f \in \mathrm{DNR}_a$ computes such reals is of course not new, since the Turing degree of a bounded DNR function is PA and so computes a random real. The extra information is the uniformity.

Proof. Fix $c > 1$. We work in the space $(ca)^\omega$. Let d be the universal left-r.e. supermartingale for this space; by scaling we may assume that $d(\langle\rangle) < 1$.

For every $\sigma \in (ca)^{<\omega}$ we compute a number $m_\sigma < \omega$ such that for all $x < c$, $J(m_\sigma, x)\downarrow = k$ if σk is the x^{th} immediate successor of σ in $(ca)^{<\omega}$ that is discovered (via some left-r.e. approximation for d) such that $d(\sigma k) \geqslant a^{|\sigma|+1}$.

The point is that if $d(\sigma) \leqslant a^{|\sigma|}$, then at most c immediate successors τ of σ can have $d(\tau) \geqslant a^{|\tau|}$ (by the supermartingale condition that the sum of $d(\tau)$ for all immediate successors τ of σ is at most $ca \cdot d(\sigma)$), thus all "heavy" extensions of σ are "captured" by J. We can thus use a function $g \in \mathcal{P}^c_{ca}$ to avoid all such extensions: given such g, inductively define $X \in (ca)^\omega$ by letting the $n+1^{\text{st}}$ bit of X be $g(m_{X\restriction n})$. Then by induction we prove that for all n, $d(X \restriction n) \leqslant a^n$.

Now let $s \geqslant 0$ and suppose that d s-succeeds on X, that is, $\left\langle d(X\restriction n)\mu^{ca}(X\restriction n)^{1-s}\right\rangle$ is unbounded. Now $\mu^{ca}(X\restriction n) = (ca)^{-n}$ and so

$$d(X\restriction n)\mu^{ca}(X\restriction n)^{1-s} \leqslant a^n(ca)^{-n(1-s)} = \left(c^{s-1}a^s\right)^n.$$

Thus if d s-succeeds on X then $c^{s-1}a^s > 1$, so $a^s > c^{1-s}$. Taking a logarithm on both sides, we get

$$\frac{s}{\log_a c} > 1 - s$$

so

$$s > 1 - \frac{1}{1+\log_a c}.$$

Thus $\dim^{ca}(X) \geqslant 1 - 1/(1+\log_a c)$. Now as c grows, $\log_a c \to \infty$ so given ε we can find a c such that the corresponding X has dimension $\dim^{ca}(X) > 1-\varepsilon$. Of course the constant function ac satisfies the regularity property of the last section, so X computes a $Y \in 2^\omega$ such that $\dim Y > 1 - \varepsilon$. □

Proposition 4.8. *For any $a \geqslant 2$, each $f \in \mathrm{DNR}_a$ computes a real X of dimension 1. This is uniform in a.*

Proof. We combine the constructions of reals of dimensions closer and closer to 1 into one construction. Let $h(n) = (n+1)a$; let d be the universal left-r.e. supermartingale for h^ω. Given $f \in \mathrm{DNR}_a$, for $n > 0$ obtain (uniformly) $g_n \in \mathcal{P}^n_{na}$. For $\sigma \in h^n$ find some $m_\sigma < \omega$ such that for $x < n+1$, $J(m_\sigma, x)\!\downarrow = k$ if σk is the x^{th} immediate successor of σ in $h^{<\omega}$ that is discovered (via some left-r.e. approximation for d) such that $d(\sigma k) \geqslant a^{|\sigma|+1}$. Again the point is that if $\sigma \in h^n$ and $d(\sigma) \leqslant a^{|\sigma|}$, then since $\sum d(\tau) \leqslant (n+1)a \cdot d(\sigma)$, where we sum over immediate successors τ of σ, there can be at most $n+1$ many such τ's such that $d(\tau) \geqslant a^{|\tau|}$. We can then define $X(n) = g_{n+1}(m_{X\restriction n})$ and inductively prove that $d(X\restriction n) \leqslant a^n$ for all n.

Now $\mu^h(X\restriction n) = a^{-n}/n!$ so for $s \geqslant 0$,

$$d(X\restriction n)\mu^h(X\restriction n)^{1-s} \le \frac{a^{sn}}{(n!)^{1-s}}.$$

If $s < 1$, then for almost all n we have $n! > a^{\frac{sn}{1-s}}$, so for almost all n we have

$$\frac{a^{sn}}{(n!)^{1-s}} < 1$$

and d cannot s-succeed on X. Thus $\dim^h(X) = 1$. Since h is dominated by 2^n, it satisfies the regularity condition and X computes a $Y \in 2^\omega$ of dimension 1. □

Finally we can paste together these constructions for all $a \geqslant 2$ and get the desired theorem.

Theorem 4.9. *There is a recursive, nondecreasing, unbounded function $j\colon \omega \to \omega \smallsetminus \{0,1\}$ such that every $f \in \mathrm{DNR}_j$ computes a real $X \in 2^\omega$ of dimension 1.*

Proof. Let $h(n) = (n+1)2^n$, and let d be the universal left-r.e. supermartingale for h^ω. For every $\sigma \in h^n$ find an input m_σ such that for all $x < 2^n$, $J(m_\sigma, x)\downarrow = k$ if σk is the x^{th} immediate successor of σ in $h^{<\omega}$ that is discovered such that $d(\sigma k) \geqslant (n+1)!$.

We know that for all $n \geqslant 1$, $\mathcal{P}^{2^n}_{h(n)} \leqslant \mathrm{DNR}_{n+1}$ uniformly in n, so there is an effective list of truth-table functionals Ψ_n such that for all $f \in \mathrm{DNR}_{n+1}$, $\Psi_n(f) \in \mathcal{P}^{2^n}_{h(n)}$. Let ψ_n be a recursive bound on the use function of Ψ_n. Let

$$m_n^* = 1 + \sup\{\langle m_\sigma, x\rangle \,:\, \sigma \in h^n \;\;\&\;\; x < 2^n\}$$

and let

$$u_n = \psi_n(m_n^*).$$

Thus for all n, if ρ is a sequence of length u_n that is a DNR_{n+1}-string (that is, $\rho \in (n+1)^{u_n}$ and for all $y < u_n$ in $\operatorname{dom} J$, $J(y) \neq \rho(y)$; equivalently, ρ is an initial segment of sequences in DNR_{n+1}) then $\Psi_n(\rho)$ is a $\mathcal{P}^{2^n}_{h(n)}$-string (an initial segment of a sequence in $\mathcal{P}^{2^n}_{h(n)}$) of length at least m_n^*. By increasing ψ_n we may assume that for all n, $u_n < u_{n+1}$. We define $j(k) = n+1$ iff $u_{n-1} \leqslant k < u_n$. So if $f \in \mathrm{DNR}_j$ then for all n, $f \upharpoonright u_n$ is a DNR_{n+1}-string and so gluing the reductions Ψ_n there is a $g \leqslant_{\mathrm{T}} f$ such that for all $n < \omega$ and all $\sigma \in h^n$,

(1) $g(m_\sigma) < h(|\sigma|)$;
(2) For all $x < 2^n$, if $(m_\sigma, x) \in \operatorname{dom} J$ then $g(m_\sigma) \neq J(m_\sigma, x)$.

We can now use g to construct $X \in h^\omega$ as in the last two constructions: $X(n) = g(m_{X\upharpoonright n})$. By induction on n we prove that $d(X \upharpoonright n) \leqslant n!$; again the point is that if $\sigma \in h^n$ and $d(\sigma) \leqslant n!$ then there are at most 2^n many immediate successors τ of σ such that $d(\tau) \geqslant (n+1)!$ and so they are all captured by J and avoided by g.

Finally we calculate dimension to show that $\dim^h(X) = 1$. We then note that h satisfies the regularity condition of Lemma 3.4 and so X computes a $Y \in 2^\omega$ of dimension 1.

Let $s < 1$ and let $\varepsilon = 1 - s > 0$. For any $\sigma \in h^n$,

$$\mu^h(\sigma) = \frac{1}{2^0 2^1 \cdots 2^{(n-1)} n!} = \frac{1}{2^{\binom{n}{2}} n!}.$$

Thus for all $n < \omega$,

$$d(X \upharpoonright n)\mu^h(X \upharpoonright n)^\varepsilon \leqslant \frac{(n!)^s}{2^{\varepsilon\binom{n}{2}}} \leqslant \frac{n!}{2^{\varepsilon\binom{n}{2}}},$$

which is bounded—indeed, it tends to 0, as $2^{\varepsilon\binom{n}{2}}$ grows faster than $2^{\delta n^2}$, for any $\delta < \varepsilon/2$, and $n!$ grows slower than $2^{n \log n}$. Thus d does not s-succeed on X. ☐

5. Computing no Martin-Löf random set

Let $j\colon \omega \to \omega \smallsetminus \{0,1\}$ be a recursive, nondecreasing, unbounded function. Our goal is to show that there is an $f \in \mathrm{DNR}_j$ that does not compute a Martin-Löf random set. To do this we force with *bushy trees*. This method originated in an unpublished preprint of Kumabe in which he proved that there is a DNR function of minimal Turing degree.[7] The preprint was circulated for several years before it was

[7] A set A has minimal Turing degree if every set computable from A is either computable or computes A; in other words, there are no Turing degrees between that of the computable sets and that of A.

rewritten and published by Kumabe and Lewis [9]. In the intervening time, Ambos-Spies, Kjos-Hanssen, Lempp and Slaman [1] had extracted ideas from Kumabe's version, specifically the machinery needed to construct a DNR function that does not compute a Martin-Löf random set. (Note that minimal degrees cannot compute Martin-Löf random sets.) Though the DNR function constructed by this method is computably bounded, the bound is not arbitrary. To build a DNR_j function, we also use an idea from Downey, Greenberg, Jockusch and Milans [5]. They showed that it is not possible to *uniformly* compute a Martin-Löf random set (or even a weakly 1-random set) from a DNR_3 function. Their method lets us contend with the possible slow growth of j.

We work entirely in $j^{<\omega}$.

Definition 5.1. A finite tree $T \subseteq j^{<\omega}$ is *n-bushy above σ* if every element of T is comparable with σ, and for every $\tau \in T$ that extends σ and is not a leaf of T, there are at least n immediate extensions of τ in T. (We say that σ is the *stem* of T.)

Bushiness is a tool that was first used in Prikry forcing to exploit complete ultrafilters. Namba forcing replaces ultrafilters with cardinality considerations in its use of bushiness. Kumabe's insight was that even in the absence of cardinality considerations when all sets considered are countable, and even despite the absence of countably complete ultrafilters on ω, a delicate use of bushiness can be used in the countable case to mimic the combinatorics of ultrafilters.

Definition 5.2. Let $\sigma \in j^{<\omega}$, and let $n < \omega$. A set $B \subseteq j^{<\omega}$ is called *n-big above σ* if there is a tree $T \subseteq j^{<\omega}$ that is n-bushy above σ, all of whose leaves lie in B.

We first make easy observations about bigness.

Lemma 5.3. *Let $\sigma \in j^{<\omega}$.*

(1) *Let $n < \omega$. If $B \subseteq j^{<\omega}$ is n-big above σ, $C \subseteq j^{<\omega}$ and $B \subseteq C$, then C is n-big above σ.*
(2) *Let $B \subseteq j^{<\omega}$. If $n < \omega$, B is n-big above σ, and $m \leqslant n$, then B is m-big above σ.*
(3) *Let $n < \omega$. For any $B \subseteq j^{<\omega}$ that is n-big above σ there is a finite $B' \subseteq B$ that is n-big above σ.*
(4) *For all $n < \omega$, $\{\sigma\}$ is n-big above σ.*

Proof. For (4), the path ending at σ is vacuously an n-bushy tree above σ. □

There are two basic lemmas we will use when working with bushy trees (these are essentially the sparse subset properties of Kumabe and Lewis [9]).

Lemma 5.4. *Let $B, C \subseteq j^{<\omega}$, let $n, m < \omega$, and let $\sigma \in j^{<\omega}$. If $B \cup C$ is $(n+m-1)$-big above σ, then either B is n-big above σ or C is m-big above σ.*

Proof. Let T be an $(n+m-1)$-bushy tree above σ with leaves in $B \cup C$. Working down from the leaves, we label each string in $\tau \in T$ such that $\tau \succeq \sigma$ with either a 'B' or a 'C'. Label a leaf in B with 'B', otherwise label it with 'C'. If all of the immediate extensions of τ have been labeled, then label it with 'B' if at least n of its extensions are labeled 'B'. Otherwise, at least m of its extensions are labeled 'C', so label τ with 'C' too. In this way, if σ is labeled with 'B', then there is an n-bushy tree above σ with leaves in B. This is the tree consisting of all prefixes of σ and all extensions of σ that are connected to it by a path labeled all 'B'. Similarly, if σ is labeled with 'C', then there is an m-bushy tree above σ with leaves in C. □

Lemma 5.5. *Let $B, C \subseteq j^{<\omega}$, let $n < \omega$ and let $\sigma \in j^{<\omega}$. Suppose that C is n-big above σ, and that B is not n-big above σ. Then there is a $\tau \in C$ extending σ such that B is not n-big above τ.*

Proof. By shrinking C to be the set of leaves of an n-bushy tree above σ, we may assume that C is finite and that every $\tau \in C$ extends σ. If the conclusion of the lemma fails, then for each $\tau \in C$ there is a tree T_τ that is n-bushy above τ, all of whose leaves lie in B. Consider $\widehat{T} = \bigcup_{\tau \in S} T_\tau$. This tree is n-bushy above σ and its leaves are in B, so B is n-big above σ. $\square$

We approximate a function $f \in \mathrm{DNR}_j$ that does not compute a random set using the method of forcing. The main idea is the following. As mentioned above, in [5] it is shown that for any $k > 2$, there is no uniform way of taking a DNR_k function and producing a random set. Jockusch [7] has shown however that every bounded DNR function computes a DNR_2 function and so computes a random set. This necessitates j being unbounded. The added complication in showing that $f \in \mathrm{DNR}_j$ does not compute a random set, via some computation procedure (Turing functional) Φ, say, is that Φ may not be total. However, as utilised in [1], Kumabe showed that bushiness can be used to force divergence, in a sense maintaining any Π^0_1 statement along the approximation. Below, we show how to modify the technique of [5] so that it can be utilized in the context of bushy trees.

Our forcing conditions are triples $(\sigma, B, n) \in j^{<\omega} \times \mathcal{P}(j^{<\omega}) \times (\omega \setminus \{0, 1\})$ such that:

(1) B is *open* in $j^{<\omega}$: if $\tau \in B$, then every extension of τ is in B;
(2) B is not n-big above σ; and
(3) $n \leq j(|\sigma|)$.

By Lemma 5.3, (1) and (2) together imply that $\sigma \notin B$. Intuitively, σ is a prefix of the DNR_j function that we are building, B is a set of bad nodes we need to avoid, and n establishes the sparseness of B. We say that (τ, C, m) *extends* (σ, B, n) (and write $(\tau, C, m) \leqslant (\sigma, B, n)$) if $\tau \succeq \sigma$ and $C \supseteq B$. (The third component does not contribute to the ordering and is really just present for bookkeeping.)

We let $\mathbb{P}_j$ denote the collection of the forcing conditions.

Given a filter $\mathcal{G} \subset \mathbb{P}_j$, let $f_{\mathcal{G}} = \bigcup_{(\sigma,B,n)\in\mathcal{G}} \sigma$. Since the conditions in $\mathcal{G}$ are all compatible, $f_{\mathcal{G}} \in j^{\leqslant\omega}$. We will show that if $\mathcal{G}$ is a sufficiently generic filter for $\mathbb{P}_j$, then $f_{\mathcal{G}}$ is in DNR_j and does not compute a random set.

Lemma 5.6. *Let $\mathcal{G} \subset \mathbb{P}_j$ be a filter. For all $(\sigma, B, n) \in \mathcal{G}$, no initial segment of $f_{\mathcal{G}}$ is in B.*

Proof. Suppose for contradiction that some $\tau \subseteq f_{\mathcal{G}}$ is in B. Let $(\rho, C, m) \in \mathcal{G}$ such that $\tau \preceq \rho$. By passing to a common extension, we may assume that $(\rho, C, m) \leqslant (\sigma, B, n)$. Hence $C \supseteq B$. Since $\tau \in B$, $\tau \preceq \rho$ and B is open in $j^{<\omega}$, we have $\rho \in B$, whence $\rho \in C$, contradicting our conclusion above. $\square$

5.1. **Totality.** For $m < \omega$, let

$$\mathcal{T}_m = \{(\sigma, B, n) \in \mathbb{P}_j : |\sigma| \geqslant m\}.$$

Lemma 5.7. *For all $m < \omega$, the set $\mathcal{T}_m$ is dense in $\mathbb{P}_j$. In fact, for all $(\sigma, B, n) \in \mathbb{P}_j$, for all $m \geqslant |\sigma|$, there is a τ extending σ of length m such that $(\tau, B, n) \in \mathbb{P}_j$.*

Proof. Let $(\sigma, B, n) \in \mathbb{P}_j$ such that $|\sigma| < m$. Let $C = j^m$. Since j is non-decreasing, C is $j(|\sigma|)$-big above σ. Since $n \leqslant j(|\sigma|)$ (property (3) in the definition of our forcing conditions), C is n-big above σ. By Lemma 5.5, there is a $\tau \in C$ extending σ such that $(\tau, B, n) \in \mathbb{P}_j$. We have $(\tau, B, n) \in \mathcal{T}_m$ and $(\tau, B, n) \leqslant (\sigma, B, n)$. ☐

It follows that if $\mathcal{G} \subset \mathbb{P}_j$ is sufficiently generic, then $f_{\mathcal{G}} \in j^\omega$.

5.2. **DNR.** Let B_{DNR} be the collection of non-DNR strings in $j^{<\omega}$:

$$B_{\mathrm{DNR}} = \{\sigma \in j^{<\omega} \,:\, (\exists e < |\sigma|)\, \sigma(e) = J(e)\}.$$

This set is open in $j^{<\omega}$.

Lemma 5.8. *B_{DNR} is not 2-big above $\langle\rangle$.*

Proof. Let T be a 2-bushy tree above $\langle\rangle$. We show that there is a leaf of T that is not in B_{DNR}. This is because if $\tau \in T \smallsetminus B_{\mathrm{DNR}}$ is not a leaf of T, there is at most one immediate extension of τ in B_{DNR}. Hence, some immediate extension of τ is also in $T \smallsetminus B_{\mathrm{DNR}}$. In this way, we can build a path through T that terminates outside of B_{DNR}. ☐

It follows that $(\langle\rangle, B_{\mathrm{DNR}}, 2) \in \mathbb{P}_j$. If $\mathcal{G} \subset \mathbb{P}_j$ is sufficiently generic and $(\langle\rangle, B_{\mathrm{DNR}}, 2) \in \mathcal{G}$, then $f_{\mathcal{G}} \in \mathrm{DNR}_j$.

5.3. **Not computing random sets.** Let $\Phi\colon j^\omega \to 2^\omega$ be a (partial) Turing functional. We want to show that if $\mathcal{G}$ is sufficiently generic, then $\Phi^{f_{\mathcal{G}}}$ is not Martin-Löf random. First, as is common in forcing arguments, including the bushy tree arguments of [9] and [1], we force totality or partiality of $\Phi^{f_{\mathcal{G}}}$. The idea here is the following: if for some x, the collection of strings $\tau \in j^{<\omega}$ that make $\Phi^\tau(x)$ converge is not big, then we can force $\Phi^{f_{\mathcal{G}}}$ to be partial. Otherwise, convergence occurs on big sets of strings, which puts us close to the situation in [5], where Φ is a truth-table functional; we then mimic the argument from that paper, in our bushy context.

For $N < \omega$ we let

$$\mathrm{Conv}_\Phi(N) = \{\tau \,:\, (\forall x < N)\, \Phi^\tau(x){\downarrow}\}$$

be the set of finite oracles that make $\Phi(x)$ defined on the first N numbers x. This set is open in $j^{<\omega}$. We let

$$\mathcal{D}_\Phi = \{(\sigma, B, n) \in \mathbb{P}_j \,:\, (\exists N)\ \mathrm{Conv}_\Phi(N) \subseteq B\}.$$

By Lemma 5.6, if $\mathcal{G} \cap \mathcal{D}_\Phi \neq \emptyset$, then $\Phi^{f_{\mathcal{G}}}$ is not total.

Lemma 5.9. *Suppose that $(\sigma, B, n) \in \mathbb{P}_j$ has no extension in $\mathcal{D}_\Phi$, and that $j(|\sigma|) \geqslant 4n$. Then for all $\varepsilon > 0$ there is an $N < \omega$ and a $C \subseteq \mathrm{Conv}_\Phi(N)$ that is n-big above σ such that*

$$|\{\Phi^\tau \restriction N \,:\, \tau \in C\}| \leqslant \varepsilon \cdot 2^N.$$

Proof. We define two sets:

- Let D be the set of $\tau \in j^{<\omega}$ such that for some N, $\mathrm{Conv}_\Phi(N)$ is not $3n$-big above τ;
- and let E be the set of $\tau \in j^{<\omega}$ such that D is n-big above τ.

For all $\tau \in j^{<\omega} \setminus E$, for all N, $\mathrm{Conv}_\Phi(N) \setminus E$ is $2n$-big above τ. To see this, first note that $D \subseteq E$, so if $\tau \notin E$, for all N, $\mathrm{Conv}_\Phi(N)$ is $3n$-big above τ. Let $N < \omega$. By Lemma 5.4, either E is n-big above τ, or $\mathrm{Conv}_\Phi(N) \setminus E$ is $2n$-big above τ. If E is n-big above τ, then by Lemma 5.5, D is n-big above τ, contradicting $\tau \notin E$.

We also claim that $\sigma \notin E$. Otherwise, by Lemma 5.5, there would be a $\tau \in D$ extending σ such that B is not n-big above τ. In that case, take $N < \omega$ such that $\mathrm{Conv}_\Phi(N)$ is not $3n$-big above τ. Lemma 5.4 would then imply that $B \cup \mathrm{Conv}_\Phi(N)$ is not $4n$-big above τ. By our assumption on σ, we have $4n \leqslant j(|\tau|)$. Hence $(\tau, B \cup \mathrm{Conv}_\Phi(N), 4n)$ would be an element of $\mathbb{P}_j$, an extension of (σ, B, n) in $\mathcal{D}_\Phi$.

We recursively define $N_m < \omega$ and $C_m \subseteq \mathrm{Conv}_\Phi(N_m) \setminus E$, a finite set that is n-big above σ such that

$$|\{\Phi^\tau \upharpoonright N_m \,:\, \tau \in C_m\}| \leqslant (3/4)^m \cdot 2^{N_m};$$

the lemma follows. Let $N_0 = 0$ and $C_0 = \{\sigma\}$. The inductive assumption holds because $\sigma \notin E$.

Given N_m and C_m, let $N_{m+1} = N_m + 2^{|C_m|} + 1$. For each $\tau \in C_m$, since $\tau \notin E$, $\mathrm{Conv}_\Phi(N_{m+1}) \setminus E$ is $2n$-big above τ.

For every $x \in [N_m, N_{m+1})$ and every $\rho \in \mathrm{Conv}_\Phi(N_{m+1}) \setminus E$, $\Phi^\rho(x)$ is defined and equals either 0 or 1. By Lemma 5.4, for each such x and $\tau \in C_m$ there is a $b(\tau, x) \in \{0,1\}$ such that

$$\{\rho \in \mathrm{Conv}_\Phi(N_{m+1}) \setminus E \,:\, \Phi^\rho(x) = b(\tau, x)\}$$

is n-big above τ.

Since $2^{|C_m|} < N_{m+1} - N_m$, there are distinct x_m and y_m in $[N_m, N_{m+1})$ such that for all $\tau \in C_m$, $b(\tau, x_m) = b(\tau, y_m)$. This is the main idea from [5], which allows us to limit the number of possible values of $\Phi^{f_\mathcal{G}} \upharpoonright N_{m+1}$: since $b(\tau, x_m) = b(\tau, y_m)$ we can force either either $\Phi^\rho(x_m) = 0$ or $\Phi^\rho(y_m) = 1$ above τ. This allows us to knock off one quarter of possible values for $\Phi^{f_\mathcal{G}} \upharpoonright [N_m, N_{m+1})$.

Let

$$A_{m+1} = \{\rho \in \mathrm{Conv}_\Phi(N_{m+1}) \setminus E \,:\, (\exists \tau \in C_m)\, \tau \preceq \rho \;\;\&\;\; [\Phi^\rho(x_m) = 0 \;\text{ or }\; \Phi^\rho(y_m) = 1]\}\,.$$

For all $\tau \in C_m$, A_{m+1} is n-big above τ, and so by Lemma 5.5, A_{m+1} is n-big above σ. We let C_{m+1} be a finite subset of A_{m+1} that is n-big above σ.

For all $\rho \in C_{m+1}$, $(\Phi^\rho(x_m), \Phi^\rho(y_m)) \neq (1,0)$. Hence

$$|\{\Phi^\rho \upharpoonright [N_m, N_{m+1}) \,:\, \rho \in C_{m+1}\}| \leqslant (3/4) \cdot 2^{N_{m+1} - N_m}.$$

We also have

$$\{\Phi^\rho \upharpoonright N_m \,:\, \rho \in C_{m+1}\} \subseteq \{\Phi^\tau \upharpoonright N_m \,:\, \tau \in C_m\}\,.$$

Hence

$$|\{\Phi^\rho \upharpoonright N_{m+1} \,:\, \rho \in C_{m+1}\}| \leqslant (3/4) 2^{N_{m+1} - N_m} \, |\{\Phi^\tau \upharpoonright N_m \,:\, \tau \in C_m\}| \leqslant (3/4)^{m+1} 2^{N_{m+1}},$$

as required. □

For $c < \omega$, let

$$\mathcal{E}_{\Phi,c} = \{(\sigma, B, n) \in \mathbb{P}_j \,:\, (\exists \nu \preceq \Phi^\sigma)\, K(\nu) \leqslant |\nu| - c\}\,.$$

Lemma 5.10. *For all $c < \omega$, $\mathcal{D}_\Phi \cup \mathcal{E}_{\Phi,c}$ is dense in $\mathbb{P}_j$.*

If $\mathcal{G} \cap \mathcal{E}_{\Phi,c} \neq \emptyset$, then $\Phi^{f_\mathcal{G}}$ is not Martin-Löf random by constant c. Hence if for all c, $\mathcal{G} \cap \mathcal{E}_{\Phi,c} \neq \emptyset$, then $\Phi^{f_\mathcal{G}}$ is not Martin-Löf random. This lemma is thus the execution of the plan described in the beginning of this subsection: we either force $\Phi^{f_\mathcal{G}}$ to be partial, if $\mathcal{G} \cap \mathcal{D}_\Phi$ is nonempty; or we force $\Phi^{f_\mathcal{G}}$ to be non-random.

Proof. Let $(\sigma, B, n) \in \mathbb{P}_j$. By Lemma 5.7, we may extend σ as long as we like, so as j is unbounded, we may assume, without loss of generality, that $4n \leqslant j(|\sigma|)$. Suppose that (σ, B, n) has no extension in $\mathcal{D}_\Phi$.

We define a KC (Kraft-Chaitin) set L. This is an r.e. set of pairs $(\sigma, k) \in 2^{<\omega} \times \omega$; the *weight* of such a set is the sum of 2^{-k}, as (σ, k) ranges over the elements of the set. The Kraft-Chaitin theorem (see [6]) states that if L is KC set whose weight is finite, then there is a constant c such that for all $(\sigma, k) \in L$, $K(\sigma) \leqslant k + c$.

For each $m > 0$, by Lemma 5.9, find an $N_m < \omega$ and a finite $C_m \subseteq \mathrm{Conv}_\Phi(N)$ that is n-big above σ such that for

$$S_m = \{\Phi^\tau \upharpoonright N_m \,:\, \tau \in C\}$$

we have $|S_m| \leqslant 2^{-2m} 2^{N_m}$. (The condition defining N_m and C_m can be verified effectively, and so such N_m and C_m can be found by an exhaustive search.)

Let

$$L_m = \{(\nu, N_m - m) \,:\, \nu \in S_m\}$$

and $L = \bigcup_{m>0} L_m$. Then L is indeed recursively enumerable. The weight of L_m is bounded by

$$|S_m| 2^{m-N_m} \leqslant 2^{-2m} 2^{N_m} 2^{m-N_m} = 2^{-m},$$

and so the weight of L is bounded by 1. Hence by the Kraft-Chaitin theorem, there is a constant k such that for all $m > 0$, for all $\nu \in S_m$, $K(\nu) \leqslant |\nu| - m + k$. Let $m = k + c$; so for all $\nu \in S_m$, $K(\nu) \leqslant |\nu| - c$. Since C_m is n-big above σ, by Lemma 5.5 there is a $\tau \in C_m$ such that $(\tau, B, n) \in \mathbb{P}_j$. This condition is an extension of (σ, B, n) in $\mathcal{E}_{\Phi,c}$. □

It follows that if $\mathcal{G} \subset \mathbb{P}_j$ is sufficiently generic, then $\Phi^{f_\mathcal{G}}$ is not Martin-Löf random. Putting everything together, the Baire category theorem ensures the existence of a sufficiently generic filter for $\mathbb{P}_j$, as we have only specified countably many dense sets we need to meet. We have thus proved the desired theorem:

Theorem 5.11. *If $j \colon \omega \to \omega \setminus \{0, 1\}$ is recursive, nondecreasing and unbounded, then there is an $f \in \mathrm{DNR}_j$ that does not compute a Martin-Löf random set.*

Remark 5.12. If $\mathcal{G}$ is sufficiently generic, then $f_\mathcal{G}$ has hyperimmune Turing degree. To see this, let $g_\mathcal{G}(m)$ be the least x such that $f_\mathcal{G}(x) \geqslant m$. Certainly $g_\mathcal{G} \leqslant_\mathrm{T} f_\mathcal{G}$. If h is any function, let $\mathcal{C}_h$ be the collection of conditions $(\tau, C, n) \in \mathbb{P}_j$ such that for some m, for all $x \leqslant h(m)$, $\tau(x) < m$. If $\mathcal{G} \cap \mathcal{C}_h \neq \emptyset$ then for some m, $g_\mathcal{G}(m) > h(m)$. The set $\mathcal{C}_h$ is dense in $\mathbb{P}_j$: let $(\sigma, B, n) \in \mathbb{P}_j$ and let $m \geqslant n$ be greater than any number in the range of σ. Note that $j^{<\omega} \cap n^{h(m)+1}$ is n-big above σ because $j(|\sigma|) \geq n$. So by Lemma 5.5, there is a $\tau \in n^{h(m)+1}$ extending σ such that $(\tau, B, n) \in \mathbb{P}_j$, and so (τ, B, n) is an extension of (σ, B, n) in $\mathcal{C}_h$. It follows that if $\mathcal{G}$ is sufficiently generic, then $g_\mathcal{G}$ is not dominated by any recursive function.

Since every hyperimmune degree computes a Kurtz random set, we cannot use $\mathbb{P}_j$ to show that for all j as above there is an $f \in \mathrm{DNR}_j$ that does not compute a Kurtz random set; this despite the fact that in [5] it is shown that from DNR_3

one cannot uniformly compute a Kurtz random set, not only a Martin-Löf random set. Furthermore, the minimal degree of Kumabe-Lewis can be made to be hyperimmune-free, and since every hyperimmune-free Kurtz random is Martin-Löf random, there is a recursively bounded DNR function that does not compute any Kurtz random set. We do not currently know if there is an unbounded j such that every $f \in \mathrm{DNR}_j$ computes a Kurtz random set. Modifying $\mathbb{P}_j$ to force with infinite bushy trees may shed light on this question.

Our description of $\mathbb{P}_j$ is maximal, in the sense that sets of any complexity are allowed for the second coordinate B of conditions (σ, B, n). This does not easily lend itself to an examination of how complicated the forcing notion is and how complicated it is to construct generic filters for $\mathbb{P}_j$. However, we can check that we only used a countable collection of conditions and that these conditions can be identified by $\mathbf{0}'$.

Again let $j\colon \omega \to \omega \setminus \{0,1\}$ be recursive, nondecreasing and unbounded. Let $\mathbb{P}_j^*$ be the collection of conditions $(\sigma, B, n) \in \mathbb{P}_j$ such that B is r.e.

Lemma 5.13. *$\mathbb{P}_j^*$ is recursive in $\mathbf{0}'$.*

Of course, we mean that the set of triples (σ, e, n) such that $(\sigma, W_e, n) \in \mathbb{P}_j^*$ is recursive in $\mathbf{0}'$. In the sequel we move seamlessly between r.e. sets and their r.e. indices.

Proof. In fact, $\mathbb{P}_j^*$ is co-r.e.: $(\sigma, B, n) \notin \mathbb{P}_j^*$ if and only if there is a finite n-bushy tree T above σ and an s such that the leaves of σ are in B_s. □

Note, also, that $(\langle\rangle, B_{\mathrm{DNR}}, 2) \in \mathbb{P}_j^*$.

Let

$$\mathfrak{D} = \{\mathcal{T}_m : m < \omega\} \cup \{\mathcal{D}_\Phi \cup \mathcal{E}_{\Phi,c} : \Phi\colon j^\omega \to 2^\omega, c < \omega\}.$$

(Recall that $\mathcal{T}_m$, defined in Section 5.1, is the collection of conditions that force that $f_{\mathcal{G}}$ is defined up to m.)

Lemma 5.14. *For all $\mathcal{C} \in \mathfrak{D}$, $\mathcal{C} \cap \mathbb{P}_j^*$ is dense in $\mathbb{P}_j^*$. Furthermore, for all $(\sigma, B, n) \in \mathbb{P}_j^*$, recursively in $\mathbf{0}'$, and uniformly in $\mathcal{C} \in \mathfrak{D}$ we can find an extension of (σ, B, n) in $\mathcal{C} \cap \mathbb{P}_j^*$.*

Proof. Let $\mathcal{C} \in \mathfrak{D}$, and let $(\sigma, B, n) \in \mathbb{P}_j^*$. If $\mathcal{C} = \mathcal{T}_m$, then by Lemma 5.7 there is a $\tau \succeq \sigma$ such that $(\tau, B, n) \in \mathcal{C}$. Of course $(\tau, B, n) \in \mathbb{P}_j^*$ and $\mathbf{0}'$ can recognise such a string τ since it only needs to check that $|\tau| \geqslant m$ and that $(\tau, B, n) \in \mathbb{P}_j^*$ (Lemma 5.13).

Suppose that $\mathcal{C} = \mathcal{D}_\Phi \cup \mathcal{E}_{\Phi,c}$ for some Turing functional $\Phi\colon j^\omega \to 2^\omega$ and $c < \omega$. We follow the proof of Lemma 5.9. We first pass to an extension in some $\mathcal{T}_m$ so we may assume that $4n \leqslant j(|\sigma|)$. The proof of Lemma 5.9 shows that either there is an $N < \omega$ and a $\tau \succeq \sigma$ such that $(\tau, B \cup \mathrm{Conv}_\Phi(N), 4n) \in \mathbb{P}_j$ (and so in $\mathcal{D}_\Phi$), or there is a $\tau \succeq \sigma$ such that $(\tau, B, n) \in \mathcal{E}_{\Phi,c}$. Note that both conditions are in $\mathbb{P}_j^*$, as $\mathrm{Conv}_\Phi(N)$ is c.e. Also, $\mathbf{0}'$ computes K, and so $\mathcal{E}_{\Phi,c} \cap \mathbb{P}_j^*$ is recursive in $\mathbf{0}'$ (uniformly in Φ and c). Similarly, an index for $B \cup \mathrm{Conv}_\Phi(N)$ can be effectively obtained from N and an index for B. Hence $\mathbf{0}'$ can search for an extension of (σ, B, n) in $\mathcal{C} \cap \mathbb{P}_j^*$ and eventually find one. □

Lemma 5.14 implies that $\mathbf{0}'$ can compute a sequence $\langle(\sigma_i, B_i, n_i)\rangle$ of conditions in $\mathbb{P}_j^*$ such that $(\sigma_0, B_0, n_0) = (\langle\rangle, B_{\mathrm{DNR}}, 2)$ and for all $\mathcal{C} \in \mathfrak{D}$ there is an $i < \omega$ such that $(\sigma_i, B_i, n_i) \in \mathcal{C}$. Let $\mathcal{G}$ be the upward closure of $\{(\sigma_i, B_i, n_i) : i < \omega\}$ in $\mathbb{P}_j$. Then $\mathcal{G}$ is a $\mathfrak{D}$-generic filter for $\mathbb{P}_j$, and so $f_{\mathcal{G}} \in \mathrm{DNR}_j$ does not compute a Martin-Löf random set. On the other hand, $f_{\mathcal{G}} = \bigcup_i \sigma_i$ is recursive in $\mathbf{0}'$. We have shown:

Theorem 5.15. *If $j\colon \omega \to \omega \smallsetminus \{0,1\}$ is recursive, nondecreasing and unbounded, then there is a Δ^0_2 function $f \in \mathrm{DNR}_j$ that does not compute a Martin-Löf random set.*

Now Theorem 4.9 implies:

Corollary 5.16. *There is a Δ^0_2 set of effective Hausdorff dimension 1 that does not compute a Martin-Löf random set.*

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