

# Totally $< \omega^\omega$ -computably enumerable degrees and $m$ -topped degrees<sup>\*</sup>

Rod Downey and Noam Greenberg

School of Mathematics, Statistics, and Computer Science  
Victoria University  
PO Box 600 Wellington  
New Zealand

## 1 Introduction

### 1.1 Degree classes

In this paper we will discuss recent work of the authors (Downey, Greenberg and Weber [8] and Downey and Greenberg [6, 7]) devoted to understanding some new naturally definable degree classes which capture the dynamics of various natural constructions arising from disparate areas of classical computability theory.

It is quite rare in computability theory to find a single class of degrees which capture precisely the underlying dynamics of a wide class of apparently similar constructions, demonstrating that they all give the same class of degrees. A good example of this phenomenon is work pioneered by Martin [22] who identified the high c.e. degrees as the ones arising from dense simple, maximal, hh-simple and other similar kinds of c.e. sets constructions. Another example would be the example of the promptly simple degrees by Ambos-Spies, Jockusch, Shore and Soare [2]. Another more recent example of current great interest is the class of  $K$ -trivial reals of Downey, Hirschfeldt, Nies and Stephan [5], and Nies [23, 24].

We remark that in each case the clarification of the relevant degree class has lead to significant advances in our basic understanding of the c.e. degrees. We believe the results we mention in the present paper fall into this category. Our results were inspired by another such example, the array computable degrees introduced by Downey, Jockusch and Stob [10, 11]. This class was introduced by those authors to explain a number of natural “multiple permitting” arguments in computability theory. The reader should recall that a degree  $\mathbf{a}$  is called array noncomputable iff for all functions  $f \leq_{wt} \emptyset'$  there is a function  $g$  computable from  $\mathbf{a}$  such that

$$\exists^\infty x (g(x) > f(x)).^1$$

---

<sup>\*</sup> Research supported by the Marsden Fund of New Zealand.

<sup>1</sup> Of course, this was not the original definition of array noncomputability, but this version from [11] captures the domination property of the notion in a way that shows the way that it weakens the notion of non-low<sub>2</sub>-ness, in that  $\mathbf{a}$  would be non-low<sub>2</sub> using the same definition, but replacing  $\leq_{wt}$  by  $\leq_T$ .

## 1.2 Totally $\omega$ -c.e. degrees

Our two new main classes are what we call the *totally  $\omega$ -c.e. degrees* and the *totally  $< \omega^\omega$ -c.e. degrees*. These classes turn out to be completely natural and relate to natural definability in the c.e. degrees as we will discuss below. We begin with the  $\omega$  case.

**Definition 1 (Downey, Greenberg, Weber [8]).** *We say that a c.e. degree  $\mathbf{a}$  is totally  $\omega$ -c.e. if for all functions  $g \leq_T \mathbf{a}$ ,  $g$  is  $\omega$ -c.e.. That is, there is a computable approximation  $g(x) = \lim_s g(x, s)$ , and a computable function  $h$ , such that for all  $x$ ,*

$$|\{s : g(x, s) \neq g(x, s + 1)\}| < h(x).$$

The reader should keep in mind that array computability is a uniform version of this notion where  $h$  can be chosen independent of  $g$ . This class captures a number of natural constructions in computability theory.

As an illustration, recall that a c.e. prefix-free set of strings  $A \in 2^{<\omega}$  presents a left c.e. real  $\alpha$  if  $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}$ , that is,  $\alpha$  is the measure of  $A$ . Now it is easy to use padding to show that every c.e. real has a presentation  $A$  which is computable (Downey [4]). On the other hand, bizarre things can happen. In [12], Downey and LaForte showed that there exists a noncomputable left c.e. real  $\alpha$ , all of whose c.e. presentations are computable. We have the following:

**Theorem 1 (Downey and Greenberg [6]).** *The following are equivalent.*

- (i)  $\mathbf{a}$  is not totally  $\omega$ -c.e.
- (ii)  $\mathbf{a}$  bounds a left c.e. real  $\alpha$  and a c.e. set  $B <_T \alpha$  such that if  $A$  presents  $\alpha$ , then  $A \leq_T B$ .

## 1.3 Natural definability

One of the really fascinating things is that this is all connected to *natural* definability issues within the computably enumerable Turing degrees. In terms of abstract results on definability, there has been significant success in recent years, culminating in Nies, Shore, Slaman [25], where the following is proven.

**Theorem 2 (Nies, Shore, Slaman [25]).** *Any relation on the c.e. degrees invariant under the double jump is definable in the c.e. degrees iff it is definable in first order arithmetic.*

The proof of Theorem 2 involves interpreting the standard model of arithmetic in the structure of the c.e. degrees without parameters, and a definable map from degrees to indices (in the model) which preserves the double jump. The beauty of this result is that it gives at one time a definition of a large class of relations on the c.e. degrees.

On the other hand, the result is somewhat unsatisfying in terms of seeking natural definitions of objects in computability theory as outlined in the paper

Shore [27]. Here we are thinking of results such as the following. (We refer the reader to Soare [28] for unexplained definitions below since they are mainly to provide background for the results of the current paper.)

**Theorem 3 (Ambos-Spies, Jockusch, Shore, and Soare [2]).** *A c.e. degree  $\mathbf{a}$  is promptly simple iff it is not cappable.*

**Theorem 4 (Downey and Lempp [13]).** *A c.e. degree  $\mathbf{a}$  is contiguous iff it is locally distributive, meaning that*

$$\begin{aligned} \forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b} (\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \\ \exists \mathbf{b}_1, \mathbf{b}_2 (\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \wedge \mathbf{b}_1 \leq \mathbf{a}_1 \wedge \mathbf{b}_2 \leq \mathbf{a}_2)) \end{aligned}$$

*holds in the c.e. degrees.*

**Theorem 5 (Ambos-Spies and Fejer [1]).** *A c.e. degree  $\mathbf{a}$  is contiguous iff it is not the top of the non-modular 5 element lattice in the c.e. degrees.*

**Theorem 6 (Downey and Shore [14]).** *A c.e. truth table degree is low<sub>2</sub> iff it has no minimal cover in the c.e. truth table degrees.*

At the present time, as articulated in Shore [27], there are very few such natural definability results.

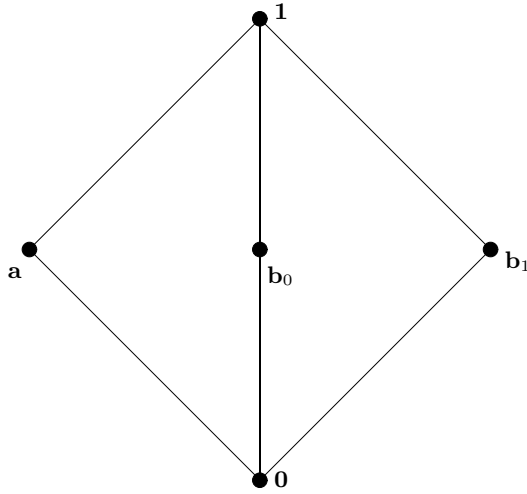
In [6–8], we gave some new natural definability results for the c.e. degrees. Moreover, these definability results are related to the central topic of lattice embeddings into the c.e. degrees as analyzed by, for instance, Lempp and Lerman [19], Lempp, Lerman and Solomon [20], and Lerman [21].

A central notion for lattice embeddings into the c.e. degrees is the notion of a *weak critical triple*. The reader should recall from Downey [3] and Weinstein [30] that three incomparable elements  $\mathbf{a}_0, \mathbf{a}_1$  and  $\mathbf{b}$  in an upper semilattice form a weak critical triple if  $\mathbf{a}_0 \cup \mathbf{b} = \mathbf{a}_1 \cup \mathbf{b}$  and there is no  $\mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_1$  with  $\mathbf{a}_0 \leq \mathbf{b} \cup \mathbf{c}$ . This notion captures the need for “continuous tracing” which is used in an embedding of the lattice  $M_5$  into the c.e. degrees (first embedded by Lachlan [17]).<sup>2</sup>

The necessity of the “continuous tracing” process was demonstrated by Downey [3] and Weinstein [30] who showed that there are initial segments of the c.e. degrees where no lattice with a (weak) critical triple can be embedded. It was also noted in Downey [3] that the embedding of (weak) critical triples seemed to be tied up with multiple permitting in a way that was similar to non-low<sub>2</sub>-ness. Indeed this intuition was verified by Downey and Shore [15] where it is shown that if  $\mathbf{a}$  is non-low<sub>2</sub> then  $\mathbf{a}$  bounds a copy of  $M_5$ .

The notion of non-low<sub>2</sub>-ness seemed too strong to capture the class of degrees which bound  $M_5$ ’s but it was felt that something like that should suffice. On the other hand, Walk [29] constructed an array noncomputable c.e. degree bounding no weak critical triple, and hence it was already known that array non-computability was not enough for such embeddings. We proved the following definitive result:

<sup>2</sup> We recall that a lattice is not join semidistributive (also called principally indecomposable) iff it contains a copy of  $M_5$  iff it contains a weak critical triple.



**Fig. 1.** The lattice  $M_5$

**Theorem 7 (Downey, Greenberg and Weber [8]).** *A degree  $\mathbf{a}$  is totally  $\omega$ -c.e. iff it does not bound a weak critical triple in the c.e. degrees. Hence, the notion of being totally  $\omega$ -c.e. is naturally definable in the c.e. degrees.*

Theorem 7 also allowed for the solution of certain technical problems from the literature.

**Corollary 1 (Downey, Greenberg and Weber [8]).** *The low degrees and the superlow degrees are not elementarily equivalent.*

*Proof.* As Schaeffer [26] and Walk [29] observe, all superlow degrees are array computable, and hence totally  $\omega$ -c.e. Thus we cannot put a copy of  $M_5$  below one. On the other hand there are indeed low copies of  $M_5$ .

**Corollary 2 (Downey, Greenberg and Weber [8]).** *There are c.e. degrees that are totally  $\omega$ -c.e. and not array computable.*

*Proof.* Walk [29] constructed an array noncomputable degree  $\mathbf{a}$  below which there was no weak critical triple. Such a degree must be totally  $\omega$ -c.e.

The class of totally  $\omega$ -c.e. degrees also captures other constructions.

**Theorem 8 (Downey, Greenberg and Weber [8]).** *A c.e. degree  $\mathbf{a}$  is totally  $\omega$ -c.e. iff there are c.e. sets  $A$ ,  $B$  and  $C$  of degree  $\leq_T \mathbf{a}$ , such that*

- (i)  $A \equiv_T B$
- (ii)  $A \not\leq_T C$
- (iii) For all  $D \leq_{\text{wtt}} A, B$ ,  $D \leq_{\text{wtt}} C$ .

## 1.4 Totally $< \omega^\omega$ -c.e. degrees

The class of totally  $< \omega^\omega$ -c.e. degrees also arises quite naturally. Recall that if  $b$  is an ordinal notation in Kleene's  $\mathcal{O}$ , then a  $\Delta_2^0$  function  $g$  is  $b$ -c.e. if there is a computable approximation  $g(x, s)$  for  $g$  such that the number of changes in the guessed value is bounded by some decreasing sequence of notations below  $b$ ; that is, there is a function  $o(x, s)$  such that for every  $x$  and  $s$ ,  $o(x, s) <_{\mathcal{O}} b$ ,  $o(x, s+1) \leq_{\mathcal{O}} o(x, s)$  and if  $g(x, s+1) \neq g(x, s)$  then  $o(x, s+1) <_{\mathcal{O}} o(x, s)$ . The definition of the class of totally  $< \omega^\omega$ -c.e. degrees involves *strong notations*, being notations for ordinals in Kleene's sense, except that we ask that below the given notation, Cantor normal form can be effectively computed. Exact formalization of this notion is straightforward for the ordinals below  $\epsilon_0$ ; such notations are computably unique, and so the corresponding class of functions is invariant under the chosen strong notation for a given ordinal; we thus call a function  $\alpha$ -c.e. if it is  $b$ -c.e. for some (all) strong notations  $b$  for  $\alpha$ . To make this definition explicit, we note how the lower levels correspond to functions that are given as increasing limits. Observe the following:

- A function  $g$  is  $\omega$ -c.e. iff there is a computable approximation  $g(x, s)$  for  $g$  such that the number of changes in the guess for  $g(x)$  is given in advance, in a computable fashion.
- A function  $g$  is  $\omega \cdot 2$ -c.e. iff there is a computable approximation  $g(x, s)$  for  $g$  such that the number of changes  $n(x)$  in the guess for  $g(x)$  has a computable approximation that changes at most once.
- Similarly, a function is  $\omega \cdot n$ -c.e. iff we may change our mind at most  $n - 1$  times about the number of possible changes.
- A function is  $\omega^2$ -c.e. iff it has some computable approximation such that the number of changes  $n(x)$  is  $\omega$ -c.e., that is, the number of times we change our mind about  $n(x)$  is computably bounded.
- Similarly, a function is  $\omega^{n+1}$ -c.e. iff it has a computable approximation for which the number  $n(x)$  of changes in the guess for  $g(x)$  is  $\omega^n$ -c.e. (So for example,  $g$  is  $\omega^3$ -c.e. iff it has an approximation where there is a computable bound on the number of times we may change our mind about the number of times we may change our mind about the number of changes of our guess for  $g(x)$ .)

A degree  $\mathbf{a}$  is totally  $< \omega^\omega$ -c.e. if every  $g \leq_T \mathbf{a}$  is  $\omega^n$ -c.e. for some  $n$ . In [6], Downey and Greenberg introduced this notion and showed that the collection of totally  $< \omega^\omega$ -c.e. degrees is naturally definable:

**Theorem 9 (Downey and Greenberg [6]).** *A c.e. degree is totally  $< \omega^\omega$ -c.e. iff it does not bound a copy of  $M_5$ .*

Again, Downey and Greenberg showed that a number of other constructions gave rise to the same class.

In the present paper, we will try to lead the reader to understanding how this class arises by showing how the class relates to the class of  $m$ -topped degrees of Downey and Jockusch [9]. Whilst we cannot get the exact classification, the analysis is revealing, as we see in section 2.

## 2 $m$ -topped degrees

Recall that a c.e. degrees  $\mathbf{a}$  is called  $m$ -topped if it contains a c.e. set  $A$  such that for all c.e.  $W \leq_T A$ ,  $W \leq_m A$ . Of course  $\mathbf{0}'$  is  $m$ -topped as it contains the halting set, but there exist incomplete  $m$ -topped degrees (Downey and Jockusch [9]); by index-set considerations, all of these are  $\text{low}_2$ .

We look at the Downey-Jockusch construction to try to understand what is needed to make it work. We must meet the requirements

$$R_e : \Phi_e^A = W_e \rightarrow W_e \leq_m A.$$

Additionally there will be a Friedberg strategy in the background making  $A$  noncomputable and some other one making sure that  $A$  is not complete.

To meet  $R_e$  we will have nodes  $\tau = \tau(e)$  on a priority tree devoted to measuring

$$\ell(\tau, s) = \max \{ x : \Phi_e^A \upharpoonright x = W_e \upharpoonright x \ [s] \}.$$

The idea is crude and simple. For a given  $z$ , at the first suitable  $\tau$ -expansionary stages  $s_z$  where  $\ell(\tau, s_z) > z$ , if  $z$  is not yet in  $W_{e, s_z}$ , we will take a fresh number  $y > s_z$  and define  $f(\tau, z) = y$ . The promise is that if  $z$  enters  $W_e$  after stage  $s_z$ , then we will put  $f(\tau, z)$  into  $A$ . Notice that  $A$  is controlling  $W_e$  and hence such a situation won't occur unless we change  $A_{s_z} \upharpoonright \varphi_e(z, s_z)$ .

Now suppose that we are trying to carry out this construction below a given degree  $\mathbf{b}$  represented by a c.e. set  $B$ . We look at this in the single requirement scenario. The action would occur as follows.

At some stage  $s_0$  we would initiate something by enumerating some number  $p$  into  $A_{s_0}$ . By that stage,  $\tau$  will have already defined  $f(\tau, z)$  for all  $z \leq n$  for some  $n$ . By the next  $\tau$ -stage we see,  $s_1$ , perhaps some  $z_1 \leq n$  entered  $W_{e, s_1}$ , causing us to now enumerate  $f(\tau, z_1)$  into  $A_{s_1}$ . In turn, this number might be below the use  $\varphi_e(z_1, s_1)$  of other  $z_i$ 's at stage  $s_1$ , and hence this process could snowball so that it re-occurs many times before all pending coding actions are finished. It will finish since we won't define new  $f(\tau, z')$  until we have a  $\tau$ -expansionary stage  $s$  where there are no pending coding actions to be done.

The point is that *each* enumeration of some  $f(\tau, z_i)$  really needs some  $B$ -permission. Thus the sequence we have began at stage  $s_0$  could actually need a sequence of more or less  $s_0$  many  $B$ -permissions to be achieved.

Indeed, things are even worse when many requirements are considered. For example, if we consider two  $\tau$ 's, say  $\tau_1, \tau_2$ , each building their own  $f(\tau_i, z)$ 's, then assuming that  $\tau_2$  has weaker priority than  $\tau_1$ ,  $\tau_1$  could recover many times before we see any  $\tau_2$ -expansionary stages. At each  $\tau_1$  expansionary stage, we would fulfill its obligation to enumerate  $f(\tau_1, z)$  into  $A$ . Now,  $\tau_1$  cannot know if  $\tau_2$  will ever recover, so that before we did any enumeration of numbers like  $f(\tau_2, z')$  we might have defined many *new*  $f(\tau_1, \hat{z})$  where  $\hat{z} > s_0$ . Now the pending action at  $\tau_2$  of enumerating some  $f(\tau_2, z')$  into  $A$  will likely cause new changes in  $W_{e_1}$  and hence yet further enumeration into  $A$  for the sake of  $\tau_1$ . This process could repeat again more or less  $s_0$  many times.

In summary, one  $R_e$  would seem to require  $f(j)$  many permissions for some computable function  $f$ , for attack number  $j$ , and two requirements would seem to need  $\sum_{i \leq f(j)} g(i, j)$  many permissions for some computable  $g(i, j)$ . Thus in some relatively natural way this construction would seem to need at least “ $\omega^\omega$  many permissions” to be carried out.

Now the construction of an  $m$ -topped degree *also* seems to need more in that once we have begun some action we must finish it. There is no way to allow us to “lose” on some  $f(\tau, z)$ . In the embeddin of  $M_5$ , we can think of the  $R_e$ ’s above as belonging to some gate of a pinball construction measuring some minimal pair requirement

$$\Phi^{A_k} = \Phi^{A_l} = h \rightarrow h \leq Q.$$

Here we will assume that the reader is familiar with the construction of a 1-3-1 using a pinball machine as in Downey and Shore [15].

The analogous action is that we have some sequence of numbers  $x, T(x, s), T^2(x, s) \dots$  that have been realized and are traveling down the machine towards the pockets. This can’t pass the gate if they are a  $k, l$  sequence. For example,  $k = 1, l = 2$  and  $x$  is targeted for  $A_2$ ,  $T(x, s)$  for  $A_1$ ,  $T^2(x, s)$  for  $A_2$  etc. They must pass one at a time. We put the last one  $p = T^n(x, s)$  (targeted, say, for  $A_1$ ) out at the gate, and give it a trace  $T^{n+1}(x, s)$  targeted for  $A_3$  and so forth as a 1-3 sequence at the gate. When the gate opens at the next expansionary stage, we would drop the balls to the first unoccupied 1-3 gate and repeat.

To achieve this, we would need to repeat this  $n$  many times one per ball at gate  $G_e$  alone. For two gates, the situation is like the above, each ball from the first gate itself generates a long 1-3 entourage, and hence needs  $g(i, j)$  many permissions for each descendent.

The critical difference between the situation for the  $M_5$  lattice and the  $m$ -topped degree, is that if some set of balls is stuck forever at some gate then that causes no real grief. However, in the  $m$ -topped case, the failure of us fulfilling some  $f(\tau, z)$  commitment is fatal. The issue seems to concern lowness; this is why we can’t get a true reversal for the class of  $m$ -topped degrees:

**Theorem 10.** *There is a degree that is not totally  $< \omega^\omega$ -c.e., but does not bound any noncomputable  $m$ -topped degree.*

*Proof.* Downey and Jockusch [9] proved that no noncomputable  $m$ -topped c.e. degree is low. On the other hand, even Lachlan’s original construction can be shown to produce a low degree that is the top of an embedding of  $M_5$ . By [7] mentioned above, such a degree cannot be totally  $< \omega^\omega$ -c.e. Of course, the low degrees form an initial segment of the c.e. degrees.

But in the present paper we will prove that the analysis above works in one direction:

**Theorem 11.** *No totally  $< \omega^\omega$ -c.e. degree bounds a noncomputable,  $m$ -topped degree.*

On the other hand, it is possible to carry out the construction of an  $m$ -topped degree at a relatively low level. The reason for the interest in the next result is that the  $m$ -topped construction was a natural candidate for a construction that needed the “full power” of non- $\text{low}_2$  permitting. The reason for this is that Downey and Shore [14] proved that a c.e. degree  $\mathbf{a}$  is  $\text{low}_2$  iff it is bounded by an incomplete  $m$ -topped degree. The following theorem shows that Theorem 11 is optimal in the hierarchy of totally  $< \alpha$ -c.e. degrees; the next level above  $< \omega^\omega$  is the class of totally  $\omega^\omega$ -c.e. degrees, the degrees that only compute  $\omega^\omega$ -c.e. functions.

**Theorem 12.** *There exists a  $m$ -topped c.e. degree that is totally  $\omega^\omega$ -c.e.*

### 3 Proof of Theorem 11

We sketch the proof of Theorem 11. As the class of totally  $< \omega^\omega$ -c.e. degrees is closed downwards, it is sufficient to show that no totally  $< \omega^\omega$ -c.e. degree is  $m$ -topped.

For a simplified start, suppose first that the given degree  $\mathbf{a}$  is totally  $\omega$ -c.e.; let  $A \in \mathbf{a}$  be a candidate for having a maximal c.e.  $m$ -degree inside  $\mathbf{a}$ . Our goal is to build a c.e. set  $V \leq_T A$  via  $\Psi^A = V$  such that we meet the requirement

$$M_e : V \not\leq_m A \text{ via } \varphi_e.$$

That is, for some  $x$ , we would have  $x \in V$  iff  $\varphi_e(x) \notin A$  (or  $\varphi_e$  is not total.)

As with all these constructions, we will build an auxiliary function  $\Delta^A = g$ . Now suppose that we knew in advance the witness to the fact that  $g$  is  $\omega$ -c.e. That is we had in advance a computable function  $f$  so that  $g$  is  $\omega$ -c.e. via some approximation  $h(x, s)$ , where the number of changes is bounded by  $f(x)$ .

We could then proceed as follows.

We choose some “permitting number”  $n$ , and a finite set  $X$  of size greater than  $f(n)$ , consisting of fresh potential diagonalisation witnesses. We wait until every  $x \in X$  is *realised*, that is,  $\varphi_e(x) \downarrow$ ; we then let  $u = \max\{\varphi_e(x) : x \in X\}$ , and define  $\psi^A(x) = u$  for all  $x \in X$  and  $\delta^A(n) = u$  as well. [Strictly speaking, we need to define both  $\delta(n)$  and  $\psi(x)$  before the realisation, because the totality of  $\Delta^A$  and  $\Psi^A$  cannot depend on  $\varphi_e$  being total; for this we use simple permitting.]

We are then ready to *attack* with some  $x \in X$  (it doesn’t matter which): we enumerate  $x$  into  $V$ . If we are unlucky, then at a later stage  $t_0$  the attack *fails*:  $\varphi_e(x)$  enters  $A$ . The way we defined  $\delta(n)$  allows us to extract a price from  $A$  in exchange for this failure: since  $\delta(n) \geq \varphi_e(x)$ , we know that the failure of the attack allows us to redefine  $\Delta^A(n)$  with new value that hasn’t been guessed before as some  $h(n, s)$ . At a later stage  $s_0$  we get a new guess  $h(n, s_0) = \Delta^A(n)[t_0]$ , and then we can attack with another  $x' \in X$ . Now note that we do not want to attack again before we get a change in  $h(n, s)$ , because the limit we have on the number of changes is used to show that some attack is eventually successful. Note that the reduction  $\Psi^A = V$  is not damaged here: we defined  $\psi(x') \geq \varphi_e(x)$ , and so

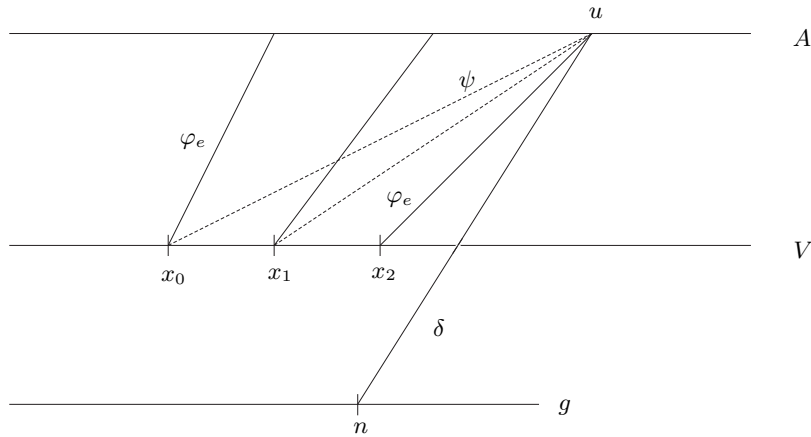


at stage  $t_0$ ,  $\psi(x') \uparrow$ ; at that stage we can define  $\Psi(x') = 1$  with anticipation of stage  $s_0$ .

This plan succeeds because:

- (i)  $h$  is indeed a correct approximation for  $g$ , and so every failure is followed by another attack; every stage  $t_0$  as above is followed by some  $s_0$ . It follows also that the definition  $\Psi(x') = 1$  made at stage  $t_0$  is correct, and so indeed  $V \leq_T A$ .
- (ii) Some attack must succeed, because  $h(n, s)$  cannot change more than  $f(n)$  many times. Hence  $M_e$  is met.

In the real construction, we don't know  $h$  and  $f$  in advance, so we list out all possibilities. We would use one  $V$  for each possible pair  $f, h$ . The point here is that *if*  $f$  is the real  $f$ , and  $h$  is the *real* witness for  $g$ , then the  $V_{h,f}$  built for  $h$  and  $f$  will have  $V_{h,f} \leq_T A$ . *But* the key point is that  $g$  is total nevertheless - we never leave  $\delta(n)$  undefined.



**Fig. 2.** The  $\omega$ -c.e. construction

Now consider the case that  $A$  is totally  $\omega^2$ -c.e. To continue the analogy above with the gates of a pinball machine construction, we see that the  $\omega$ -c.e. case corresponds exactly to the failure of a single node  $\tau$  to meet  $R_e$ . A set that has totally  $\omega^2$ -c.e. degree may be able to win on all single gates alone, but fails to meet the combined requirements  $R_{e_0}$  and  $R_{e_1}$ . The analogy suggests that we need to build *two* sets  $V_{e_0}$  and  $V_{e_1}$  and succeed on one of them. We now describe how the necessities of the construction lead us to require these two sets.

Again we construct an auxiliary function  $g = \Delta^A$  and guess at an approximation  $h(x, s)$  for  $g$ , which is accompanied by a bounding function  $o(x, s)$  which gives, for every  $x$ , a non-increasing sequence of ordinals below  $\omega^2$ ; every change

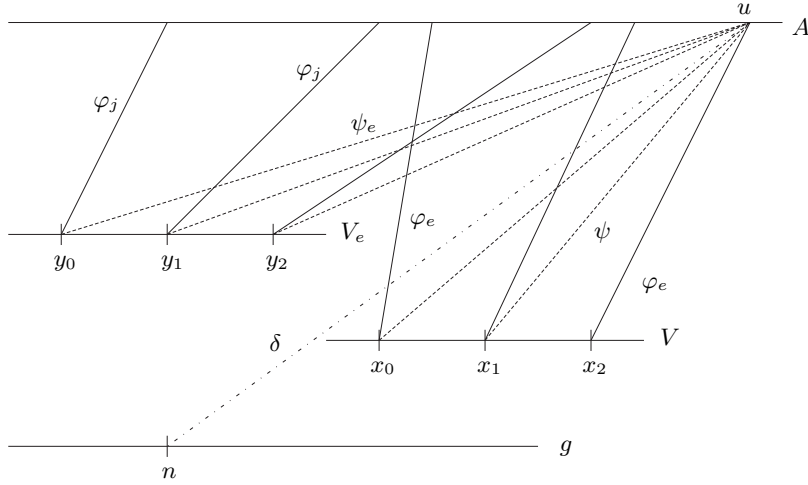
in the guess for  $g(x)$  is matched by a change in  $o(x, s)$ . Consider the following naïve plan. At first, we get  $o(x, 0) = \omega \cdot k_0 + k_1$ ; we set up a collection of potential witnesses  $X$  of size greater than  $k_1$  and repeatedly attack with these witnesses as before. Since each attack is related to a decrease of  $o(x, s)$ , before we run out, we have a new value  $\omega \cdot l_0 + l_1$  where  $l_0 < k_0$ , so we'd like to have at least  $l_1$  more new witnesses to throw into  $X$ . Of course this cannot work as it would translate to an argument that no totally  $b$ -c.e. degree (for any notation  $b$ ) can be  $m$ -topped, contradicting Theorem 12. [For the naïve plan to “work” we need to work with some  $b$ , otherwise we may repeat the process infinitely many times and  $\Delta^A(n)$  would be undefined.] The problem is one of timing: before  $X$  runs out, we can appoint  $l_1$  new witnesses; but at some point we need to wait for them to get realised. This has to happen before some stage as  $s_0$  above where we can redefine  $\delta(n)$  to be at least  $\varphi_e(x)$  for all *new* witnesses  $x$ . This means that before, say, we make the last attack with an old witness, we first need to wait for realisation of the new witnesses. But if  $\varphi_e$  is not total then this may never happen, and this spoils the reduction  $\Psi^A = V$ . Here the nonuniformity, familiar from these constructions, creeps in: the solution is to build a backup set  $V_e$  that is only needed if we fail to meet  $M_e$  for the main set  $V$ . All work regarding  $V_e$  (including the definition of a reduction  $\Psi_e^A = V_e$ ) is based on the assumption that  $\varphi_e$  is total. Thus, when we run out of old witnesses, we appoint new witnesses, wait for realisation, and then attack with a  $V_e$ -witness; when this attack fails,  $\delta(n)$  frees up and we can redefine it as larger as is required to start working with the new witnesses.

Here's the construction for the  $\omega^2$ -case, assuming that we guessed  $h$  and  $o$  correctly. We show how to meet the requirement

$M_{e,j}$ : Either  $M_e$  holds, or  $V_e$  is not 1-1 reducible to  $A$  via  $\varphi_j$ .

The algorithm is as follows:

1. Appoint a permitting number  $n$ . Let  $o(n, 0) = \omega \cdot k_0 + k_1$ . Appoint a set of witnesses  $Y$ , targeted for  $V_e$ , of size greater than  $k_0$ . Wait for realisation, i.e. for  $\varphi_j(y) \downarrow$  for all  $y \in Y$ .
2. Let  $u_Y = \max\{\varphi_j(y) : y \in Y\}$ ; let  $\psi_e(y) = u_Y$  for all  $y \in Y$  and let  $\delta(n) = u_Y$  as well. Appoint a set of witnesses  $X$ , targeted for  $V$ , of size greater than  $k_1$ . Wait for realisation, i.e. for  $\varphi_e(x) \downarrow$  for all  $x \in X$ . [In the meanwhile we can define  $\psi(x) = u_Y$  for all  $x \in X$ .]
3. Attack with some  $y \in Y$ : enumerate it into  $V_e$ . Wait for the failure of the attack, i.e. for  $\varphi_j(y)$  to enter  $A$ .
4. Let  $u_X = \max\{\varphi_e(x) : x \in X\}$  and  $u = \max\{u_X, u_Y\}$ . Redefine  $\delta(n) = u$  and  $\psi(x) = u$  for  $x \in X$  and  $\psi_e(y) = u$  for  $y \in Y$ . However, reserve one  $x \in X$  for attack, and wait for a new guess  $h(n, s) = \Delta^A(n)$ .
5. Attack with  $x$ : enumerate it into  $V$ . Wait for the failure of this attack, i.e. for  $\varphi_e(x)$  to enter  $A$ ; repeat as in the  $\omega$ -case, until  $X$  runs out.
6. Upon the failure of the attack of the last  $x \in X$ , we have a new number  $l_1$  as above; we appoint a new  $X$  with more than  $l_1$  many fresh witnesses; we let  $\delta(n) = u_Y = \psi(x)$  for  $x \in X$ . We wait for realisation of all  $x \in X$ .



**Fig. 3.** The  $\omega^2$ -c.e. construction

7. We then attack with another  $y \in Y$ ; repeat as in step (3) and onwards.

For  $\omega^n$ , we need  $n$  sets  $V, V_{e_0}, V_{e_0, e_1}, \dots$ , nested by layers of nonuniformity; the idea is the same. In the more complex case of  $\Delta^B = g$  being only  $< \omega^\omega$ -c.e., we must guess for which  $n$  it is  $\omega^n$ -c.e., and the witnesses for this. Again this is typical.

## 4 Proof of Theorem 12

In this last section we sketch the proof of Theorem 12. In fact this comes from analyzing the complexity of the natural construction of Downey and Jockusch [9], carefully controlling where numbers are assigned as follows.

We enumerate a set  $A$ . We meet the following requirements:

$P_e$ :  $\varphi_e$  is not the characteristic function of  $A$ .

$R_e$ : If  $\Phi_e(A) = W_e$  then  $W_e \leq_1 A$ .

$Q_e$ : If  $\Psi_e(A)$  is total then it is  $\omega^\omega$ -c.e.

Here  $\langle \Psi_e \rangle$  is a listing of all Turing functionals, and  $\langle \varphi_e \rangle$  is a listing of all partial computable functions.  $\langle \langle \Phi_e, W_e \rangle \rangle$  is a listing of all pairs of Turing functionals and c.e. sets. Note that the fact that  $A$  is totally  $\omega^\omega$ -c.e. guarantees its incompleteness; in fact, it guarantees that it is  $\text{low}_2$ . Recall that the key is that (in the setting of  $R_e$ )  $\Phi_e(A)$  controls  $W_e$  in the sense that if at some stage  $t$  we have  $\Phi_e(A, x) = 0 = W_e(x) [t]$  and  $s > t$ , then if  $A$  did not change below the use  $\phi_e(A, x)[t]$  between  $t$  and  $s$  then  $x \notin W_e [s]$ .

The construction is done on a tree of strategies, with every level working for a single requirement. The tree of strategies is  $2^{<\omega}$  (and as usual, the priority

ordering is the lexicographic one); we identify 0 with the infinite outcome for a node working for  $R_e$  or  $Q_e$ , and with the positive satisfaction for  $P_e$ . Recall that a node  $\sigma$  that works for  $R_e$  builds a recursive function  $f_\sigma$  that attempts to be a one-one reduction of  $\Phi_e(A)$  to  $A$ .

At stage  $s$ , we construct (by induction) the path of accessible nodes; for an accessible node  $\sigma$ , we describe  $\sigma$ 's action and which successor (if any) is next accessible.

**$\sigma$  works for  $P_e$ :** If  $\sigma$  has no follower, appoint a fresh follower  $x$ ; let  $\sigma \smallfrown 1$  be accessible. If  $\sigma$  has a follower  $x$  then there are three cases:

1. If we already have  $x \in A$  (so  $P_e$  is satisfied) we let  $\sigma \smallfrown 0$  be accessible.
2. If it is not the case that  $\varphi_e(x) \downarrow = 0$ , then let  $\sigma \smallfrown 1$  be accessible.
3. If  $\varphi_e(x) \downarrow = 0$ , but  $x$  has not yet been enumerated into  $A$ , then we enumerate  $x$  into  $A$  and let  $\sigma \smallfrown 0$  be accessible.

**$\sigma$  works for  $R_e$ :** First, we need to correct potential inconsistencies between  $W_e = \Phi_e(A)$  and  $f_\sigma^{-1}A$ . For every  $x$  such that  $f_\sigma(x)$  is defined,  $x \in W_e[s]$  and  $f_\sigma(x) \notin A[s]$ , enumerate  $f_\sigma(x)$  into  $A$ .

If some numbers were enumerated into  $A$ , then we end the stage; we initialise all nodes that lie to the right of  $\sigma \smallfrown 0$ , but not nodes that extend  $\sigma \smallfrown 0$ .

Assuming that we did not end the stage, we define  $\ell(\sigma)[s]$  be the length of agreement between  $\Phi_e(A)$  and  $W_e$ . Let  $t < s$  be the last stage at which  $\sigma \smallfrown 0$  was accessible ( $t = 0$  if no such stage exists); if  $\ell(\sigma)[s] > t$  then let  $\sigma \smallfrown 0$  be accessible, otherwise let  $\sigma \smallfrown 1$  be accessible. In the first case, for every  $x < \ell(\sigma)[s]$  for which  $f_\sigma(x)$  is not yet defined, define it (with fresh value).

**$\sigma$  works for  $Q_e$ :** Let  $t < s$  be the last stage at which  $\sigma \smallfrown 0$  was accessible. If  $\text{dom } \Psi_e(A)[s] > t$  then let  $\sigma \smallfrown 0$  be accessible; otherwise let  $\sigma \smallfrown 1$  be accessible.

If the stage was not halted by the time we got to a node of length  $s$ , we end the stage and initialise all nodes that are weaker than the last accessible node. [This means that all followers are cancelled, and all functions are restarted from scratch.]

**Verification.** The existence of a true path is standard. On the true path, each node is eventually never initialised. The point here is that if  $\sigma$  works for  $R_e$  and  $\sigma \smallfrown 1$  is on the true path then only finitely many values  $f_\sigma(x)$  are defined, so after some stage,  $\sigma$  does not halt the stage (and initialise  $\sigma \smallfrown 1$ ) because it enumerates some  $f_\sigma(x)$  into  $A$ . It follows that every  $P_e$  requirement is met. It is also easy to see that each  $R_e$  requirement is met: if  $\sigma$  on the true path works for  $R_e$  (and the hypothesis  $\Phi_e(A) = W_e$  holds), then for every  $x$ ,  $f_\sigma(x)$  is eventually defined, and enumerated into  $A$  iff  $x$  enters  $W_e$ ; this is because  $\sigma \smallfrown 0$  is accessible infinitely many times.

It thus remains to show that each  $Q_e$  is met; fix  $e < \omega$ , assume that  $Z = \Psi_e(A)$  is total, and let  $\sigma$  be the node on the true path that works for  $Q_e$ ; we know that the next node on the true path must be  $\sigma \smallfrown 0$ . Let  $r^*$  be a stage after which  $\sigma$  is never initialised.

Let  $d < \omega$ ; we describe how to approximate  $Z(d)$  in an  $\omega^\omega$ -c.e. fashion. The approximation itself is simple: at a stage  $s > r^*$  at which  $\sigma \smallfrown 0$  is accessible and  $d < \text{dom } \Psi_e(A)[s]$ , we guess that  $Z(d) = \Psi_e(A, d)[s]$ . The point is of course to find the ordinal bound on the number of possible injuries to these computations. Of course such a computation can only be injured by nodes that are compatible with  $\sigma \smallfrown 0$ .

Recall that the key to this construction (as is in Fejer's branching degree construction, Slaman's density and other constructions) is the establishment of natural barriers and the preservation of the sets below these barriers by a concert of all agents involved. Let  $s_0$  be a stage at which  $\sigma \smallfrown 0$  is accessible and such that  $d < \text{dom } \Psi_e(A)[s_0]$ . Suppose, for example, that at stage  $s_0$ , there is only one node  $\tau$  compatible with  $\sigma \smallfrown 0$  which works for some  $R_{e'}$  and has any value  $f_\sigma(x)$  defined (say  $\sigma \smallfrown 0 \subseteq \tau$ ) and that there is only one node  $\rho \supseteq \tau \smallfrown 0$  that works for some  $P_{e''}$  and has a follower  $y$  defined at  $s_0$ . Until the computation  $\Psi_e(A, d)[s_0]$  is injured (possibly at stage  $s_0$ , but possibly later), all new values  $f_\tau(x)$  and followers  $y$  appointed for any node are greater than the use  $\psi_e(d)[s_0]$ , and so the injury has to result from action by either  $\tau$  or  $\rho$ . To begin with, some such injuries can happen by  $\tau$  enumerating values  $f_\tau(x)$  into  $A$ ; the number of such injuries is bounded by  $s_0$ . After each such injury, nodes to the right of  $\tau \smallfrown 0$  are initialised, and nodes extending  $\tau$  are not accessible, so the next injury still must come from  $\tau$  or  $\rho$ . Eventually, new values  $f_\tau(x)$  are defined at a stage  $s_1$  at which  $\tau \smallfrown 0$  is accessible. The barrier mechanism now comes into place: these values are defined only for  $x < \ell(\tau)[s_1]$ , and the only node that has a number smaller than the  $\Phi_{e'}$ -use is  $\rho$ . By  $\Phi_{e'}(A)$ 's controlling of  $W_{e'}$ , no  $x < \ell(\tau)[s_1]$  will enter  $W_{e'}$  (and no  $f_\tau(x)$  will enter  $A$ ) unless  $\rho$  acts at some  $s_2 \geq s_1$ . At that stage some new cascading by  $\tau$  may begin, yielding at most  $s_2$ -many new injuries for  $\Psi_e(A, d)$ . Thus the approximation for  $Z(d)$  is  $\omega \cdot 2$ -c.e. If there are further  $P$ -nodes  $\rho$  then we get  $\omega \cdot n$ -c.e.

However, if we have two  $R$ -nodes  $\tau_0$  and  $\tau_1$  (say  $\tau_0 \subset \tau_1$ ), then the effect is multiplicative (in a reverse fashion). After each time  $\tau_1$  enumerates a number into  $A$ , the total  $\tau_0, \tau_1$ -equilibrium is damaged and a barrage of new numbers can be enumerated into  $A$  by  $\tau_0$ . The result (together with several  $P$ -nodes weaker than  $\tau_1$ ) is an  $\omega^2 \cdot n$ -c.e. approximation. As  $d$  increases, more and more nodes  $\tau$  have values  $f_\tau(x)$  defined when  $\Psi_e(A, d)$  is first encountered, which means that we get  $\omega^k$ -c.e. approximations where  $k \rightarrow \infty$ . Overall, we get an  $\omega^\omega$ -approximation for  $Z$ .

## References

1. Ambos-Spies, K, and P. Fejer, *Embeddings of  $N_5$  and the contiguous degrees*, Annals of Pure and Applied Logic , 112 , (2001), 151-188
2. Ambos-Spies K., C. Jockusch, R. Shore, and R. Soare, *An algebraic decomposition of recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees*, Trans. Amer. Math. Soc., Vol. 281 (1984), 109-128.
3. R. Downey, Lattice nonembeddings and initial segments of the recursively enumerable degrees, Annals Pure and applied Logic, **49** (1990) 97-119.

4. Downey, R., *Some Computability-Theoretical Aspects of Reals and Randomness*, in *The Notre Dame Lectures* (P. Cholak, ed) *Lecture Notes in Logic* Vol. 18, Association for Symbolic Logic, 2005, 97-148.
5. Downey, R., D. Hirschfeldt, A. Nies, and F. Stephan, *Trivial reals*, extended abstract in *Computability and Complexity in Analysis* Malaga, (Electronic Notes in Theoretical Computer Science, and proceedings, edited by Brattka, Schröder, Weihrauch, FernUniversität, 294-6/2002, 37-55), July, 2002. Final version appears in *Proceedings of the 7th and 8th Asian Logic Conferences*, (Downey, R., Ding Decheng, Tung Shi Ping, Qiu Yu Hui, Mariko Yasuugi, and Guohua Wu (eds)), World Scientific, 2003, viii+471 pages. 2003, 103-131.
6. Downey, R. and N. Greenberg, *Totally  $\omega$  computably enumerable degrees II : Left c.e. reals*, in preparation.
7. Downey, R. and N. Greenberg *Totally  $\omega^\omega$ -computably enumerable degrees I: lattice embeddings*, in preparation.
8. Downey, R. and N. Greenberg and R. Weber, *Totally  $\omega$  computably enumerable degrees I: bounding critical triples* submitted.
9. Downey, R., and C. Jockusch, *T-degrees, jump classes and strong reducibilities*, *Trans. Amer. Math. Soc.*, **301** (1987) 103-136.
10. Downey, R., C. Jockusch, and M. Stob, *Array nonrecursive sets and multiple permitting arguments*, in *Recursion Theory Week* (Ambos-Spies, Muller, Sacks, eds.) *Lecture Notes in Mathematics* 1432, Springer-Verlag, Heidelberg, 1990, 141-174.
11. Downey, R., C. Jockusch, and M. Stob, *Array nonrecursive degrees and genericity*, in *Computability, Enumerability, Unsolvability* (Cooper, Slaman, Wainer, eds.), London Mathematical Society Lecture Notes Series 224, Cambridge University Press (1996), 93-105.
12. Downey, R. and G. LaForte, *Presentations of computably enumerable reals*, *Theoretical Computer Science*, Vol. 284 (2002), 539-555.
13. Downey, R. and S. Lempp, *Contiguity and distributivity in the enumerable degrees*, *Journal of Symbolic Logic*, Vol. 62 (1997), 1215-1240. (Corrigendum in *ibid* Vol 67 (2002) 1579-1580.)
14. Downey, R., and R. Shore, Degree theoretical definitions of low<sub>2</sub> recursively enumerable sets, *J. Symbolic Logic*, **60** (1995) 727-756.
15. Downey, R., and R. Shore, *Lattice embeddings below a non-low<sub>2</sub> recursively enumerable degree*, *Israel Journal of Mathematics*, **94** (1996), 221-246.
16. Ishmukhametov, S., *Weak recursive degrees and a problem of Spector*, in *Recursion Theory and Complexity*, (ed. M. Arslanov and S. Lempp), de Gruyter, (Berlin, 1999), 81-88.
17. Lachlan, A, Embedding nondistributive lattices in the recursively enumerable degrees, in *Conference in Mathematical Logic, London 1970* (ed. W. Hodges) *Lecture notes in mathematics*, **255**, Springer-Verlag, New York (1972) 149-177.
18. A Lachlan and R. Soare, Not every finite lattice is embeddable in the recursively enumerable degrees, *Advances in Math*, **37** (1980) 78-82.
19. S. Lempp, and M. Lerman, *A finite lattice without critical triple that cannot be embedded into the computably enumerable Turing degrees*, *Annals of Pure and Applied Logic*, Vol. 87 (1997), 167-185.
20. S. Lempp, M. Lerman, and D. Solomon, *Embedding finite lattices into the computably enumerable degrees - a status survey*, to appear.
21. M. Lerman, The embedding problem for the recursively enumerable degrees, in *Recursion Theory* (ed. A. Nerode and R. Shore), American Math. Soc., Providence, R. I., U. S. A., (1995) 13-20.

22. Martin, D., *Classes of recursively enumerable sets and degrees of unsolvability*, Z. Math. Logik Grundlag. Math. 12 (1966) 295-310.
23. Nies, A., *Reals which compute little*, to appear, *Proceedings of CL 2002*.
24. Nies, A., *Lowness properties and randomness*, Advances in Mathematics 197, 1 (2005), 274- 305..
25. Nies, A., R. Shore and T. Slaman, *Interpretability and definability in the recursively enumerable degrees*, Proc. Lon. Math. Soc. (3) 77 (1998), 241-291
26. Schaeffer, B., *Dynamic notions of genericity and array noncomputability*, Ann. Pure Appl. Logic, Vol. 95(1-3) (1998), 37-69.
27. Shore, R., *Natural definability in degree structures*, in *Computability Theory and Its Applications: Current Trends and Open Problems*, P. Cholak, S. Lempp, M. Lerman and R. A. Shore eds., Contemporary Mathematics, AMS, Providence RI, 2000, 255-272.
28. Soare, R., *Recursively enumerable sets and degrees* (Springer, Berlin, 1987).
29. Walk, S., *Towards a definition of the array computable degrees*, Ph. D. Thesis, University of Notre Dame, 1999.
30. S. Weinstein, Ph. D. Thesis, University of California, Berkeley (1988).